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# Point and counterpoint between Mathematical Physics and Physical Mathematics

PGL Leach<sup>1,1</sup> and MC Nucci<sup>2</sup>

<sup>1</sup> Department of Mathematics and Statistics, University of Cyprus, CY 1678 Lefkosia, Cyprus

<sup>2</sup> Dipartimento di Matematica e Informatica, Università di Perugia, 06123 Perugia, Italy

E-mail: leach@ucy.ac.cy; leachp@ukzn.ac.za; leach@math.aegean.gr; nucci@unipg.it

**Abstract.** In recent years there has been a resurgence of interest in problems dating back for over half a century. In particular we refer to the questions of the consistency of quantisation and nonlinear canonical transformations and the quantisation of higher-order field theories. We present resolutions to these questions based upon considerations of symmetry. This enables one to examine these problems within the context of existing theory without the need to introduce new and exotic theories.

**Keywords:** Quantisation; Lie symmetry; Noether symmetry; reduction of order;  $PT$ -symmetry

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In the recent literature there has been a resurgence of interest in two questions relating to the quantisation of classical systems. By coincidence the two questions arose at roughly the same time almost sixty years ago. One question related to the interchange of the actions of making a canonical transformation and performing quantisation on an Hamiltonian. In a lengthy and detailed paper of 1951 van Hove [31] demonstrated that in general there could not be consistency under the interchange. In 1950 Pais and Uhlenbeck [27] introduced a fourth-order field-theoretic model which, when quantised, led to the presence of eigenstates of negative norm. In both cases there was a conflict between the expectation of consistency of description, which is what Physics demands, and the results which Mathematics produced.

It is a commonplace to describe Mathematics as the Queen and the Servant of Science, by which one generally thinks of Physics since it is the most mathematically developed of the sciences. The description is quite eloquent and yet one can wonder if its implication is generally appreciated. One can relate to the concept of ‘Servant’ in which role Mathematics provides the tools for the resolution of problems in Physics in that the Mathematics is subservient to the realistic demands of Physics. Independently of Science Mathematics can evolve along manifold paths into areas which are not obviously connected to the reality as observed in Nature. Nevertheless one must expect, nay demand, a mathematical description of a physical problem

<sup>1</sup> permanent address: School of Mathematical Sciences, University of KwaZulu-Natal, Private Bag X54001 Durban 4000, Republic of South Africa

to be consistent with the Physics as it is observed experimentally. No matter the method of description of a physical system the bottom line should always be the same.

Hamilton [13] developed his theory of Mechanics. Dirac [9] found that Hamilton's theory was most acceptable for his operator theory of Quantum Mechanics. van Hove demonstrated that the generality permitted in the canonical transformations of Hamilton's theory caused problems with quantisation<sup>2</sup>. In the last decade Calogero (with Graffi in one paper) [6, 7, 8] examined a number of nonlinear oscillators classically related by nonlinear canonical transformations and demonstrated some of the problems associated with a consistent quantisation. In an attempt to bring consistency between Mathematics and Physics Nucci *et al.* [24] proposed that the criterion for a valid quantisation procedure between two Hamiltonians related by a canonical transformation be the preservation of the Lie point symmetries of the corresponding Schrödinger equations and demonstrated this with several examples taken from the then current literature.

Hamilton's theory is essentially a first-order theory in that the canonical equations of motion derived from the Hamiltonian are of first order. A standard way to obtain the Hamiltonian is to make use of a Legendre transformation of a first-order Lagrangian. However, the Euler-Lagrange theory is not confined to first-order Lagrangians. Only a few years after Hamilton's presentation of his theory Ostrogradsky [26] proposed a procedure which would enable the construction of an Hamiltonian function from an higher-order Lagrangian. The field-theoretic model of Pais-Uhlenbeck is of the fourth order and has a second-order Lagrangian. They are, respectively,

$$\ddot{z} + (\Omega_1^2 + \Omega_2^2) \dot{z} + \Omega_1^2 \Omega_2^2 z = 0 \quad (1)$$

and

$$L = \frac{1}{2} \gamma \left\{ \dot{z}^2 - (\Omega_1^2 + \Omega_2^2) z \dot{z} + \Omega_1^2 \Omega_2^2 z^2 \right\}. \quad (2)$$

In [17] and [18] the constraint method of Dirac was used to construct a Hamiltonian. The result was the same as if the method of Ostrogradsky had been employed.

The problem with the Hamiltonian

$$H = \frac{1}{2} \left\{ \frac{p_y^2}{\gamma} + 2p_z y + \gamma (\Omega_1^2 + \Omega_2^2) y^2 - \gamma \Omega_1^2 \Omega_2^2 z^2 \right\} \quad (3)$$

obtained in this way was that it possessed 'ghosts' when considered from the point of view of Quantum Mechanics in the usual formulation of Dirac. Not surprisingly the existence of unwanted properties did not bring joy to physicists. The Mathematics may well have been impeccable, but the Physics was definitely peccable. Bender and Mannheim [4] sought to obviate the problem of ghosts by discarding the approach of Dirac of some eighty years ago and to approach the problem of a correct quantisation of the model of Pais-Uhlenbeck using the methods of the theory of  $\mathcal{PT}$  symmetry rather than the Hermitian approach which underpins Dirac's route to quantisation<sup>3</sup>. Their analysis was successful in that ghosts no longer bedevilled the problem.

<sup>2</sup> This leaves aside the question of the correct quantisation of a function of the classical momentum and position [2]. Naturally the two questions are related, but for the nonce we concentrate upon the difficulties with any particular quantisation scheme.

<sup>3</sup> One should bear in mind that an essential feature of the Hermitian approach is that the eigenvalues of the operators are necessarily real. However, this does not mean that nonhermitian operators cannot have real eigenvalues and so be physically acceptable. A potentially interesting application has been recently proposed to deal with the almost eternally vexed question of the quantum mechanics of a system with dissipation such as

A critical point, which seems to have been generally overlooked in previous treatments of this problem, is the process whereby the fourth-order equation of Pais-Uhlenbeck and its second-order Lagrangian are reduced to an Hamiltonian. The method, as we mentioned above, is due to Ostrogradsky. However, it is not the only method available. Indeed one could opine that the method of Ostrogradsky is a quite complicated way to implement the methods of reduction of order. In [25] we used the method proposed by Nucci [21], which has seen some quite successful applications such as in [19], [22], [23], [10], [12], to reduce the fourth-order model of Pais-Uhlenbeck to obtain a first-order Lagrangian. The quantum mechanical treatment of the Hamiltonian derived from this Lagrangian saw no ghosts. We wrote the fourth-order equation, (1), as a system of four first-order equations and then recombined the variables to produce the Lagrangian

$$L = \frac{1}{2} \left( \dot{r}_1^2 + \dot{r}_2^2 - \Omega_1^2 r_1^2 - \Omega_2^2 r_2^2 \right), \quad (4)$$

where

$$z = r_1 - r_2 \quad \text{and} \quad \dot{z} = -\Omega_1^2 r_1 + \Omega_2^2 r_2. \quad (5)$$

It is obvious that the Hamiltonian is

$$H = \frac{1}{2} \left( p_1^2 + p_2^2 + \Omega_1^2 r_1^2 + \Omega_2^2 r_2^2 \right) \quad (6)$$

and that it is quite free of ghosts in the quantum-mechanical context. We emphasise that there was no need to depart from the context of Dirac's formulation of quantum mechanics.

We have the situation in which two analyses of the same fourth-order equation with its associated second-order Lagrangian have led to two quite distinct quantum-mechanical treatments. Were the results of the treatments the same, one would not be perturbed since one would expect consistency from two different descriptions of the same physical system. However, there were some quite notable differences in results and it is incumbent upon anyone either to resolve the differences or to explain why they exist.

The method of Ostrogradsky as applied to the Lagrangian, (2), proceeds as follows according to the report in Whittaker [32]. Commencing with the Lagrangian, (2), we obtain the two momenta

$$\begin{aligned} p_1 &= -\gamma \left( \Omega_1^2 + \Omega_2^2 \right) \dot{z} - \gamma \ddot{z} \\ p_2 &= \gamma \ddot{z} \end{aligned}$$

the damped linear oscillator [1] by making use of the fractional calculus introduced a quarter of a millennium ago by Euler subsequent to a proposal by l'Hôpital towards the end of the seventeenth century and building upon the treatment of Lagrangians and Hamiltonian functions for systems with certain forms of dissipation by Riewe [29]. The operators are obviously nonhermitian, but the eigenvalues are certainly real. One recalls that Bateman [3] proposed a method – maybe better known to physicists through the works of Caldirola [5] and Kanai [14] – to obtain a Lagrangian and hence Hamiltonian for this problem. Subsequent quantisation led to results which were regarded as impossibly nonphysical. The approach in [1] still remains to be subjected to thorough examination by the scientific community, but it does provide a route which at least does not tread the doubtful path initiated by Bateman. Indeed Bateman's approach could be cited as an instance of Mathematics behaving as a Queen. Schuch [30] provided an alternate approach based on the Caldirola-Kanai formalism which was more acceptable physically. A key feature to his treatment was introduction of a nonunitary transformation which obviated the need to confront the nonphysical aspects of earlier treatments. Somehow it would seem that the avoidance of physically unacceptable conclusions necessitates a departure from Dirac's formulism when dealing with dissipation.

and the corresponding canonical coordinates are  $q_1 = z$  and  $q_2 = \dot{z}$ . Using the prescription of Ostrogradsky as given by Whittaker for the number of variables in the present situation we obtain the Hamiltonian

$$H = -\frac{1}{2}\gamma \left\{ \frac{p_2^2}{\gamma^2} - (\Omega_1^2 + \Omega_2^2) q_2^2 + \Omega_1^2 \Omega_2^2 q_1^2 \right\} + p_1 q_2. \quad (7)$$

One observes that the four coordinates introduced by Ostrogradsky involve the dependent variable and its derivatives according to a prescription. This is not greatly different from the method of reduction of order and yet one finds a rather dramatic difference in the outcome. For the benefit of the reader without access to either the original paper by Ostrogradsky or a succinct account such as presented by Whittaker we summarise Ostrogradsky's method for recasting a problem from the calculus of variations in one independent variable with an higher-order Lagrangian into Hamiltonian form.

Let  $L(t, x, \dot{x}, \ddot{x}, \dots, x^{(n)})$ , where  $x$  can be a multivariable and  $n$  a multi-index in a general treatment, but we confine our attention to a single dependent variable which may occur up to the  $n$ th derivative with respect to the independent variable,  $t$ , be a Lagrangian for which the Euler-Lagrange equation of the Calculus of Variations is

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) + \dots + (-1)^n \frac{d^n}{dt^n} \left( \frac{\partial L}{\partial x^{(n)}} \right). \quad (8)$$

To obtain an Hamiltonian representation a first-order Lagrangian is required. Ostrogradsky defines the momenta as thus.

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{x}} \right) + \dots + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}} \left( \frac{\partial L}{\partial x^{(n)}} \right) \\ p_2 &= \frac{\partial L}{\partial \ddot{x}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \overset{\cdot\cdot\cdot}{x}} \right) + \dots + (-1)^{n-2} \frac{d^{n-2}}{dt^{n-2}} \left( \frac{\partial L}{\partial x^{(n)}} \right) \\ &\vdots \\ p_n &= \frac{\partial L}{\partial x^{(n)}} \end{aligned} \quad (9)$$

which ensures that all derivatives present in the Lagrangian are covered. The canonical coordinates are defined according to

$$q_1 = x, q_2 = \dot{x}, \dots, q_n = x^{(n-1)}. \quad (10)$$

The Hamiltonian function is then defined according to

$$H = -L + p_1 q_2 + p_2 q_3 + \dots + p_{n-1} q_n + p_n x^{(n)}, \quad (11)$$

which reduces to the standard prescription when the Lagrangian is of the first order. When Hamilton's Principle is applied, we obtain the standard equations for an Hamiltonian describing a system of  $n$  degrees of freedom, namely

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j} \quad \text{and} \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}. \quad (12)$$

The formalism is fitting and yet we have found that the transition to quantum mechanics has led to a flawed description of the physical properties of the original system.

The first question to be resolved is at what point in this transition has there been a departure from a faithful description of the mechanics. Is it in the variation of the classical description implied by the application of the method of Ostrogradsky or does the problem arise afterwards when one attempts to do quantum mechanics? We apply Hamilton's equations of motion, (12), to the Hamiltonian, (7). We obtain

$$\begin{aligned}\dot{q}_1 &= q_2 & \dot{p}_1 &= \gamma\Omega_1\Omega_2q_1 \\ \dot{q}_2 &= -\frac{p_2}{\gamma} & \dot{p}_2 &= -\gamma(\Omega_1^2 + \Omega_2^2)q_2 - p_1.\end{aligned}$$

Recalling that  $q_1 = z$  we keep differentiating the first equation and substituting from the others until we indeed return to (1).

This particular problem does not provide a counterexample to the validity of the method of Ostrogradsky in the context of classical mechanics. Consequently we infer that there is a problem in the transition to quantum mechanics. In a formal sense the quantisation of the Hamiltonian, (7), is not a difficult procedure. There is not even a case to be made for this method or that method of quantisation since there are no products of noncommuting operators. We can only conclude that the problem arises in the manner in which the method of Ostrogradsky affects the subsequent quantisation procedure. We can compare this method in a manner of speaking with the problems encountered with the quantisation of simple mechanical systems after they have been subjected to nonlinear canonical transformation. There has been more than a little literature on the subject over the last half century for which a relevant sampling is [31, 6, 7, 8, 24]. Of particular relevance to the present discussion is the proposal made in [24] in which the point was made, with supporting examples, that an essential feature of the quantisation procedure appeared to be the preservation of symmetry. This immediately begs the question of what kind of symmetry should be preserved. In this respect we are guided by a particular experience which is the relationship between the Noether point symmetries of the classical Lagrangian and the Lie point symmetries of the corresponding Schrödinger equation [16, 2, 15, 20]. Apart from the generic Lie point symmetries, *i.e.* those corresponding to the solutions of the linear Schrödinger equation and its very linearity, a one-to-one correspondence is observed between the two classes of symmetry. When that nexus has been broken, the results of the quantisation have been embarrassing [8].

It seems that the method of Ostrogradsky has something in common with the application of nonlinear canonical transformations to Hamiltonian systems and their subsequent quantisation. That has been demonstrated to cause a difficulty. This is not a good thing. It is ever the desire of a describer of physical processes to provide a consistent description, no matter the viewpoint. Following our success [24] with the correct quantisation of a variety of nonlinear oscillators rendered nonlinear by means of nonlinear canonical transformations from the standard representation of a linear oscillator we propose in this paper a procedure to rectify the problems of the appearance of ghosts in the reduction and subsequent quantisation of higher-order field theories. The philosophy of the procedure is quite simple. Preserve the symmetries!

We consider the Lagrangian of equation (1) namely

$$L = \frac{1}{2} \left\{ \dot{z}^2 - (\Omega_1^2 + \Omega_2^2) z^2 + \Omega_1^2 \Omega_2^2 z^2 \right\} + \frac{d}{dt} F(t, z, \dot{z}) \quad (13)$$

where  $F$  is the gauge function. Equation (1) admits a six-dimensional Lie point symmetry algebra generated by

$$\begin{aligned}\Gamma_1 &= \partial_t & \Gamma_2 &= z\partial_z \\ \Gamma_3 &= \cos(\Omega_1 t)\partial_z & \Gamma_4 &= -\sin(\Omega_1 t)\partial_z \\ \Gamma_5 &= \cos(\Omega_2 t)\partial_z & \Gamma_6 &= -\sin(\Omega_2 t)\partial_z,\end{aligned}$$

while the Lagrangian (13) admits five Noether point symmetries and consequently five first integrals, namely

$$\begin{aligned}\Gamma_1 &\implies I_1 = \frac{1}{2}(-\Omega_1^2\Omega_2^2z^2 - \Omega_1^2\dot{z}^2 - \Omega_2^2z^2 - 2z\ddot{z} + \ddot{z}^2) \\ \Gamma_3 &\implies I_3 = (\Omega_2^2z + \ddot{z})\sin(\Omega_1 t)\Omega_1 + (\Omega_2^2\dot{z} + \dot{\ddot{z}})\cos(\Omega_1 t) \\ \Gamma_4 &\implies I_4 = (\Omega_2^2z + \ddot{z})\cos(\Omega_1 t)\Omega_1 - (\Omega_2^2\dot{z} + \dot{\ddot{z}})\sin(\Omega_1 t) \\ \Gamma_5 &\implies I_5 = (\Omega_1^2z + \ddot{z})\sin(\Omega_2 t)\Omega_2 + (\Omega_1^2\dot{z} + \dot{\ddot{z}})\cos(\Omega_2 t) \\ \Gamma_6 &\implies I_6 = (\Omega_1^2z + \ddot{z})\cos(\Omega_2 t)\Omega_2 - (\Omega_1^2\dot{z} + \dot{\ddot{z}})\sin(\Omega_2 t).\end{aligned}$$

*En passant* we remark that these first integrals, except  $I_1$ , could not be obtained if the gauge function  $F$  was taken to be a constant and yet for some strange reason there are writers who would have that so.

If we suitably combine the first integrals  $I_3$ ,  $I_4$ ,  $I_5$  and  $I_6$  as  $\frac{1}{2}(I_3^2 + I_4^2 + I_5^2 + I_6^2)$ , then the autonomous first integral,

$$I_{\text{aut}} = \frac{1}{2} \left[ (\Omega_2^2z + \ddot{z})^2\Omega_1^2 + (\Omega_2^2\dot{z} + \dot{\ddot{z}})^2 + (\Omega_1^2z + \ddot{z})^2\Omega_2^2 + (\Omega_1^2\dot{z} + \dot{\ddot{z}})^2 \right], \quad (14)$$

is obtained. If we make the obvious transformations

$$q_1 = \ddot{z} + \Omega_2^2z \quad q_2 = \dot{\ddot{z}} + \Omega_1^2z \quad (15)$$

$$p_1 = \dot{\ddot{z}} + \Omega_2^2\dot{z} \quad p_2 = \ddot{\ddot{z}} + \Omega_1^2\dot{z}, \quad (16)$$

from (14) we obtain the Hamiltonian

$$H \equiv I_{\text{aut}} = \frac{1}{2}[p_1^2 + p_2^2 + \Omega_1^2q_1^2 + \Omega_2^2q_2^2] \quad (17)$$

and the corresponding canonical equations are

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = -\Omega_1^2q_1, \quad \dot{p}_2 = -\Omega_2^2q_2. \quad (18)$$

This is the appropriate Hamiltonian<sup>4</sup> for the quantization of the fourth-order field-theoretic model of Pais-Uhlenbeck, (1).

This shows that Noether's Theorem does provide a guide for the transition from classical to quantum mechanics and that it has a definite advantage over Ostrogradsky's method when it comes to the nonexistence of ghosts and the necessity to exorcise them by means of the development of new theories. Indeed Ostrogradsky in his original paper<sup>5</sup> was not so much interested in Mechanics since in his setting the kinetic energy came to be negative, something

<sup>4</sup> We have put the parameter  $\gamma = 1$ .

<sup>5</sup> We wonder how many of the authors who have cited this work have actually "wasted their time" and read it. We did [28].

that Whittaker corrected in his book. We recall our remarks above regarding Mathematics and Physics!

The application of the Legendre transformation to the Hamiltonian (17) gives the Lagrangian

$$L = \frac{1}{2}[\dot{q}_1^2 + \dot{q}_2^2 - (\Omega_1^2 q_1^2 + \Omega_2^2 q_2^2)] \quad (19)$$

and the corresponding Euler-Lagrange equations

$$\ddot{q}_1 = -\Omega_1^2 q_1, \quad \ddot{q}_2 = -\Omega_2^2 q_2 \quad (20)$$

which admit a seven-dimensional algebra of Lie point symmetries<sup>6</sup> generated by the following operators

$$\begin{aligned} \Gamma_1 &= \partial_t, & \Gamma_2 &= q_1 \partial_{q_1}, & \Gamma_3 &= \cos(\Omega_1 t) \partial_{q_1}, & \Gamma_4 &= -\sin(\Omega_1 t) \partial_{q_1}, \\ \Gamma_5 &= q_2 \partial_{q_2}, & \Gamma_6 &= \cos(\Omega_2 t) \partial_{q_2}, & \Gamma_7 &= -\sin(\Omega_2 t) \partial_{q_2}. \end{aligned} \quad (21)$$

We note that the Lagrangian, (19), possesses *five* Noether point symmetries which is in curious agreement with the number of Noether point symmetries admitted by (13).

We started with the observation that Mathematics and Physics are not the same. One can do things mathematically which make no sense physically. In the application of Mathematics to the resolution of problems in Physics one should always bear in mind that in this situation Mathematics is the humble and obedient servant of Science.

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<sup>6</sup> The minimal number for a system of two autonomous linear second-order ordinary differential equations [11].



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