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Uniform $\{C_h, S(C_h)\}$-Factorizations of $K_n - I$ for Even h

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Abstract: Let H be a connected subgraph of a graph G. An H-factor of G is a spanning subgraph of G whose components are isomorphic to H. Given a set $\mathcal{H}$ of mutually non-isomorphic graphs, a uniform $\mathcal{H}$-factorization of G is a partition of the edges of G into H-factors for some $H \in \mathcal{H}$. In this article, we give a complete solution to the existence problem of uniform $\mathcal{H}$-factorizations of $K_n - I$ (the graph obtained by removing a 1-factor from the complete graph K_n) for $\mathcal{H} = \{C_h, S(C_h)\}$, where C_h is a cycle of length an even integer $h \geq 4$ and $S(C_h)$ is the graph consisting of the cycle C_h with a pendant edge attached to each vertex.

Keywords: Graph decompostion; factor; uniform factorization

MSC: 05B30

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1. Introduction

Let $V(G)$ and $E(G)$ denote the vertex set and the edge set, respectively, of a graph G. Let K_n be the complete graph on n vertices, C_h be a cycle of length h (briefly, h-cycle) and $S(C_h)$ be an h-sun, i.e., the graph consisting of the cycle C_h with a pendant edge attached to each vertex. For missing notions and terms that are not explicitly defined in this paper, we point the reader to [1] and its online updates. If $\mathcal{H}$ is a set of mutually non-isomorphic connected graphs, an *$\mathcal{H}$-decomposition* of a graph G is a partition of $E(G)$ into subgraph (*blocks*) that are isomorphic to some element of $\mathcal{H}$. An *$\mathcal{H}$-factor* of G is a spanning subgraph of G, i.e., a subgraph of G with the same vertex set as G, the connected components of which are isomorphic to some element of $\mathcal{H}$. An *$\mathcal{H}$-factorization* of G is an $\mathcal{H}$-decomposition of G whose set of blocks admits a partition into $\mathcal{H}$-factors. An $\mathcal{H}$-factorization of G is also known as a *resolvable* $\mathcal{H}$-decomposition of G and an $\mathcal{H}$-factor of G can be called a *parallel class* of G. When $\mathcal{H} = \{H\}$, then we simply write H-factor and H-factorization. An $\mathcal{H}$-factorization of a graph G is said to be *uniform* if each factor is an H-factor for some $H \in \mathcal{H}$, sometimes referred to as a *uniformly resolvable* $\mathcal{H}$-decomposition of G. A K_2-factorization of G is better known as a 1-*factorization* of G and its factors are said to be 1-*factors*; a 1-factor of K_n is a set of $\frac{n}{2}$ mutually vertex disjoint edges of K_n, and a 1-factorization of K_n exists if and only if n is even [2]. When $\mathcal{H}$ is a set of cycles, then an *$\mathcal{H}$-factorization* of G is known as a 2-*factorization* of G and its factors are said 2-*factors*. A 2-factorization of K_n whose cycles all have the same length h, i.e., a C_h-factorization of K_n, exists if and only if $3 \leq h \leq n$; n and h are odd; and $n \equiv 0 \pmod h$ [3]. A decomposition of a graph G of order n into Hamilton cycles (i.e., cycles of length n) is trivially a C_n-factorization where each cycles is a factor.

In the context of graph factorizations, more precisely cycle factorizations, the *Oberwolfach* and *Hamilton–Waterloo problems are the most famous. The first was first posed in 1967 by G. Ringel and asked whether it were possible to seat n mathematicians at m round tables at $(n-1)/2$ dinners so that each mathematician sat next to everyone else exactly once. This problem can be formalized as graph factorizations as follows. If $p_1, p_2, \ldots, p_m$ denotes the sizes of the m round tables, then a solution to the Oberwolfach Problem is a factorization of K_n, where each factor* *has m components that are isomorphic to the cycles of length $p_1, p_2, \ldots, p_m$, $\sum_{i=1}^{m} p_i = n$. It is well known that such a factorization can exist only if n is odd. For even n, it is common to decompose $K_n - I$, which is the graph obtained by removing a 1-factor from K_n. The* unifrom Oberwolfach Problem*, i.e., the case when all cycles of a factor have the same length, has been completely solved by Alspach and Häggkvist* [4] and Alspach, Schellenberg, Stinson and Wagner [3]. A variation of the Oberwolfach Problem is the Hamilton–Waterloo Problem, which requires that the dinners take place at two different venues. In this case,the factors of K_n (when n is odd) or $K_n - I$ (when n is even) can be either one of two types: more specifically, factors having s components that are isomorphic to cycles of length $p_1, p_2, \ldots, p_s$ or factors having t components that are isomorphic to cycles of length $q_1, q_2, \ldots, q_t$ (clearly, $\sum_{i=1}^{s} p_i = \sum_{i=1}^{t} q_i = n$). If the tables in one venue sit p mathematicians and those ones in the other venue sit q mathematicians, then we speak of the *uniform Hamilton–Waterloo Problem*, which asks for a decomposition of K_n or $K_n - I$ into C_p-factors and C_q-factors. For the Hamilton-Waterloo and the non-uniform Oberwolfach problems, many partial results are known, but a complete solution is far from being achieved.

Existence problems for $\mathcal{H}$-factorizations of K_n or $K_n - I$ have been investigated and many results have been obtained, especially in the uniform case. For instance, when $\mathcal{H}$ contains two complete graphs of order $k \leq 5$ [5–8], when $\mathcal{H}$ contains two or three paths of order $2 \leq k \leq 4$ [9,10], for $\mathcal{H} = \{K_2, S(C_k)\}$ [11], for $\mathcal{H} = \{K_2, K_{1,3}\}$ [12,13], for $\mathcal{H} = \{K_2, K_{1,4}\}$ [14], for $\mathcal{H} = \{C_{2k}, K_{1,2k}\}$ [15], for $\mathcal{H} = \{C_{2k}, P_{2k+1}\}$ [16].

This article fits in the context of a series of papers where authors investigate the existence of uniform $\mathcal{H}$-factorizations of K_n or $K_n - I$ in the case that $\mathcal{H}$ contains a cycle (see [15,16]). Here, we are interested in the case when $\mathcal{H} = \{C_h, S(C_h)\}$, where h is an even integer greater or equal to 4. For brevity, a uniform $\{H_1, H_2, \ldots H_l\}$-factorization of G with r_i H_i-factors, $i = 1, 2, \ldots l$, is denoted by $\mathsf{URD}(G; H_1^{r_1}, H_2^{r_2}, \ldots, H_l^{r_l})$; when $G = K_n$, we simply write $\mathsf{URD}(n; H_1^{r_1}, H_2^{r_2}, \ldots, H_l^{r_l})$. It is known that no $\mathsf{URD}(n; C_h^0, S(C_h)^{r_2})$ exists [11]; no $\mathsf{URD}(n; C_h^{r_1}, S(C_h)^0)$ exists because n must be odd and divisible by h; a $\mathsf{URD}(K_n - I; C_h^{r_1}, S(C_h)^0)$ exists if and only if $n \equiv 0 \pmod 2$, and h divides n [17]. When n is even, no $\mathsf{URD}(n; C_h^{r_1}, S(C_h)^{r_2})$ exists because the resolvability implies $2(r_1 + r_2) = n - 1$. Therefore, we investigate the existence of uniform $\{C_h, S(C_h)\}$-factorizations of $K_n - I$. A $\{C_h, S(C_h)\}$-factorization of $K_n - I$ is denoted by $\mathsf{URD}^*(n; C_h^{r_1}, S(C_h)^{r_2})$ (with the obvious meaning of r_i, $i = 1, 2$). Moreover, since h and $2h$ must divide n, we assume $n \equiv 0 \pmod{2h}$.

The goal of this paper is to characterize the existence of $\mathsf{URD}^*(n; C_h^{r_1}, S(C_h)^{r_2})$ in the previously defined cases; namely, $h \equiv 0 \pmod 2$, $h \geq 4$, and $n \equiv 0 \pmod{2h}$. Our main result, given in the last section, proves that such decompositions exist if and only if the pair (r_1, r_2) belongs to the set $I(n) = \{(\frac{n-2}{2} - 2x, 2x),\ \ x = 0, 1, \ldots, \frac{n}{4} - 1\}$.

To obtain our main result, we first describe in Section 2 two contructions, GDD and filling, which allow the obtaining of decompositions for more general cases from small decompositions. In Section 3 we prove the necessary conditions for the existence of $\mathsf{URD}^*(n; C_h^{r_1}, S(C_h)^{r_2})$, and we factorize certain graphs of small order. By means of these results, in Section 4 we prove that the necessary conditions are also sufficient for the existence of a $\mathsf{URD}^*(n; C_h^{r_1}, S(C_h)^{r_2})$. (Section 4, essentially, contains the statement of our main theorem by combining the partial results of the previous sections).

2. General Constructions

In what follows, $K_{u(g)}$ denotes the complete multipartite graph with u partite sets of size g. An $\mathcal{H}$-decomposition of $K_{u(g)}$ is known as a *group divisible decomposition* (briefly, $\mathcal{H}$-GDD) of type g^u; the partite sets are called *groups*. An $\mathcal{H}$-decomposition of K_n can be regarded as an $\mathcal{H}$-GDD of type 1^n. When $\mathcal{H} = \{H\}$ we simply write H-GDD. In what follows, a (uniformly) resolvable $\mathcal{H}$-GDD is denoted by $\mathcal{H}$-(U)RGDD. More specifically, an $\{H_1, H_2, \ldots, H_l\}$-URGDD with r_i H_i-factors is denoted by $\text{URGDD}(H_1^{r_1}, H_2^{r_2}, \ldots, H_l^{r_l})$. When $\mathcal{H} = \{H\}$, for uniformity and convenience, we will also use the notation $\text{RGDD}(H^r)$;

in this case, by using the double counting technique (counting in two ways the size of block set) it is not hard to see that the r number of H-factors is

$$r = \frac{g(u-1)|V(H)|}{2|E(H)|}.$$

Let G be a given graph. For any positive integer t, $G_{(t)}$ denotes the graph with vertex set $V(G) \times \mathbb{Z}_t$ and edge set $\{\{x_i, y_j\} : \{x, y\} \in E(G), i, j \in \mathbb{Z}_t\}$, where the subscript notation a_i denotes the pair (a, i). We say that the graph $G_{(t)}$ is obtained from G by expanding each vertex t times. When $G = K_m$, the graph $G_{(t)}$ is the complete equipartite graph

$$K_{\underbrace{t, t, \ldots, t}_{m \text{ times}}}$$

with m partite sets of size t and denoted by $K_{m(t)}$. Analogously, $C_{m(t)}$ denotes the graph $G_{(t)}$ where G is an m-cycle.

Given two pairs (r_1, r_2) and (r'_1, r'_2) of non-negative integers, define $(r_1, r_2) + (r'_1, r'_2) = (r_1 + r'_1, r_2 + r'_2)$. Given two sets I and I' of pairs of non-negative integers and a positive integer α, then $I + I'$ denotes the set

$$\{(r_1, r_2) + (r'_1, r'_2) : (r_1, r_2) \in I, (r'_1, r'_2) \in I'\}.$$

Moreover, we denote $\alpha * I$ the set that has elements that are all pairs of non-negative integers obtained by adding any α elements of I (repetitions of elements of I are allowed).

To obtain our main result we combine the GDD construction (see Theorem 1) and the filling construction (see Theorem 2). The first construction gives RGDDs with appropriate parameters, while the second one allows us to obtain uniformly resolvable decompositions of $K_n - I$ by filling with suitable URDs groups of the RGDDs given by the GDD Construction, which can be obtained from the more general construction described in [15].

Theorem 1 (GDD Construction)**.** *Let t be a positive integer and $\mathcal{G}$ be a Γ-RGDD of type g^u, where Γ is a set of graphs of order at least 2. If for any fixed factor F_i, $i = 1, 2, \ldots, \alpha$, of $\mathcal{G}$ there exists a URD$(B_{(t)}; H_1^{\bar{r}_1}, H_2^{\bar{r}_2})$, for every $B \in F_i$ and for every $(\bar{r}_1, \bar{r}_2) \in I_i$, then so does a URGDD$(H_1^{r_1}, H_2^{r_2})$ of type $(gt)^u$, for every $(r_1, r_2) \in I_1 + I_2 + \cdots + I_\alpha$.*

The filling construction is a minor variation of the namesake construction in [15].

Theorem 2 (Filling Construction)**.** *If there exists a URGDD$(C_h^{r_1}, S(C_h)^{r_2})$ of type g^u, for every $(r_1, r_2) \in I$, and a URD$^*(g; C_h^{r'_1}, S(C_h)^{r'_2})$, for every $(r'_1, r'_2) \in I'$, then so does a URD$^*(ug; C_h^{\bar{r}_1}, S(C_h)^{\bar{r}_2})$, for every $(\bar{r}_1, \bar{r}_2) \in I' + I$.*

Proof. For any two fixed pairs $(r_1, r_2) \in I$ and $(r'_1, r'_2) \in I'$, it is sufficient to start from a URGDD$(C_h^{r_1}, S(C_h)^{r_2})$ of type g^u and on each group G_i, $i = 1, 2, \ldots, u$, place a copy of a URD$^*(g; C_h^{r'_1}, S(C_h)^{r'_2})$. The resulting decomposition is a URD$^*(gu; C_h^{r_1+r'_1}, S(C_h)^{r_2+r'_2})$ whose factors are those of the starter URGDD plus the factors originating from the union of u factors, each of them taken in a different group. □

We conclude this section by quoting the following results for a later use.

Lemma 1 ([18])**.**

(i) *For all odd $n \geq 3$, K_n can be decomposed into $\frac{n-1}{2}$ Hamilton cycles; for all even $n \geq 2$, K_n can be decomposed into $\frac{n-2}{2}$ Hamilton cycles and a 1-factor.*

(ii) *For all even $m \geq 2$, $K_{2(m)}$ can be decomposed into $\frac{m}{2}$ Hamilton cycles; for all odd $m \geq 1$, $K_{2(m)}$ can be decomposed into $\frac{m-1}{2}$ Hamilton cycles and a 1-factor.*

Lemma 2 ([19]). *Let $m \geq 3$ and $u \geq 2$. There exists a C_m-RGDD of type g^u if and only if $g(u-1) \equiv 0 \pmod 2$, $gu \equiv 0 \pmod m$, $m \equiv 0 \pmod 2$ if $u = 2$, and $(g,u,m) \neq (2,3,3)$, $(2,6,3)$, $(6,2,6)$, or $(6,3,3)$.*

3. Necessary Conditions and Basic Decompositions

Throughout the paper, we assume $h \equiv 0 \pmod 2$, and $h \geq 4$. In this section, we start by giving necessary conditions for the existence of a $\mathrm{URD}^*(n; C_h^{r_1}, S(C_h)^{r_2})$ and then give a direct construction for small decompositions that will be used as ingredients in the GDD and filling constructions.

Lemma 3. *Let $n \equiv 0 \pmod{2h}$. If there exists a $URD^*(n; C_h^{r_1}, S(C_h)^{r_2})$ then $(r_1, r_2) \in I(n) = \{(\frac{n-2}{2} - 2x, 2x),\ \ x = 0, 1, \ldots, \frac{n}{4} - 1\}$.*

Proof. The resolvability implies $2r_1 + 2r_2 = n-2$. For any fixed vertex v, since v has degree $d(v) = n-2$ in the graph $K_n - I$ and a C_h-factor is a regular graph of degree 2, v is incident to $2r_1$ edges in the union of the C_h-factors and incident to $n-2-2r_1$ in the union of the $S(C_h)$-factors (since $n-2-2r_1$ is a non-negative integer, $r_1 \leq \frac{n-2}{2}$). Let x and y denote the number of $S(C_h)$-factors where v appears with degree 1 and 3, respectively. Combining the conditions

$$x + y = r_2 \ \text{ and } \ x + 3y = n - 2 - 2r_1$$

with the equality $2r_1 + 2r_2 = n-2$ gives $x = y$ and so $r_2 = 2x$ and $r_1 = \frac{n-2}{2} - 2x$, where $x \leq \lfloor \frac{n-2}{4} \rfloor$ (because r_1 is a non-negative integer) and so $x \leq \frac{n}{4} - 1$. □

From now on, we denote by $(a_1, a_2, \ldots, a_h)$ the h-cycle on $\{a_1, a_2, \ldots, a_h\}$ with edge set $\{\{a_1,a_2\}, \{a_2,a_3\}, \ldots, \{a_{k-1},a_k\}, \{a_k,a_1\}\}$, and by $(a_1, a_2, \ldots, a_h; b_1, b_2, \ldots, b_h)$ the h-sun consisting of the h-cycle $\{a_1, a_2, \ldots, a_h;\}$ and the 1-factor $\{\{a_1,b_1\}, \{a_2,b_2\}, \ldots, \{a_h,b_h\}\}$ (see Figure 1).

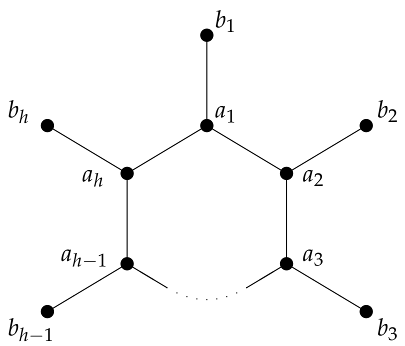

Figure 1. h-sun.

Lemma 4. *A $URD(C_{h(2)}; C_h^{r_1}, S(C_h)^{r_2})$ exists for $(r_1, r_2) \in \{(2,0), (0,2)\}$.*

Proof. Starting from $C_h = (1, 2, \ldots, h)$, on $\{1, 2, \ldots, h\} \times \mathbb{Z}_2$ consider the factors of $C_{h(2)}$ listed below.

- $(2,0)$:
  $\{(1_0, 2_0, \ldots, h_0), (1_1, 2_1, \ldots, h_1)\}, \{(1_0, 2_1, 3_0, 4_1, \ldots, (h-1)_0, h_1), (1_1, 2_0, 3_1, 4_0, \ldots, (h-1)_1, h_0)\}$.
- $(0,2)$:
  $\{(1_0, 2_0, \ldots, h_0; 2_1, 3_1, \ldots, h_1, 1_1)\}, \{(1_1, 2_1, 3_1, \ldots, h_1; 2_0, 3_0 \ldots, h_0, 1_0)\}$.

□

Lemma 5. *If $h \equiv 0 \pmod 4$, then an $RGDD(C_h^{\frac{h}{2}})$ of type h^2 exists.*

Proof. Let $h = 4p$, $p \geq 1$. Apply the GDD construction with $t = 2$ to an RGDD(C_{4p}^{p}) of type $(2p)^2$ (which exists by Lemma 1) to obtain an RGDD(C_{4p}^{2p}) of type $(4p)^2$ by using a URD($C_{h(2)}; C_h^2, S(C_h)^0$) from Lemma 4. □

Lemma 6. *If $h \equiv 2 \pmod 4$, then a URGDD($C_h^{\frac{h-2}{2}}, K_2^2$) of type h^2 exists.*

Proof. Let $h = 4p + 2$, $p \geq 1$. Apply the GDD construction with $t = 2$ to a URGDD(C_{4p+2}^{p}, K_2^1) of type $(2p+1)^2$ (which exists by Lemma 1) and obtain a URGDD(C_{4p+2}^{2p}, K_2^2) of type $(4p+2)^2$ by using a URD($C_{h(2)}; C_h^2, S(C_h)^0$) from Lemma 4. □

Starting from a copy of an RGDD of type h^2 fixed in Lemma 5 or in Lemma 6 (depending on $h \equiv 0 \pmod 4$ or $h \equiv 2 \pmod 4$, respectively), it is possible to give a solution to our problem for the smallest possible order $n = 2h$, first by filling the groups with copies of a URD($h; C_h^{\frac{h-2}{2}}, K_2^1$) from Lemma 1 to obtain the maximum possible number of C_h-factors and subsequently by using a technique that consists in destroying two suitable factors of cycles and replacing them with two $S(C_h)$-factors. To illustrate this technique, we give the following example.

Example 1. *A URD*$^*(8; C_4^{r_1}, S(C_4)^{r_2})$ *for* $(r_1, r_2) = (3, 0), (1, 2)$. Start from an RGDD($C_4^2$) with groups $G_1 = \{a_1, b_1, c_1, d_1\}$ and $G_2 = \{a_2, b_2, c_2, d_2\}$ and factors $\mathcal{C}_1 = \{(a_1, a_2, b_1, b_2), (c_1, c_2, d_1, d_2)\}$ and $\mathcal{C}_2 = \{(a_1, c_2, b_1, d_2), (c_1, a_2, d_1, b_2)\}$. For $i = 1, 2$, fill the group G_i with a copy of a URD($4; C_4^1, K_2^1$); let $\mathcal{C}'_i = \{(a_i, b_i, c_i, d_i)\}$ and $F_i = \{\{a_i, c_i\}, \{b_i, d_i\}\}$ be the 4-cycle and 1-factor, respectively. Trivially, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}' = \mathcal{C}'_1 \cup \mathcal{C}'_2$ give a C_4-factorization of $K_4 - I$, where $I = F_1 \cup F_2$, and so a URD$^*(8; C_4^3, S(C_4)^0)$ is constructed. In order to obtain a URD$^*(8; C_4^1, S(C_4)^2)$, it is sufficient to combine $\mathcal{C}' = \mathcal{C}'_1 \cup \mathcal{C}'_2$ with $\mathcal{C}_1$ and obtain two $S(C_4)$-factors as follows. The edges of the cycles of $\mathcal{C}_1$ can be partitioned into the following 1-factors:

$$F'_1 = \{\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}, \{d_1, d_2\}\}$$
$$F'_2 = \{\{a_2, b_1\}, \{b_2, a_1\}, \{c_2, d_1\}, \{d_2, c_1\}\}.$$

Now, the edges of F'_1 can be attached to the cycle of $\mathcal{C}'_1$ and obtain the sun $\mathcal{S}_1 = \{(a_1, b_1, c_1, d_1; a_2, b_2, c_2, d_2)\}$, while the edges of F'_2 can be used to complete the cycle of $\mathcal{C}'_2$ and obtain $\mathcal{S}_2 = \{(a_2, b_2, c_2, d_2; b_1, a_1, d_1, c_1)\}$. The required factorization of $K_4 - I$ is fixed by $\mathcal{C}_2$, $\mathcal{S}_1$, $\mathcal{S}_2$.

The technique to construct Example 1 can be used to give a solution for the order $n = 2h$.

Lemma 7. *Let $h \equiv 0 \pmod 4$. A URD*$^*(2h; C_h^{r_1}, S(C_h)^{r_2})$ *exists for every* $(r_1, r_2) \in I(2h)$.

Proof. Start from an RGDD($C_h^{\frac{h}{2}}$) of type h^2 (from Lemma 5); let G_1 and G_2 be its groups and $\mathcal{C}_i$, for $i = 1, 2, \ldots, \frac{h}{2}$, be the C_h-factors. Filling the groups by using copies of a URD($h; C_h^{\frac{h-2}{2}}, K_2^1$) (which exists by Lemma 1) gives a URD$^*(2h; C_h^{h-1}, S(C_h)^0)$, whose C_h-factors are $\mathcal{C}_i$, for $i = 1, 2, \ldots, \frac{h}{2}$, and $\mathcal{C}'_i = \mathcal{C}_i^{(1)} \cup \mathcal{C}_i^{(2)}$, for $i = 1, 2, \ldots, \frac{h-2}{2}$, where $\mathcal{C}_i^{(1)}$ and $\mathcal{C}_i^{(2)}$ are factors of X_{G_1} and X_{G_2}, respectively. For any integer $0 \leq x \leq \frac{h-2}{2}$, replacing x pairs of cycle factors of type $\{\mathcal{C}_i, \mathcal{C}'_i\}$, for $i \in \{1, 2, \ldots, \frac{h-2}{2}\}$, with $2x$ $S(C_h)$-factors gives a URD$^*(2h; C_h^{h-1-2x}, S(C_h)^{2x})$ for every $0 \leq x \leq \frac{h-2}{2}$, i.e., a URD$^*(2h; C_h^{r_1}, S(C_h)^{r_2})$ for every $(r_1, r_2) \in I(2h)$. □

Lemma 8. *Let $h \equiv 2 \pmod 4$. A URD*$^*(2h; C_h^{r_1}, S(C_h)^{r_2})$ *exists for every* $(r_1, r_2) \in I(2h)$.

Proof. Consider the URGDD($C_h^{\frac{h-2}{2}}, K_2^2$) of type h^2 Lemma 3 in Lemma 6; denote its C_h-factors by $\mathcal{C}_i$, for $i = 1, 2, \ldots, \frac{h-2}{2}$, and the two 1-factors by F_1 and F_2. Filling the groups, say

G_1 and G_2, by using copies of a URD$(h; C_h^{\frac{h-2}{2}}, K_2^1)$ from Lemma 1 gives a URD$(2h; C_h^{h-2}, K_2^3)$, whose cycle factors are $\mathcal{C}_i$, for $i = 1, 2, \ldots, \frac{h-2}{2}$, and $\mathcal{C}'_i = \mathcal{C}_i^{(1)} \cup \mathcal{C}_i^{(2)}$, for $i = 1, 2, \ldots, \frac{h-2}{2}$, where $\mathcal{C}_i^{(1)}$ and $\mathcal{C}_i^{(2)}$ are factors of X_{G_1} and X_{G_2}, respectively. Combining $\mathcal{C}_1^{(1)}$ and $\mathcal{C}_1^{(2)}$ with F_1 and F_2, respectively, gives two $S(C_h)$-factors (see point 1 of Remark 1) so that a URD$^*(2h; C_h^{h-3}, S(C_h)^2)$ is obtained. Now, for any integer $0 \leq x \leq \frac{h-4}{2}$, replacing x pairs of cycle factors of type $\{\mathcal{C}_i, \mathcal{C}'_i\}$, for $i \in \{2, \ldots, \frac{h-2}{2}\}$, with $2x$ $S(C_h)$-factors gives a URD$^*(2h; C_h^{h-3-2x}, S(C_h)^{2+2x})$, for every $0 \leq x \leq \frac{h-4}{2}$, i.e., a URD$^*(2h; C_h^{r_1}, S(C_h)^{r_2})$ for every $(r_1, r_2) \in I(2h) \setminus \{(h-1, 0)\}$. Finally, the case $(r_1, r_2) = (h-1, 0)$ corresponds to a C_h-factorization of $K_{2h} - I$, which is known to exist [17]. □

The method applied to construct Example 1 and prove Lemmas 7 and 8 suggest some useful considerations, which justify a more general technique to obtain sun factors from cycle factors (Here, different lengths of the cycles are allowed for a possible later use in studying the more general existence problem for $\mathcal{H}$-decompositions where $\mathcal{H}$ contains cycles and suns of different orders). For brevity, let K_X denote the complete graph on X and K_{X_1,X_2} denote the complete bipartite graph with partite sets X_1 and X_2.

Remark 1. Let X_1 and X_2 be disjoint sets of the same size m. Then:

1. A 2-factorization $\mathcal{C}$ of K_{X_1} can be combined with a 1-factor of K_{X_1,X_2} and obtain a sun factor $\mathcal{S}$ of $K_{X_1 \cup X_2}$. Note that the orders of the suns of $\mathcal{S}$ depend on the lengths of the cycles of $\mathcal{C}$ and $\mathcal{S}$ is an $S(C_h)$-factor when all the cycles of $\mathcal{C}$ have the same length h.
2. A 2-factorization $\mathcal{C}$ of K_{X_1,X_2} with cycles all of even length can be decomposed into two 1-factors of K_{X_1,X_2}. (Indeed, the graph $\mathcal{C}$ is edge-colorable with two colors and each color class is a 1-factor).
3. If $\mathcal{C}$ is a 2-factorization of K_{X_1,X_2} with cycles all of even length and $\mathcal{C}_1$ and $\mathcal{C}_2$ are 2-factorizations of K_{X_1} and K_{X_2}, respectively, then by combining $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ it is possible to obtain two sun factors of $K_{X_1 \cup X_2}$, the components of which are h-suns if all the cycles of $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same length h.

As an example, we apply what observed was in Remark 1 to settle the case $(h, n) = (6, 24)$, which will be useful for proving sufficiency in the next section.

Example 2. *A URD*$^*(24; C_6^{r_1}, S(C_6)^{r_2})$ *for every* $(r_1, r_2) \in I(24)$. First, construct an RGDD(C_6^6) of type 12^2 by considering the orbit of $C = (0, 13, 4, 21, 2, 23)$ under $\mathbb{Z}_{24}$, which can be partitioned into the six C_6-factors:

$$\mathcal{C}_i = \{C + i + 6j : j = 0, 1, 2, 3\}, \ \ i = 0, 1, 2, 3, 4, 5.$$

The groups are the cosets of $H = 2\mathbb{Z}_{24}$ in $\mathbb{Z}_{24}$, i.e., $G_1 = H$ and $G_2 = H + 1$. Then, filling the groups by using copies of a URD$^*(12; C_6^5)$ from Lemma 8 gives a URD$^*(24; C_6^{11})$, the factors of which are $\mathcal{C}_i$, $i = 0, 1, 2, 3, 4, 5$, and $\mathcal{C}'_i = \mathcal{C}_i^{(1)} \cup \mathcal{C}_i^{(2)}$, for $i = 1, 2, 3, 4, 5$, where $\mathcal{C}_i^{(1)}$ and $\mathcal{C}_i^{(2)}$ are factors of X_{G_1} and X_{G_2}, respectively. Now, for every $i \in \{1, 2, 3, 4, 5\}$, $\mathcal{C}_i$ and $\mathcal{C}'_i = \mathcal{C}_i^{(1)} \cup \mathcal{C}_i^{(2)}$ can be replaced by two $S(C_6)$-factors (see point 3 of Remark 1). Therefore, for any integer $0 \leq x \leq 5$, by combining x pairs of cycle factors of type $\{\mathcal{C}_i, \mathcal{C}'_i\}$, it is possible to construct $2x$ sun factors and so get a URD$^*(24; C_6^{11-2x}, S(C_6)^{2x})$ for every $0 \leq x \leq 5$; that is, a URD$^*(24; C_6^{r_1}, S(C_6)^{r_2})$ for every $(r_1, r_2) \in I(24)$.

4. Main Result

By using the basic results in Section 3 we are now able to prove the sufficiency.

Proposition 1. *Let* $n \equiv 0 \pmod{2h}$. *A URD*$^*(n; C_h^{r_1}, S(C_h)^{r_2})$ *exists for every* $(r_1, r_2) \in I(n)$.

Proof. Let $n = 2hu$, $u \geq 1$. For $u = 1$, the thesis follows by Lemmas 7 and 8. For $(h, u) = (6, 2)$, it follows by Example 2. For any $(h, u) \neq (6, 2)$, $u \geq 2$, apply the GDD construction with $t = 2$ to a C_h-RGDD of type h^u, which exists by Lemma 2 and has $\alpha = \frac{h}{2}(u-1)$ factors. The input designs are from Lemma 4 and the resulting design is a URGDD$(C_h^{\bar{r}_1}, S(C_h)^{\bar{r}_2})$ of type $(2h)^u$ where

$$(\bar{r}_1, \bar{r}_2) \in \left[\frac{h}{2}(u-1)\right] * \{(2,0),(0,2)\}$$

Finally, apply the filling construction by using copies of a URD$^*(2h; C_h^{r'_1}, S(C_h)^{r'_2})$ with $(r'_1, r'_2) \in I(2h)$ and obtain a URD$^*(2hu; C_h^{r_1}, S(C_h)^{r_2})$ for every

$$\begin{aligned}(r_1, r_2) \in I(2h) + \left[\frac{h}{2}(u-1)\right] * \{(2,0),(0,2)\} = \\ = \{(h-1-2x, 2x) : x = 0, 1, \ldots, \frac{h}{2} - 1\} + \{(h(u-1) - 2y, 2y) : y = 0, 1, \ldots, \frac{h}{2}(u-1)\} = \\ = \{(hu - 1 - 2(x+y), 2(x+y) : x + y = 0, 1, \ldots, \frac{hu}{2} - 1\} = I(n).\end{aligned}$$

□

Combining Lemma 3 with Proposition 1 gives our main result.

Main Theorem. *Let $n \equiv 0 \pmod{2h}$. There exists a URD$^*(n; C_h^{r_1}, S(C_h)^{r_2})$ if and only if $(r_1, r_2) \in I(n)$.*

Proof. By Lemma 3 the necessary conditions follow, while Poposition 1 gives the sufficiency. □

5. Conclusions

As pointed out in the Introduction, our main result fits in the context of a series of papers; namely [15,16], those where the authors investigated the existence of $\mathcal{H}$-factorizations of K_n or $K_n - I$ when $\mathcal{H}$ contained a cycle. More specifically, in these papers the length of the cycle was even and determining necessary and sufficient conditions for odd lengths was still an open problem of interest for further research.

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