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Relativistic Free Schrödinger Equation for Massive Particles in Schwartz Distribution Spaces

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Abstract: In this work, we pose and solve, in tempered distribution spaces, an open problem proposed by Schrödinger in 1925. In particular, on the Schwartz distribution spaces, we define the linear continuous quantum operators associated with relativistic Hamiltonians of massive particles—particles with rest mass different from 0 and evolving in the four-dimensional Minkowski vector space $\mathbb{M}_4$. In other words, upon the tempered distribution state-space $\mathcal{S}'(\mathbb{M}_4, \mathbb{C})$, we have found the most natural way to introduce the free-particle relativistic Hamiltonian operator and its corresponding Schrödinger equation (together with its conjugate equation, standing for antiparticles). We have found the entire solution space of our relativistic linear continuous evolution equation by completely solving a division problem in tempered distribution space. We define the Hamiltonian (Schwartz diagonalizable) operator as the principal square root of a strictly positive, Schwartz diagonalizable second-order differential operator (linked with the "Klein–Gordon operator" on the tempered distribution space $\mathcal{S}'_4$). The principal square root of a Schwartz nondefective operator is defined in a straightforward way—following the heuristic fashion of some classic and greatly efficient quantum theoretical approach—in the paper itself.

Keywords: energy conservation; relativistic Schrödinger equation; Schwartz distribution spaces; relativistic Hamiltonian

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1. Introduction

1.1. Historical Introduction and Motivations

The problem of the relativistic description of fermions (half-spin particles) was historically solved by Dirac in the twenties of the last century by the formulation of his celebrated equation. The Dirac equation can be rightfully considered the relativistic counterpart of the Pauli equation, although it strangely acts on the state space S^4, instead of the Pauli state space S^2, where S is the Schrödinger equation state space. In the literature, the problem of finding the relativistic counterpart of Schrödinger's equation is still not solved, nor does it seem fully understood, as it has been simplified to the problem of a relativistic equation for spin-1 particles (and subsequent ones). In this paper, we found an extremely natural way to formulate (essentially) one (Lorentz-invariant) relativistic equation for all the above cases, starting from the original idea of Schrödinger himself and from some fruitful developments of Laurent Schwartz distribution theory.

Indeed, the classic Dirac equation deals with half-spin particles; on the contrary, the first relativistic Schrödinger equation we propose here deals with zero-spin relativistic particles. When applying the same equation to biwaves (bitempered distributions) or triwaves (tritempered distributions), we simply obtain the free evolution equation for "positive" fermions (Dirac particles with positive energy) and "positive" spin-1 particles. The equation we propose and solve, in its Lorentz-invariant form, is

$$\left(\sqrt{\boldsymbol{\mu}^2(P)} - (m_0 c)\mathbb{I}\right)\psi = 0, \tag{1}$$

where the following holds:

- The domain of the equation is intended to be the space of all tempered distributions ψ obtainable by a continuous superposition of all De Broglie waves $(\beta_p)_p$, characterized by strictly positive Minkowsky square norms $\boldsymbol{\mu}(p,p) > 0$;
- The square root operation is intended in its positive definition (see Appendix B);
- P is the operator 4-momentum;
- $\boldsymbol{\mu}$ is the Minkowsky metric tensor, and $\boldsymbol{\mu}^2$ represents its associated quadratic form;
- The square root of a strictly positive operator takes its proper place here, since the quadratic form $\boldsymbol{\mu}^2$ calculated upon the 4-momentum operator P is a strictly positive operator upon our chosen domain (its eigenvalues upon the chosen basis vectors β_p are all strictly positive and, indeed, they are equal to the Minkowsky square norm of the four-vector p);
- $\mathbb{I}$ stands for the identity operator upon the chosen domain.

As a matter of fact, the problem of a relativistic description of a quantum mechanical particle has not been completely solved, historically, by Dirac, not in the 1920s nor later. The story is much more complicated, interesting and rich.

In the beginning, Schrödinger studied the problem of finding an equation that admitted the relativistic De Broglie waves (not only these special harmonic waves but them in particular) as possible solutions. He tried to use the Einstein relativistic dispersion relation, and then he tried to build up a differential equation by substituting (in the standard way) the four-momentum by the partial derivative operators.

Unfortunately, such a simple substitution leads to an "operator-form", which is not a differential operator, because of the problematic square root; it was a case practically intractable at that time for many reasons (the theory of fractional differential operators and pseudodifferential operators is recent and not even imaginable in those days, and it is based on distribution theory, which was fully developed and accepted in the fifties of the last century).

This failure induced Schrödinger to abandon the idea of obtaining a relativistic equation for quantum particles (those not endowed with spin, and he never introduced the spin in the question) and to propose its celebrated (but nonrelativistic) equation (which, by the way, is not at all—in its applications or in its meaning—surpassed or canceled by the Dirac equation; it remains a fundamental tool and useful piece of quantum mechanics nowadays).

We underline that the half-spin particles can also be introduced in the nonrelativistic setting by using the square S^2 of the Schrödinger state space S and the Pauli matrices, which act naturally upon the state space S^2. Indeed, when the electromagnetic field is absent, the nonrelativistic analogous of the nonrelativistic Schrödinger equation for spin $1/2$ particles is the famous Pauli equation (which, in its standard form, is nothing more than the nonrelativistic Schrödinger equation acting on a "Pauli biwave" function: an element of the space S^2).

In 1928, Dirac came across an equation, which addressed the relativistic particles with $1/2$ spin, only by a fortuitous algebraic coincidence: in order to eliminate the Einstein–Schrödinger problematic square root of the dispersion relation. The equation revealed the fourth-power S^4 of the Schrödinger state-space S enough. Dimension 4 in the Dirac equation is decomposed in 2 + 2: the first dimension 2 for the half-spin particles of positive energy and the second dimension 2 for the half-spin particles of the negative energy (whatever this meant).

On the other hand, the relativistic de Broglie waves are not solutions of the nonrelativistic Schrödinger equation, and Schrödinger recognized immediately that its equation should be a first approximation of some other equation definable by the sum of a power series of differential operators (regrettably, nothing about that series was understandable at that time, not even the Hilbert space approach was in place at that time, and the problem of studying that operator series goes far beyond the Hilbert space theory). For this purpose, we observe that the differential operators used by Schrödinger (in their classic equation) are not even specified in their domain: Schrodinger considered these operators simply defined on "differentiable functions", without considering any well-structured function space or good topology on them; so, the problem of studying this series of operators became intractable and utterly meaningless.

Clearly, to many theoretical physicists of that time, the basic problem of finding a relativistic Schrödinger equation for spinless particles was of capital importance in the development of quantum mechanics. Many fathers of quantum mechanics faced (and failed) such a problem; among them, we remember Pauli, Dirac, Klein, Gordon, Heisenberg and many others.

As we mentioned before, Dirac proposed a very smart algebraic way to circumnavigate the square root problem, but it worked in S^4 (we desire to emphasize again that the method proposed by Dirac solves the fermion problem in the fourth power of the Schrödinger state-space S of a zero-spin particle, not the original problem itself in S).

Moreover, the Dirac method introduces some negative energy states to be dangerously interpreted (we need to observe here that—from a strictly mathematical point to view—the presence of negative energy is highly questionable because the Einstein dispersion relation uses the "positive square root" operation, not the complex one; so, when we calculate this "positive square root", we must discard any negative outcome by the very definition of the operation involved, but this is another story).

At this moment, we do not want to underline the good and bad sides of the Dirac equation; we desire to underline that it is not the much-desired relativistic Schrödinger equation for zero-spin particles, and it addresses explicitly the half-spin particle evolution problem in an unexpected space S^4 instead of S^2 (the natural space of action for the Pauli matrices).

We desire to (also) recall the approach of Klein and Gordon, who directly considered the argument under the problematic square root and found an evolution-type equation for zero-spin particles (0, yes indeed); however, as we know, that equation offers other kinds of issues and many interpretative questions from a probabilistic point of view.

We do not want to emphasize the famous issues that such an equation provides, we desire only to underline that the Kleine Gordon equation is not the desired relativistic Schrödinger equation. In some sense, it is the square of the dreamed relativistic Schrödinger equation.

Again, Dirac, using a very smart algebraic trick, solved the relativistic energy momentum relation issue in the power 4 state-space S of zero-spin particles, S^4, which shall reveal itself as the state space of fermions and no longer the space of the zero-spin particles, but he did not prove that the issue can only be fulfilled using (4,4) matrices.

To prove that the problem can be conveniently transformed, changed, stated and solved (without square roots of operators) in S^4 does not prove that the problem cannot be somehow stated and solved in S.

We did not consider, initially, the fermion problem, because it was solved by Dirac; now, however, we have found a new simpler and natural relativistic Schrödinger form of the Dirac equation in S^2 (the Pauli matrix state space).

Actually, here, we propose the original problem proposed by Schrödinger (exactly the original one, not transformed or changed) in the space of tempered distributions (which therefore becomes the zero-spin particle state space).

We think that the majority of the scientific community has simply discarded the problem, because it was apparently mathematically intractable, and has preferred to consider the problem not closable.

Our paper clarifies many misunderstood aspects of relativistic quantum mechanics, not only because it solves the fundamental problem proposed more than 100 years ago but also because it answers the following questions:

Is there a relativistic counterpart to the original (without spin) nonrelativistic Schrödinger equation? Why is the Klein Gordon equation not the relativistic Schrödinger equation and we are we unable to prove the probability current conservation in relativistic quantum mechanics?

In fact, by means of the relativistic Schrödinger equation for zero-spin particles, we can now prove the conservation of probability currents.

By the way, the new equation, in any case (for any spin), is accompanied by a conjugate equation, which admits as solutions negative energy solutions (or time-reverse solution).

The relativistic Schrödinger equation allows to avoid wrong or vague explanations about the developments of the Schrödinger equation, Dirac equation, Pauli equation, or Klein–Gordon equation.

For the moment, because of mysterious and vague reasons, we should believe that the nonrelativistic Schrödinger equation can be defined correctly, while its simple relativistic twin cannot be defined, without considering other problems or features such as non-zero-spin particles and so on.

1.2. The Role of Schwartz Distribution Theory

The mathematical precontext of our analysis is the Laurent Schwartz distribution theory—in particular, that based on the Minkowski space-time. For distribution theory, we refer to Schwartz [1–6], Barros-Neto [7], Dieudonné [8,9], Bourbaki [10–12], Horváth [13], Kesavan [14], Boccara [15], Lang [16], Trèves [17], Yosida [18] and Zeidler [19].

Tempered distribution spaces are the topological dual of the Schwartz function spaces (with respect to their natural Schwartz topologies). In other terms, tempered distribution spaces contains all the bras of Schwartz functions (rapidly smoothing decreasing functions with all their derivatives), functions considered as kets, with respect to their Schwartz topology. Schwartz function spaces are not Hilbert spaces, but they are endowed with the classic Dirac inner product (L^2 product) but not with the (too much rigid) topology induced by the Dirac product itself. The bra spaces of Schwartz function spaces are "much larger" than their respective ket spaces and contain all the Dirac delta "functions" of $\mathbb{R}^n$ (once $\mathbb{R}^n$ is fixed as the domain of the Schwartz functions). One of the basic features of the tempered distribution spaces is that they can be generated by all their respective deltas via superpositions in the sense of Schwartz, for instance, the position operator eigenvectors generate all possible zero-spin quantum states.

Tempered distribution spaces contain nonseparable Hilbert subspaces induced by any quantum mechanics observable with a continuous Schwartz eigenbasis: tempered distribution spaces are much more than Hilbert spaces and much more useful and solve a lot of apparent non-normalizability issues.

1.3. Schwartz Linear Algebra and Physical Hilbert Space

Distribution theory allows us to provide an efficient and deep calculus method for quantum mechanics, inspired by the renowned Dirac Calculus [20].

Specifically, we move inside the realm of Schwartz Linear Algebra (see [21–23]), which helps us to reach a deeper comprehension of the physical structures studied in quantum mechanics [24]. In particular, in Schwartz Linear Algebra, we consider new state spaces for quantum systems. State spaces are, often erroneously, identified only with separable Hilbert spaces. On the contrary, the existence of the so-called "non-normalizable" states—which are revealed to be fundamental for the quantization—shows that state spaces should be a sort of disjoint union of different Hilbert spaces. In the literature, sometimes, the state spaces are called "physical Hilbert spaces" [25], without providing a clear mathematical definition. We have already shown in the past [26] that "physical Hilbert spaces" can smoothly be identified with distribution spaces on suitable Euclidean spaces or Minkovski spaces, depending on the nature of the quantum system considered. Such distribution spaces should be endowed with some algebraic–topological structures, such as continuous operations of superposition, extended Dirac products, partial inner products, etc.

1.4. Extended Dirac Products and Associated Nonseparable Hilbert Structures

The extended Dirac products determine new usual scalar products upon some distinguished subspaces of distribution space, attaching to them nonseparable Hilbert structures.

These new nonseparable Hilbert products clarify (definitely) the probability role of the so-called "non-normalizable states", which indeed become normalizable with respect to suitable Hilbert products. As two fundamental examples, the Dirac delta distributions and the celebrated "Unitary" De Broglie waves become, respectively, Hilbert basis of two new nonseparable Hilbert spaces (subspaces of tempered distribution space) and, consequently, they assume the right status of normalizable states. That normalizability seems completely natural and even obliged, since the physical usual probability interpretation of such states dictates that those states represent certainty (for position and momentum measurements, respectively); in other words , those states represent the basic elements in any probability theory: the elemental certainty bricks of any probabilistic approach.

1.5. Continuous Superpositions

The new operation of continuous superposition revealed the right tool in order to build—in a mathematically meaningful way—the Dirac "extended Linear Algebra" in distribution spaces, adopting the Schwartz topological linear structures of distribution spaces.

More precisely, we verified that the natural algebraic–topological structure of distribution spaces allows to define an extension of the finite linear combination when the index sets of distribution families are continuous sets, especially when the systems of coefficients possess a continuous infinity of terms different from zero.

Some features of Schwartz Linear Algebra (for instance, the informal use of superpositions and the extended Dirac products) could be identified, here and there, in the works of Antoniou and Prigogine [27], Balescu [28], Pauli [29], Penrose [30] and Prigogine [31–33].

1.6. Some Previous Studies

For what concerns the square root of operators, we suggest reading [34,35]. Previous studies about relativistic quantum theory are presented in [36–39]. For what concerns the classic quantum theory, we refer to [40–45], while for the celebrated Dirac equation, we refer to the studies of Dirac [46,47], Foldy [48–50] and Case [51]. For what concerns the Hermite expansions of tempered distributions, we considered [52,53].

2. Theoretical Background: Relativistic Hamiltonian

For what concerns our approach to relativistic dynamics, see Appendix A. We here start directly by recalling the definition of relativistic Hamiltonian.

Definition 1. *The relativistic Hamiltonian*

$$\mathcal{H} : \mathbb{R}^{2n} \to \mathbb{R},$$

of a free material particle evolving upon a spatial manifold of dimension n is defined by

$$\mathcal{H}(\boldsymbol{q},\boldsymbol{p}) = \sqrt{|c\boldsymbol{p}|^2 + (m_0 c^2)^2} = c\sqrt{|\boldsymbol{p}|^2 + \boldsymbol{p}_0^2}\,,$$

for every position-momentum pair $(\boldsymbol{q},\boldsymbol{p})$ in the phase-space $\mathbb{R}^{2n}$. In the above definition:

- *The positive real number m_0 stands for the rest mass of the particle;*
- *$|.|$ represents the Euclidean norm in $\mathbb{R}^n$;*
- *The momentum*

  $$\boldsymbol{p}_0 = m_0 c$$

  stands for what we call the "rest-momentum" of the particle (in the present formalization, it is a real number);
- *The energy*

  $$c\boldsymbol{p}_0 = m_0 c^2$$

  stands for the so-called "rest-energy" of the particle.

Since the above Hamiltonian function $\mathcal{H}$ actually depends only upon the second variable (momentum variable), we can introduce the 4-momentum variable complex function

$$H : \mathbb{R} \times \mathbb{R}^n \to \mathbb{C} : \ (E/c, \boldsymbol{p}) \mapsto \ H_{\boldsymbol{p}} := c\sqrt{|\boldsymbol{p}|^2 + \boldsymbol{p}_0^2} \ = c\,|(\boldsymbol{p}_0, \boldsymbol{p})|\,,$$

which we call the *Hamiltonian of the particle* as well.

Above, we used the symbol $|.|$ for both the Euclidean norm in $\mathbb{R}^n$ and in $\mathbb{R}^{1+n}$.

We are ever so slightly changing the framework to allow an easy quantization of the Hamiltonian function $\mathcal{H}$.

In the following, we denote the 4-momentum vector space by $\mathbb{P}_4$ and the Minkowski 4-vector space by $\mathbb{M}_4$. The de Broglie–Fourier transformations (and, in particular, the Minkowski–Fourier transformation) will provide us topological isomorphism between the tempered distribution spaces $\mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ and $\mathcal{S}'(\mathbb{P}_4, \mathbb{C})$.

Remark 1. *In the presence of a smooth 4-positional potential V, the "complex" Hamiltonian H would become the function*

$$H : \mathbb{M}_4 \times \mathbb{P}_4 \to \mathbb{C} : \ (x, p) \mapsto c\,|(\boldsymbol{p}_0, \boldsymbol{p})| + V(x)\,,$$

which is smooth if the rest mass is positive. In this work, we assume no potentials are defined around.

3. Theoretical Core: Relativistic Hamiltonian Operator in $\mathcal{S}'(\mathbb{M}_4)$

We observe that if the rest mass of the particle is different from 0, then the Hamiltonian function

$$H : \mathbb{P}_4 \to \mathbb{C} : (E/c, \boldsymbol{p}) \mapsto c\sqrt{|\boldsymbol{p}|^2 + (m_0 c)^2}$$

is smooth (despite the presence of the squareroot function) and "slowly increasing at infinity" (at infinity, it reveals linearity: it is asymptotically linear).

Smoothness and a slow increase will allow us to conveniently define the associated quantum Hamiltonian operator $\hat{H}$ by a well-established theorem from Schwartz Linear Algebra. However, we explicitly show the construction and basic properties of the observable.

Definition 2 (Quantization of the relativistic Hamiltonian)**.** *We define the Hamiltonian operator*

$$\hat{H} : \mathcal{S}'(\mathbb{M}_4) \to \mathcal{S}'(\mathbb{M}_4)$$

by the Schwartz integral (superposition)

$$\hat{H}(\psi) = \int_{\mathbb{P}_4} H(\psi)_\beta\, \beta := [H(\psi)_\beta] \circ \hat{\beta},$$

for every tempered wave (i.e., tempered scalar field) ψ. Here, the following holds:

- *$\beta = (\beta_p)_{p \in \mathbb{P}_4}$ represents the de Broglie family, the family of regular bounded tempered distributions defined by*
$$\beta_p = \left[e^{(\imath/\hbar)(\boldsymbol{p}\cdot\mathbf{x}\ -\ Et)}\right],$$
- *$(\psi)_\beta$ is the coordinate distribution of the wave ψ with respect to the de Broglie family β. It is the unique tempered wave living in $\mathcal{S}'(\mathbb{P}_4)$, such that*
$$\psi = \int_{\mathbb{P}_4} (\psi)_\beta \beta = (\psi)_\beta \circ \hat{\beta}.$$
- *$H(\psi)_\beta$ is the product of the function H times the distribution ψ_β;*
- *$\hat{\beta}$ is the operator associated with the Schwartz family β, defined by*
$$\hat{\beta} : \mathcal{S}(\mathbb{M}_4) \to \mathcal{S}(\mathbb{P}_4) : \hat{\beta}(\phi)(p) = \beta_p(\phi),$$

for every $\phi \in \mathcal{S}(\mathbb{M}_4)$ *and for each* $p \in \mathbb{P}_4$.

3.1. Analysis of the Definition

In the above expression, we used an integral, in the sense of distribution, of the family (vector valued test function) β. We clarify here briefly some terms of the above definition.

1. $\beta = (\beta_p)_{p \in \mathbb{P}_4}$ represents the de Broglie family, the family of regular bounded tempered distributions defined by

$$\beta_p = [e^{(\imath/\hbar)(\boldsymbol{p}\mathbf{x} - E\mathrm{t})}] = [e^{(\imath/\hbar)(-E/c,\, \boldsymbol{p})\cdot(c\mathrm{t},\, \mathbf{x})}] = [e^{-(\imath/\hbar)\boldsymbol{\mu}(p,\mathrm{x})}],$$

   here, the following holds:

   - $\mathbf{x}, \mathrm{t}, \mathrm{x}$ are the standard coordinates—canonical projections—in $\mathbb{M}_4$, viewed as complex functions; specifically, x represents the canonical immersion of $\mathbb{M}_4$ in $\mathbb{C}^4$ and $\mathrm{x} = (c\mathrm{t}, \mathbf{x})$;
   - $\boldsymbol{\mu} : \mathbb{P}_4 \times \mathbb{M}_4 \to \mathbb{C}$ is the Minkowski pairing defined by

$$\boldsymbol{\mu}(p, x) = p^0 x^0 - \boldsymbol{p} \cdot \boldsymbol{x} .$$

2. $(\psi)_\beta$ is the coordinate system of the wave function ψ with respect to the de Broglie family β. $(\psi)_\beta$ is the unique tempered wave living in $\mathcal{S}'(\mathbb{P}_4)$, such that

$$\psi = \int_{\mathbb{P}_4} (\psi)_\beta \beta.$$

   Indeed, we could prove that the operator $\hat{\beta}$ is a topological isomorphism of $\mathcal{S}(\mathbb{M}_4)$ onto $\mathcal{S}(\mathbb{P}_4)$. Therefore, the coordinate distribution ψ_β can be immediately obtained by the inverse of the operator $\hat{\beta}$ as

$$\psi_\beta = \psi \circ (\hat{\beta})^- .$$

   Observe, also, that

$$\psi = {}^t\hat{\beta}(\psi_\beta),$$

   and the transpose operator ${}^t\hat{\beta}$ is a topological isomorphism of $\mathcal{S}'(\mathbb{P}_4)$ onto $\mathcal{S}'(\mathbb{M}_4)$.
3. The term $H(\psi)_\beta$ is the (standard defined) product of the smooth function $H \in \mathcal{O}_M(\mathbb{P}_4)$ times the representation $\psi_\beta \in \mathcal{S}'(\mathbb{P}_4)$.
4. In other terms, the Schwartz integral

$$\int_{\mathbb{P}_4} : \mathcal{S}'(\mathbb{P}_4) \times \mathcal{S}(\mathbb{P}_4, \mathcal{S}'(\mathbb{M}_4)) \to \mathcal{S}'(\mathbb{M}_4) : (a, v) \mapsto \int_{\mathbb{P}_4} av$$

   is defined by

$$< \int_{\mathbb{P}_4} av, \phi > = < a, v(\phi) >,$$

   for every test function $\phi \in \mathcal{S}(\mathbb{M}_4)$, as soon as the wave family $v = (v_p)_{p \in \mathbb{P}_4}$ (of tempered waves in $\mathcal{S}'(\mathbb{M}_4)$) reveals "scalarly Schwartz"—sending every Schwartz test function ϕ to a Schwartz test function $v(\phi)$ by

$$(v\phi)(p) := v_p(\phi),$$

   for every index $p \in \mathbb{P}_4$.

Remark 2. *The product of the smooth, slowly increasing complex function* H *times the coordinate system* $(\psi)_\beta$ *is, according to the standard manner, well-defined by*

$$(H\psi_\beta)(\omega) := \psi_\beta(H\omega),$$

for every test function ω, and it belongs to the space of tempered distributions defined upon the 4-momentum space $\mathbb{P}_4$.

3.2. Minkowski Transforms Induced by de Broglie Families

In this brief section, we show the connection between the (non-normalized) de Broglie family β and the (non-normalized) Minkowski–Fourier transform $\mathcal{M}$ on the test function space $\mathcal{S}(\mathbb{M}_4)$: we show that $\mathcal{M}$ is the operator $\hat{\beta}$ induced by the family β.

Theorem 1. *Let*

$$\mathcal{M} : \mathcal{S}(\mathbb{M}_4) \to \mathcal{S}(\mathbb{P}_4) \ : \phi \mapsto < f, \phi > ,$$

stand for the (nonunitary) Minkowski–Fourier transform, where

- *f is the family of smooth bounded complex functions $f_p : \mathbb{M}_4 \to \mathbb{C}$, defined by*

  $$f_p(x) = e^{-(\iota/\hbar)\boldsymbol{\mu}(p,x)} ,$$

  for each 4-momentum p and 4-position x;
- *The action $< f, \phi >$ of the family f upon the test function ϕ by the pairing $< ., . >$ is defined by*

  $$< f, \phi > (p) = \int_{\mathbb{M}_4} f_p \phi \, d\mathrm{x}.$$

Then, the Minkowski transform $\mathcal{M}$ is the operator $\hat{\beta}$ induced canonically by the family β.

Proof. We notice that the wave β_p is the regular tempered distribution generated just by the smooth bounded complex function $f_p : \mathbb{M}_4 \to \mathbb{C}$, defined by

$$f_p(x) = e^{(\iota/\hbar)(\boldsymbol{p}\boldsymbol{x} \, - \, Et)} = e^{(\iota/\hbar)(-E/c, \, \boldsymbol{p})\cdot(ct, \, \boldsymbol{x})} = e^{-(\iota/\hbar)\boldsymbol{\mu}(p,x)} ,$$

for each 4-momentum $p = (E/c, \boldsymbol{p}) \in \mathbb{P}_4$ and 4-position $x = (ct, \boldsymbol{x}) \in \mathbb{M}_4$. This means

$$\begin{aligned} \hat{\beta}(\phi)(p) \ &:= \ \beta_p(\phi) = \\ &= \ [f_p](\phi) = \\ &= \ \int_{\mathbb{M}_4} f_p \phi \, d\mathrm{x} = \\ &= \ \int_{\mathbb{M}_4} \phi \, e^{-(\iota/\hbar)\boldsymbol{\mu}(p,\mathrm{x})} \, d\mathrm{x} = \\ &= \ \mathcal{M}(\phi)(p), \end{aligned}$$

for every 4-momentum p. □

Remark 3. *We desire to underline that the natural extension of $\mathcal{M}$ to the corresponding tempered distribution spaces is the transpose of the Schwartz adjoint (transpose) of $\mathcal{M}$. The Schwartz transpose of $\mathcal{M}$ is characterized by*

$$< \mathcal{M}\phi, \omega > = < \phi, \mathcal{M}^*\omega >,$$

it is the adjoint of $\mathcal{M}$ with respect to the pairing $< ., . >$. Therefore, we see that

$$\tilde{\mathcal{M}} : \mathcal{S}'(\mathbb{M}_4) \to \mathcal{S}'(\mathbb{P}_4) \ : \ u \mapsto u \circ \mathcal{M}^*,$$

with

$$\mathcal{M}^* : \mathcal{S}(\mathbb{P}_4) \to \mathcal{S}(\mathbb{M}_4) \ : \ \mathcal{M}^*(\omega)(x) = \int_{\mathbb{P}_4} \omega \, e^{-(\iota/\hbar)\boldsymbol{\mu}(.,x)} \, d\mathrm{p}.$$

While, the inverse of the Minkowski transform is

$$\mathcal{M}^{-1} : \mathcal{S}(\mathbb{P}_4) \to \mathcal{S}(\mathbb{M}_4) \ : \ \mathcal{M}^{-1}(\omega)(x) = N \int_{\mathbb{P}_4} \omega \, e^{(\imath/\hbar)\boldsymbol{\mu}(.,x)} \, d\mathrm{p},$$

for a convenient real positive normalization constant N.

From the last two relations of the above remark 3, we immediately deduce

$$\mathcal{M}^{-1} = N\bar{\mathcal{M}}^* = N\mathcal{M}^\dagger.$$

In other terms, *the Minkowski transform is a unitary operator up to a normalization constant N.*

Clearly, we immediately obtain

$$(\sqrt{N}\mathcal{M})^{-1} = (\sqrt{N}\mathcal{M})^\dagger.$$

Thus, the operator

$$U = \sqrt{N}\mathcal{M}$$

is unitary, and we call it the unitary Minkowsky transform; as it is well known, the normalization constant amounts to

$$\sqrt{N} = \frac{1}{(2\pi\hbar)^2}.$$

3.3. Principal Properties of the Hamiltonian Operator

Now, let us see the principal theorem of the section.

Theorem 2. *The relativistic Hamiltonian operator $\hat{H}$ reveals the unique linear continuous operator on $\mathcal{S}'(\mathbb{M}_4)$ sending each de Broglie wave β_p to the tempered distribution $H_{\boldsymbol{p}}\beta_p$, where $\boldsymbol{p}$ denotes the spatial part of p. Moreover, we see the following:*

- *$\hat{H}$ reveals to be Schwartz diagonalizable (Schwartz nondefective): there exists a Schwartz basis of $\mathcal{S}'(\mathbb{M}_4)$ constituted by eigenvectors of $\hat{H}$;*
- *The operator $\hat{H}$ reveals to be regular and Hermitian in the Schwartz sense: it could be restricted to an endomorphism of the test function space $\mathcal{S}(\mathbb{M}_4)$, and its restriction reveals Hermitian with respect to the standard Dirac inner product of $\mathcal{S}(\mathbb{M}_4)$.*

Proof. We proceed by steps.

1. *The family β is an eigenfamily of $\hat{H}$.* Indeed, we immediately see

$$\begin{aligned}
\hat{H}(\beta_p) &= \int_{\mathbb{P}_4} H(\beta_p)_\beta \, \beta = \\
&= \int_{\mathbb{P}_4} H\delta_p \, \beta = \\
&= \int_{\mathbb{P}_4} H(p)\delta_p \, \beta = \\
&= \int_{\mathbb{P}_4} H_{\boldsymbol{p}}\delta_p \, \beta = \\
&= H_{\boldsymbol{p}} \int_{\mathbb{P}_4} \delta_p \, \beta = \\
&= H_{\boldsymbol{p}}\beta_p,
\end{aligned}$$

for every 4-momentum $p = (E/c, \boldsymbol{p}) \in \mathbb{P}_4$.

2. The operator $\hat{H}$ is linear and continuous. Indeed, it is a composition of three linear continuous operators:

$$\hat{H} = {}^t\hat{\beta} \circ M_H \circ {}^t\hat{\beta}^{-1} \,:\, \hat{H}(\psi) = {}^t\hat{\beta}\Big(M_H\Big({}^t\hat{\beta}^{-1}(\psi)\Big)\Big),$$

where M_H represents the multiplication operator by H.

3. The operator $\hat{H}$ is Schwartz diagonalizable. Indeed, first of all, any linear and continuous operator reveals to be Schwartz linear (characterization of Schwartz linear operators). Schwartz linearity means that

$$\hat{H}\left(\int_{\mathbb{R}^n} av\right) = \int_{\mathbb{R}^n} a\hat{H}(v),$$

for every natural number n, tempered distribution $a \in \mathcal{S}'_n$ and for every Schwartz family v in $\mathcal{S}'(\mathbb{M}_4)$, indexed by $\mathbb{R}^n$. Now, the eigenfamily β is indeed a Schwartz basis of $\mathcal{S}'(\mathbb{M}_4)$, just because the associated operator $\hat{\beta}$ reveals a bijective operator (the Minkowski transform is a topological isomorphism).

4. The operator $\hat{H}$ is the unique linear continuous operator with eigen-system (H, β). Indeed, the family β is total in $\mathcal{S}'(\mathbb{M}_4)$; therefore, there exists at most one linear continuous operator on $\mathcal{S}'(\mathbb{M}_4)$ assigning a fixed family w in $\mathcal{S}'(\mathbb{P}_4)$ to the family β. Alternatively, we could remember that two Schwartz linear operators equal upon the same Schwartz basis are everywhere equal.

5. The operator $\hat{H}$ is Hermitian in the Schwartz sense. Let $\mathcal{M}$ be the unitary Minkowski transform upon $\mathcal{S}(\mathbb{M}_4)$ and β the corresponding normalized de Broglie family. We proved already that

$$\hat{H} = {}^t\hat{\beta} \circ {}^tM_H \circ {}^t\hat{\beta}^{-1} = {}^t\mathcal{M} \circ {}^tM_H \circ {}^t\mathcal{M}^{-1},$$

where M_H denotes the multiplication operator in $\mathcal{S}(\mathbb{M}_4)$ by the function H. From the above relation, we deduce, by topological transposition (remember the inverse order rule), that

$${}^t\hat{H} = \mathcal{M}^{-1} \circ M_H \circ \mathcal{M}.$$

Hence, by Schwartz transposition—remembering again the inverse order rule of transposition and the symmetry of the Minkowski transform with respect to the test function pairing—we obtain the endomorphism restriction of $\hat{H}$ to the test function space:

$$\hat{H}_{\mathcal{S}} = ({}^t\hat{H})^* = \mathcal{M} \circ M_H \circ \mathcal{M}^{-1}.$$

Now, by applying the Hermitian transposition (remembering the inverse order rule and the unitary character of $\mathcal{M}$), we obtain

$$\hat{H}^\dagger_{\mathcal{S}} = (\mathcal{M}^{-1})^\dagger \circ M_H \circ (\mathcal{M})^\dagger = \mathcal{M} \circ M_H \circ \mathcal{M}^{-1} = \hat{H}_{\mathcal{S}};$$

so that the restriction

$$\hat{H}_{\mathcal{S}} := ({}^t\hat{H})^* : \mathcal{S}(\mathbb{M}_4) \to \mathcal{S}(\mathbb{M}_4)$$

—of the Hamiltonian $\hat{H}$—reveals, indeed, to be self-adjoint with respect to the Dirac inner product upon $\mathcal{S}(\mathbb{M}_4)$. □

Remark 4. *We invite readers to observe that the eigenvalue of the observable $\hat{H}$, corresponding to the eigenstate β_p, depends only upon the relativistic momentum $\boldsymbol{p}$ and not upon the energy component of p, since—as we have already well-underlined before—*

$$H(E/c, \boldsymbol{p}) = H_{\boldsymbol{p}}$$

for every $(E/c, \boldsymbol{p}) \in \mathbb{P}^4$.

3.4. Schrödinger Equation of a Free Particle

Our quantum evolution equation remains the *general Schrödinger equation*

$$\mathcal{E}_H \ : \ \imath\hbar\, \partial_t \psi = \hat{H}\psi, \tag{2}$$

framed within the tempered distribution space $\mathcal{S}'(\mathbb{M}_4)$ built upon the Minkowski 4-vector space $\mathbb{M}_4$.

Alternatively, in space-time coordinates, we can write

$$\imath\hbar\, \partial_0 \psi = (1/c)\ \hat{H}\psi,$$

where

$$\partial_0 : \mathcal{S}'(\mathbb{M}_4) \to \mathcal{S}'(\mathbb{M}_4) : \psi \to (1/c)\frac{\partial}{\partial \mathrm{t}}\psi$$

is the (globally defined and continuous) partial derivative operator—of the tempered distribution space $\mathcal{S}'(\mathbb{M}^4)$—with respect to the "time variable"

$$c\mathrm{t} : \mathbb{M}_4 \to \mathbb{C} : (ct, \boldsymbol{x}) \mapsto ct + 0\imath.$$

Using the above notations, we can write the Lorentz-invariant form of our relativistic Schrödinger Equation (2):

$$\left(\imath\sqrt{\boldsymbol{\mu}^2(\partial)} - \frac{m_0 c}{\hbar}\mathbb{I}\right)\psi = 0. \tag{3}$$

The domain of the above Equation (3) is the tempered distribution space $\mathcal{S}'(\mathbb{M}_4)$ for 0-spin particles, the second power $\mathcal{S}'(\mathbb{M}_4)^2$ for half-spin particles and the cube $\mathcal{S}'(\mathbb{M}_4)^3$ for 1-spin particles. The standard-spin operator matrices act naturally on the above three spaces, respectively.

3.5. Energy Operator in $\mathcal{S}'(\mathbb{M}_4)$

Observe that, applying the momentum operator

$$P = -\imath\hbar\,\frac{\partial}{\partial \mathbf{x}} : \mathcal{S}'(\mathbb{M}_4) \to [\mathcal{S}'(\mathbb{M}_4)]^3 : \psi \mapsto -\imath\hbar\left(\frac{\partial}{\partial \mathbf{x}_j}\psi\right)_{j\in I_4^*}$$

to the De Broglie waves β_p, we obtain

$$P(\beta_{E/c,\boldsymbol{p}}) = \boldsymbol{p}\, \beta_{E/c,\boldsymbol{p}}\ .$$

While, applying the energy operator

$$\hat{E} = \imath\hbar\frac{\partial}{\partial \mathrm{t}} : \mathcal{S}'(\mathbb{M}_4) \to \mathcal{S}'(\mathbb{M}_4),$$

we obtain

$$\hat{E}(\beta_{E/c,\boldsymbol{p}}) = \imath\hbar\ \frac{\partial}{\partial \mathrm{t}}\ (\beta_{E/c,\boldsymbol{p}}) = E\ \beta_{E/c,\boldsymbol{p}},$$

for every 4-momentum $(E/c, \boldsymbol{p})$.

Moreover, concerning the explicit form of energy operator $\hat{E}$, we know that there exists one unique continuous operator $\hat{E} \in \mathcal{L}(\mathcal{S}'(\mathbb{M}_4))$ such that

$$\hat{E}\beta = \mathrm{E}\beta,$$

where E/c represents the first projection of the 4-momentum vector space $\mathbb{P}_4$.

The operator $\hat{E}$ expands as follows, in the De Broglie basis β:

$$\hat{E}(\psi) = \int_{\mathbb{P}_4} \mathrm{E}(\psi)_\beta \, \beta,$$

for every tempered distribution ψ. We can, consequently, formulate the following proposition.

Proposition 1. *The value of the energy operator $\hat{E}$, at any tempered wave ψ, equals the superposition of the De Broglie family β with respect to the tempered distribution $\mathrm{E}(\psi)_\beta$—obtained from the product of the energy projection*

$$\mathrm{E} : \mathbb{P}_4 \to \mathbb{C} : p \mapsto cp^0$$

times the representation of ψ in the basis β itself.

Interlude: The 4-Position Operator

We conclude this section with a note on symmetry. Energy and momentum operators find their counterpart, with respect to the Minkowski transform, in the 4-position operator:

$$X : \mathcal{S}'(\mathbb{M}_4) \to \mathcal{S}'(\mathbb{M}_4)^4 : \psi \mapsto \mathrm{x}\psi,$$

where x represents (again) the immersion of the Minkowski vector space $\mathbb{M}_4$ in $\mathbb{C}^4$. The *time-location operator* X^0/c corresponds directly to the energy operator $\hat{E}$ by "Minkowski conjugation" or Heisenberg pairing. The eigenbasis of X is the Dirac basis δ, indexed by $\mathbb{M}_4$ itself.

4. Results: Solution Space of the Schrödinger Equation

In this section, we desire to focus upon the subspace S of the waves ψ that could be actual states of the free particle, from an energy point of view, in the sense that they fulfill the Einstein energy relation

$$E = c\sqrt{|p|^2 + (m_0 c)^2}\,, \tag{4}$$

which, in operator form, can be translated into the Schrödinger relation

$$\hat{E}\psi = \hat{H}\psi.$$

This section also presents a natural definition, by means of superpositions, of the restriction of the Hamiltonian operator $\hat{H}$ to the subspace $\mathcal{X}$ Schwartz generated by those eigenvectors of $\hat{H}$ satisfying the relativistic Schrödinger equation.

4.1. The Family χ

Our relativistic Hamiltonian operator of a free particle reveals, in particular, to be measurable upon the De Broglie waves

$$\chi_{\boldsymbol{p}} = \beta_{(H_{\boldsymbol{p}}/c,\, \boldsymbol{p})},$$

of definite momentum $\boldsymbol{p}$ and energy $H_{\boldsymbol{p}}$. Note that these waves respect the Einstein energy relation in operator form

$$\hat{E}\chi_{\boldsymbol{p}} = H_{\boldsymbol{p}}\chi_{\boldsymbol{p}}.$$

We already proved, in fact, that the eigenvalue of the operator $\hat{H}$, corresponding to eigenvector $\chi_{\boldsymbol{p}}$, equals the Einstein value $H_{\boldsymbol{p}}$.

Clearly, we are induced to think that the Schwartz span of the family χ will play a special role in the resolution of the relativistic Schrödinger equation; we shall see that our intuition reveals to not be so bad, although the solution space of the relativistic Schrödinger equation will offer us some surprises.

In order to consider the Schwartz span of χ, we observe that the family χ is a Schwartz family. Indeed, observe, first of all, that the set of all 4-momenta p satisfying the Einstein energy equation is a three-dimensional smooth manifold G:

$$G = \{(E/c, \boldsymbol{p}) \in \mathbb{P}_4 : E = H_{\boldsymbol{p}}\}.$$

Now, introducing the parameterization

$$h : \mathbb{R}^3 \to \mathbb{P}_4 : \boldsymbol{p} \to (H_{\boldsymbol{p}}/c, \boldsymbol{p}),$$

we obtain

$$\chi(\phi)(\boldsymbol{p}) = \chi_{\boldsymbol{p}}(\phi) = \beta_{(H_{\boldsymbol{p}}/c,\, \boldsymbol{p})}(\phi) = \beta(\phi)(h(\boldsymbol{p})),$$

for every test function ϕ. Hence, we see

$$\chi(\phi) = \beta(\phi) \circ h.$$

In other terms, the image of ϕ by χ is the Schwartz function $\beta\phi : \mathbb{P}_4 \to \mathbb{C}$ composed with the parameterization

$$h : \mathbb{R}^3 \to \mathbb{P}_4 : \boldsymbol{p} \to (H_{\boldsymbol{p}}/c, \boldsymbol{p}).$$

This composition is surely smooth, and it is Schwartz because the parameterization is asymptotically flat and of order 1.

We remark that χ is Schwartz linearly independent because it is a section of a another Schwartz linearly independent family. Henceforth, we can talk about the coordinate distribution ψ_χ of any point in its Schwartz span.

Now, we can prove our first step towards the resolution of the equation $\mathcal{E}_H$.

Proposition 2. *The Schwartz span of the family χ is contained in the solution space S of the relativistic Schrödinger equation.*

Proof. Indeed,

$$\hat{E}\psi = \hat{E}\int_{\mathbb{R}^3} (\psi)_\chi \, \chi = \int_{\mathbb{R}^3} (\psi)_\chi \, \hat{E}\chi = \int_{\mathbb{R}^3} (\psi)_\chi \, \hat{H}\chi = \hat{H}\int_{\mathbb{R}^3} (\psi)_\chi \, \chi = \hat{H}\psi,$$

for every ψ in the Schwartz span of χ. □

4.2. The Operator $\hat{\mathbf{H}}$

Let us introduce the function

$$\mathbf{H} : \mathbb{R}^3 \to \mathbb{C} : \boldsymbol{p} \mapsto H_{\boldsymbol{p}}.$$

If the rest mass m_0 of our free particle is strictly positive, the operator $\hat{\mathbf{H}}$ corresponding to the Hamiltonian $\mathbf{H}$ and to the Schwartz family

$$\chi = (\beta_{(H_{\boldsymbol{p}}/c,\, \boldsymbol{p})})_{\boldsymbol{p} \in \mathbb{R}^3}$$

is defined by

$$\hat{\mathbf{H}} : \mathcal{X} \to \mathcal{X} : \psi \mapsto \int_{\mathbb{R}^3} \mathbf{H}\, (\psi)_\chi \, \chi,$$

for every wave ψ in the Schwartz linear span $\mathcal{X} = {}^{\mathcal{S}}\mathrm{span}\chi$. Observe that the above superposition belongs indeed to $\mathcal{X}$, because it is a superposition of the family χ.

The operator $\hat{\mathbf{H}}$ is characterized, on the subspace $\mathcal{X}$ of the tempered distribution space $\mathcal{S}'(\mathbb{M}_4)$, by

$$\hat{\mathbf{H}}\chi_{\boldsymbol{p}} = \mathbf{H}(\boldsymbol{p})\, \chi_{\boldsymbol{p}} \; = H_{\boldsymbol{p}}\, \chi_{\boldsymbol{p}},$$

for every momentum $\boldsymbol{p}$ in $\mathbb{R}^3$; and it reveals, obviously, the endomorphism restriction of $\hat{H}$ to the subspace $\mathcal{X}$ (the bilateral restriction is possible just because $\mathcal{X}$ is invariant under $\hat{H}$).

4.3. Solution of the Schrödinger Equation

Let us show now the complete solution of the relativistic Schrödinger equation.

Theorem 3. *Let $\mathcal{E}_H$ be the (relativistic) Schrödinger equation corresponding to the relativistic Hamiltonian H. Then, we see the following:*

1. *The closed subspace*
$$S = \mathrm{sol}(\mathcal{E}_H),$$
the solution space of the relativistic Schrödinger equation, contains the closed (Schwartz) linear span of the section
$$\chi : \boldsymbol{p} \mapsto \chi_{\boldsymbol{p}} = \beta_{(H_{\boldsymbol{p}}/c,\boldsymbol{p})}$$
of the De Broglie family
$$(E/c, \boldsymbol{p}) \ \mapsto \ \beta_{E/c,\boldsymbol{p}} = [e^{(\imath/\hbar)(\boldsymbol{p}\cdot\mathbf{x}-Et)}];$$
2. *The solution space of the Schrödinger equation is the solution of the distribution division problem*
$$(E - H)(\psi)_\beta = 0,$$
with $\psi \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$, where $(\psi)_\beta \in \mathcal{S}'(\mathbb{P}_4, \mathbb{C})$ is the representative of the tempered wave ψ—in the 4-momentum space $\mathbb{P}_4$—according to the de Broglie family β;
3. *The solution space of the relativistic Schrödinger equation is contained in the space of all tempered distributions $\psi \in \mathcal{S}'(\mathbb{M}_4, \mathbb{C})$ whose component system ψ_β vanishes outside of the inverse graph of the function $\mathbf{H}/c$;*
4. *The solution space of the Schrödinger equation $\mathcal{E}_H$ strictly contains the closed linear span $\bar{\chi}$;*
5. *In two dimensions, the solution space S, represented by β in $\mathbb{P}_2$, is the sum*
$$\mathrm{sol}_{\mathbb{P}_2}(\mathcal{E}_H) = \overline{\mathrm{span}(\delta_G)} + \mathrm{span}(\delta'_{(m_0c,0)});$$
6. *In four dimensions, the solution space S, represented by β in $\mathbb{P}_4$, is*
$$\mathrm{sol}_{\mathbb{P}_4}(\mathcal{E}_H) = \overline{\mathrm{span}(\delta_G)} + \sum_{j=1}^{3} \mathrm{span}\left(\frac{\partial}{\partial \mathbf{p}}_j \delta_p\right)_{p \in P_j}.$$
where
$$P_j = \{(H_{\boldsymbol{p}}, (\mathbb{I}_3 - \mathbb{I}_3^j)\boldsymbol{p}) \in \mathbb{P}_4 : \boldsymbol{p} \in \mathbb{R}^3\},$$
with $\mathbb{I}_3$ being the identity 3-matrix and $\mathbb{I}_3^j$ being the elementary matrix whose only term different from zero is a unit placed in position (j, j).

Proof. The proof follows the points of the thesis.

1. We already proved that the Schwartz linear span $\mathcal{X}$ is contained in the solution space S; therefore, the subspace S reveals to be nonempty and infinite-dimensional. Since S is a closed subspace (since it is the kernel of the linear continuous operator $\hat{E} - \hat{H}$), the closure of the Schwartz linear span of χ still remains inside S.

2. Let us prove the characterization of S as a division problem. To this end, consider any wave $\psi \in \mathcal{S}'_4$. The distribution wave ψ lives in the solution space S if it satisfies the Einstein–Schrödinger equation
$$\hat{E}\psi = \hat{H}\psi;$$

explicitly, the above equation reads

$$\int_{\mathbb{P}_4} \mathrm{E}(\psi)_\beta \beta = \int_{\mathbb{P}_4} H(\psi)_\beta \beta,$$

where

$$\mathrm{E} : \mathbb{P}_4 \to \mathbb{C} : p = (E/c, \boldsymbol{p}) \mapsto E$$

stands for the first canonical coordinate of the 4-momentum space $\mathbb{P}_4$ multiplied by c. Therefore, because of the Schwartz linear independence of the de Broglie family β, we deduce that the wave ψ lives in S if and only if

$$\mathrm{E}(\psi)_\beta = H(\psi)_\beta,$$

that is, if

$$(\mathrm{E} - H)(\psi)_\beta = 0_{\mathcal{S}'}.$$

3. From the above equality, we infer that a necessary (not sufficient) condition for ψ to belong in S is that the coordinate system $(\psi)_\beta$ must vanish upon the complement of the level zero (null set) of the smooth function

$$\mathrm{E} - H : \mathbb{P}_4 \to \mathbb{C} : (E/c, \boldsymbol{p}) \mapsto E - \mathbf{H}(\boldsymbol{p}).$$

We recall here that a distribution u vanishing outside a closed set C means that u sends to 0 every test function with compact support contained in the complement of C. Now, the above level zero is the set

$$G = \{(E/c, \boldsymbol{p}) \in \mathbb{P}_4 : E/c = (\mathbf{H}/c)(\boldsymbol{p})\}.$$

Thus, we observe that the level zero above is nothing more than the inverse graph (symmetric graph) of the function $\mathbf{H}/c$, a graph which lives in the 4-momentum space $\mathbb{P}_4$. Moreover, we can write

$$\psi = \int_{\mathbb{P}_4} (\psi)_\beta \, \beta = \int_{\Omega} (\psi)_\beta \, \beta,$$

for every open neighborhood Ω of the inverse graph

$$G := \mathrm{gr}^-(\mathbf{H}/c).$$

We observe that the distribution ψ reveals to be supported by G and can act upon the smooth vector family β.

4. We desire to prove now that the closed linear span of the family χ remains strictly included in the solution space S. First of all, we note that a wave ψ belongs to the Schwartz span $\mathcal{X}$ if

$$\psi_\beta \in {}^{\mathcal{S}}\mathrm{span}(\delta_G),$$

where

$$\delta_G : G \to \delta(G) : p \mapsto \delta_p$$

represents the section of the Dirac family δ determined by the graph G. Consequently, the wave ψ belongs to the closed span $\bar{\mathcal{X}}$ if

$$\psi_\beta \in \overline{{}^{\mathcal{S}}\mathrm{span}(\delta_G)}.$$

Now, let M be the subset

$$\delta(G) = \{\delta_p\}_{p \in G}.$$

From the "biorthogonal theorem", we know that

$$\overline{{}^{\mathcal{S}}\mathrm{span}(\delta_G)} = (M^*)^*,$$

where (following Dieudonne) by * we denote the orthogonal in the duality $(\mathcal{S}'_4, \mathcal{S}_4)$. Explicitly, the orthogonal M^* is the subspace of all test functions ϕ which are orthogonal to the family δ_G—in other terms, all the test functions vanishing over G. Therefore, the biorthogonal M^{**} is the subspace of all waves vanishing at the test functions sending each point of the graph G to 0. Clearly, we can find waves ψ belonging to the solution space S but with their representations ψ_β (in the basis β) not belonging to M^{**}. Indeed, for a fixed 3-momentum $\boldsymbol{p}$, consider, for example, the partial derivative

$$\omega^j_{\boldsymbol{p}} = \frac{\partial}{\partial \mathbf{p}_j} \delta_{(H_{\boldsymbol{p}}/c, \boldsymbol{p})},$$

for some $j \in I^*_4 = \{1, 2, 3\}$. Clearly, the distribution $\omega^j_{\boldsymbol{p}}$ vanishes outside of G and, consequently, the wave

$$\psi^j_{\boldsymbol{p}} = \int_{\mathbb{P}_4} \omega^j_{\boldsymbol{p}}\, \beta,$$

could reveal a solution of $\mathcal{E}_H$; indeed, it is, if $\boldsymbol{p}_j = 0$. Nevertheless, $\omega^j_{\boldsymbol{p}}$ does not vanish upon each test function which simply sends every point of G to 0: since the distribution $\omega^j_{\boldsymbol{p}}$ takes also into account the values of the partial j-derivative of test functions and not only the mere values of the test functions. Hence, the wave $\psi^j_{\boldsymbol{p}}$ does not belong to $\bar{\mathcal{X}}$.

5. Let us solve the equation in $\mathbb{M}_2$. First, note that for every $\boldsymbol{p}$ different from 0, the only solutions of our division problem supported by the point $(H_{\boldsymbol{p}}/c, \boldsymbol{p})$ are the distributions proportional to the basis vector $\delta_{(H_{\boldsymbol{p}}/c, \boldsymbol{p})}$. We need only to examine the case $\boldsymbol{p} = 0$. In dimension 1, we know well that

$$f\delta_0 = f(0)\delta_0, \;\; f\delta'_0 = -f'(0)\delta_0 + f(0)\delta'_0, \;\; f\delta''_0 = f''(0)\delta_0 - 2f'(0)\delta'_0 + f(0)\delta''_0.$$

Now, at the point $p^* = (m_0 c, 0)$, we see, for $f = E - H$,

$$f(p^*) = 0, \;\; \frac{\partial}{\partial \mathbf{p}} f(p^*) = 0, \;\; \frac{\partial^2}{\partial \mathbf{p}^2} f(p^*) \neq 0,$$

more specifically

$$\frac{\partial^{2k}}{\partial \mathbf{p}^{2k}} f(p^*) \neq 0, \;\; \frac{\partial^{2k-1}}{\partial \mathbf{p}^{2k-1}} f(p^*) = 0$$

for $k > 0$. Therefore, our equation reveals to also be verified by the elements of the straight line

$$\mathrm{span}(\delta'_{p^*}).$$

The solution space of the division problem fu is the sum

$$S_\beta = \overline{\mathrm{span}(\delta_G)} + \mathrm{span}(\delta'_{(m_0 c, 0)}).$$

6. To find find the wave ψ represented by the doublet $\delta'_{(m_0 c, 0)}$ (we indicate the partial derivation with respect to the 1-momentum argument by a prime), we see

$$\psi = \int_{\mathbb{P}_2} \delta'_{(m_0 c, 0)}\, \beta = {}^t\hat{\beta}(\delta'_{(m_0 c, 0)}) = \mathcal{M}(\delta'_{(m_0 c, 0)}),$$

where $\mathcal{M}$ represents the Minkowski–Fourier transform

$$\mathcal{M} : \mathcal{S}'(\mathbb{P}_2) \to \mathcal{S}'(\mathbb{M}_2) : u \mapsto u \circ \hat{\beta}.$$

It reveals to be defined by

$$< \mathcal{M}(u), \phi > = < u, \beta(\phi) > = < u, (\beta_p \phi)_{p \in \mathbb{P}_2} > = < u, \left(\int_{\mathbb{M}_2} \beta_p \phi \right)_{p \in \mathbb{P}_2} > .$$

The test function $\beta\phi$ inside the brackets acts as follows:

$$p \mapsto \int_{\mathbb{M}_2} e^{(\imath/\hbar)\boldsymbol{\mu}(p,\mathrm{x})} \phi \, d\mathrm{x},$$

if we apply the distribution δ'_{p^*}, we obtain

$$\delta'_{p^*}(\beta\phi) = -(\beta\phi)'(p^*) = -(\imath/\hbar) \int_{\mathbb{M}_2} \mathbf{x} e^{(\imath/\hbar)\boldsymbol{\mu}(p^*,\mathrm{x})} \phi \, d\mathrm{x}.$$

From the above equality, we try

$$\psi = -(\imath/\hbar)[\mathbf{x} e^{(\imath/\hbar)\boldsymbol{\mu}(p^*,\mathrm{x})}],$$

that is the regular tempered distribution generated by the $\mathcal{O}_M(\mathbb{M}_2)$ function

$$-(\imath/\hbar)\mathbf{x} f_{p^*} = -(\imath/\hbar)\mathbf{x} e^{-(\imath/\hbar)(m_0 c^2)\mathrm{t}},$$

where, as usual, $\mathbf{x}$ and t are the spatial and time complex projections

$$\mathbf{x} : (ct, \boldsymbol{x}) \mapsto \boldsymbol{x} + 0\imath \, , \quad \mathrm{t} : (ct, \boldsymbol{x}) \mapsto t + 0\imath.$$

7. Analogously, the solution space of our relativistic equation in the 4-momentum space $\mathbb{P}_4$ is

$$\mathrm{sol}_{\mathbb{P}_4}(\mathcal{E}_H) = \overline{\mathrm{span}(\delta_G)} + \sum_{j=1}^{3} \mathrm{span}\left(\frac{\partial}{\partial \mathbf{p}}_j \delta_p \right)_{p \in P_j} .$$

where

$$P_j = \{ (H_{\boldsymbol{p}}, (\mathbb{I}_3 - \mathbb{I}_3^j)\boldsymbol{p}) \in \mathbb{P}_4 : \boldsymbol{p} \in \mathbb{R}^3 \},$$

with $\mathbb{I}_3$ is the identity 3-matrix and $\mathbb{I}_3^j$ is the elementary matrix whose only term different from zero is a unit placed in position (j, j). The proof ends here. □

5. Conclusions and Epilogue

The present manuscript deals with the classic problem of finding a self-consistent "square root" Hamiltonian for a relativistic zero-spin particle. Indeed, we desire to point out that we have also extended the same equation to the square and cube powers of tempered distribution space (upon Minkowski space), say S', obtaining two other Lorentz-invariant relativistic equations, which also address also 1/2-spin and 1-spin particles, again with a mass different from 0. In other terms, we have recognized that the same equation form is valid in the description of Fermions (for instance), obtaining the positive energy spinor solutions (two components out of four) of the celebrated Dirac equation, while the negative energy spinor solutions come from the associated conjugate relativistic Schrödinger equation (in the suitable power 2 of distribution space S').

We specifically propose a proficient approach based on the tempered distribution theory (developed by mathematicians like Schwartz and Dieudonné) and, in particular, we use the main definitions and results of Schwartz Linear Algebra and Schwartz Spectral theory, introduced by D. Carfì in [23].

We desire to underline that the general distribution treatment of QM and QFT is widely adopted, more or less consciously, (Schwartz, Simon, Zeidler, Shankar, Dirac and so on) but, on one hand, the distribution theory is used more as a tool than a foundational approach and, on the other, the use is limited to a calculus perspective more than to a structural standpoint, as is the Hilbert space theory, for example.

5.1. Square Root Operator in QM and QFT

The idea of a square root operator is not new, and every QFT book starts with an introduction to this topic. We desire to clarify the good points of our approach with respect to the previous approaches to square roots and functional calculus in QM and QFT.

We want to observe here that, essentially—in the QM literature—we find the following ways to introduce the square roots of operators, both in quantum mechanics and quantum field theory:

- Practical and "ad hoc" methodologies, mostly based on the presumed strict parallelism between infinite-dimensional complex Hilbert spaces and finite-dimensional complex Euclidean spaces. Such methodologies reveal to be particularly efficient when adopted in tanded with experimental knowledge and heuristic techniques, but they are completely lacking a solid mathematical theory or crystalline universal definitions. Clearly, for the correct statement and resolution of a differential equation, such methods cannot be enough, because—for instance—we do not even know where we are working and what kind of functional objects we need to consider in order to find the actual solutions of the equation, nor do we know the topological properties (if any) of the involved operators:
- Spectral theories in separable Hilbert spaces;
- Spectral theories in separable Banach spaces;
- Algebraic methods based on finite dimensional matrices that, essentially, avoid the problematic definition of the square roots of differential operators by constructing higher-dimensional "perfect squares" operators which lie "under square roots". These methodologies allow us to introduce and solve a higher-dimensional differential problem associated with the original fractional differential equation and (somehow) also give the solutions of the original problem. The Dirac method of formulating its celebrated equation falls in this category, as well as Foldy's approach (with its generalizations and variants from many authors).

5.2. Necessity of Distribution Spaces and Topology for Relativistic QM

We immediately observe that the above classic Hilbert/Banach space methods cannot be used for the position and resolution of the free relativistic Schrödinger equation, because the very fundamental solutions of this equation cannot be framed in a Hilbert or Banach space context, as we explain later.

Now, we can justify the necessity—the absolute necessity—to use distribution theory, for (at least) two main reasons:

1. If we lock ourselves down in separable Hilbert space theory, we cannot hope to satisfactorily solve (from a physical point of view) the relativistic Schrödinger equation for free particles. The simple reason is that the very main (and generating) solutions of the relativistic Schrödinger equation for free particles cannot be considered as elements of a separable Hilbert Space:
    - First of all, if we desire to consider the standard L^2 product, we immediately observe that the de Broglie waves do not belong to the space of square integrable functions.
    - Moreover, the set of all harmonic waves is a continuous set (not a discrete one) and, if we select its "naturally orthonormal" subfamily (that generating the unitary Minkowsky–Fourier transform as integral kernel), we are again obtaining a continuous family that should be orthogonal by right, from a physical perspective, but cannot be as such in any separable Hilbert space (even different from L^2); we cannot find continuous orthonormal families in a separable Hilbert space, only discrete orthonormal systems!

- Furthermore, even forcing the matter and considering a Hilbert space generated by all those "unitary orthogonal waves", we would obtain a nonseparable Hilbert space, which would enormously complicate the matter from a functional calculus point of view, because we have no reasonable or natural spectral theory for nonseparable Hilbert spaces. We are not saying, here, that we should not use nonseparable Hilbert spaces in Quantum mechanics, but we see that they do not help in the formulation and resolution of the relativistic Schrödinger equation.

The analogous problems we would risk to face if we lock ourselves down in separable Normed Space theory are asfollows: it is very hard to keep, in a unique functional theory, a reasonable and convenient separable norm with a good spectral theory and the presence of the continuous family of de Broglie waves.
This is a general problem in quantum mechanics and quantum field theory: when we consider harmonic waves and related differential equations (or operator equations), we, theoretical physicists, actually do not use Hilbert Space theory—and we (somehow) know it—instead, we use smooth function theory, differentiable function theory, working, essentially, with calculus techniques and distribution theory.
Here, we have another general problem of Hilbert space theory in QM: even when we solve the classic Dirac free equation, the manipulations and resolutions proceed, essentially, in a differentiable theory context. Indeed, the basis solutions of the free Dirac equation are bispinors constructed by harmonic waves, and then, automatically, we work out of the Hilbert space theory.
Moreover, when we work out of the Hilbert space theory, we also work out of the spectral theory on Hilbert spaces.
Consequently, we cannot expect to find a correct and unambiguous definition of the square root of an operator by Hilbert space techniques, if the domain of such an operator should contain the de broglie waves, because in this case, we are playing outside of any separable Hilbert space.
It is not the case (and it does not surprise at all) that Dirac's equation was solved by smooth calculus and finite algebraic methods rather than infinite-dimensional Hilbert space techniques.
In order to define the square roots of differential operators in a quantum mechanical context (where we need to manage harmonic waves, eigenstates of position operator, continuous spectra and so on. . .), we need a spectral theory constructed elsewhere, not in Hilbert spaces.

2. In some way, quantum mechanics needs to coordinate and put together two apparently incompatible aspects: the state space of a quantum system can be generated by both discrete and continuous bases: the position and momentum basis $((|x>), (|p>))$ are continuous, while the the Hermite function basis is discrete.
Very often, we read "let's solve the harmonic oscillator problem in the position basis", or "let's solve the harmonic oscillator problem in the momentum basis", which are continuous basis (in some sense to be correctly defined), only to see, after a while, that the harmonic oscillator is solved by the discrete Hermite function basis of L^2 (it would be better to say of the Schwartz function Space S).
How can the position basis and momentum basis (that completely stay out of L^2) generate the same state space generated by the L^2 Hermite function basis?
In what sense can a continuous family of vectors generate a functional space?
The position eigenstates are not even functions, they are measures.
In what space are we moving?
Is the state space separable or not?
What is its Hilbert dimension, $\aleph_0$ or $\aleph_1$?
How could a separable Hilbert (or Banach) space contain "non-normalizable" vectors and continuous orthogonal families of non-normalizable vectors that, from a physical point of view, simply represent the certainty to observe a specific result?

In tempered distribution spaces, we know that the Hermite function family is a discrete basis in a rightful algebraic–topological sense; it is a total family, and it is also a basis in a generalized Hilbert sense (with respect to the tempered distribution topology).Moreover, the position and momentum basis lives in S' and generate S' in the Schwartz Linear Algebraic sense: S' is a separable topological vector space, it is wonderfully generated by a discrete and continuous basis, in two different rigorous and operative meanings and, by the way, exactly the meaning used practically by quantum physicists, in a more heuristic way.
In addition to this, in the distribution approach, any quantum mechanics observable are a continuous and everywhere-defined operator, while in the Hilbert space approach, we almost surely face discontinuous (unbounded) operators and very strange, unnatural domains, even for the most simple observables (position, momentum, ladder operators, number operator and so on and so forth).
Consequently, in Hilbert spaces, we face any kind of difficulties, even to add or multiply two straightforward operators such as a derivative operator and the position operator—which show different domains—and that without, and well before, coming to ask "what the principal square root of a discontinuous, non-everywhere-defined, not properly hermitian, densely defined (or perhaps closable) operator is".

5.3. Schwartz Linear Algebra

What does our new Schwartz Linear Algebra theory clarify compared with previous methods?

The new theory definitely clarifies where we are working, in quantum theories and quantum field theory, especially in relativistic quantum mechanics.

We clarify that the state spaces of quantum objects are not Hilbert spaces (if we exclude the finite-dimensional spaces that, anyway, are subspaces of the tempered distribution space S') but powers of tempered distribution spaces.

Fortunately, tempered distribution spaces contain a lot of good inner product spaces, useful in the evaluation of probabilities and expectation values for quantum mechanics. By those inner products, we finally understand the possible way to properly "normalize" the eigenvectors of the position and momentum operators (eigenstates that should be normalizable because of their straightforward physical meaning: they simply represent certainty).

First of all, we solve the problem of evolution in the tempered distribution space. Then, when necessary, we find the right inner product subspace of tempered distribution space in which we calculate the probabilities and expectation values.

5.4. No Pathological Features Associated with the Distributional Square Root Operator

To prove that the new formalism has no pathological features, we need general theorems and correct definitions.

The principal square root of (continuous) Schwartz diagonalizable strictly positive operators is always well-defined; it never shows pathological features.

The principal square root of (continuous) Schwartz diagonalizable positive (non-negative, just to clarify) operators is always well-defined, but it needs more work and caution. It shows no pathological features, but the domain is a subspace of the tempered distribution space. In this more "dangerous" case, we need to restrict our attention (domain) only to the subspace generated by the eigenvectors not belonging to the zero eigenvalue, roughly speaking.

Fortunately, to solve our problem of a relativistic Schrödinger equation for massive particles, we need only the first simpler definition of a square root of strictly positive operators.

By the way, in the definition of the square root of Schwartz nondefective strictly positive operators, we do not find any complication, because we have a general theorem of Schwartz Linear Algebra assuring us of the following:

"for any choice of a smooth, slowly increasing function a and any Schwartz basis b of the tempered distribution space, there exists a Schwartz linear operator A such that the

Schwartz basis b is an eigensystem of the operator A, corresponding to the eigenvalue family a."

Therefore, in order to define the square root of a strictly positive operator, we can go (with practical physicists) and affirm that: "the principal square root of a strictly positive operator A is the only Schwartz diagonalizable operator which has the square root of the eigenvalue system a as its eigenvalue system corresponding to the same Schwartz eigenbasis of A".

Indeed, this is the most natural definition of the square root of an operator in the finite-dimensional case.

What about the problem of external fields which makes the previous operators often badly defined?

Well, a good point of the theory is that we can discern perfectly among the cases in which we can define the square root of an operator, or not.

For example, if we prove that the perturbed operator inside the square root is Schwartz diagonalizable and strictly positive, then we know that the square root operator is well-defined. And we can work inside the realm of tempered distributions to find, for example, actual converging approximations of the square root operator, in the right topology, with the right object in the right context. The rest is usual, well-established, mathematical hard work.

We desire to observe that a lot of problems arise when one is working with operators and other functional objects without knowing to what space they belong, without knowing the topology one is dealing with: it is almost sure that one will end up in a mess.

5.5. Special Cases to Compare the New Approach with Previous Ones

We give an indicative example in the "discrete case"—we already observed that in the continuous case we have no match, because separable Hilbert spaces are intolerant to continuous orthonormal families in every respect and meaning.

Consider for example any ladder operator L. We know well that there exists an explicit discrete L^2-orthonormal eigenbasis b of L.

We know that every element f of L^2 is of the form

$$f = \sum < b|f > b,$$

where the denumerable sum is intended in the L^2 topology. So, we would like to apply L to both sides,

$$L(f = \sum < b|f > b),$$

in order to obtain the natural spectral decomposition of L:

$$L(f) = \sum < b|f > L(b) = \sum < b|f > l(b)b, \tag{5}$$

where $L(b) = l(b)b$. Unfortunately, we cannot, because L is not L^2-continuous, and hence, the operator L cannot overcome the L^2-sum. Consequently, we cannot even define a functional calculus for the operator L, in its natural form:

$$g(L)(f) := \sum < b|f > g(l(b))b, \tag{6}$$

because the definition is sick even when g is the identity function.

Note indeed that the summations in (5) and (6) are not always well-defined or convergent in Hilbert spaces.

We lose, in Hilbert space, the natural way to define a functional calculus, and also the natural way to define the principal square root when $l(b) > 0$, by simply putting

$$g = \sqrt{\cdot}\,.$$

On the other hand, if we work in S', L is continuous and the above series expansion should be interpreted in the distribution sense; so, any ladder operator L is compatible with the Hermite distribution expansion and an efficient functional calculus can be defined—in the natural unsurpassed way—and we define the square roots of operators by the simple relation

$$\sqrt{L}(f) := \sum < b|f > \sqrt{l(b)}b,$$

as soon as the function l is strictly positive. The above definitions should be interpreted in the distribution topology (the easy pointwise topology).

So, we see that even in a realm apparently covered by Hilbert spaces, the distribution approach is better, easier and by far more complete.

Another example is a "student problem", which is very hard in Hilbert space (indeed, not meaningfully solvable there) and extremely simple in distribution spaces: "find the square root of number operator N".

The immediate obvious answer would be that, since

$$N = \sum n\ |n >< n|, \tag{7}$$

where $(|n >)$ is the Hermite function basis, then

$$\sqrt{N} = \sum \sqrt{n}\ |n >< n|. \tag{8}$$

The problem here is again that the number of operators N is L^2-unbounded (discontinuous) and, consequently, the above relation (7) is not true, and we do not even know when the above summations are convergent in Hilbert space.

So, in order to solve that "student problem" in L^2, we need general spectral theory for unbounded operators in separable Hilbert spaces, spectral measures, general defined spectrum of operators, resolvent operators, unnatural domains and other amenities.

On the contrary, the above two relations (7), (8) are straightforwardly true in Schwartz distribution space (without any problem of domains, convergence or continuity), and, consequently, the problem is immediately solved–in distribution spaces–in its most natural and desirable physical form.

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Appendix A. Hamiltonian 4-Vector Formulation

In this relativistic Hamiltonian context, we need to define the classic 4-vectors associated with the relativistic free particle using only its Hamiltonian function and the momentum variable.

Definition A1 (Lorentz factor determined by relativistic three-momentum)**.** *Consider a free particle with nonzero rest mass*

$$m_0 = \boldsymbol{p}_0/c.$$

We define the Lorentz factor corresponding to any relativistic 3-momentum $\boldsymbol{p} \in \mathbb{R}^3$ *as the ratio*

$$\gamma_{\boldsymbol{p}} := \frac{H_{\boldsymbol{p}}}{c\boldsymbol{p}_0} = \frac{|(\boldsymbol{p}_0, \boldsymbol{p})|}{\boldsymbol{p}_0}.$$

Definition A2. *In the conditions of the above definition, we define the following:*

- *The relativistic mass*

$$m^0_{\boldsymbol{p}} := \gamma_{\boldsymbol{p}} m_0,$$

corresponding to $\boldsymbol{p}$*;*

- *The relativistic velocity*

$$\mathbf{v}_{\boldsymbol{p}} := \boldsymbol{p}/m^0_{\boldsymbol{p}}\,,$$

corresponding to the momentum $\boldsymbol{p}$*;*

- *The 4-velocity*

$$v_{\boldsymbol{p}} := \gamma_{\boldsymbol{p}}\,(c, \mathbf{v}_{\boldsymbol{p}})\,,$$

corresponding to the momentum $\boldsymbol{p}$*;*

- *The 4-momentum*

$$p_{\boldsymbol{p}} = m_0 v_{\boldsymbol{p}} = (\gamma_{\boldsymbol{p}}\,\boldsymbol{p}_0, \boldsymbol{p})\,,$$

corresponding to the momentum $\boldsymbol{p}$*.*

Definition A3. *It also reveals to be convenient for introducing the 4-energy and 4-mass of our particle*

$$E_{\boldsymbol{p}} := (\gamma_{\boldsymbol{p}}\,m_0c^2, c\boldsymbol{p}) = c\,p_{\boldsymbol{p}}\,, \quad m_{\boldsymbol{p}} := (\gamma_{\boldsymbol{p}}\,m_0, \boldsymbol{p}/c) = E_{\boldsymbol{p}}/c^2.$$

All the above-defined 4-quantities induce contravariant 4-vectors, in the sense that they respect (follow) the Lorentz transformation when passing from a Lorentz frame to another.

We desire to remark that it is revealed to be impossible to define the 4-position of the particle or other related observables by only using the momentum-related 4-vectors.

Proposition A1. *We observe that*

$$\boldsymbol{\mu}(m_{\boldsymbol{p}}) = m_0,\ \boldsymbol{\mu}(v_{\boldsymbol{p}}) = c,\ \boldsymbol{\mu}(p_{\boldsymbol{p}}) = m_0c,\ \boldsymbol{\mu}(E_{\boldsymbol{p}}) = m_0c^2,$$

for every relativistic momentum vector $\boldsymbol{p} \in \mathbb{R}^3$*, while*

$$H_{\boldsymbol{p}} = E^0_{\boldsymbol{p}} = \boldsymbol{E}_0\gamma_{\boldsymbol{p}} = \boldsymbol{\mu}(E_{\boldsymbol{p}})\gamma_{\boldsymbol{p}},$$

where $\boldsymbol{E}_0 = m_0c^2$ *is the rest energy of the particle.*

Remark A1. *The definition of the relativistic Hamiltonians of free particles appears simplified by the use of the* Euclidean norm $|.|$ *of the hybrid space* $\mathbb{R} \times \mathbb{P}^3$ *of all vectors* $(\boldsymbol{p}_0, \boldsymbol{p})$ *(which are not proportional to the 4-momenta* $p_{\boldsymbol{p}}$ *of the particle) defined by the rest momentum* m_0c *and by the relativistic momentum* $\boldsymbol{p}$*. Indeed, we already have seen that*

$$H(p) = c|(\boldsymbol{p}_0, \boldsymbol{p})|,$$

for every 4-momentum p*, with*

$$p = (\gamma_{\boldsymbol{p}} m_0 c, \boldsymbol{p}).$$

Appendix B. Square Root of Strictly Positive Operators

Appendix B.1. Strictly Positive Operators

A Schwartz diagonalizable operator A is called strictly positive (negative) if any its eigenvalues are positive (negative).

For example, the operator

$$\hat{H}^2 : \mathcal{S}_2' \to \mathcal{S}_2' : \hat{H}^2 = c^2(P^2 + (m_0c)^2\mathbb{I})$$

is strictly positive iff $m_0 > 0$.

Appendix B.2. Square Root of Strictly Positive Operators

Let $\mathcal{A}$ be a subspace of the space $\mathcal{S}'_{1+n}$ admitting a Schwartz basis α of topological dimension d. Let

$$A : \mathcal{A} \to \mathcal{A},$$

be a Schwartz diagonalizable operator, let α be a Schwartz eigenbasis of the operator A and let a be the smooth system of eigenvalues corresponding to the eigenbasis α. We know that

$$A\alpha_s = a(s)\alpha_s,$$

for every s in $\mathbb{R}^d$; moreover, we can immediately prove that

$$A(\psi) = \int_{\mathbb{R}^d} a.(\psi)_\alpha \, \alpha,$$

for every $\psi \in \mathcal{A}$.

Definition A4 (Principal root)**.** *In the above conditions, if the operator A is strictly positive, we call the principal root of A the unique operator $\sqrt{A}$ defined over $\mathcal{A}$ by*

$$\sqrt{A}(\psi) = \int_{\mathbb{R}^d} \sqrt{a}.(\psi)_\alpha \, \alpha,$$

for every $\psi \in \mathcal{A}$. Here, for every $y \in \mathbb{R}_>$, the square root $\sqrt{y}$ is the unique positive real z such that $z^2 = y$.

If the operator A is strictly negative, we call the *principal root of* A the unique operator $\sqrt{A}$ defined over $\mathcal{A}$ by

$$\sqrt{A}(\psi) = \int_{\mathbb{R}^d} \sqrt{a}.(\psi)_\alpha \, \alpha,$$

for every $\psi \in \mathcal{A}$. Here, for every $y \in \mathbb{R}_<$, the square root $\sqrt{y}$ is the unique positive imaginary number z such that $z^2 = y$.

We desire to observe that if A is strictly positive, then its opposite $-A$ is strictly negative, and we have

$$\sqrt{-A} = \iota\sqrt{A}.$$

Appendix B.3. Hamiltonian as a Square Root

Now, we can define the relativistic Hamiltonian $\hat{H}$ via the square root operation.

We define $\hat{H}$ as the everywhere-defined operator

$$\hat{H} : \mathcal{S}'_4 \to \mathcal{S}'_4 \; : \; \hat{H} = c\sqrt{|P|^2 + (m_0 c)^2 \mathbb{I}}.$$

It is evident that the energy operator $\hat{E}$ equals the above Hamiltonian operator $\hat{H}$ over the subspace $\mathcal{X}$.

Indeed, observe that

$$\hat{E}(\beta_{E/c,\boldsymbol{p}}) = E\beta_{E/c,\boldsymbol{p}},$$

for every 4-momentum $(E/c, \boldsymbol{p})$ in $\mathbb{P}^4$, while

$$\hat{H}(\beta_{E,\boldsymbol{p}}) = c\sqrt{|\boldsymbol{p}|^2 + (m_0 c)^2} \beta_{E/c,\boldsymbol{p}},$$

for every $(E/c, \boldsymbol{p})$ in $\mathbb{R}^2$. Therefore, when assigned a 4-momentum pair $(E/c, \boldsymbol{p})$, the above two distributions coincide if

$$E = c\sqrt{|\boldsymbol{p}|^2 + (m_0 c)^2},$$

that is, over the section of the family β indexed by the inverse graph of the Hamiltonian function H.

References

1. Schwartz, L. *Oeuvres Scientifiques I, II, III*; American Mathematical Society: Providence, RI, USA, 2011.
2. Schwartz, L. *Mathematics for the Physical Sciences*; Hermann and Addison–Wesley: Boston, MS, USA, 1966.
3. Schwartz, L. *Analyse Hilbertienne*; Hermann: Paris, France, 1979.
4. Schwartz, L. *Application of Distributions to the Theory of Elementary Particles in Quantum Mechanics*; Gordon and Breach: New York, NY, USA, 1968.
5. Schwartz, L. *Théorie des Distributions*; Hermann: Paris, France, 1966.
6. Schwartz, L. *Functional Analysis*; New York University, Courant Institute of Mathematical Sciences: New York, NY, USA, 1964.
7. Barros-Neto, J. *An Introduction to the Theory of Distributions*; Marcel Dekker: New York, NY, USA, 1973.
8. Dieudonné, J. La dualité dans les espaces vectoriels topologiques. *Ann. Sci. l'École Norm. Supérieure* **1942**, *59*, 107–139. [CrossRef]
9. Dieudonné, J.; Schwartz, L. La dualité dans les espaces $(\mathcal{F})$ and $(\mathcal{LF})$. *Ann. l'institut Fourier* **1949**, *1*, 61–101. [CrossRef]
10. Bourbaki, N. *Topologie Générale. (Fascicule de Résultats)*; Hermann: Paris, France, 1953.
11. Bourbaki, N. *Topological Vector Spaces*; Hermann: Paris, France, 1955.
12. Bourbaki, N. *Intégration. Chapitre* 1 *à* 4; Hermann: Paris, France, 1965.
13. Horváth, J. *Topological Vector Spaces and Distributions*; Addison-Wesley Publishing Company: Boston, MA, USA, 1966; Volume 1.
14. Kesavan, S. *Topics in Functional Analysis and Applications*; Wiley: New Delhi, India, 1989.
15. Boccara, N. *Functional Analysis: An Introduction for Physicists*; Academic Press: Boston, MA, USA, 1990.
16. Lang, S. *Real and Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 1993.
17. Trèves, F. *Topological Vector Spaces, Distributions and Kernels*; Dover Books on Mathematics; Dover Publications: Mineola, NY, USA, 2006.
18. Yosida, K. *Functional Analysis*, 6th ed.; Classics in Mathematics; Springer: Berlin/Heidelberg, Germany, 1996.
19. Zeidler, E. *Applied Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 1995; Volume 1,
20. Dirac, P. *The Principles of Quantum Mechanics*; The Clarendon Press: Oxford, UK, 1930.
21. Carfì, D. Topological characterizations of $\mathcal{S}$-linearity. *AAPP | Phys. Math. Nat. Sci.* **2007**, *85*, 1–16. [CrossRef]
22. Carfì, D. $\mathcal{S}$-diagonalizable operators in Quantum Mechanics. *Glas. Math.* **2005**, *40*, 261–301. . [CrossRef]
23. Carfì, D. Dirac-orthogonality in the space of tempered distributions. *J. Comput. Appl. Math.* **2003**, *153*, 99–107. [CrossRef]
24. Carfì, D. $\mathcal{S}$-linear operators in quantum Mechanics and in Economics. *Appl. Sci. (APPS)* **2004**, *6*, 7–20.
25. Shankar, R. *Principles of Quantum Mechanics*; Plenum Press: New York, NY, USA, 1994.
26. Carfì, D. Quantum statistical systems with a continuous range of states. In *Applied and Industrial Mathematics in Italy, Proceedings of the 7th Conference,Venice, Italy, 20—24 September 2004*; Primicerio, M., Spigler, R., Valente, V., Eds.; Series on Advances in Mathematics for Applied Sciences; World Scientific: Singapore, 2005; Volume 69, pp. 189–200. [CrossRef]
27. Antoniou, I.; Prigogine, I. Intrinsic irreversibility and integrability of dynamics. *Physica* **1993**, *A192*, 443–464. [CrossRef]
28. Balescu, R. *Equilibrium and Nonequilibrium Statistical Mechanics*; Wiley & Sons: New York, NY, USA, 1975.
29. Pauli, W. *Die allgemeinen Prinzipien der Wellenmechanik*; Springer: Berlin/Heidelberg, Germany, 1958.
30. Penrose, R. Quantum Mechanics: Foundations. In *Encyclopedia of Mathematical Physics*; Elsevier: Amsterdam, The Netherlands, 2006; pp. 260–265.
31. Prigogine, I. *Non-Equilibrium Statistical Mechanics*; Wiley: New York, NY, USA, 1962.
32. Prigogine, I. *From Being to Becoming: Time and Complexity in the Physical Sciences*; Freeman: San Francisco, CA, USA, 1980.
33. Prigogine, I. *Le Leggi Del Chaos*; Laterza: Rome, Italy, 1993.
34. Namsrai, K.; von Geramb, H. Square-Root Operator Quantization and Nonlocality: A Review. *Int. J. Theor. Phys.* **2001**, *40*, 1929–2010. [CrossRef]
35. Gill, T.L.; Zachary, W.W. Analytic representation of the square-root operator. *J. Phys. A Math. Gen.* **2005**, *38*, 2479. [CrossRef]
36. Gill, T.; Zachary, W.; Lindesay, J. The Classical Electron Problem. *Found. Phys.* **2001**, *31*, 1299–1355. [CrossRef]
37. Gill, T.; Zachary, W. Time-ordered operators and Feynman–Dyson algebras. *J. Math. Phys.* **1987**, *28*, 1459–1470. [CrossRef]
38. Gill, T.; Zachary, W. Foundations for relativistic quantum theory. I. Feynman's operator calculus and the Dyson conjectures. *J. Math. Phys.* **2002**, *43*, 69–93. [CrossRef]
39. Simulik, V. Relativistic wave equations of arbitrary spin in quantum mechanics and field theory, example spin s = 2. *J. Phys. Conf. Ser.* **2017**, *804*, 012040. [CrossRef]
40. Silenko, A.J.; Zhang, P.; Zou, L. Fundamental operators in Dirac quantum mechanics. *J. Phys. Conf. Ser.* **2020**, *1435*, 1–8. . [CrossRef]
41. Schrödinger, E. Quantizierung als Eigenwertproblem. *Ann. Phys.* **1926**, *81*, 109. [CrossRef]
42. Gordon, W. Der Comptoneffekt nach der Schrödingerschen Theorie. *Z. Phys.* **1926**, *40*, 117–133. [CrossRef]
43. Klein, O. Quantentheorie und fünfdimensionale Relativitätstheorie. *Z. Phys.* **1926**, *37*, 895–906. [CrossRef]
44. Fock, V. Zur Schrödingerschen Wellenmechanik. *Z. Phys.* **1926**, *38*, 242–250. [CrossRef]
45. de Donder, T.; van den Dungen, F.H. La quantification déduite de la Gravifique einsteinienne. In *Comptes Rendus*; Max-Planck-Gesellschaft zur Förderung der Wissenschaften: Munich, Germany, 1926; Volume 183, pp. 22–24.

46. Dirac, P.A.M. The quantum theory of the electron. *Proc. R. Soc. Lond.* **1928**, *A117*, 610–624. . [CrossRef]
47. Dirac, P.A.M. The quantum theory of the Electron. Part II. *Proc. R. Soc. Lond.* **1928**, *A118*, 351–361. [CrossRef]
48. Foldy, L.L. *Quantum Theory of Radiation III and High Energy Physics*; Academic Press: New York, NY, USA, 1962.
49. Foldy, L.L. Synthesis of Covariant Particle Equations. *Phys. Rev.* **1956**, *102*, 568. [CrossRef]
50. Foldy, L.L.; Wouthuysen, S.A. On the Dirac Theory of Spin 1/2 Particles and Its Non-Relativistic Limit. *Phys. Rev.* **1950**, *78*, 29. [CrossRef]
51. Case, K.M. Hamiltonian Form of Integral Spin Wave Equations. *Phys. Rev.* **1955**, *100*, 1513. [CrossRef]
52. Chihara, H.; Furuya, T.; Koshikawa, T. Hermite expansions of some tempered distributions. *J. Pseudo-Differ. Oper. Appl.* **2018**, *9*, 105–124. [CrossRef]
53. Catuogno, P.; Olivera, C. Tempered generalized functions and Hermite expansions. *Nonlinear Anal. Theory Methods Appl.* **2011**, *74*, 479–493. [CrossRef]