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Bi-Squashing $S_{2,2}$-Designs into $(K_4 - e)$-Designs

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Abstract: A double-star S_{q_1,q_2} is the graph consisting of the union of two stars, K_{1,q_1} and K_{1,q_2}, together with an edge joining their centers. The spectrum for S_{q_1,q_2}-designs , i.e., the set of all the $n \in \mathbb{N}$ such that an S_{q_1,q_2}-design of the order n exists, is well-known when $q_1 = q_2 = 2$. In this article, $S_{2,2}$-designs satisfying additional properties are investigated. We determine the spectrum for $S_{2,2}$-designs that can be transformed into $(K_4 - e)$-designs by a double squash (bi-squash) passing through middle designs whose blocks are copies of a bull (the graph consisting of a triangle and two pendant edges). Here, the use of the difference method enables obtaining cyclic decompositions and determining the spectrum for cyclic $S_{2,2}$-designs that can be purely bi-squashed into cyclic $(K_4 - e)$-designs (the middle bull designs are also cyclic).

Keywords: graph decomposition; double-star; squash

MSC: 05B30; 05C51

1. Introduction

For any graph G, let $V(G)$ and $E(G)$ be the vertex-set and the edge-set of G, respectively. Throughout the paper, K_v denotes the complete graph on v vertices, while $K_v \setminus K_h$ denotes the graph with $V(K_v)$ as a vertex-set and $E(K_v) \setminus E(K_h)$ as an edge-set (this graph is sometimes referred to as a complete graph of the order v with a *hole* of size h). The graph $K_{n_1,n_2,\ldots,n_t}$ is the complete multi-partite graph with t parts of size $n_1, n_2, \ldots, n_t$; the complete multi-partite graph with u parts of size g is more simply denoted by $K_{u(g)}$.

Let G and H be simple finite graphs, a *G-decomposition* of H is a pair $(X, \mathcal{B})$ where $X = V(H)$, and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of G whose edges partition $E(H)$. When $H = K_n$, we also refer to such a decomposition as a *G-design* of the order n. A G-decomposition of $K_{u(g)}$ is known as a *group divisible design* (G-GDD in short) of type g^u; the parts of size g are called the *groups* of the GDD. A G-decomposition of H is *cyclic* if there exists a labeling of $V(H)$ with the elements of the group of integers modulo $\mathbb{Z}_n$ such that the label permutation $x \to x+1$ preserves the blocks of the decomposition. A G-decomposition of H is *balanced* if each vertex of H occurs in the same number of blocks. If a G-decomposition is cyclic, then it is balanced.

Fixing a graph G, a natural problem that arises is to determine the *spectrum* for G-designs (or, more simply, for the graph G), which is the set of all $n \in \mathbb{N}$ such that a G-design of the order n exists. If a G-design of the order n exists, then some necessary conditions must be satisfied ([1]): $|V(G)| \leq n$; $\frac{n(n-1)}{2|E(G)|} \in \mathbb{N}$ (the number of blocks); and $\frac{n-1}{d} \in \mathbb{N}$, where d is the gcd of the degrees of the vertices in G. In addition, if a G-design of the order n is balanced, then $\frac{(n-1)|V(G)|}{2|E(G)|} \in \mathbb{N}$ (the number of blocks in which each vertex of K_n occurs). The spectrum problem has been investigated for a large number of graphs, and numerous articles have dealt with the existence of G-designs, including several surveys (see [2]). A great deal of work has also been conducted on variations and generalizations regarding G-designs, and on G-designs with additional properties. The interest in graph

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decompositions is motivated by several applications in many areas of mathematics, as well as in other disciplines, including computer science and social science. The significance of graph decompositions is well-explained in [3].

In the context of graph decompositions, one of the mostly investigated families of graphs have been trees. A *tree* is a connected graph without cycles. Special trees are caterpillar graphs. A *caterpillar* is a graph consisting of a path on $r \geq 2$ vertices $x_1, x_2, \ldots, x_r$ and r stars K_{1,q_i} $(q_i \geq 0)$, $i = 1, 2, \ldots, r$ such that the central vertex of the star K_{1,q_i} is attached to the i-th vertex x_i of the path. By definition, the set of caterpillars includes all the paths, stars, and double-stars. A *double-star* is the caterpillar graph, usually denoted by S_{q_1,q_2}, which consists of the union of two stars, K_{1,q_1} and K_{1,q_2}, together with an edge joining their centers. Caterpillar graphs have increased in popularity since the 1970s and are still being studied. Currently, they are a useful tool in coding theory, especially in the development of technologies for post-quantum cryptography, but they are also popular in chemistry, where they are used to describe the structure of molecules. In particular, a special class of trees (phylogenetic trees) is used in computational biology to represent the evolutionary relationships of a set of extant species, and the most basic piece of phylogenetic information is the *quartet graph*, which is the term used in phylogenetics to refer to the double-star $S_{2,2}$ (see [4]). The spectrum problem for trees with at most nine vertices has been completely solved by Huang and Rosa ([5]). Numerous existence results for G-designs, in particular when G is a tree, have been proved by the use of graph labelings, which were introduced by Rosa ([6]) in 1967 and are very useful in constructing cyclic decompositions (see [7]).

In this paper, we determine the spectrum for $S_{2,2}$-designs satisfying additional properties. It is well-known that the spectrum for $S_{2,2}$-designs is precisely the set of $n \equiv 0, 1 \pmod 5$, $n \geq 6$ ([2]). Here, we determine the spectrum for $S_{2,2}$-designs that can be transformed into $(K_4 - e)$-designs by a double squash passing through bull designs (with regard to the problems concerning the possibility of transforming a G-design of the order n into a G'-design of the same order, the reader should refer to, for example, [8–12]). In order to solve our problem, we extensively apply the *difference method* (see [13]), an efficient tool to obtain cyclic designs and, in general, to describe graph decompositions.

In what follows, we will denote the following:

- the double-star $S_{2,2}$ consisting of the *central* edge $\{a, b\}$ and the four pendant edges $\{a, a_1\}$, $\{a, a_2\}$, $\{b, b_1\}$, and $\{b, b_2\}$ by $(a, b; a_1, a_2, b_1, b_2)$;
- the *bull* graph consisting of the triangle (a, b, c) and the pendant edges $\{a, a_1\}$ and $\{b, b_1\}$ by $(a, b, c; a_1, b_1)$;
- the graph $K_4 - e$ obtained from the complete graph on the vertices a, b, c, d by deleting the edge $\{c, d\}$ by $(a, b, c; d)$.

A double-star S is said to be *squashed* into a bull if we identify a pair of pendant vertices not in the same star and name one of them with the other. In turn, a bull can be squashed into a $K_4 - e$ by identifying the pair of pendant vertices. If the double-star S is squashed into a $K_4 - e$ by two consecutive squashes, then we say that S is *bi-squashed* (see Figure 1).

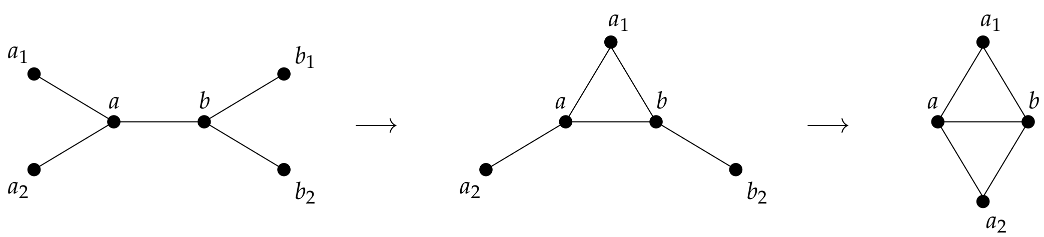


Figure 1. Bi-squash.

We remark that the double-star $(a, b; a_1, a_2, b_1, b_2)$ can be squashed in eight different ways depending on the pair of vertices we squash and on the vertex we keep. For instance, in Figure 1, we have applied the squash $b_1 \mapsto a_1$ (i.e., we rename b_1 with a_1) and obtain $(a, b, a_1; a_2, b_2)$, but we could also apply the squash $a_1 \mapsto b_1$ and obtain $(a, b, b_1; a_2, b_2)$.

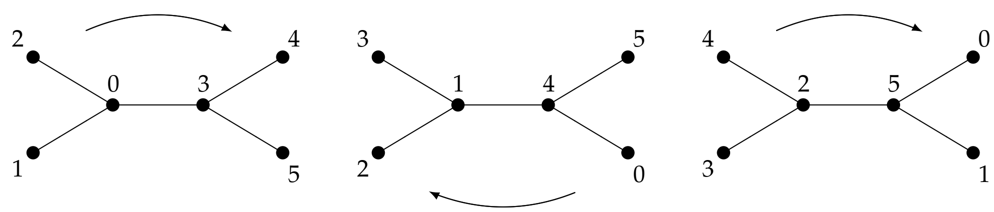
Therefore, for each pair of pendant vertices not in the same star, we have two different bulls. Likewise, to the bull $(a, b, a_1; a_2, b_2)$, we can apply the squash $b_2 \mapsto a_2$ or $a_2 \mapsto b_2$ and obtain $(a, b, a_1; a_2)$ or $(a, b, a_1; b_2)$, respectively.

Let $(X, \mathcal{B})$ be an $S_{2,2}$-design of the order n. We say that $(X, \mathcal{B})$ can be bi-squashed into a $(K_4 - e)$-design of the order n if it is possible to squash each block $B \in \mathcal{B}$ into a bull $S_1(B)$ and then to squash $S_1(B)$ into a copy $S_2(B)$ of a $K_4 - e$ such that the resulting collections $S_1(\mathcal{B}) = \{S_1(B) : B \in \mathcal{B}\}$ and $S_2(\mathcal{B}) = \{S_2(B) : B \in \mathcal{B}\}$ are the block-set of a bull design and a $(K_4 - e)$-design, respectively.

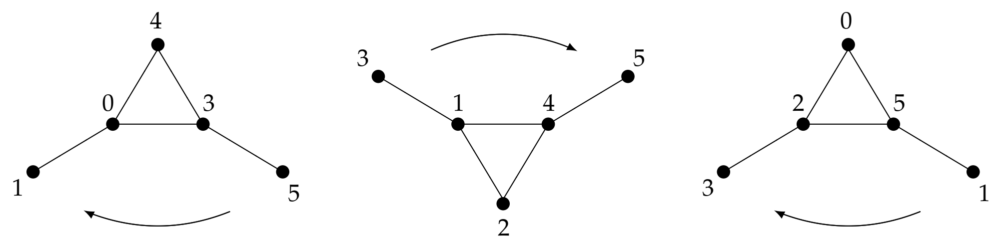
It is well-known that a $(K_4 - e)$-design of the order n exists if and only if $n \equiv 0, 1 \pmod 5$, $n \geq 6$ (see [11], for example). The bull designs share the spectrum with both $S_{2,2}$-designs and $(K_4 - e)$-designs (see [14]; in addition, a bull design of the order $n = 5$ exists). Moreover, it is easy to see that the necessary conditions for the existence of a cyclic G-design of the order n are $n \equiv 1 \pmod 5$ when $G = S_{2,2}, K_4 - e$ and $n \equiv 1, 5 \pmod{10}$ when G is a bull. Therefore, a cyclic $S_{2,2}$-design could be bi-squashed into a cyclic $(K_4 - e)$-design, but the middle bull design might not be cyclic (see Example 1).

Example 1 (A cyclic $S_{2,2}$-design of the order 6 that can be bi-squashed into a cyclic $(K_4 - e)$-design)**.** *Let* $X = \mathbb{Z}_6$ *and*

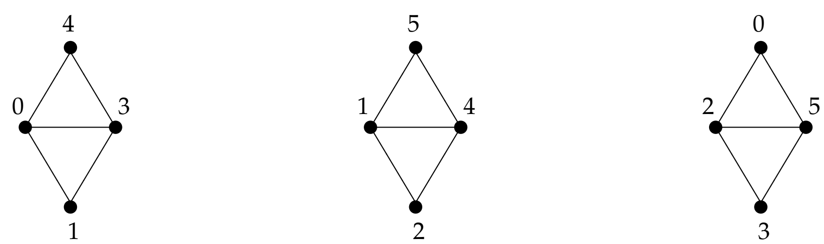
$\mathcal{B}$:



$S_1(\mathcal{B})$:



$S_2(\mathcal{B})$:



When all three decompositions involved in the process of squashing are cyclic, we will say that the cyclic $S_{2,2}$-decomposition has been *purely* bi-squashed into a cyclic $(K_4 - e)$-decomposition. In this paper, as the main result, we prove the following theorem.

Theorem 1 (Main Theorem)**.** *For every* $n \equiv 0, 1 \pmod 5$, $n \geq 6$, *there exists an* $S_{2,2}$*-design of the order* n *that can be bi-squashed into a* $(K_4 - e)$*-design of the order* n*. Moreover, for every* $n \equiv 1 \pmod{10}$, *there exists a cyclic* $S_{2,2}$*-design of the order* n *that can be purely bi-squashed into a cyclic* $(K_4 - e)$*-design of the order* n*.*

We remark that the reverse process of a squash can also be considered (called *detachment*) where a vertex of degree 2 is split into two pendant vertices. So, starting from a copy of $K_4 - e$, we can obtain a bull graph and then a double-star $S_{2,2}$ by two consecutive detachments (*bi-detachment*). A similar result as Theorem 1 could be reformulated in terms of bi-detachment by going from $(K_4 - e)$-designs to $S_{2,2}$-designs through bull designs.

2. Preliminaries

In this section, we provide a solution for small orders and provide ad hoc decompositions to use as ingredients in the following construction.

Theorem 2 (Filling Construction)**.** *Let h be a non-negative integer and n, g, and u be positive integers such that $n = gu + h$. If there exist*

- *an $S_{2,2}$-GDD of type g^u that can be bi-squashed into a $(K_4 - e)$-GDD;*
- *an $S_{2,2}$-decomposition of $K_{g+h} \setminus K_h$ that can be bi-squashed into a $(K_4 - e)$-decomposition of $K_{g+h} \setminus K_h$; and*
- *an $S_{2,2}$-design of the order $g + h$ that can be bi-squashed into a $(K_4 - e)$-design;*

then so does an $S_{2,2}$-design of the order n that can be bi-squashed into a $(K_4 - e)$-design.

Proof. Let $(X, \mathcal{B})$ be an $S_{2,2}$-GDD of type g^u that can be bi-squashed into a $(K_4 - e)$-GDD; say G_i, $i = 1, 2, \ldots, u$, its groups. Let H be a set of size h such that $H \cap X = \emptyset$. For each $i = 2, 3 \ldots, u$, let $(G_i \cup H, \mathcal{B}_i)$ be an $S_{2,2}$-decomposition of $K_{g+h} \setminus K_h$ (with H as hole) that can be bi-squashed into a $(K_4 - e)$-decomposition of $K_{g+h} \setminus K_h$. By the assumption, on $G_1 \cup H$, we can also construct an $S_{2,2}$-design $(G_1 \cup H, \mathcal{B}_1)$ of the order $g + h$ that can be bi-squashed into a $(K_4 - e)$-design. It is easy to check that $(H \cap X, \mathcal{B} \cup (\cup_{i=1}^{u} \mathcal{B}_i))$ is the required design. $\square$

Remark 1. *The "filling" technique allows us to construct an $S_{2,2}$-design of the order $n + h$ that can be bi-squashed into a $(K_4 - e)$-design whenever we have an $S_{2,2}$-decomposition of $K_{n+h} \setminus K_h$ and an $S_{2,2}$-design of the order h, which are both bi-squashable.*

From now on, in order to say that a block $B = (a, b; a_1, a_2, b_1, b_2)$ is bi-squashed by $b_1 \mapsto a_1$ (first squash) and then by $b_2 \mapsto a_2$ (second squash), we will write $B = (a, b; \dot{a}_1, \ddot{a}_2, b_1, b_2)$. Likewise, by the notation $B = (a, b; \dot{a}_1, a_2, b_1, \ddot{b}_2)$, we will mean that B is bi-squashed by $b_1 \mapsto a_1$ (first squash) and then by $a_2 \mapsto b_2$ (second squash). Note that, in $B = (a, b; \dot{a}_1, \ddot{a}_2, b_1, b_2)$, we keep vertices belonging to the same star (we speak of a *block of type I*), while in $B = (a, b; \dot{a}_1, a_2, b_1, \ddot{b}_2)$ the vertices kept belong to different stars (*block of type II*). Although each double-star can be squashed into eight different bulls and each bull into two different copies of a $K_4 - e$, the two above notations will be sufficient to list the blocks of an $S_{2,2}$-decompostion and say they can be bi-squashed into a $(K_4 - e)$-decomposition, without listing the bull-blocks and the $(K_4 - e)$-blocks they have been squashed into. As an example, in the following lemma, we list the blocks of the $S_{2,2}$-design of the order 6 described in Example 1 (here, the blocks are all of type II).

Lemma 1. *There exists a cyclic $S_{2,2}$-design of the order 6 that can be bi-squashed into a cyclic $(K_4 - e)$-design.*

Proof. Let $X = \mathbb{Z}_6$ and $\mathcal{B} = \{(3, 0; \dot{4}, 5, 2, \ddot{1}), (1, 4; \dot{2}, 3, 0, \ddot{5}), (5, 2; \dot{0}, 1, 4, \ddot{3})\}$. $\square$

In what follows, if G is a graph whose vertices belong to $\mathbb{Z}_n$, then we will call *orbit of G under $\mathbb{Z}_n$* the set of the *translates* of G, i.e., $Orb(G) = \{G + i : i \in \mathbb{Z}_n\}$, where $G + i$ is the graph with $V(G + i) = \{a + i : a \in V(G)\}$ and $E(G + i) = \{\{a + i, b + i\} : \{a, b\} \in E(G)\}$. If the orbit of G under $\mathbb{Z}_n$ has cardinality n, then the orbit is *full*; otherwise, it is *short*. If $(\mathbb{Z}_n, \mathcal{B})$ is a cyclic G-decomposition of a graph H, then $\mathcal{B}$ can be partitioned into orbits and described by a set of orbit representatives (*base blocks*). Likewise if $V(G) \subseteq \mathbb{Z}_n \times \mathbb{Z}_t$, then by $G + i$ we mean the graph obtained from G by $(j,k) \longmapsto (j+i,k)$ and we speak of translates and orbit of G under $\mathbb{Z}_n$ with obvious meaning of the terms. If, further, $V(G) \subseteq \mathbb{Z}_n \times \mathbb{Z}_t \cup \{\infty\}$, then we can again speak of translates and orbit of G under $\mathbb{Z}_n$ by means of $(j,k) \longmapsto (j+i,k)$ and $\infty + 1 \longmapsto \infty$. In what follows, the element $(j,k) \in \mathbb{Z}_n \times \mathbb{Z}_t$ will be denoted by j_k.

Lemma 2. *For $n = 10, 20$, there exists an $S_{2,2}$-decomposition of $K_n \setminus K_6$ that can be bi-squashed into a $(K_4 - e)$-decomposition of $K_n \setminus K_6$.*

Proof. Let $H = \{a_1, a_2, \ldots, a_6\}$. For $n = 10$, let $X = \mathbb{Z}_4 \cup H$ and consider the blocks $(0,1; \dot{a}_1, \ddot{a}_2, a_5, a_6)$, $(2,3; \dot{a}_1, \ddot{a}_2, a_3, a_4)$, $(1,2; \dot{a}_3, \ddot{a}_4, a_5, a_6)$, $(0,3; \dot{a}_3, \ddot{a}_4, a_1, a_2)$, $(0,2; \dot{a}_5, \ddot{a}_6, a_3, a_4)$, and $(3,1; \dot{a}_5, \ddot{a}_6, a_1, a_2)$. For $n = 20$, let $X = \mathbb{Z}_{14} \cup H$ and consider the orbit of $(2,0; \dot{5}, \ddot{6}, 9, 8)$ under $\mathbb{Z}_{14}$, together with the blocks $(2j, 1+2j; \dot{a}_1, \ddot{a}_2, a_5, a_6)$, $(2+2j, 1+2j; \dot{a}_3, \ddot{a}_4, a_1, a_2)$, and $(8+2j, 1+2j; \dot{a}_5, \ddot{a}_6, a_3, a_4)$ for $j = 0,1,2,3,4,5,6$. □

Lemma 3. *There exists an $S_{2,2}$-design of the order $n = 10, 20$ that can be bi-squashed into a $(K_4 - e)$-design.*

Proof. It follows from Lemmas 1 and 2 together with Remark 1. □

The following lemma provides a solution for the smallest order for which there exists a cyclic $S_{2,2}$-design that can be purely bi-squashed into a cyclic $(K_4 - e)$-design. Fron now on, when we will speak of type of a base block, we will mean that all its translates have the same type, unless specified otherwise.

Lemma 4. *There exists a cyclic $S_{2,2}$-design of the order $n = 11$ that can be purely bi-squashed into a cyclic $(K_4 - e)$-design.*

Proof. Let $X = \mathbb{Z}_{11}$ and take $(1,0; \dot{3}, \dddot{5}, 8, 6)$ as a base block. □

Lemma 5. *There exists an $S_{2,2}$-decomposition of $K_{15} \setminus K_5$ that can be bi-squashed into a $(K_4 - e)$-decomposition of $K_{15} \setminus K_5$.*

Proof. Let $X = \mathbb{Z}_{10} \cup H$, $H = \{a_1, a_2, a_3, a_4, a_5\}$, and consider the blocks

$(1,6; \dot{a}_4, 8, 5, \ddot{a}_5)$, $(6,2; \dot{3}, a_3, 4, \ddot{10})$, $(7,2; \dot{4}, a_1, 3, \ddot{9})$ $(6,9; \dot{a}_2, 10, 1, \ddot{a}_3)$, $(a_1, 6; \dot{5}, 9, a_4, \dddot{7})$, $(a_1, 1; \dot{2}, \ddot{10}, 5, a_3)$, $(3,9; \dot{1}, \ddot{a}_1, a_2, 7)$, $(4,8; \dot{6}, \ddot{a}_1, a_2, a_5)$, $(2,5; \dot{a}_2, \ddot{a}_3, a_4, a_5)$, $(3,8; \dot{a}_2, \ddot{a}_3, 6, 10)$, $(10,7; \dot{a}_2, \ddot{a}_3, a_4, a_5)$, $(4,1; \dot{a}_2, \ddot{a}_3, 2, 10)$, $(10,5; \dot{a}_4, \ddot{a}_5, a_2, a_3)$, $(3,7; \dot{a}_4, \ddot{a}_5, a_2, a_3)$, $(2,8; \dot{a}_4, \ddot{a}_5, 5, a_1)$, $(9,4; \dot{a}_4, \ddot{a}_5, 5, 10)$, $(7,1; \dot{5}, \ddot{8}, a_2, a_5)$, $(3,4; \dot{5}, \ddot{10}, a_4, a_5)$, $(9,8; \dot{5}, \ddot{10}, a_4, a_3)$.

□

Lemma 6. *There exists an $S_{2,2}$-design of the order 15 that can be bi-squashed into a $(K_4 - e)$-design.*

Proof. Let $X = \mathbb{Z}_7 \times \mathbb{Z}_2 \cup \{\infty\}$ and consider the orbits of $(0_1, 1_1; \dot{3}_1, \ddot{0}_0, 6_1, 6_0)$, $(2_1, 5_0; \dot{6}_0, \dddot{3}_0, \infty, 0_0)$, and $(5_1, 0_0; \dot{\infty}, 4_0, 1_0, \dddot{3}_0)$ under $\mathbb{Z}_7$. □

Lemma 7. *There exists a cyclic $S_{2,2}$-design of the order 21 that can be purely bi-squashed into a cyclic $(K_4 - e)$-design.*

Proof. Let $X = \mathbb{Z}_{21}$ and consider the base blocks $(1,0; \dot{7}, \dddot{5}, 14, 16)$ and $(11,0; \dot{9}, \ddot{8}, 12, 13)$. □

Lemma 8. *There exists an $S_{2,2}$-design of the order 25 that can be bi-squashed into a $(K_4 - e)$-design.*

Proof. Let $X = \mathbb{Z}_{12} \times \mathbb{Z}_2 \cup \{\infty\}$ and consider the orbits of $(7_1, 0_0; \dot{10}_1, \dddot{2}_1, 4_1, 8_1)$, $(1_1, 0_1; \dot{8}_0, \ddot{4}_0, 2_0, 10_0)$, $(6_0, 0_1; \dot{1}_0, \ddot{9}_0, \infty, 11_0)$, and $(0_0, 0_1; \dot{\infty}, \ddot{11}_0, 1_0, 9_0)$ under $\mathbb{Z}_{12}$, along with the twelve blocks $(6_k, 0_k; \dot{8}_k, 10_k, 4_k, \ddot{2}_k) + i$, $(2_k, 8_k; \dot{4}_k, 6_k, 0_k, \ddot{10}_k) + i$, $(10_k, 4_k; \dot{0}_k, 2_k, 8_k, \ddot{6}_k) + i$, where $i = 0, 1$, $i \in \mathbb{Z}_{12}$ and $k \in \mathbb{Z}_2$. □

Lemma 9. *For every $k \geq 3$, there exists a cyclic $S_{2,2}$-GDD of type 10^k that can be purely bi-squashed into a cyclic $(K_4 - e)$-GDD.*

Proof. Let $X = \mathbb{Z}_{10k}$, $k \geq 3$. For $k = 3$, consider the base blocks $(0, 5; \dot{4}, \ddot{16}, 6, 24)$ and $(0, 20; \dot{13}, \ddot{22}, 27, 18)$. For $k = 4$, consider the base blocks $(0, 33; \dot{3}, \ddot{34}, 23, 32)$, $(0, 38; \dot{15}, \ddot{9}, 21, 27)$, and $(0, 35; \dot{14}, \ddot{13}, 16, 17)$. For $k \geq 5$, consider the following $k - 1$ base blocks, which are all of type I:

$$(3k + 2 + i,\ 0\,;\ 4 + 2i,\ 5k - 3 - i,\ 10k - 4 - 2i,\ 5k + 3 + i),\ i = 0, 1, \ldots, k - 5,\ i \neq \lfloor \tfrac{k-4}{2} \rfloor,\ i \in \mathbb{Z}_{10k},$$
$$(3k + 1,\ 0\,;\ 5k - 2,\ 5k - 1,\ 5k + 2,\ 5k + 1),$$
$$(4k - 2,\ 0\,;\ k - 1,\ 2k - 4,\ 9k + 1,\ 8k + 4),$$
$$(2,\ 0\,;\ 3,\ 4k + 1,\ 10k - 3,\ 6k - 1),$$
$$\left(\left\lfloor \tfrac{9k-1}{2} \right\rfloor,\ 0\,;\ \left\lfloor \tfrac{5k+1}{2} \right\rfloor,\ \left\lfloor \tfrac{13k+1}{2} \right\rfloor,\ \left\lfloor \tfrac{15k}{2} \right\rfloor,\ \left\lfloor \tfrac{7k}{2} \right\rfloor\right).$$

The orbits of the above base blocks provide the required GDD, whose groups are the cosets of the subgroup $H = k\mathbb{Z}_{10k}$ in $\mathbb{Z}_{10k}$, i.e., $G_i = k\mathbb{Z}_{10k} + i$ for $i = 0, 1, \ldots, k - 1$. □

3. Main Result

By using the basic results in Section 2, we are now able to obtain our main result, i.e., to prove that the spectrum for (cyclic) $S_{2,2}$-designs that can be (purely) bi-squashed into (cyclic) $(K_4 - e)$-designs is precisely the set of all $n \equiv 0, 1 \pmod 5$ (respectively, $n \equiv 1 \pmod{10}$), $n \geq 6$.

As an additional result, for every $n \equiv 6 \pmod{20}$, we construct a cyclic $S_{2,2}$-design of the order n that can be (not purely) bi-squashed into a cyclic $(K_4 - e)$-design of the order n (leaving open the problem of constructing such a design for $n \equiv 16 \pmod{20}$). To begin with, we prove this result, which will also be useful in the proof of Theorem 1.

Proposition 1. *For every $n \equiv 6 \pmod{10}$, there exists an $S_{2,2}$-design of the order n that can be bi-squashed into a cyclic $(K_4 - e)$-design of the order n. Moreover, for every $n \equiv 6 \pmod{20}$, there exists a cyclic $S_{2,2}$-design of the order n that can be bi-squashed into a cyclic $(K_4 - e)$-design of the order n.*

Proof. Write $n = 10k + 6$, $k \geq 0$. Let $X = \mathbb{Z}_{10k+6}$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1$ is the union of the orbits of the following base blocks of type I

$$B_i = (3k + 3 + i,\ 0;\ 2 + 2i,\ 5k + 2 - i,\ 10k + 4 - 2i,\ 5k + 4 + i),\ i = 0, 1, \ldots, k - 1,$$

while the definition of $\mathcal{B}_2$ depends on the parity of k. If k is even, then $\mathcal{B}_2$ is the short orbit of the block $B = (5k + 3, 0; 7k + 4, 8k + 5, 3k + 2, 2k + 1)$, whose translates are all of type II as long as you rewrite and partition them as follows:

$$(5k + 3 + j,\ j;\ 7k + 4 + j,\ 8k + 5 + j,\ 3k + 2 + j,\ 2k + 1 + j),\ j = 0, 2, \ldots, 5k + 2,$$
$$(j,\ 5k + 3 + j;\ 2k + 1 + j,\ 3k + 2 + j,\ 8k + 5 + j,\ 7k + 4 + j),\ j = 1, 3, \ldots, 5k + 1.$$

If k is odd, then $\mathcal{B}_2$ is the set of the following blocks of type II

$$\mathcal{P}_1 : (j,\ 5k + 3 + j;\ 2k + 1 + j,\ 8k + 5 + j,\ 3k + 2 + j,\ 7k + 4 + j),\ j = 0, 1, \ldots, 2k,$$
$$\mathcal{P}_2 : (7k + 4 + j,\ 2k + 1 + j;\ 4k + 2 + j,\ j,\ 5k + 3 + j,\ 9k + 5 + j),\ j = 0, 1, \ldots, k,$$
$$\mathcal{P}_3 : (8k + 5 + j,\ 3k + 2 + j;\ 5k + 3 + j,\ k + 1 + j,\ 6k + 4 + j,\ j),\ j = 0, 1, \ldots, k - 1,$$
$$\mathcal{P}_4 : (9k + 5 + j,\ 4k + 2 + j;\ 6k + 3 + j,\ 7k + 4 + j,\ 2k + 1 + j,\ k + j),\ j = 0, 1, \ldots, k.$$

Note that, as a result of a squash, a graph G loses an edge, say $\epsilon^-(G)$, and obtains a new edge, say $\epsilon^+(G)$; for a set Γ of graphs, let $E^-(\Gamma) = \{\epsilon^-(G) : G \in \Gamma\}$ and $E^+(\Gamma) = \{\epsilon^+(G) : G \in \Gamma\}$. Now, it is easy to check that by the first squash

$$\begin{aligned} &E^+(\mathcal{P}_1) = E^-(\mathcal{P}_2) \cup E^-(\mathcal{P}_3),\\ &E^+(\mathcal{P}_2) = E^-(\mathcal{P}_4),\\ &E^+(\mathcal{P}_3) \cup E^+(\mathcal{P}_4) = E^-(\mathcal{P}_1), \end{aligned}$$

and by the second squash

$$\begin{aligned} &E^+(S_1(\mathcal{P}_1)) = E^-(S_1(\mathcal{P}_2)) \cup E^-(S_1(\mathcal{P}_3)),\\ &E^+(S_1(\mathcal{P}_2)) = E^-(S_1(\mathcal{P}_4)),\\ &E^+(S_1(\mathcal{P}_3)) \cup E^+(S_1(\mathcal{P}_4)) = E^-(S_1(\mathcal{P}_1)), \end{aligned}$$

so that $\mathcal{B}_2$ can be bi-squashed into the orbit of $(0, 5k+3, 2k+1; 7k+4)$. $(X, \mathcal{B})$ is an $S_{2,2}$-design of the order $n = 10k+6$ that can be bi-squashed into a cyclic $(K_4 - e)$-design. If k is even, then $(X, \mathcal{B})$ is also cyclic. □

Remark 2. *In the proof of Proposition 1, the union of the orbits of B_i, $i = 0, 1, \ldots, k-1$, and the short orbit of B provide a cyclic $S_{2,2}$-design of the order $n \equiv 16 \pmod{20}$ that can be squashed into a cyclic $(K_4 - e)$-design by two simultaneous squashes; i.e., no middle bull design can be obtained.*

Proof of Theorem 1 (Main Theorem). Distinguish the following congruence classes.

(a) $n \equiv 0 \pmod 5$. For $n = 10, 15, 20, 25$, the result follows from Lemmas 3, 6, and 8. For $n \geq 30$, write $n = 10k + h$, where $k \geq 3$ and $h = 0, 5$. Let $X = \mathbb{Z}_{10k} \cup H$, $|H| = h$, $H \cap \mathbb{Z}_{10k} = \emptyset$. Apply the Filling Construction to the $S_{2,2}$-GDD of type 10^k provided by Lemma 9 by using as ingredients copies of an $S_{2,2}$-decomposition of $K_{10+h} \setminus K_h$ from Lemmas 3 (for $h = 0$) or 5 (for $h = 5$) and an $S_{2,2}$-design of the order $10 + h$ provided by Lemmas 3 or 6.

(b) For $n \equiv 1 \pmod 5$, consider the following two subcases.

- $n \equiv 6 \pmod{10}$. It follows from Proposition 1.
- $n \equiv 1 \pmod{10}$. Write $n = 10k + 1$, $k \geq 1$. For $k = 1, 2$, the result follows from Lemmas 4 and 7. For $k \geq 3$, let $X = \mathbb{Z}_{10k+1}$ and consider the base blocks of type I

$$\begin{aligned} &(3k+2+i,\ 0;\ 2+2i,\ 5k-1-i,\ 8k+1-2i,\ 5k+2+i),\ i = 0, 1, \ldots, k-3,\\ &(1,\ 0;\ 2k+2,\ 2k-1,\ 8k-1,\ 8k+2),\ (2k,\ 0;\ 5k+1,\ 6k+1,\ 5k,\ 4k), \end{aligned}$$

whose translates are the blocks of a cyclic $S_{2,2}$-design of the order $10k+1$, which can be purely bi-squashed into a cyclic $(K_4 - e)$-design of the order $10k+1$.

4. Conclusions

In this article, we determine the spectrum for $S_{2,2}$-designs that can be bi-squashed into $(K_4 - e)$-designs. A complete solution is also provided for the existence problem of a cyclic $S_{2,2}$-design that can be purely bi-squashed into a cyclic $(K_4 - e)$-design (which means that the middle bull design is also cyclic), while a partial answer is provided for the existence problem of a cyclic $S_{2,2}$-design that can be (not purely) bi-squashed into a cyclic $(K_4 - e)$-design because we prove that such a design exists for every $n \equiv 6 \pmod{20}$, while the problem of its existence is still open for $n \equiv 16 \pmod{20}$.

Open Problem 1. *Determine the set of all $n \equiv 16 \pmod{20}$ such that there exists a cyclic $S_{2,2}$-design of the order n that can be bi-squashed into a cyclic $(K_4 - e)$-design of the order n.*

Finally, it is an open problem to determine the spectrum for $S_{2,2}$-decompositions of λK_n (the complete graph whose edges are replicated λ times) that can be bi-squashed into

$(K_4 - e)$-decompositions of λK_n. Moreover, decompositions with the additional property to be cyclic could be investigated and an analogous result to Theorem 1 could be obtained.

Open Problem 2. *Fixing any integer $\lambda > 1$, determine the set of all $n \in \mathbb{N}$ such that there exists a (cyclic) $S_{2,2}$-decomposition of λK_n that can be (purely) bi-squashed into a (cyclic) $(K_4 - e)$-decomposition of λK_n.*

As a final note, we want to point out that, by applying the Theorem 1, Proposition 1, and Remark 2, we also provide a complete solution to the existence problem of cyclic G-designs when $G = S_{2,2}$, $K_4 - e$ by proving that a cyclic G-design of the order n exists if and only if $n \equiv 1 \pmod 5$, $n \geq 6$.

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