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NONSMOOTH CRITICAL POINT THEORY AND APPLICATIONS TO NONLINEAR DIFFERENTIAL PROBLEMS

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Declaration of Authorship

I, VALERIA MORABITO, declare that this thesis titled, "Nonsmooth critical point theory and applications to nonlinear differential problems" and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
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Abstract

Department of Mathematics and Computer Sciences, Physical Sciences and Earth Sciences

Doctor of Philosophy

Nonsmooth critical point theory and applications to nonlinear differential problems

by VALERIA MORABITO

The motivation for nonsmooth calculus arises from the limitations of classical analysis, where differentiability assumptions often exclude many relevant models. Real world problems in engineering, economics, and optimization frequently involve irregular, discontinuous, or nondifferentiable data. Nonsmooth analysis provides a rigorous framework to extend variational methods and optimization tools beyond the smooth setting. In particular, it establishes a fundamental theoretical basis for the study of variational problems with discontinuous nonlinearities, as well as for problems where the nonlinear term is represented by multifunctions or formulated in the form of hemivariational inequalities.This thesis is situated within this context, developing tools and results concerning the existence and multiplicity of solutions to differential equations, differential inclusions, and hemivariational inequalities, where classical methods based on differentiability are not applicable. The main tools are Clarke's generalized gradient and nonsmooth extensions of variational methods, in particular critical point theory. Within this framework, a central role is played by the nonsmooth version of the Mountain Pass Theorem as well as by abstract multiplicity results, such as two and three critical point theorems due to Bonanno, D'Aguì and Winkert [20] and Bonanno and Marano [26], which ensure the existence of multiple solutions under weak regularity and polynomial growth assumptions.

The thesis addresses several classes of nonlinear problems. First, the existence of two generalized weak solutions is established in two different settings: an elliptic differential inclusion driven by the Laplace operator with Dirichlet boundary conditions, and a second order ordinary differential inclusion with periodic boundary conditions. In both cases, an explicit interval of parameters is identified for which the problem admits a solution.

Next, two problems with highly discontinuous nonlinearities are investigated. The first is a Sturm Liouville type problem with sign-changing weight functions and mixed boundary conditions, where the application of a three critical points theorem guarantees the existence of at least three generalized solutions. The second one concerns a Neumann problem involving the p-Laplacian and discontinuous nonlinearities, for which the existence of two weak solutions with opposite energy sign is proven.

Finally, a fourth-order hemivariational inequality is studied, and the existence of three distinct solutions is established.

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Contents

To Ludovica

Introduction

This doctoral thesis presents original results concerning the existence and multiplicity of solutions for differential problems with discontinuous nonlinearities, as well as for differential inclusions and hemivariational inequalities. The analysis of these problems is carried out through variational methods, which are based on reformulating differential problems as the search for critical points of suitable functionals defined on appropriate function spaces.

To address such issues, it is necessary to make use of nonsmooth analysis, an extension of classical analysis that overcomes the limitations imposed by regularity and differentiability assumptions. Nonsmooth analysis relies on the introduction of generalized derivatives, such as the generalized gradient and Clarke's subdifferential, which makes it possible to deal with locally Lipschitz functionals that lack classical derivatives.

The thesis recalls the theoretical foundations of this framework, emphasizing its historical and conceptual motivations. The need for generalized derivatives arose from the inadequacy of smooth tools to describe many real-world applications in engineering, economics, and applied sciences, where models frequently involve irregular, discontinuous, or nondifferentiable data. Within this perspective, nonsmooth calculus provides not only the natural setting for the problems under investigation, but also the essential analytical tools to establish the existence and multiplicity results developed in this work.

Variational methods are based on the principle of associating each differential problem to a functional, known as the energy functional, $I : X \to \mathbb{R}$, defined on a suitably chosen Banach space X. The critical points of I, that is, the points where the classical (or, as will be seen, generalized) derivative vanishes, correspond to weak solutions of the original problem. This strategy finds its origins in the Calculus of Variations, specifically in Weierstrass theorem, which guarantees the existence of absolute minima for continuous functions defined on compact subsets of $\mathbb{R}$.

When extended to infinite dimensional spaces, typically Sobolev spaces, this principle is generalized via Tonelli theorem, forming the basis of the Direct Methods in the Calculus of Variations. The theorem relies on two key properties: coercivity (the behavior of the functional tending to infinity as the norm of the argument grows) and lower semicontinuity (which ensures the stability of the minimum under weak convergence). These conditions guarantee the existence of at least one minimizer, allowing existence results to be established without explicitly solving the Euler-Lagrange equation. However, the effectiveness of variational methods extends far beyond the search for minima. The introduction of compactness conditions (such as the Palais–Smale condition) and topological results (notably the celebrated Mountain Pass theorem, established by A. Ambrosetti and P. Rabinowitz in 1973 [2]) made it possible to investigate critical points that are not necessarily local or global minima, thereby broadening the applicability of these methods to problems with more complex functional geometries and to unbounded functionals.

This approach, which is based on the analysis of the geometric structure of the functional rather than the direct study of its derivative, led to numerous generalizations and refinements of classical theorems. Variants of the Mountain Pass theorem had already been formulated by M. Morse and C.B. Tompkins [71], M. Shiffman [87], and R. Courant [41]. A more general version of the Mountain Pass theorem, encompassing even a limiting case excluded from the original formulation, was developed by P. Pucci and J. Serrin [78, 79, 80]. These results contribute to expanding the theoretical framework by providing new sufficient conditions for the existence of critical points.

The variational method also has numerous interdisciplinary applications. In classical physics, it underpins fundamental principles such as the principle of least action, which governs the dynamic evolution of physical systems. In engineering, it is used to model complex structures and materials. In economics, it is used to solve constrained optimization problems involving discontinuities. In computational sciences and control theory it is used to analyze and design dynamic systems subject to abrupt regime changes, thresholds, or impacts.

The methodological framework adopted in this thesis thus lies within the scope of variational methods, which, however, must be suitably reformulated in order to address nonsmooth problems, such as those involving discontinuous nonlinearities, differential inclusions, or hemivariational inequalities. Beyond these classes, nonsmooth variational techniques also find applications in quasi-variational inequalities, complementarity problems, optimal control with nonsmooth constraints, and more generally in nondifferentiable optimization and impulsive differential equations. For this reason, in addition to being situated within the context of variational methods, this thesis also belongs to the broader field of nonsmooth analysis, a branch of functional analysis that developed in the 1970s following the pioneering paper of F. H. Clarke. Motivated by the desire to extend recent advances in optimization, where the systematic replacement of differentiability assumptions with convexity hypotheses had already yielded significant results through the papers of Rockafellar [84, 85], Clarke introduced in his 1975 paper, *Generalized gradients and applications* [38], the concepts of generalized derivative and generalized gradient. He later elaborated and developed these notions in his monograph *Optimization and Nonsmooth Analysis* [39], providing a solid theoretical foundation for extending classical differential calculus to locally Lipschitz functionals, thereby overcoming the limitations imposed by classical differentiability and enabling the formulation of a critical point theory in highly irregular settings.

Following these developments, K.-C. Chang extended several fundamental results of classical theory to the nonsmooth setting, including the aforementioned Mountain Pass theorem [81, Theorem 2.2] and the Saddle Point theorem [81, Theorem 4.6], using Clarke's gradient as a key tool (see Theorems 3.3 and 3.4 in [34]). These contributions marked the beginning of the systematic application of variational methods in nonsmooth environments. Further progress in this direction was made by D. Motreanu and C. Varga [73], who generalized a result by Y. Du [45, Theorem 2.1] by weakening the geometric condition of the Mountain Pass theorem to include the limiting case of equality. At the same time, G. Barletta and S.A. Marano [11] provided a nonsmooth version of the dual formulation of the theorem originally proposed by N. Ghoussoub and D. Preiss [50], as well as its generalization [81, Theorem 5.3].

A structured framework for the study of nonsmooth functionals, and in particular for the nonsmooth theory of critical points, was proposed by D. Motreanu and P.D. Panagiotopoulos in the so-called Motreanu–Panagiotopoulos nonsmooth setting [72]. In this approach, the total functional is expressed as a sum $I = \Phi + \Psi$, where Φ is a locally Lipschitz functional responsible for the discontinuities, while Ψ is convex, proper, and lower semicontinuous, and introduces a regular structure according to the theory of convex analysis. The critical analysis of I is carried out by combining Clarke's generalized gradient for Φ with the subdifferential properties of Ψ, thus enabling the application of variational methods even in the absence of classical differentiability. This framework has paved the way for numerous generalizations and refinements.

Thanks to the papers by B. Ricceri [82, 83] and to the extensions developed by S.A. Marano and D. Motreanu [65, 66], the theory of critical points for nonsmooth functionals has undergone significant further developments. In this direction lie the contributions of G. Bonanno and P. Candito [17], G. Bonanno and S.A. Marano [26], G. Bonanno, G. D'Aguì, and P. Winkert [20, 21], and R. Livrea and S.A. Marano [62], who established existence and multiplicity theorems for locally Lipschitz functionals, even in the presence of strong discontinuities and under weak compactness assumptions. Besides the explicitly mentioned contributions, these works contain numerous essential references that allow for further exploration of the theoretical and applicative framework of critical point theory in the nonsmooth setting.

Some of these results, which represent the main abstract tools underlying the analysis developed in this thesis, are explicitly collected and presented in Chapter 2, so as to provide the reader with a clear and accessible reference framework. It is worth emphasizing that there exists a strong interest within the mathematical community in the study of existence and multiplicity results for differential problems and differential inclusions through nonsmooth analysis, and many authors have significantly contributed to this line of research. In particular, variational methods have proved to be powerful tools for establishing the existence and multiplicity of solutions to problems with discontinuous nonlinearities. To provide an indicative, though not exhaustive, example, significant contributions in this direction have been provided by authors such as Barletta, Bonanno, Candito, D'Aguì, Marano, Motreanu, Papageorgiou, Winkert, and others, as documented in the references cited above [4, 10, 12, 16, 17, 18, 20, 21, 22, 23, 24, 27, 28, 43, 63, 69]. In addition to variational methods, differential inclusions have also been studied by means of fixed point theorems, Leray–Schauder theory, monotone operator techniques, and various approximation schemes. Examples of such approaches can be found in [8, 35, 36, 47, 48, 60, 75]). However, variational methods have also provided an effective framework for the study of differential inclusions. In particular, starting from [34], these methods have been successfully applied to elliptic partial second order inclusions involving set-valued mappings of the type

$$F(u) = [f^-(u), f^+(u)],$$

namely, the interval determined pointwise by the lower and upper limits of a discontinuous single-valued function, and later extended to the case of general set-valued reactions. Significant contributions in this direction have been given by Bonanno, Iannizzotto, Marano and Motreanu, among others. Some recent papers illustrating these developments can be found in [18, 25, 29, 54, 55, 56, 57, 58, 77].

The results obtained in this thesis are situated precisely within this line of research, offering original contributions both in methodology and in applications. They establish general criteria for the existence and multiplicity of solutions, and in some cases also for their positivity. A central contribution concerns differential inclusions with a general nonlinear term. Traditionally, as mentioned above, such problems have been addressed through fixed point theorems, which yield existence results but no information on multiplicity. Overall, these findings extend the existing theoretical framework and open the way for further developments in nonsmooth analysis and variational methods. The thesis is organized as follows.

In Chapter 1, after a preliminary overview of the basic concepts and main notations concerning Lebesgue and Sobolev spaces, as well as some fundamental results from nonlinear analysis, the tools of set-valued analysis and the essential notions of generalized gradient theory, as developed by Clarke, are introduced.

Chapter 2 is devoted to presenting classical results from variational methods, such as the Direct Methods theorem and the Ambrosetti–Rabinowitz theorem, followed by some recent generalizations in the field of critical point theory, useful for studying the existence of one, two, three, or infinitely many solutions. These generalizations are due to Bonanno, D'Aguì and Winkert, Bonanno and Marano, and Marano and Motreanu.

Chapter 3 is devoted to the study of differential inclusions in two different settings and under various boundary conditions. In the first section, we focus on the elliptic differential inclusion

$$\begin{cases} -\Delta u \in \lambda G(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

where Dirichlet boundary conditions are imposed. Under suitable growth assumptions on the nonlinear term, we prove the existence of at least two positive solutions. This result is then applied to an elliptic problem with discontinuous nonlinearity, with particular attention to their behavior near zero, identifying a range of the parameter λ for which two positive solutions exist. The main tool employed is an abstract two critical points theorem developed in [20]. These results are taken from [29], obtained in collaboration with G. Bonanno, D. O'Regan, and B. Vassallo.

The second section studies a second order ordinary differential inclusion with periodic boundary conditions:

$$\begin{cases} u''(t) + u(t) \in \lambda G(u(t)) & t \in (0,T), \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases} \tag{P_λ}$$

where, assuming that the multifunction G is upper semicontinuous with nonempty, compact, and convex values and satisfies the Ambrosetti–Rabinowitz type condition (AR), we prove the existence of an interval of parameters λ for which the problem admits at least two generalized weak solutions. This abstract existence result, based on [20], is illustrated by an explicit example involving a discontinuous nonlinearity satisfying all the assumptions required. The results in this section are from [4], in collaboration with E. Amoroso, G. Bonanno, and G. Klun.

Chapter 4 is devoted to the study of two different problems involving nonlinearities f belonging to the class $\mathcal{H}$, which are commonly known in the literature as highly discontinuous functions. Such functions are suitable for modeling phenomena characterized by abrupt changes, such as phase transitions, population dynamics, fluid mechanics problems, or economic and social systems with threshold effects. More precisely, highly discontinuous functions are those $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ that are measurable in x, locally essentially bounded in t, and may exhibit discontinuities on sets of measure zero. Moreover, additional assumptions are made on the essential lower and upper limits, f^- and f^+.

The first part focuses on the following Sturm Liouville type ordinary differential problem with mixed boundary conditions:

$$\begin{cases} -u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x, u(x)), \quad x \in (a, b), \\ u(a) = 0, \quad u'(b) = 0. \end{cases} \tag{M_λ}$$

Under a suitable sign assumption on the weight function δ (see assumption (4.1)), the problem, which has a non-variational structure, has been reformulated in variational terms by following the approach introduced in [3]. In particular, a norm equivalent to the standard one is introduced in the Sobolev space $W^{1,2}([a, b])$, incorporating the variable weights in the differential equation and thus enabling the application of variational methods. A notable aspect of this analysis is that the weight function $\delta(x)$ is allowed to change sign on $[a, b]$. By applying a three critical points theorem due to Bonanno and Marano [26], the existence of at least three generalized solutions is established. Two particular cases are examined in detail: the autonomous case and the case where the nonlinearity is nonnegative on a suitable interval. These results are contained in [69].

The second part studies the elliptic problem with Neumann boundary conditions

$$\begin{cases} -\Delta_p u + \delta(x)|u|^{p-2}u = \lambda f(x, u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{N_λ}$$

where the nonlinearity is highly discontinuous. Under suitable growth assumptions on f, the existence of two weak solutions is proved. A special focus is given to the case in which the nonlinearity exhibits superlinear growth near the origin. The approach relies on an abstract result ensuring the existence of two critical points due to Bonanno, D'Aguì and P. Winkert [20] and the results presented here, obtained in collaboration with G. D'Aguì and P. Winkert, are taken from [43].

Chapter 5 is devoted to the study of the existence of solutions for a fourth-order hemivariational inequality. In particular, the main result, obtained in collaboration with G. Bonanno and B. Di Bella in [22], ensures the existence of at least three solutions for the following problem:

Find

$$u \in W^{2,2}([0, 1]) \cap W_0^{1,2}([0, 1])$$

such that

$$\begin{aligned} &- \int_0^1 [p(t)\, u''(t)\, v''(t) - q(t)\, u'(t)\, v'(t) + r(t)\, u(t)\, v(t)]\, \mathrm{d}t \\ &+ \lambda \int_0^1 F^\circ(t, u(t); v(t))\, \mathrm{d}t \geq 0 \end{aligned} \tag{E_λ}$$

for every $v \in W^{2,2}([0, 1]) \cap W_0^{1,2}([0, 1])$.

Assuming suitable conditions on the nonlinear term, we establish an explicit range of values of λ that guarantee three distinct solutions. Furthermore, the autonomous setting is examined, and an application to a fourth-order differential equation is discussed.

Chapter 1

Introduction to Nonsmooth Analysis

Variational methods and critical point theory provide a fundamental framework for the analysis of differential problems. Traditionally, these methods have been developed in the setting of C^1- functionals defined on Banach or Hilbert spaces. Within this classical framework, the notion of a critical point is based on the vanishing of the derivative of the so-called energy functional, and several results - such as Ekeland's variational principle, the Mountain Pass theorem, and the Palais–Smale compactness condition - ensure the existence and multiplicity of solutions. However, numerous theoretical and applied problems naturally lead to the study of functionals that are not differentiable, often due to discontinuities, nonconvexities, or irregular constraints. To address such situations, nonsmooth analysis emerged as an extension of classical differential calculus, enabling the treatment of functionals which, although generally not differentiable in the classical sense, satisfy weaker regularity conditions such as Lipschitz continuity. Clarke [37, 38, 39] played a key role in the development of nonsmooth analysis by introducing the generalized gradient, or Clarke's subdifferential, which is now a fundamental tool in the theory of generalized differential calculus. This framework preserves many of the core features of classical differential calculus, while extending its applicability to nonsmooth settings. Later, Chang [34] significantly advanced this theory by adapting variational methods and critical points results, originally formulated for smooth functionals, to the nonsmooth context.

The purpose of this chapter is to introduce the basic notions of nonsmooth analysis. We will present the main definitions and properties of the generalized gradient, and discuss its role in extending classical variational techniques to a broader class of problems, including those with discontinuous nonlinearities and multivalued differential inclusions.

To ensure a self-contained exposition and clearly define the functional and analytic setting, we begin by recalling some essential preliminaries on Lebesgue and Sobolev spaces, followed by a brief review of selected tools from multivalued analysis. These foundational elements will serve as the groundwork for the subsequent developments of nonsmooth critical point theory.

1.1 Preliminaries on Lebesgue and Sobolev spaces

This section is devoted to recalling some fundamental definitions and classical results concerning Lebesgue and Sobolev spaces, which will be extensively used in the sequel. These notions are standard and can be found in classical references such as [1], [31] and [51].

Throughout, we adopt the following notation:

- $(X, \|\cdot\|_X)$ denotes a Banach space,
- X^* denotes its topological dual,
- $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X^* and X.

Let $\Omega \subset \mathbb{R}^N$ be an open set. We denote by $\mathcal{M}(\Omega)$ the set of all real-valued measurable functions on Ω. For any $1 \leq p < +\infty$, the Lebesgue space $L^p(\Omega)$ is defined as

$$L^p(\Omega) = \left\{ u \in \mathcal{M}(\Omega) : \int_\Omega |u(x)|^p \, \mathrm{d}x < +\infty \right\},$$

endowed with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{1/p}.$$

For $p = \infty$, we define

$$L^\infty(\Omega) = \{ u \in \mathcal{M}(\Omega) : |u| \leq C \text{ a.e. in } \Omega \text{ for some constant C } \geq 0 \},$$

endowed with the essential supremum norm

$$\|u\|_\infty = \inf \{ C \geq 0 : |u(x)| \leq C \text{ for a.a. } x \in \Omega \}.$$

The following propositions summarize the main properties of Lebesgue spaces, see for instance [30, Chapter 4].

Proposition 1.1.1. *Let $1 \leq p \leq +\infty$. Then:*

- *$(L^p(\Omega), \|\cdot\|_p)$ is a Banach space;*
- *$L^p(\Omega)$ is reflexive and uniformly convex for $1 < p < +\infty$;*
- *$L^p(\Omega)$ is separable for $1 \leq p < +\infty$.*

For $1 \leq p \leq \infty$, we define the conjugate exponent p' by

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Proposition 1.1.2 (Hölder's inequality). *Let $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, with $1 \leq p \leq \infty$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_1 \leq \|f\|_p \, \|g\|_{p'}.$$

In particular, if $p = 1$, then $p' = +\infty$, and the inequality becomes

$$\|fg\|_1 \leq \|f\|_1 \, \|g\|_\infty \quad \text{for all } f \in L^1(\Omega), \ g \in L^\infty(\Omega);$$

We now turn to Sobolev spaces. We recall that $C^1_c(\Omega)$ denotes the space of continuously differentiable functions with compact support in Ω. Moreover, in this context, we also introduce the notion of the weak derivative.

Definition 1.1.1 (Weak derivative). *Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in L^p(\Omega)$, with $1 \leq p \leq \infty$. We say that $g_i \in L^p(\Omega)$ is the i-th* weak derivative *of u if*

$$\int_\Omega u(x) \, \frac{\partial \varphi(x)}{\partial x_i} \, \mathrm{d}x = - \int_\Omega g_i(x) \, \varphi(x) \, \mathrm{d}x \quad \text{for all } \varphi \in C^1_c(\Omega), \text{ for all } i = 1, \ldots N.$$

In this case, we write $\frac{\partial u}{\partial x_i} = g_i$.

Definition 1.1.2 (Sobolev spaces)**.** *Let* $1 \le p < \infty$. *The Sobolev space* $W^{1,p}(\Omega)$ *is defined as*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{\partial u(x)}{\partial x_i} \in L^p(\Omega) \text{ for all } i = 1, ..., N \right\}.$$

This space is endowed with the standard norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p$$

where $\|\nabla u\|_p = \||\nabla u|\|_p$. An equivalent norm is given by

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p}.$$

These norms are equivalent in the sense that they induce the same topology on $W^{1,p}(\Omega)$. Furthermore, in correspondence of $p = +\infty$ we introduce

$$W^{1,\infty}(\Omega) = \left\{ u \in L^\infty(\Omega) : \frac{\partial u(x)}{\partial x_i} \in L^\infty(\Omega) \text{ for all } i = 1, \dots, N \right\},$$

endowed with the usual norm

$$\|u\|_\infty = \max\left\{ \|u\|_\infty, \left\|\frac{\partial u}{\partial x_1}\right\|_\infty, \left\|\frac{\partial u}{\partial x_2}\right\|_\infty, \dots, \left\|\frac{\partial u}{\partial x_N}\right\|_\infty \right\}.$$

We recall some key properties of these spaces, for details see [30, Chapter 9].

Proposition 1.1.3. *The Sobolev space* $W^{1,p}(\Omega)$ *is:*

- *a Banach space for any* $1 \le p \le +\infty$;
- *reflexive for any* $1 < p < +\infty$;
- *separable for any* $1 \le p < +\infty$;
- *a Hilbert space for* $p = 2$.

We denote by p^* the critical Sobolev exponent of p in Ω, given by

$$p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } p < N, \\ \text{any } h \in (p, +\infty) & \text{if } p \ge N. \end{cases}$$

Theorem 1.1.1 (Sobolev Embeddings)**.** *Let* $\Omega \subset \mathbb{R}^N$ *be an open bounded domain with Lipschitz boundary, and let* $1 \le p \le \infty$. *Then the following continuous embeddings hold:*

- *If* $1 \le p < N$, *then*
$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [1, p^*].$$
- *If* $p = N$, *then*
$$W^{1,N}(\Omega) \hookrightarrow L^q(\Omega) \quad \text{for all } q \in [p, \infty).$$
- *If* $p > N$, *then*
$$W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega).$$

Theorem 1.1.2 (Rellich–Kondrachov theorem). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Then the following compact embeddings hold:*

- *If $1 \leq p < N$, then*
$$W^{1,p}(\Omega) \hookrightarrow\!\!\!\!\rightarrow L^q(\Omega), \quad \textit{for all } q \in [1, p^*[.$$
- *If $p = N$, then*
$$W^{1,N}(\Omega) \hookrightarrow\!\!\!\!\rightarrow L^q(\Omega) \quad \textit{for all } q \in [1, \infty).$$
- *If $p > N$, then*
$$W^{1,p}(\Omega) \hookrightarrow\!\!\!\!\rightarrow C(\overline{\Omega}).$$

Let $C_c^\infty(\Omega)$ denote the space of infinitely differentiable functions with compact support in Ω. The Sobolev space $W_0^{1,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ in the norm $\|\cdot\|_{1,p}$. A classical result, known as Poincaré inequality (stated below), guarantees that this space can be equivalently endowed with a simpler norm, depending only on the gradient.

Proposition 1.1.4 (Poincaré inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain. There exists a constant $C > 0$, depending on p, N, Ω, such that for all $u \in W_0^{1,p}(\Omega)$*
$$\|u\|_p \leq C\|\nabla u\|_p$$
and the norm $\|\cdot\|_{1,p,0} = \|\nabla \cdot\|_p$ is an equivalent norm on $W_0^{1,p}(\Omega)$.

We conclude this section by recalling some classical results from functional analysis which will be used throughout the thesis.

Theorem 1.1.3 (Hahn–Banach theorem). *Let X be a real vector space, $p : X \to \mathbb{R}$ a sublinear functional, and $f : Y \to \mathbb{R}$ a linear map defined on a subspace $Y \subset X$, such that $f(y) \leq p(y)$ for all $y \in Y$. Then there exists a linear extension $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_Y = f$ and $\tilde{f}(x) \leq p(x)$ for all $x \in X$.*

Theorem 1.1.4 (Riesz Representation theorem for L^p Spaces). *Let $\Omega \subset \mathbb{R}^N$ be a Lebesgue measurable set and let $p \in (1, +\infty)$. For every bounded linear functional $T \in (L^p(\Omega))^*$, there exists a unique $g \in L^q(\Omega)$ such that*
$$T(f) = \int_\Omega f(x)g(x)\,\mathrm{d}x \quad \textit{for all } f \in L^p(\Omega).$$
In particular,
$$\|T\| = \sup_{\substack{f \in L^p(\Omega) \\ \|f\|_p \leq 1}} |T(f)| = \|g\|_q.$$

Theorem 1.1.5 (Banach–Alaoglu theorem). *Let X be a normed space. Then the closed unit ball in the dual space X^* is compact in the weak*- topology.*

Lemma 1.1.1 (Fatou Lemma). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions $f_n : \Omega \to [0, +\infty]$. Then*
$$\int_\Omega \liminf_{n\to\infty} f_n(x)\,\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}x.$$

Lemma 1.1.2 (Reverse Fatou Lemma)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $f_n \le g$ a.e. in Ω for all $n \in \mathbb{N}$, where $g \in L^1(\Omega)$. Then*

$$\int_\Omega \limsup_{n\to\infty} f_n(x)\,\mathrm{d}x \ge \limsup_{n\to\infty} \int_\Omega f_n(x)\,\mathrm{d}x.$$

Theorem 1.1.6 (Dominated Convergence theorem)**.** *Let $A \subset \mathbb{R}^N$ be measurable and let $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(A)$ be a sequence of measurable functions converging pointwise almost everywhere to a function f. Assume that there exists $g \in L^1(A)$ such that*

$$|f_j(x)| \le g(x) \quad \text{for all } j \in \mathbb{N} \text{ and for a.a. } x \in A.$$

Then $f \in L^1(A)$ and

$$\lim_{j\to\infty} \int_A f_j(x)\,\mathrm{d}x = \int_A \lim_{j\to\infty} f_j(x)\,\mathrm{d}x.$$

1.2 Set-valued maps

This section offers a concise introduction to the essential concepts and main results related to the measurability and continuity of set-valued mappings, commonly referred to as multivalued functions or multifunctions. These notions play a central role in the theory of differential inclusions and variational analysis, particularly when dealing with discontinuous nonlinearities or nonsmooth phenomena. In the sequel, we shall exploit several properties of multifunctions - such as measurability, upper semicontinuity, and compactness of values - to ensure the well-posedness of certain problems and to apply critical point theory in the nonsmooth setting. For a deeper exploration and detailed proofs, we refer the reader to the monographs [7], [59] and [68].

Definition 1.2.1 (Multifunction)**.** *Let X and Y be two nonempty sets. A* multifunction *(or* set-valued mapping*) is a map*

$$F\colon X \to 2^Y,$$

where 2^Y denotes the power set of Y, i.e., the set of all subsets of Y.

Definition 1.2.2 (Direct image)**.** *Let $F\colon X \to 2^Y$ be a multifunction and let $A \subseteq X$. The* direct image *of A under F is defined as*

$$F(A) = \bigcup_{x\in A} F(x).$$

Remark 1.2.1. *It is worth noting that the standard definition of image for single-valued functions $f\colon X \to Y$,*

$$f(A) = \{f(x) : x \in A\}$$

cannot be applied to multifunctions $F\colon X \to 2^Y$, since this would yield

$$F(A) = \{F(x) : x \in A\},$$

which is not appropriate as $F(x)$ is itself a set.

For single-valued functions, the inverse image of a set $B \subseteq Y$ is defined by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

In the case of multifunctions, the following two notions of inverse images are introduced.

Definition 1.2.3. *Let $f\colon X \to 2^Y$ be a multifunction and let $B \subset Y$. Then*

(*a*) *the weak (or lower) inverse image is*

$$F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

(*b*) *the strong (or upper) inverse image is*

$$F^{+}(B) = \{x \in X : F(x) \subset B\}.$$

Definition 1.2.4 (Graph of a multifunction)**.** *Let $F\colon X \to 2^Y$ be a multifunction. The graph of the multifunction F is the set*

$$\operatorname{gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Definition 1.2.5 (Inverse multifunction)**.** *Let $F : X \to 2^Y$ be a multifunction. The inverse multifunction of F, denoted by $I_F : Y \to 2^X$, is defined by*

$$I_F(y) = F^{-1}(y) = \{x \in X : y \in F(x)\}, \quad \textit{for all } y \in Y.$$

Definition 1.2.6 (Domain of a multifunction)**.** *Let $F\colon X \to 2^Y$ be a multifunction. The domain of F is defined as the set*

$$dom(F) = \{x \in X : F(x) \neq \emptyset\}.$$

Theorem 1.2.1. *Let $\mathcal{F}(X, Y)$ denote the set of all multifunctions from a set X to the power set 2^Y, i.e., $\mathcal{F}(X, Y) = \{F : X \to 2^Y\}$. Let $2^{X\times Y}$ denote the power set of the Cartesian product $X \times Y$. Then $\mathcal{F}(X, Y)$ and $2^{X\times Y}$ have the same cardinality, that is, there exists a bijection between them.*

Proof. Define a mapping $T : \mathcal{F}(X, Y) \to 2^{X\times Y}$ by assigning to each multifunction F its graph:

$$T(F) = \operatorname{gr}(F).$$

To prove that T is injective, suppose $F, G \in \mathcal{F}(X, Y)$ and $F \neq G$. Then there exists $x \in X$ such that $F(x) \neq G(x)$, and hence there exists $y \in F(x) \setminus G(x)$, which implies that $(x, y) \in \operatorname{gr}(F)$ but $(x, y) \notin \operatorname{gr}(G)$, so $T(F) \neq T(G)$. Thus, T is injective.
To prove surjectivity, let $C \subseteq X \times Y$ be arbitrary. Define a multifunction $F : X \to 2^Y$ by

$$F(x) = \{y \in Y : (x, y) \in C\}, \quad \text{for all } x \in X.$$

Then $\operatorname{gr}(F) = C$, so $T(F) = C$, showing that T is surjective. Therefore, T is a bijection between $\mathcal{F}(X, Y)$ and $2^{X\times Y}$ and the claim follows. □

Define the map

$$\varphi : X \times Y \to Y \times X, \quad \varphi(x, y) = (y, x).$$

Note that φ is a set-theoretic bijection. Moreover, for every multifunction $F : X \to 2^Y$, we have

$$\varphi(\operatorname{gr}(F)) = \operatorname{gr}(I_F),$$

Therefore, the graph of F is equipotent to the graph of its inverse, due to the bijectivity of φ.

Remark 1.2.2. *Let $X \times Y$ and $Y \times X$ be topological spaces endowed with the product topology. Then φ is a homeomorphism, i.e., it is bijective, continuous, and its inverse is continuous. Therefore, $\mathrm{gr}(F)$ is homeomorphic to $\mathrm{gr}(I_F)$, and hence, one can study the topological properties (such as connectedness, compactness, etc.) of the former by examining the latter. If $X \times Y$ and $Y \times X$ are vector spaces, φ is a linear homeomorphism, preserving subspaces and convexity.*

Definition 1.2.7 (Composed multifunction)**.** *Let X, Y, and Z be non-empty sets, and let $F : X \to 2^Y$ and $G : Y \to 2^Z$ be multifunctions. The composed multifunction $H : X \to 2^Z$ is defined by*

$$H(x) = G(F(x)), \quad \text{for all } x \in X.$$

Definition 1.2.8 (Intersection multifunction)**.** *Let X and Y be non-empty sets, and let $F, G : X \to 2^Y$ be multifunctions. The intersection multifunction $F \cap G : X \to 2^Y$ is defined by*

$$(F \cap G)(x) = F(x) \cap G(x), \quad \text{for all } x \in X.$$

We now introduce the key concepts of continuity for multifunctions.

Definition 1.2.9. *Let X and Y be Hausdorff topological spaces, and let $F : X \to 2^Y$ be a multifunction. Then:*

(i) *F is called upper semicontinuous at $x_0 \in X$ if, for every open set $A \subset Y$ such that $F(x_0) \subset A$, there exists a neighborhood $U(x_0)$ such that $F(U(x_0)) \subseteq A$ for all $x \in U(x_0)$. We say that F is upper semicontinuous (u.s.c.) if it is upper semicontinuous at every $x_0 \in X$.*

(ii) *F is called lower semicontinuous at $x_0 \in X$ if, for every open set $A \subset Y$ such that $F(x_0) \cap A \neq \emptyset$, there exists a neighborhood $U(x_0)$ such that $F(x) \cap A \neq \emptyset$ for all $x \in U(x_0)$. We say that F is lower semicontinuous (l.s.c.) if it is lower semicontinuous at every $x_0 \in X$.*

(iii) *F is called continuous (or Vietoris continuous) at $x_0 \in X$ if it is both upper semicontinuous and lower semicontinuous at x_0. We say that F is continuous (or Vietoris continuous) if it is continuous at every $x_0 \in X$.*

The following two propositions present equivalent conditions for the semicontinuity of multifunctions.

Proposition 1.2.1. *The following statements are equivalent:*

1. *F is lower semicontinuous;*
2. *$F^+(C)$ is closed in X for every $C \subset Y$ closed;*
3. *$F^-(A)$ is open in X for every $A \subset Y$ open.*

Proof. We proceed by proving that the statements are mutually equivalent. First, assume that F is l.s.c..
Let $A \subseteq Y$ be open, and let $x_0 \in F^-(A)$. By the definition of F^-, this means that $F(x_0) \cap A \neq \emptyset$. Since F is lower semi-continuous, there exists a neighborhood $U(x_0)$ such that for all $x \in U$, we have $F(x) \cap A \neq \emptyset$. Thus, $U \subseteq F^-(A)$, implying that $F^-(A)$ is open in X. Next, suppose that $F^-(A)$ is open for every open set $A \subseteq Y$. Let $C \subseteq Y$ be closed. We aim to show that $F^+(C)$ is closed in X. Consider the complement of $F^+(C)$, which is $X \setminus F^+(C) = F^-(Y \setminus C)$. Since $Y \setminus C$ is open, it follows from the assumption that $F^-(Y \setminus C)$ is open in X, and hence $X \setminus F^+(C)$ is

open. Therefore, $F^+(C)$ is closed in X.
Now, assume that $F^+(C)$ is closed in X for every closed set $C \subseteq Y$. To show that $F^-(A)$ is open for every open set $A \subseteq Y$, we consider the complement $X \setminus F^-(A)$. We observe that $X \setminus F^-(A) = F^+(Y \setminus A)$, where $Y \setminus A$ is closed. By the assumption that $F^+(C)$ is closed for every closed set $C \subseteq Y$, it follows that $F^+(Y \setminus A)$ is closed, and thus $X \setminus F^-(A)$ is closed. Consequently, $F^-(A)$ is open in X.
Finally, we consider the case where $F^+(C)$ is closed for every closed set $C \subseteq Y$. The argument is analogous to the previous one, where we interchange the roles of F^- and F^+, establishing the desired result. $\square$

Proposition 1.2.2. *The following statements are equivalent:*

1. *F is upper semicontinuous;*
2. *$F^+(A)$ is open in X for every $A \subseteq Y$ open;*
3. *$F^-(C)$ is closed in X for every $C \subseteq Y$ closed.*

The proof of this proposition proceeds analogously to the proof of the previous one. Now, we consider set-valued mappings of the form $F(x) = [f(x), g(x)]$, where f and g are single-valued mappings defined on a topological space X with $f(x) \leq g(x)$ for all $x \in X$. The following propositions explores the relationship between the semicontinuity of F and that of its endpoint functions f and g.

Proposition 1.2.3. *Let X be a topological space. Given $F : X \to 2^{\mathbb{R}}$ as above, the following statements are equivalent:*

1. *F is upper semicontinuous;*
2. *f is lower semicontinuous and g is upper semicontinuous.*

Proposition 1.2.4. *Let X be a topological space. Given $F : X \to 2^{\mathbb{R}}$ as above, the following statements are equivalent:*

1. *F is lower semicontinuous;*
2. *f is upper semicontinuous and g is lower semicontinuous.*

Proposition 1.2.5. *Let X be a topological space. Given $F_1(x) = [f(x), +\infty[$ and $F_2(x) =]-\infty, g(x)]$. Then*

$$F_1 \text{ is l.s.c.} \Leftrightarrow f \text{ is u.s.c} \quad \text{and} \quad F_2 \text{ is u.c.s} \Leftrightarrow g \text{ is u.c.s.}\ .$$

Remark 1.2.3. *Note that F is l.s.c. (u.s.c.) in the sense of Definition 1.2.8, whereas f and g are lower and upper semicontinuous, respectively, in the classical (single-valued) sense. That is, for a function $h : X \to \mathbb{R}$ and a point $x_0 \in X$, we say:*

- *h is lower semicontinuous at x_0 if, for every $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $h(x) > h(x_0) - \varepsilon$ for all $x \in U$, equivalently,*

$$\liminf_{x \to x_0} h(x) \geq h(x_0);$$

- *h is upper semicontinuous at x_0 if, for every $\varepsilon > 0$, there exists a neighborhood U of x_0 such that $h(x) < h(x_0) + \varepsilon$ for all $x \in U$, equivalently,*

$$\limsup_{x \to x_0} h(x) \leq h(x_0).$$

Moreover, the following characterizations hold:

- h is lower semicontinuous $\Leftrightarrow$ for every $r \in \mathbb{R}$, the closed sublevel set $f^{-1}(]-\infty, r])$ is closed $\Leftrightarrow$ for every $r \in \mathbb{R}$, the open superlevel set $f^{-1}(]r, +\infty[)$ is open;
- h is upper semicontinuous $\Leftrightarrow$ for every $r \in \mathbb{R}$, the open sublevel set $f^{-1}(]-\infty, r[)$ is open $\Leftrightarrow$ for every $r \in \mathbb{R}$, the closed superlevel set $f^{-1}([r, +\infty[)$ is closed.

Theorem 1.2.2. *Let X be a compact topological space and Y a topological space. Suppose that $F : X \to 2^Y$ is upper semicontinuous and takes compact values, that is, for every $x \in X$, the set $F(x) \subset Y$ is compact (note that the empty set is also considered compact). Then*

$$F(X) = \bigcup_{x \in X} F(x)$$

is a compact subset of Y.

Proof. Let $F : X \to 2^Y$ be an upper semicontinuous multifunction with compact values, and assume that X is compact. We aim to show that the set

$$F(X) = \bigcup_{x \in X} F(x)$$

is compact in Y.

Let $\{\Omega_i\}_{i \in I}$ be an open cover of $F(X)$, with each $\Omega_i \subseteq Y$ open, i.e.,

$$F(X) \subseteq \bigcup_{i \in I} \Omega_i.$$

Our goal is to extract a finite subcover.

Since $F(x)$ is compact for each $x \in X$, there exists a finite subset $I_x \subseteq I$ such that

$$F(x) \subseteq \bigcup_{j \in I_x} \Omega_j.$$

Define the set

$$F^+\left(\bigcup_{j \in I_x} \Omega_j\right) = \left\{x' \in X \mid F(x') \subseteq \bigcup_{j \in I_x} \Omega_j\right\}.$$

By upper semicontinuity of F, this set is open in X. Therefore, the family

$$\left\{F^+\left(\bigcup_{j \in I_x} \Omega_j\right)\right\}_{x \in X}$$

forms an open cover of X.

By compactness of X, there exist finitely many points $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} F^+\left(\bigcup_{j \in I_{x_i}} \Omega_j\right).$$

Applying F to both sides yields

$$F(X) = \bigcup_{i=1}^{n} F\left(F^{+}\left(\bigcup_{j\in I_{x_i}} \Omega_j\right)\right) \subseteq \bigcup_{i=1}^{n}\bigcup_{j\in I_{x_i}} \Omega_j.$$

Since each I_{x_i} is finite, the union on the right-hand side is finite. Hence, $\{\Omega_j\}_{j\in\bigcup_{i=1}^{n} I_{x_i}}$ is a finite subcover of $F(X)$. Therefore, $F(X)$ is compact. □

Remark 1.2.4 (Counterexample to lower semicontinuity)**.** *The conclusion of Theorem 1.2.2 does not hold if we replace upper semicontinuity with lower semicontinuity. We provide a counterexample by constructing a lower semicontinuous multifunction $F : [0,1] \to 2^{\mathbb{R}}$ with compact values such that the image $F([0,1])$ is not compact in $\mathbb{R}$.*

Define the function $g : [0,1] \to \mathbb{R}$ by

$$g(x) := \begin{cases} \frac{1}{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function g is lower semicontinuous on $[0,1]$ (as any function that attains its infimum is l.s.c.), hence $-g$ is upper semicontinuous. Now, define the multifunction

$$F(x) = [-g(x), g(x)].$$

Using Proposition 1.2.4, F is lower semicontinuous on $[0,1]$, and F(x) is with compact values. However, for $x > 0$ sufficiently small, $g(x) = \frac{1}{x}$ becomes arbitrarily large, so that

$$F([0,1]) = \bigcup_{x\in[0,1]} [-g(x), g(x)] = \mathbb{R}.$$

Thus, $F([0,1])$ is not compact in $\mathbb{R}$, despite F being lower semicontinuous and taking compact values. This shows that the assumption of upper semicontinuity in Theorem 1.2.2 is essential.

Another classical result concerning single-valued functions $f\colon X \to Y$ that naturally invites generalization to multifunctions is the following: if f is continuous and Y is a Hausdorff topological space, then the graph $\mathrm{gr}(f)$ is closed. The following theorem provides a partial generalization for set-valued mappings with closed values.

Theorem 1.2.3. *Let X be a topological space and Y a regular topological space.*[1] *Let $F : X \to 2^{Y}$ be a set-valued mapping with closed values that is upper semicontinuous. Then the graph*

$$\mathrm{gr}(F) := \{(x,y) \in X \times Y : y \in F(x)\}$$

is closed in $X \times Y$.

Proof. We prove that the complement of $\mathrm{gr}(F)$ is open. Let $(x_0, y_0) \in (X \times Y) \setminus \mathrm{gr}(F)$, so that $y_0 \notin F(x_0)$. Since Y is regular and $F(x_0)$ is closed, there exist disjoint open sets $\Omega_1, \Omega_2 \subset Y$ such that

$$y_0 \in \Omega_1, \quad F(x_0) \subset \Omega_2, \quad \text{and} \quad \Omega_1 \cap \Omega_2 = \emptyset.$$

[1] A topological space Y is said to be *regular* if for every point $y \in Y$ and every closed set $A \subset Y$ with $y \notin A$, there exist disjoint open sets containing y and A, respectively.

By upper semicontinuity of F at x_0, there exists an open neighborhood $U \subset X$ of x_0 such that for all $x \in U$, $F(x) \subset \Omega_2$. Consider the open set $U \times \Omega_1 \subset X \times Y$. For every $(x, y) \in U \times \Omega_1$, we have $y \in \Omega_1$ and $F(x) \subset \Omega_2$, and since $\Omega_1 \cap \Omega_2 = \emptyset$, it follows that $y \notin F(x)$, i.e., $(x, y) \notin \mathrm{gr}(F)$. Hence, $U \times \Omega_1 \subset (X \times Y) \setminus \mathrm{gr}(F)$, which shows that the complement of $\mathrm{gr}(F)$ is open. Therefore, $\mathrm{gr}(F)$ is closed. $\square$

1.3 Theory and calculus of Clarke's generalized gradient

In this section, we introduce the concepts of generalized directional derivative and Clarke's generalized gradient for a class of locally Lipschitz functions. For a comprehensive treatment, the reader may refer to [32], [34],[39],[68] and [72].
To do so, we recall the following definitions.

Definition 1.3.1 (Lipschitz function)**.** *Let U be a subset of X. A function $f : U \to \mathbb{R}$ is said to be Lipschitz on U , if there exists a constant $L > 0$ such that*

$$|f(z) - f(y)| \leq L\|z - y\|_X \quad \text{for all } z, y \in U.$$

The constant L is called the Lipschitz constant.

Definition 1.3.2 (Locally Lipschitz function)**.** *Let U be a subset of X. A function $f : U \to \mathbb{R}$ is said to be locally Lipschitz on U if for every point $x \in U$ there exists a neighborhood V of x and a constant $L_V > 0$ such that*

$$|f(z) - f(y)| \leq L_V\|z - y\|_X \quad \text{for all } z, y \in V.$$

The constant L_V is called the local Lipschitz constant of f near x.

Clearly, a function $f : U \to \mathbb{R}$ defined on a subset U of X, which is Lipschitz on bounded subsets of U, is locally Lipschitz. The converse, however, is not generally true.

We are interested in exploring the differential properties of such a function f. It is important to point out that f need not be differentiable at a point x, nor does it necessarily admit directional derivatives at x.

Definition 1.3.3 (Generalized directional derivative)**.** *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function and fix two points $x, v \in X$. The generalized directional derivative (in the sense of Clarke) of f at x in the direction v is given by*

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{f(y + tv) - f(y)}{t},$$

or equivalently,

$$f^\circ(x; v) = \limsup_{\substack{h \to 0 \\ t \to 0^+}} \frac{f(x + h + tv) - f(x + h)}{t}.$$

It is important to note that, unlike the classical directional derivative, the generalized directional derivative is always well-defined. Indeed, locally Lipschitz assumption ensures that f is bounded above by $L_V\|v\|_X$ for y sufficiently close to x and t sufficiently close to 0. We have the following basic properties.

Proposition 1.3.1. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. Then:*

(a) *the function* $v \mapsto f^\circ(x;v)$ *is subadditive, positively homogeneous, convex on* X *and satisfies*
$$|f^\circ(x;v)| \le L_V \|v\|_X;$$

(b) *the function* $(x,v) \mapsto f^\circ(x;v)$ *is upper semicontinuous;*

(c) *the function* $x \mapsto f^\circ(x;v)$ *is continuous;*

(d) $f^\circ(x;-v) = (-f^\circ)(x;v)$.

Proof. Let's begin by verifying (a). Thanks to Lipschitz condition, for y sufficiently near x and y sufficiently near 0, one has
$$\frac{f(y+tv)-f(y)}{t} \le L_V \|v\|_X.$$

So, the lim sup of this bounded quantity exist finite and $|f^\circ(x;v)| \le L_V\|v\|_X$. To prove the subadditivity, we have
$$\begin{aligned} f^\circ(x;v+w) &= \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{f(y+t(v+w))-f(y)}{t} \\ &\le \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{f(y+tv+tw)-f(y+tw)}{t} \\ &+ \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{f(y+tw)-f(y)}{t} \\ &= f^\circ(x;v)+f^\circ(x;w). \end{aligned}$$

Moreover, for any $\lambda > 0$
$$f^\circ(x;\lambda v) = \lambda \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{f(y+tv)-f(y)}{\lambda t} = \lambda f^\circ(x;v).$$

Then, positivity homogeneity is also proven, and (a) is established.
Now, we prove (b). Let $\{x_n\}$ and $\{v_n\}$ be two sequence converging to x and v respectively. For each n, there exist $y_n \in X$ and $t_n > 0$ such that $\|y_n - x_n\|_X + t_n < \dfrac{1}{n}$. Thus, $y_n \to x$ and $t_n \to 0^+$. One has
$$\begin{aligned} f^\circ(x_n,v_n) - 1 &\le \frac{f(y_n+t_nv_n)-f(v_n)}{t_n} \\ &= \frac{f(y_n+t_nv)-f(y_n)}{t_n} + \frac{f(y_n+t_nv_n)-f(y_n+t_nv)}{t_n}. \end{aligned}$$

We observe that, as proved in (a), the last term is bounded by $L\|v_n - v\|_X$, so for $n \to +\infty$ we obtain
$$\limsup_{n\to+\infty} f^\circ(x_n,v_n) \le f^\circ(x,v)$$

and upper semicontinuity is obtained. To prove (c), we show that the function $v \mapsto f^\circ(x,v)$ is Lipschitz. Indeed, for y near x and t near 0,

$$\begin{aligned} f(y+tv)-f(y) &= f(y+tv)-f(y+tw)+f(y+tw)-f(y) \\ &\leq f(y+tw)-f(y)+K\|y-w\|_X. \end{aligned}$$

Dividing by t and passing to the upper limits as $y \to x$ and $t \to 0^+$, one has

$$f^\circ(x,v) \leq f^\circ(x,w)+K\|v-w\|_X.$$

Finally, to prove (d), we calculate

$$\begin{aligned} f^\circ(x,-v) &= \limsup_{t\to 0^+} \frac{f(y-tv)-f(y)}{t} \\ &= \limsup_{t\to 0^+} \frac{f(u)-f(u+tv)}{t} \\ &= \limsup_{t\to 0^+} \frac{(-f)(u+tv)-(-f)(u)}{t} = (-f)^\circ(x,v). \end{aligned}$$

□

Clearly, when f is differentiable at x, $f^\circ(x,v)$ reduces to the usual directional derivative

$$f'(x;v) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

Definition 1.3.4 (Regular function). *A locally Lipschitz function $f : X \to \mathbb{R}$ is said to be regular at a point $x \in X$ if*

(i) *the directional derivative $f'(x;v)$ exists, for every $v \in X$;*

(ii) $f^\circ(x;v) = f'(x;v)$.

For example, if $f : X \to \mathbb{R}$ is convex and continuous, then it is regular.

Definition 1.3.5 (Generalized gradient). *Let $f\colon X \to \mathbb{R}$ be a locally Lipschitz function. The generalized gradient (or Clarke's subdifferential) of f at $x \in X$ is the subset of the dual space X^* defined by*

$$\partial f(x) = \{\zeta \in X^* : f^\circ(x;v) \geq \langle \zeta, v\rangle, \quad \textit{for all } v \in X\}.$$

Remark 1.3.1. *Note that, if f is a convex and continuous function, the generalized gradient coincides with the subdifferential in the sense of convex analysis.*

Example 1.3.1. *Consider the function $f(x) = |x|$, defined on $X = \mathbb{R}$. The function f is Lipschitz continuous, which follows directly from the triangle inequality. We compute the generalized gradient of f in the sense of Clarke. For $x > 0$, the function is differentiable, and the directional derivative is*

$$f^\circ(x;v) = \lim_{t\to 0^+} \frac{|x+tv|-|x|}{t} = \lim_{t\to 0^+} \frac{x+tv-x}{t} = v.$$

Thus, the generalized gradient is the singleton

$$\partial f(x) = \{1\}.$$

Similarly, for $x < 0$, we obtain

$$f^\circ(x;v) = \lim_{t\to 0^+} \frac{|x+tv|-|x|}{t} = \lim_{t\to 0^+} \frac{-(x+tv)+x}{t} = -v,$$

and hence

$$\partial f(x) = \{-1\}.$$

At the nondifferentiable point $x = 0$, we compute:

$$f^\circ(0;v) = \limsup_{t\to 0^+} \frac{|tv|-0}{t} = \limsup_{t\to 0^+} \frac{|tv|}{t} = |v|.$$

So, we determine the generalized gradient by solving:

$$\partial f(0) = \{\zeta \in \mathbb{R} : \zeta v \le |v| \ \text{ for all } v \in \mathbb{R}\}.$$

This condition is equivalent to $\zeta \in [-1,1]$. Indeed, for all $v \in \mathbb{R}$,

$$\zeta v \le |v| \quad \iff \quad -1 \le \zeta \le 1.$$

Therefore,

$$\partial f(0) = [-1,1].$$

Next proposition lists some important properties of the generalized gradient.

Proposition 1.3.2. *Let f be a locally Lipschitz function. Then, for any $x \in X$, the following properties hold:*

(i) *The set $\partial f(x)$ is a convex, nonempty, and weak*- compact subset of X^*, and*

$$\|\zeta\|_{X^*} \le L_V, \quad \text{for all } \zeta \in \partial f(x),$$

where $L_V > 0$ is the Lipschitz constant of f near x.

(ii) *For all $v \in X$,*

$$f^\circ(x;v) = \max\{\langle \zeta, v\rangle : \zeta \in \partial f(x)\}.$$

(iii) *The mapping $x \mapsto \partial f(x)$ has a weak*- closed graph from X into X^*.*

(iv) *The mapping $x \mapsto \partial f(x)$ is upper semicontinuous from X into X^* endowed with the weak* topology.*

Proof. Assertion (i) is a consequence of Definition 1.3.3, Proposition 1.3.1, and the Hahn-Banach theorem 1.1.3.

Assertion (ii) follows also from the Hahn-Banach theorem. In particular, suppose that for some $v \in X$, $f^\circ(x;v)$ exceeds the given maximum (it cannot be less, by definition of $\partial f(x)$). According to the Hahn-Banach theorem, there exists a linear functional ζ majorized by $f^\circ(x;\cdot)$ and coinciding with it at v. It follows that $\zeta \in \partial f(x)$, hence

$$f^\circ(x;v) > \langle \zeta, v\rangle = f^\circ(x;v),$$

which is a contradiction, thus establishing (ii). To prove (iii), let $(u_n) \subset X$ satisfy $u_n \to u$ in X and let $\zeta_n \in \partial f(u_n)$ with $\zeta_n \rightharpoonup^* \zeta$ in X^*. We show that $\zeta \in \partial f(u)$. By Definition 1.3.5, we have

$$\langle \zeta_n, v\rangle \le f^\circ(u_n;v) \quad \text{for all } v \in X,$$

which, together with the weak*- convergence of (ζ_n) and the upper semicontinuity of the map $x \mapsto f^\circ(x;v)$ (cf. Proposition 1.3.1 (b)), implies

$$\langle \zeta, v\rangle = \lim_{n\to\infty} \langle \zeta_n, v\rangle \leq \limsup_{n\to\infty} f^\circ(u_n;v) \leq f^\circ(u;v) \quad \text{for all } v \in X,$$

and thus $\zeta \in \partial f(u)$. □

Proposition 1.3.3. *Let $\Omega \subseteq X^*$ be a nonempty, convex, and weak*- compact subset. Then,*

$$\partial f(x) \subseteq \Omega \quad \textit{if and only if} \quad f^\circ(x;v) \leq \max\{\langle w, v\rangle : w \in \Omega\} \quad \textit{for all } v \in X.$$

Proof. Clearly, if $\partial f(x) \subseteq \Omega$, then the inequality holds since

$$f^\circ(x;v) = \max_{\zeta \in \partial f(x)} \langle v, \zeta\rangle \leq \max_{w\in\Omega} \langle v, w\rangle.$$

Conversely, suppose there exists $\zeta \in \partial f(x)$ such that $\zeta \notin \Omega$. By the Hahn-Banach theorem, there exists an element $v \in X$ such that

$$\langle v, \zeta\rangle > \sup_{w\in\Omega} \langle v, w\rangle,$$

which contradicts the hypothesis that

$$f^\circ(x;v) = \max_{\zeta' \in \partial f(x)} \langle v, \zeta'\rangle \leq \max_{w\in\Omega} \langle v, w\rangle.$$

Hence, $\partial f(x) \subseteq \Omega$. □

As established in Proposition 1.3.2, knowing the generalized gradient $\partial f(x)$ is equivalent to knowing the generalized directional derivative $f^\circ(x;\cdot)$, as each can be recovered from the other. In particular, the map $v \mapsto f^\circ(x;v)$ coincides with the support function of the set $\partial f(x)$. Recall that the support function of a nonempty subset C of X is the function $\sigma_C \colon X^* \to C$ such that

$$\sigma_C(\zeta) = \{\sup \langle \zeta, x\rangle : x \in C\}.$$

Proposition 1.3.4. *The following properties hold:*

(a) *$\zeta \in \partial f(x)$ if and only if $f^\circ(x;v) \geq \langle \zeta, v\rangle$ for all $v \in X$.*

(b) *Let $(x_i) \subset X$ and $(\zeta_i) \subset X^*$ be sequences such that $\zeta_i \in \partial f(x_i)$ for all $i \in \mathbb{N}$, $x_i \to x$, and $\zeta \in X^*$ is a cluster point of (ζ_i) in the weak*- topology. Then $\zeta \in \partial f(x)$.*

(c) *For every $\delta > 0$,*

$$\partial f(x) = \bigcap_{\delta>0} \left(\bigcup_{y\in x+\delta B} \partial f(y) \right).$$

(d) *If X is finite-dimensional, then the mapping $x \mapsto \partial f(x)$ is upper semicontinuous at x.*

Proof. Assertion (a) follows directly from the definition of the generalized gradient $\partial f(x)$ (see Definition 1.3.5).

To prove (b), let $v \in X$ be arbitrary. Since $\zeta_i \in \partial f(x_i)$, it follows from (a) that

$$f^\circ(x_i;v) \geq \langle \zeta_i, v\rangle.$$

Passing to the limit along a subsequence (without relabeling), we obtain

$$\langle \zeta, v\rangle = \lim_{i\to\infty}\langle \zeta_i, v\rangle \leq \limsup_{i\to\infty} f^\circ(x_i; v) \leq f^\circ(x; v),$$

where the last inequality follows from the upper semicontinuity of $f^\circ(\cdot; v)$ (cf. Proposition 1.3.1 (b)). Since v is arbitrary, it follows from (a) that $\zeta \in \partial f(x)$. Note that this is equivalent to say that the multifunction ∂f is weak*- closed. Assertion (c) is an immediate consequence of (b). To prove (d), suppose by contradiction that ∂f is not upper semicontinuous at x. Then there exist sequences $(x_i) \to x$ in X and $(\zeta_i) \to \zeta$ in X^* such that $\zeta_i \in \partial f(x_i)$ for all i, but $\zeta \notin \partial f(x)$ and this contradicts (b). Hence, ∂f is upper semicontinuous at x. □

We outline the fundamental calculus rules for the generalized gradient.

Proposition 1.3.5. *Let $f, g\colon X \to \mathbb{R}$ be locally Lipschitz functions, $\lambda \in \mathbb{R}$, and let $u \in X$. Then*

(p_1) $\partial(\lambda f)(u) = \lambda \partial f(u)$;

(p_2) *the inclusion*

$$\partial(f+g)(u) \subseteq \partial f(u) + \partial g(u)$$

holds for all $u \in X$, equivalently

$$(f+g)^\circ(x; v) \leq f^\circ(x; v) + g^\circ(x; v)$$

for all $x, v \in X$;

(p_3) *the function*

$$m_f(x) = \min_{w\in\partial f} \|w\|_{X^*}$$

is well-defined for every $x \in X$, and it is lower semicontinuous, that is,

$$\liminf_{x\to x_0} m_f(x) \geq m_f(x_0).$$

Proof. The following results are proved in [34] and [37], but we include complete proofs here for the reader's convenience.

We check (p_1). The statement is trivial for $\lambda = 0$.
Considering $\lambda > 0$, we have

$$\begin{aligned}
\zeta \in \partial(\lambda f)(u) &\iff \langle \zeta, v\rangle \leq (\lambda f)^\circ(u; v) \quad \text{for all } v \in X\\
&\iff \langle \tfrac{1}{\lambda}\zeta, v\rangle \leq \tfrac{1}{\lambda}(\lambda f)^\circ(u; v) = f^\circ(u; v) \quad \text{for all } v \in X\\
&\iff \tfrac{1}{\lambda}\zeta \in \partial f(u)\\
&\iff \zeta \in \lambda\partial f(u).
\end{aligned}$$

If $\lambda < 0$, using Proposition 1.3.1 (a), for all $v \in X$, we get

$$\begin{aligned}
\zeta \in \partial(\lambda f)(u) &\iff \langle \zeta, v\rangle \leq (\lambda f)^\circ(u; v)\\
&\iff \langle \tfrac{1}{\lambda}\zeta, v\rangle = -\tfrac{1}{\lambda}\langle \zeta, -v\rangle \leq -\tfrac{1}{\lambda}(\lambda f)^\circ(u; -v) = f^\circ(u; v)\\
&\iff \tfrac{1}{\lambda}\zeta \in \partial f(u)\\
&\iff \zeta \in \lambda\partial f(u).
\end{aligned}$$

To prove (p_2), let $\zeta \in \partial(f+g)(u)$. Then, for all $v \in X$,

$$\langle \zeta, v\rangle \le f^\circ(u;v) + g^\circ(u;v). \tag{1.1}$$

Assume by contradiction that $\zeta \notin \partial f(u) + \partial g(u)$. By the separation theorem, there exists $w \in X$ such that

$$\begin{aligned}\langle \zeta, w\rangle &> \max\{\langle z, w\rangle : z \in \partial f(u) + \partial g(u)\}\\ &= \max\{\langle z_1, w\rangle : z_1 \in \partial f(u)\} + \max\{\langle z_2, w\rangle : z_2 \in \partial g(u)\}\\ &= f^\circ(u;w) + g^\circ(u;w).\end{aligned}$$

This contradicts (1.1), hence the inclusion (p_2) holds. We now proceed to prove (p_3). Since $\partial f(x)$ is a nonempty, weak*-compact, and convex subset of X^*, and the norm function $w \mapsto \|w\|_{X^*}$ is weak*-lower semicontinuous and bounded from below, it follows that for each $x_0 \in X$ there exists $w_0 \in \partial f(x_0)$ such that

$$\|w_0\|_{X^*} = \min\{\|w\|_{X^*} : w \in \partial f(x_0)\} = m_f(x_0).$$

The lower semicontinuity of the function $m_f(x)$ is proved as follows: if there exists a sequence $(x_n) \to x_0$ such that

$$\liminf_{n\to\infty} m_f(x_n) < m_f(x_0),$$

and a corresponding sequence $w_n \in \partial f(x_n)$ satisfying

$$\|w_n\|_{X^*} = m_f(x_n),$$

then we could choose a subsequence $w_{n_k} \to w_0 \in \partial f(x_0)$ (due to the upper semicontinuity of the set-valued mapping $x \mapsto \partial f(x)$), but

$$\liminf_{k\to\infty} \|w_{n_k}\|_{X^*} = \liminf_{n\to\infty} m_f(x_n) < m_f(x_0) \le \|w_0\|_{X^*},$$

which is a contradiction.

□

Remark 1.3.2. *If f and g are regular at $u \in X$, then $f+g$ is regular at u and the inclusion in (p_2) becomes an equality. Indeed, for every $\zeta \in \partial f(u) + \partial g(u)$, one has*

$$\begin{aligned}\langle \zeta, v\rangle &\le f^\circ(u;v) + g^\circ(u;v)\\ &= f'(u;v) + g'(u;v)\\ &= (f+g)'(u;v)\\ &\le (f+g)^\circ(u;v),\end{aligned}$$

for all $v \in X$. Hence, $\zeta \in \partial(f+g)(u)$.

We also state the following condition for extremal points of a locally Lipschitz function, established in [37, Proposition 6].

Proposition 1.3.6. *If $u \in X$ is a local minimum or maximum point for a locally Lipschitz function $f\colon X \to \mathbb{R}$, then $0 \in \partial f(u)$.*

Proof. Assume u is a local minimum for f, i.e.,

$$f(u) \le f(y) \quad \text{for all } y \in B(u,\delta),$$

where $B(u, \delta)$ is the ball centered at u with radius δ. For any direction $v \in X$ and sufficiently small $t > 0$, we have

$$f(y + tv) \geq f(y), \quad \text{for all } y \text{ close to } u.$$

Dividing by t and taking the upper limit as $y \to u$ and $t \to 0^+$, it follows that

$$f^\circ(u; v) = \limsup_{\substack{y \to u \\ t \to 0^+}} \frac{f(y + tv) - f(y)}{t} \geq 0, \quad \text{for all } v \in X.$$

This means $0 \in \partial f(u)$. If u is a local maximum, apply the same argument to $-f$. $\square$

1.4 The generalized gradient in function spaces

In this subsection, we study the regularity properties of a functional defined via an integrand that may be discontinuous with respect to the real variable.
Let $\varphi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a function and for each fixed $x \in \Omega$ define Φ as

$$\Phi(x, t) = \int_0^t \varphi(x, \xi)\, \mathrm{d}\xi.$$

The function φ is assumed to satisfy the following conditions:

1. measurability in the spatial variable x;
2. local essential boundedness in the second variable, with possible discontinuities in t for almost every $x \in \Omega$;
3. the growth condition
$$|\varphi(x, t)| \leq C_1 + C_2 |t|^\sigma, \tag{1.2}$$
for some constants $C_1, C_2 > 0$ and exponent $\sigma \geq 0$.

Since $t \mapsto \varphi(x, t)$ is locally essentially bounded for almost every $x \in \Omega$, Φ is locally Lipschitz continuous with respect to the second variable. Thus, both its generalized directional derivative $\Phi^\circ(x, \cdot)$ and its generalized gradient $\partial\Phi(x, \cdot)$ in the sense of Clarke are well-defined. In particular,

$$\Phi^\circ(x, t; z) = \limsup_{\substack{h \to 0 \\ \lambda \to 0^+}} \frac{\Phi(x, t + h + \lambda z) - \Phi(x, t + h)}{\lambda},$$

and

$$\partial\Phi(x, t) = \{\xi \in \mathbb{R} \mid \xi z \leq \Phi^\circ(x, t; z) \text{ for all } z \in \mathbb{R}\}.$$

Define for $\delta > 0$:

$$\varphi_\delta^-(x, t) = \operatorname*{ess\,inf}_{|t-s|<\delta} \varphi(x, s), \quad \varphi_\delta^+(x, t) = \operatorname*{ess\,sup}_{|t-s|<\delta} \varphi(x, s),$$

and

$$\varphi^-(x, t) = \lim_{\delta \to 0} \varphi_\delta^-(x, t), \quad \varphi^+(x, t) = \lim_{\delta \to 0} \varphi_\delta^+(x, t).$$

We also denote by

$$\varphi(x, t + 0) = \lim_{h \to 0^+} \varphi(x, t + h), \quad \varphi(x, t - 0) = \lim_{h \to 0^+} \varphi(x, t - h)$$

the one-sided pointwise limits of the function $\varphi(x,t)$ with respect to the second variable. The following result, stated in [34, Section 2], holds.

Proposition 1.4.1. *Let φ be as defined above. Then*

$$\partial\Phi(x,t) \subset [\varphi^-(x,t), \varphi^+(x,t)].$$

If, in addition, the limits $\varphi(x, t \pm 0)$ exist for each $t \in \mathbb{R}$, then

$$\partial\Phi(x,t) = [\varphi^-(x,t), \varphi^+(x,t)].$$

Proof. We observe that

$$\begin{aligned}\Phi^\circ(x,t;z) &= \limsup_{\substack{h\to 0\\ \lambda\to 0^+}} \frac{1}{\lambda}\int_{t+h}^{t+h+\lambda z} \varphi(x,s)\,\mathrm{d}s \\ &\leq \begin{cases} \varphi^+(x,t)\cdot z, & \text{if } z>0, \\ \varphi^-(x,t)\cdot z, & \text{if } z<0. \end{cases}\end{aligned}$$

from it follows that

$$\partial\Phi(x,t) \subset [\varphi^-(x,t), \varphi^+(x,t)].$$

If we additionally assume that the limits $\varphi(x, t\pm 0)$ exist for each $t \in \mathbb{R}$, then it holds that

$$\varphi^-(x,t) = \min\{\varphi(x,t-0), \varphi(x,t+0)\}, \quad \varphi^+(x,t) = \max\{\varphi(x,t-0), \varphi(x,t+0)\}.$$

In particular, for all $z \in \mathbb{R}$,

$$\varphi(x, t\pm 0)\cdot z \leq \Phi^\circ(x,t;z),$$

and since $\partial\Phi(x,t)$ is a closed interval, we conclude that

$$\partial\Phi(x,t) = [\varphi^-(x,t), \varphi^+(x,t)].$$

□

Example 1.4.1. *Consider the function*

$$f(t) = \begin{cases} \dfrac{t^2}{2}\cos\left(\dfrac{1}{t}\right), & \text{if } t \neq 0, \\ 0, & \text{if } t = 0. \end{cases}$$

The function f is continuous on $\mathbb{R}$ and differentiable for all $t \neq 0$. Its classical derivative is

$$f'(t) = t\cos\left(\frac{1}{t}\right) + \frac{1}{2}\sin\left(\frac{1}{t}\right), \quad t \neq 0.$$

To compute the Clarke generalized gradient $\partial f(0)$, we consider the set of all limit points of $f'(t)$ as $t \to 0$. Since $\cos(1/t)$ and $\sin(1/t)$ oscillate between -1 and 1, and $t \to 0$, the term $t\cos(1/t)$ tends to zero, while $\frac{1}{2}\sin(1/t)$ oscillates between $-\frac{1}{2}$ and $\frac{1}{2}$. Therefore, the Clarke subdifferential at $t = 0$ is

$$\partial f(0) = \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

We here recall the definitions of Baire measurability and $\mathcal{N}$ - measurability. Let Y be a topological space and $\mathcal{B}(Y)$ denote the Borel σ-algebra on Y.

Definition 1.4.1. *A function $f : Y \to \mathbb{R}$ is said to be Baire measurable if it belongs to the smallest σ-algebra that makes all continuous real-valued functions on Y measurable. Equivalently, f is Baire measurable if it is the pointwise limit of a sequence of continuous functions. That is, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous functions $f_n : Y \to \mathbb{R}$ such that*

$$f(x) = \lim_{n\to\infty} f_n(x) \quad \textit{for all } x \in Y.$$

This σ-algebra is known as the Baire σ-algebra.

Definition 1.4.2. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$. We say that f is $\mathcal{N}$-measurable if, for every real-valued measurable function u defined on Ω, the function*

$$x \mapsto f(x, u(x))$$

is measurable on Ω.

Following the notation of M. A. Krasnosel'ski et al. (see [61]), $\mathcal{N}$-measurable operators are also called superpositionally measurable. From the definition, it is easy to see that the sum, difference, and product of two $\mathcal{N}$-measurable functions are measurable, and that the upper (lower) limit of a sequence of $\mathcal{N}$-measurable functions is again $\mathcal{N}$-measurable. For example, Carathéodory functions and Baire measurable functions are $\mathcal{N}$-measurable.

In general, superpositional measurability is not guaranteed. Therefore, we shall make use of the following assumption:

$$\varphi^-(x,t) \quad \text{and} \quad \varphi^+(x,t) \quad \text{are } \mathcal{N} \text{ measurable.} \tag{1.3}$$

Assumption (1.3) is not too strong. If $\varphi(x,t)$ does not depend on x, then $\partial\Phi(t) = [\varphi^-, \varphi^+]$. Since the map $t \mapsto \partial\Phi(t)$ is upper semicontinuos (see Proposition 1.3.2), thank to Proposition 1.2.3, $\varphi^+(t)$ is an upper semicontinuous function and $\varphi^-(t)$ is a lower semicontinuos function. Both are Baire measurable, and therefore $\mathcal{N}$-measurable. If $\varphi(x,t)$ is Baire measurable in $(\Omega \times \mathbb{R})$, and we suppose that $\varphi(x,t)$ is monotone non-decreasing in t, then

$$\partial\varphi(x,t) = [\varphi^-(x,t), \varphi^+(x,t)].$$

Then, $\varphi(x, t \pm 0)$ is also Baire measurable and the functions $\varphi^-(x,t)$ and $\varphi^+(x,t)$ are $\mathcal{N}$-measurable. Next, we define

$$g(u) = \int_\Omega \Phi(x, u(x))\, \mathrm{d}x = \int_\Omega \int_0^{u(x)} \varphi(x,t)\, \mathrm{d}t\mathrm{d}x \tag{1.4}$$

Next theorem establishes some properties of the operator g.

Theorem 1.4.1. *[34, Theorem 2.1] Let g defined as in (1.4). Assume that the function $\varphi(x,t)$ satisfies (1.2) and (1.3). Then $g(u)$ is locally Lipschitz in $L^{\sigma+1}(\Omega)$ and*

$$\partial g(u) \subset \partial_t \Phi(\cdot, u(\cdot)) = [\varphi^-(x,t), \varphi^+(x,t)]. \tag{1.5}$$

Proof. Let $w \in L^{\sigma+1}(\Omega)$ and $\delta > 0$. Using growth condition in (1.2) and Hölder inequality

$$\begin{aligned}
|g(u) - g(v)| &= \left| \int_\Omega \Phi(x, u(x))\mathrm{d}x - \int_\Omega (\Phi(x, v(x))\mathrm{d}x \right| \\
&\le \int_\Omega \left| \int_{v(x)}^{u(x)} \varphi(x,t)\mathrm{d}t \right| \mathrm{d}x \\
&\le \int_\Omega \left(\int_{v(x)}^{u(x)} C_1 + C_2|t|^\sigma \mathrm{d}t \right) \mathrm{d}x \\
&= C_1|\Omega|^{\frac{\sigma}{\sigma+1}} \|u - v\|_{\sigma+1} + \int_\Omega C_2 \left(\int_{v(x)}^{u(x)} |t|^\sigma \mathrm{d}t \right) \mathrm{d}x,
\end{aligned} \tag{1.6}$$

for all $u, v \in B(w, \delta)$. By the Mean Value theorem, follows that

$$\int_\Omega \left(\int_{v(x)}^{u(x)} |t|^\sigma \mathrm{d}t \right) \mathrm{d}x = \int_\Omega |\zeta_x(x)|^\sigma \cdot |u(x) - v(x)|\mathrm{d}x$$

with $\zeta_x(x) \in [\min\{u(x), v(x)\}, \max\{u(x), v(x)\}]$. By applying Hölder inequality again, we obtain

$$\int_\Omega |\zeta_x(x)|^\sigma \cdot |u(x) - v(x)|\mathrm{d}x \le \|\zeta_x\|^{\sigma}_{\frac{\sigma}{\sigma+1}} \cdot \|u - v\|_{\sigma+1}. \tag{1.7}$$

Moreover, note that $\zeta_x \in B(w, 3\delta)$. Indeed

$$\|\zeta_x - w\|_{\sigma+1}(\Omega) \le \|\zeta_x - u\|_{L^{\sigma+1}(\Omega)} + \|u - w\|_{\sigma+1}(\Omega),$$

by the triangle inequality. Moreover,

$$\|\zeta_x - u\|_{L^{\sigma+1}(\Omega)} \le \|u - v\|_{L^{\sigma+1}(\Omega)}.$$

Since $u, v \in B(w, \delta)$, combining these estimates, yields

$$\|\zeta_x - w\|_{L^{\sigma+1}(\Omega)} \le \|u - v\|_{L^{\sigma+1}(\Omega)} + \|u - w\|_{L^{\sigma+1}(\Omega)} < 2\delta + \delta = 3\delta,$$

which shows that $\zeta_x \in B(w, 3\delta)$. Then, combining (1.6) and (1.7), one has

$$|g(u) - g(v)| \le \left(C_1|\Omega|^{\frac{\sigma}{\sigma+1}} + C_2 \max_{k \in B(w,3\delta)} \|k\|^{\sigma}_{\sigma+1} \right) \|u - v\|_{\sigma+1},$$

which means that g is locally Lipschtiz in $L^{\sigma+1}(\Omega)$. Now we prove the inclusion (1.5). Since g is locally Lipschitz, its generalized directional derivative $g^\circ(u; v)$ is well-defined. By definition of the Clarke subdifferential, we have

$$g^\circ(u; v) = \limsup_{\substack{h_i \to 0 \\ \lambda_i \to 0^+}} \int_\Omega \frac{1}{\lambda_i} \int_{h_i(x)}^{h_i(x)+\lambda_i v(x)} \varphi(x, \xi + u(x))\, \mathrm{d}\xi\, \mathrm{d}x,$$

where $h_i \in L^{\sigma+1}(\Omega)$ and $h_i \to 0$ strongly. Without loss of generality, we may assume $h_i(x) \to 0$ almost everywhere. Then we can pass to the pointwise limit in the inner integral and obtain

$$\begin{aligned}
g^\circ(u;v) &\le \int_\Omega \Phi^\circ(x;u(x);v(x))\,dx \\
&= \int_\Omega \max\{\langle w,v\rangle : w \in \partial_t \Phi(x,u(x))\}\,\mathrm{d}x \\
&= \int_\Omega \max\langle v,w\rangle : w \in [\varphi^-(x,u(x)),\varphi^+(x,u(x))]\}\,\mathrm{d}x \\
&= \int_{v(x)>0} v(x)\varphi^+(x,u(x))\,\mathrm{d}x + \int_{v(x)<0} v(x)\varphi^-(x,u(x))\,\mathrm{d}x.
\end{aligned}$$

Now let $w \in \partial g(u)$. We show that, for almost every $x \in \Omega$,

$$\varphi^-(x,u(x)) \le w(x) \le \varphi^+(x,u(x)).$$

Suppose, by contradiction, that there exists a measurable set $E \subset \Omega$ with $|E| > 0$ such that

$$w(x) < \varphi^-(x,u(x)) \quad \text{for all } x \in E. \tag{1.8}$$

Take $v(x) = -\chi_E(x)$. Then, on the one hand:

$$\langle w,v\rangle = \int_\Omega w(x)v(x)\,\mathrm{d}x = -\int_E w(x)\,\mathrm{d}x.$$

On the other hand, from the expression of $g^\circ(u;v)$, since $v(x) = -1$ on E, we get:

$$g^\circ(u;v) \le -\int_E \varphi^-(x,u(x))\,\mathrm{d}x$$

and, from hypothesis (1.8)

$$\int_E w(x)\mathrm{d}x < \int_E \varphi^-(x,u(x))\mathrm{d}x.$$

Hence,

$$\langle w,v\rangle = -\int_E w(x)\mathrm{d}x > -\int_e \varphi^-(x,u(x))\mathrm{d}x \ge g^\circ(u;v),$$

which contradicts the definition of $w \in \partial g(u)$, since we must have $\langle w,v\rangle \le g^\circ(u;v)$. Therefore, we must have

$$w(x) \ge \varphi^-(x,u(x)) \quad \text{a.e. in } \Omega.$$

With similar argument we obtain $w(x) > \varphi^+(x,u(x))$, concluding the proof that

$$w(x) \in [\varphi^-(x,u(x)),\varphi^+(x,u(x))] \quad \text{for a.a. } x \in \Omega.$$

□

Lemma 1.4.1. *[34, Lemma 2.1] Suppose that X, Y are two Banach spaces, X is reflexive with $X \hookrightarrow Y$, i.e., $X \subset Y$ and the embedding map is continuous. Assume that X is dense in Y. Let $f\colon Y \to \mathbb{R}$ be function such that*

(a_1) *f is locally Lipschitz continuos;*

(a_2) *$\hat{f} = f_{|X}$;*

(a_3) *f is convex.*

Then, we have

$$\partial f(x) = \partial \hat{f}(x).$$

Proof. Assumption (a$_3$), as stated in Remark 1.3.1, ensures that the generalized gradient of f coincides with the subdifferential in the sense of convex analysis and it is Let $w \in \partial f(x) \subset Y^*$. Then, by definition,

$$f(u) \geq f(x) + \langle w, u - x \rangle \quad \text{for all } u \in Y.$$

In particular, this inequality holds for all $u \in X \subset Y$. Therefore,

$$f(u) \geq f(x) + \langle w|_X, u - x \rangle \quad \text{for all } u \in X,$$

where $w|_X \in X^*$ denotes the restriction of w to X. This shows that $w_{|x} \in \partial \hat{f}(x)$, and hence

$$\partial f(x) \subset \partial \hat{f}(x).$$

Now, we prove the opposite inclusion. Let $w \in \partial \hat{f}(x) \subset X^*$. By definition:

$$\hat{f}(u) \geq \hat{f}(x) + \langle w, u - x \rangle \quad \text{for all } u \in X.$$

Since X is dense in Y and the embedding is continuous, the Hahn–Banach theorem (see Theorem 1.1.3) ensures that any functional $w \in X^*$ can be extended to a continuous linear functional $\tilde{w} \in Y^*$ such that:

$$\tilde{w}|_X = w.$$

Let $y \in Y$. Since X is dense in Y, there exists a sequence $\{z_n\} \subset X$ such that $z_n \to y$ in Y. By convexity of f, we have for all n:

$$f(z_n) \geq f(x) + \langle \tilde{w}, z_n - x \rangle.$$

Taking the limit as $n \to \infty$, and using the continuity of f and $\tilde{w}$, we obtain:

$$f(y) \geq f(x) + \langle \tilde{w}, y - x \rangle.$$

Hence, $\tilde{w} \in \partial f(x)$. We have shown that for every $w \in \partial \hat{f}(x) \subset X^*$, there exists $\tilde{w} \in \partial f(x) \subset Y^*$ such that $\tilde{w}|_X = w$. Therefore,

$$\partial \hat{f}(x) \subset \partial f(x).$$

□

From the previous lemma follows the next theorem.

Theorem 1.4.2. *[34, Theorem 2.2] Under the assumptions of Lemma 1.4.1, but without requiring that f satisfies assumption* (a$_3$)*, then*

$$\partial \hat{f}(x) \subset \partial f(x)$$

still holds.

Proof. Note that, from Proposition (1.3.1) it follows that the function $v \mapsto f^\circ(x; v)$ is a continuous convex function on Y and that

$$f^\circ(x; \cdot)_{|X} \geq \hat{f}^\circ(x, \cdot).$$

As we have pointed out in Remark 1.3.1, the generalized gradients coincides with the subdifferential in the sense of convex analysis for convex functions. So the conclusion follows directly from Lemma 1.4.1. □

Corollary 1.4.1. *[34, Corollary p.111] Let $g(u)$ be defined as in (1.6), with $\varphi(x,t)$ satisfying (1.2) and $\sigma \leq \dfrac{N+2}{N-2}$. Suppose that (1.3) holds. Then the function $g(u)$ is locally Lipschitz continuous on $W_0^{1,2}(\Omega)$, and*

$$\partial g(u) \subset [\varphi^-(x,u(x)), \varphi^+(x,u(x))] \qquad \textit{for a.e. } x \in \Omega.$$

Proof. We set:

- $X = W_0^{1,2}(\Omega)$,
- $Y = L^{\sigma+1}(\Omega)$,
- $g \colon Y \to \mathbb{R}$, and $\hat{g} := g|_X \colon X \to \mathbb{R}$.

We already know that g is locally Lipschitz continuous on $L^{\sigma+1}(\Omega)$ and satisfies

$$\partial g(u) \subset [\varphi^-(x,u(x)), \varphi^+(x,u(x))] \qquad \text{for a.a. } x \in \Omega.$$

Now, since $\sigma + 1 \leq 2^* := \dfrac{2N}{N-2}$, the Sobolev embedding

$$W_0^{1,2}(\Omega) \hookrightarrow L^{\sigma+1}(\Omega)$$

is continuous. Moreover, $W_0^{1,2}(\Omega)$ is a reflexive Banach space and is densely embedded in $L^{\sigma+1}(\Omega)$. Therefore, we can apply Theorem 1.4.2, which gives the desired inclusion for $\partial g(u)$. In the case where g is convex, the conclusion also follows from Lemma 1.4.1. □

Chapter 2

Nonsmooth Analysis in variational methods and critical point theory

This chapter recalls some classical results of the Calculus of Variations, reformulated within the nonsmooth framework, such as the Palais-Smale compactness condition and the Mountain Pass theorem. These extensions provide the necessary foundation for applying critical point theory in contexts where classical differentiability is not guaranteed. Several theorems are also presented, ensuring the existence of one or more critical points for specific classes of nonsmooth functionals and serving as the main tools for the analysis of existence and multiplicity of solutions of nonlinear differential problems.

2.1 Classical results for locally Lipschitz functionals

Variational methods represent a fundamental approach for analyzing nonlinear problems, particularly those arising in partial differential equations and in the Calculus of Variations. The central idea is to reformulate the original problem as the search for critical points of an appropriate functional defined on a suitable function space. A fundamental tool in classical variational analysis is the Direct Methods theorem, which combines the geometric structure of the functional with compactness properties of the underlying space to provide a sufficient condition for the existence of an absolute minimum.

Theorem 2.1.1 (Direct Methods theorem)**.** *Let X be a reflexive Banach space equipped with norm $\|\cdot\|_X$, and let $Y \subset X$ be a weakly closed subset of X. Suppose that $F : Y \to \mathbb{R} \cup \{+\infty\}$ is coercive on Y with respect to X, that is:*

(i) $\displaystyle\lim_{\|u\|\to+\infty} F(u) = +\infty, \qquad u \in Y$

and sequentially weakly lower semicontinuous on Y with respect to X, namely:

(ii) for all $u \in Y$ and for all sequences $\{u_n\} \subset Y$ such that $u_n \rightharpoonup u$ in X, it holds that

$$F(u) \leq \liminf_{n\to+\infty} F(u_n).$$

Then F is bounded from below on Y, and it attains its minimum on Y; that is, there exists $\bar{u} \in Y$ such that

$$F(\bar{u}) = \min_{u\in Y} F(u).$$

Proof. Set $\alpha_0 = \inf_{u\in Y} F(u)$ and let $\{u_n\} \subset Y$ be a minimizing sequence, i.e., a sequence such that

$$\lim_{n\to+\infty} F(u_n) = \alpha_0.$$

By the coercivity of F, the sequence $\{u_n\}$ is bounded in Y. Since X is reflexive, by the Eberlein–Šmulian theorem (see [30]) we may assume, up to a subsequence, that u_n weakly converges to some $u \in X$. Moreover, since Y is weakly closed, it follows that $u \in Y$. Using the weak lower semicontinuity of F, we obtain

$$F(u) \leq \liminf_{n \to +\infty} F(u_n) = \alpha_0,$$

which concludes the proof. □

However, many functionals of interest are not smooth and do not admit classical derivatives. To address such situations, nonsmooth analysis extends the traditional framework by introducing generalized concepts, such as Clarke's subdifferential. These tools allow critical point theory to be applied in nonsmooth settings, making it possible to identify generalized solutions to the associated variational problems. A fundamental result in this context is the *Ekeland's Variational Principle*, given by Ivar Ekeland in 1974 [46]. It allows one to select approximate minimizers possessing stronger variational properties, thereby overcoming difficulties related to the lack of smoothness or compactness. This principle plays a central role in establishing both existence and multiplicity results in nonsmooth variational problems.

In the following, we state Ekeland's Variational Principle in its general form.

Theorem 2.1.2 (Ekeland's Variational Principle)**.** *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R}$ be a lower semicontinuous function on X and bounded from below on X. Then, for every $\varepsilon > 0$ and every $u \in X$ such that*

$$f(u) \leq \inf_{x \in X} f(x) + \varepsilon,$$

there exists $v \in X$ such that:

1. $f(v) \leq f(u)$;
2. $d(v, u) \leq 1$;
3. $f(v) < f(w) + \varepsilon\, d(v, w) \qquad$ *for all* $w \in X,\ w \neq v$.

Clearly, if f is a proper and locally Lipschitz continuous function, then it is also lower semicontinuous. This follows from the fact that local Lipschitz continuity implies continuity, and a proper function (i.e., a function that is finite and attains its minimum on some subset of its domain) is bounded from below. Then, the Ekeland's variational principle still holds under these conditions.
We now recall two fundamental variants of the Palais-Smale condition, suitably adapted to the nonsmooth setting. This condition provide a compactness criterion essential for applying variational methods.

Definition 2.1.1. *(Chang [34]) Let $F : X \to \mathbb{R}$ be a locally Lipschitz function, and let $c \in \mathbb{R}$. We say that F satisfies the Palais-Smale condition at level c, briefly $(PS)_c$, if every sequence $\{u_n\}$ in X fulfilling*

1. $F(u_n) \to c$,
2. $m_f(x) = \min_{w \in \partial f(u_n)} \|w\|_{X^*} \to 0$,

has a convergent subsequence in X. If this condition is satisfied only in the region where $f \geq \alpha > 0$ (resp. $f \leq \alpha < 0$) for all $\alpha > 0$, we say that f satisfies $(PS)^+$ (resp. $(PS)^-$).

Definition 2.1.2. *(Motreanu-Panagiotopoulos [72]) Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional, and let $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex, and lower semicontinuous function. The functional $\Phi + \Psi \colon X \to \mathbb{R} \cup \{+\infty\}$ satisfies the Palais-Smale condition if every sequence $\{u_n\} \subset X$ with $\Phi(u_n) + \Psi(u_n)$ bounded and for which there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \to 0$, such that*

$$\Phi^\circ(u_n)(v - u_n) + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|_X, \qquad \text{for all } v \in X,$$

contains a (strongly) convergent subsequence in X.

Remark 2.1.1. *If $\Psi = 0$, Definition 2.1.2 reduces to Definition 2.1.1.*

Within this framework, the Mountain Pass theorem stands out as a fundamental result in critical point theory. Initially formulated for smooth functionals, it provides a variational principle that characterizes critical values via minimax methods, rather than identifying local extrema. Intuitively, the theorem represents the concept of the lowest point along a path traversing a "mountain range" between two valleys.

In the current setting, we focus on a generalization introduced by Chang, designed for nonsmooth analysis. This extension applies to locally Lipschitz functionals and combines a compactness-type condition with an appropriate geometrical structure. The theorem guarantees the existence of a critical point, interpreted through Clarke's generalized gradient, by means of a minimax argument, thus preserving the essential mountain pass geometry even without classical differentiability.

Theorem 2.1.3 (Chang Nonsmooth Mountain Pass theorem)**.** *Let X be a reflexive Banach space, and let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying the $(PS)^+$ condition. Assume that:*

(1) $f(0) = 0$, and there exist constants $\rho > 0$, $\alpha > 0$ such that

$$f(u) \geq \alpha > 0 \quad \text{per ogni } u \in X \text{ con } \|u\|_X = \rho;$$

(2) There exists $e \in X$, with $\|e\|_X > \rho$, such that

$$f(e) = 0.$$

Then, the functional f attains a positive critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0,\ \gamma(1) = e\}.$$

Remark 2.1.2. *The geometry of the mountain pass provides valuable information about the multiplicity of critical points. Specifically, if the functional is bounded from below, this geometric structure implies the existence of at least two local minima. Conversely, when the functional is unbounded from below, the mountain pass geometry still guarantees the existence of at least one local minimum. This shows that the Mountain Pass theorem is, in fact, a multiplicity result. By combining the Mountain Pass theorem with classical results on local minima, we can derive the existence of multiple solutions to a wide range of variational problems. In particular, when the functional is bounded from below we obtain at least three distinct critical points. On the other hand, if the functional is unbounded from below we can* *still conclude the existence of at least two distinct critical points. In the next section, we will examine in detail the conditions under which we obtain existence results for one, two, three or infinitely many critical points.*

2.2 Existence of critical points for locally Lipschitz functionals of type $\Phi - \Psi$

After introducing some basic tools, this section presents a series of theorems on the existence of critical points for locally Lipschitz functionals. We first establish conditions ensuring the existence of at least one critical point through local minimum principles, and then extend the analysis to results yielding two or three critical points by combining local minimum methods with mountain pass techniques and by imposing suitable geometric assumptions. We conclude with results guaranteeing infinitely many critical points, obtained via variational methods specifically adapted to the nonsmooth setting. These results will serve as a fundamental tool for the study of the problems addressed in the following chapters.

Let $\Phi, \Psi\colon X \to \mathbb{R}$ be two locally Lipschitz continuous functionals in a real Banach space $(X, \|\cdot\|_X)$. We define $I = \Phi - \Psi$. We further fix two numbers $r_1, r_2 \in [-\infty, +\infty]$ such that $r_1 < r_2$. The following definition is a special version of the Palais-Smale condition, given in [21].

Definition 2.2.1. *We say that the function $I : X \to \mathbb{R}$ satisfies the Palais-Smale condition cut off lower at r_1 and upper at r_2(abbreviated as the ${}^{[r_1]}(PS)^{[r_2]}$-condition) if any sequence $\{u_n\} \subset X$ satisfying*

1. *$I(u_n)$ is bounded,*
2. *there exists a sequence $\{\varepsilon_n\} \subset \mathbb{R}^+$ with $\varepsilon_n \to 0^+$ such that*
$$I^\circ(u_n; v) \geq -\varepsilon_n \|v\|_X \quad \text{for all } v \in X,$$
3. *$r_1 < \Phi(u_n) < r_2$ for all $n \in \mathbb{N}$,*

has a convergent subsequence.

In the case where $r_1 = -\infty$ and $r_2 \in \mathbb{R}$, we write $(\mathrm{PS})^{[r_2]}$. If $r_1 \in \mathbb{R}$ and $r_2 = +\infty$, we write ${}^{[r_1]}(\mathrm{PS})$. It is easy to see that if $r_1 = \infty$ and $r_2 = +\infty$, the definition above reduces to the well-known (PS)-condition for locally Lipschitz continuous functions given in Definition 2.1.2. For the convenience of the reader, we now recall the version of the (PS)-condition tailored to the functional I, which will be used in the sequel. In addition, we provide the definition of the Palais–Smale condition at level μ, which will be instrumental in the arguments developed in Chapters 4 and 5.

Definition 2.2.2. *We say that the functional I satisfies the Palais–Smale condition (briefly, the (PS)-condition) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that*

(P_1) *$\{I(u_n)\}_{n\in\mathbb{N}}$ is bounded,*

(P_2) *$I^\circ(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_X \geq 0$, with $\varepsilon_n \to 0^+$ and for all $v \in X$ and $n \in \mathbb{N}$,*

has a convergent subsequence.

Definition 2.2.3. *We say that the functional I satisfies the Palais–Smale condition at level $\mu \in \mathbb{R}$ (briefly, the $(PS)_\mu$-condition) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that*

(P_1^μ) $\{I(u_n)\}_{n\in\mathbb{N}} \to \mu$,

(P_2^μ) $I^\circ(u_n; v - u_n) + \varepsilon_n \|v - u_n\|_X \geq 0$, *with* $\varepsilon_n \to 0^+$ *and for all* $v \in X$ *and* $n \in \mathbb{N}$,

has a convergent subsequence.

We should also mention that if I fulfills the ${}^{[r_1]}(PS)^{[r_2]}$-condition, then it satisfies the ${}^{[s_1]}(PS)^{[s_2]}$-condition for all $s_1, s_2 \in [-\infty, +\infty]$ such that $r_1 \leq s_1 < s_2 \leq r_2$. Particularly, if I satisfies the usual (PS)-condition for locally Lipschitz continuous functions, then it fulfills the ${}^{[s_1]}(PS)^{[s_2]}$-condition for all $s_1, s_2 \in [-\infty, +\infty]$ with $s_1 < s_2$.

We recall the following lemma due to Bonanno [14, Lemma 3.1], which is a consequence of Ekeland's Variational Principle.

Lemma 2.2.1. *Let X be a real Banach space and let $I : X \to \mathbb{R}$ be a locally Lipschitz continuous functional bounded from below. Then, for every minimizing sequence $(u_n) \subset X$ of I, there exists another minimizing sequence $(v_n) \subset X$ of I such that:*

1. $I(v_n) \leq I(u_n)$ *for all* $n \in \mathbb{N}$;
2. $I^\circ(v_n; h) \geq -\varepsilon_n \|h\|_X$ *for all* $h \in X$, *for all* $n \in \mathbb{N}$, *where* $\varepsilon_n \to 0^+$.

First, we present a result by Bonanno, D'Aguì and Winkert [21], which ensures the existence of a nonzero local minimum for locally Lipschitz functionals. We also discuss some related consequences of this theorem.

Theorem 2.2.1. *[21, Theorem 2.3] Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functionals. Define*

$$I = \Phi - \Psi.$$

Assume that there exist $x_0 \in X$ and $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < \Phi(x_0) < r_2$ such that

$$\sup_{u\in\Phi^{-1}(]r_1,r_2[)} \Psi(u) \leq r_2 - \Phi(x_0) + \Psi(x_0), \tag{2.1}$$

and

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u) \leq r_1 - \Phi(x_0) + \Psi(x_0), \tag{2.2}$$

Furthermore, suppose that I satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition. Then, there exists a critical point $u_0 \in \Phi^{-1}(]r_1, r_2[)$ such that

$$I(u_0) \leq I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[),$$

with u_0 being a critical point of I.

Proof. We first put

$$C = r_2 - \Phi(x_0) + \Psi(x_0) \tag{2.3}$$

to define

$$\Phi_{r_1}(u) := \max\{\Phi(u), r_1\}, \quad \Psi_C(u) := \min\{\Psi(u), C\} \tag{2.4}$$

and

$$J = \Phi_{r_1} - \Psi_C.$$

Note that J is locally Lipschitz continuous and bounded from below. Taking into account Lemma 2.2.1, let $(u_n) \subseteq X$ be a minimizing sequence of J, i.e.

$$\lim_{n\to\infty} J(u_n) = \inf_X J,$$

we find a sequence $(v_n) \subseteq X$ of J such that

$$\lim_{n\to\infty} J(v_n) = \inf_X J \quad \text{and} \quad J^\circ(v_n;h) \geq -\varepsilon_n \|h\|_X$$

for all $h \in X$, for all $n \in \mathbb{N}$, with $\varepsilon_n \to 0^+$.
If $J(x_0) = \inf_X J$, then from (2.1) and (2.4) we infer, for $u \in \Phi^{-1}(]r_1, r_2[)$, $\Psi(u) \leq C$ and therefore, $J(u) = I(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$. Thus,

$$I(x_0) = J(x_0) \leq J(u) = I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[).$$

Let us now suppose that $\inf_X J < J(x_0)$. Then, we find $n_0 > 0$ such that

$$J(v_n) < J(x_0) \quad \text{for all } n > n_0.$$

Claim: $r_1 < \Phi(v_n) < r_2 \quad$ for all $n > n_0$. First observe that

$$\Phi(v_n) - \Psi_C(v_n) \leq \Phi_{r_1}(v_n) - \Psi_C(v_n) < \Phi(x_0) - \Psi(x_0).$$

Hence

$$\Phi(v_n) < \Psi_C(v_n) + \Phi(x_0) - \Psi(x_0) \leq C + \Phi(x_0) - \Psi(x_0) = r_2.$$

Suppose now that $\Phi(v_n) \leq r_1$. This yields

$$r_1 - \Psi(v_n) = \Phi_{r_1}(v_n) - \Psi(v_n) < \Phi(x_0) - \Psi(x_0),$$

or equivalently,

$$r_1 - \Phi(x_0) + \Psi(x_0) < \Psi(v_n).$$

Owing to (2.2), we obtain $\Phi(v_n) > r_1$, which is a contradiction. This proves the claim.
Then, from the claim and (2.1) we obtain $J(v_n) = I(v_n)$ and $J^\circ(v_n;h) = I^\circ(v_n;h)$ for all $n > n_0$ and for all $h \in X$. Therefore,

$$\lim_{n\to\infty} I(v_n) = \lim_{n\to\infty} J(v_n) = \inf_X J, \quad \text{and} \quad I^\circ(v_n;h) \geq -\varepsilon_n\|h\| \quad \text{for all } h \in X.$$

Since I satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition, it follows that, up to a subsequence, (v_n) converges strongly to some $v^* \in X$. Thus,

$$I(v^*) = \inf_X J \leq J(u) = I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[).$$

In summary, we have

$$I(v^*) \leq I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[). \tag{2.5}$$

Due to the claim and the continuity of Φ, it holds that $v^* \in \Phi^{-1}([r_1, r_2])$. If $v^* \in \Phi^{-1}(]r_1, r_2[)$, the conclusion of the theorem follows directly from (2.5). If $\Phi(v^*) = r_1$, then (2.2) implies

$$I(v^*) = r_1 - \Psi(v^*) \geq r_1 - \sup_{\Phi(u) \leq r_1} \Psi(u) \geq \Phi(x_0) - \Psi(x_0) = I(x_0).$$

This combined with (2.5) gives

$$I(x_0) \leq I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[),$$

and the assertion of the theorem is proved. If $\Phi(v^*) = r_2$, we have, due to $I(v^*) = J(v^*)$,

$$r_2 - \Psi(v^*) = r_2 - \Psi_C(v^*),$$

which implies $\Psi(v^*) = \Psi_C(v^*) \leq C$. Suppose now that $I(v^*) < I(x_0)$. Then from (2.3) and $\Psi(v^*) \leq C$, we get

$$I(v^*) = r_2 - \Psi(v^*) \geq r_2 - C = \Phi(x_0) - \Psi(x_0) = I(x_0),$$

which is a contradiction. Therefore, $I(v^*) = I(x_0)$, and (2.5) yields

$$I(x_0) \leq I(u) \quad \text{for all } u \in \Phi^{-1}(]r_1, r_2[).$$

This finishes the proof of the theorem, taking into account that each local minimum is also a critical point of I (see, for instance, [65, Proposition 2.1]). □

As a consequence we recall the following result (see [20, Theorem 2.3]).

Theorem 2.2.2. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions with Φ being bounded from below. Put*

$$I = \Phi - \Psi$$

and assume that there exist $x_0 \in X$ and $r \in \mathbb{R}$ satisfying $\Phi(x_0) < r$ such that

$$\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) \leq r - \Phi(x_0) + \Psi(x_0).$$

Furthermore, suppose that I satisfies the $(PS)^{[r]}$-condition.
Then, there exists $u_0 \in \Phi^{-1}((-\infty, r))$ such that $I(u_0) \leq I(u)$ for all $u \in \Phi^{-1}(]-\infty, r[)$, with u_0 being a critical point of I.

Proof. The assertion of the theorem follows directly from theorem 2.2.1 by setting $r_1 \in \mathbb{R}$ such that $r_1 < \inf_X \Phi$ and $r_2 = r$. Here we use the convention $\sup_{\emptyset} \Psi = -\infty$. □

Now, we define

$$I_\lambda = \Phi - \lambda\Psi \quad \text{with } \lambda > 0.$$

Moreover, we set:

$$\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \tag{2.6}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, and

$$\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1} \tag{2.7}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$.

Theorem 2.2.3. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions. Suppose that there exist two numbers $r_1, r_2 \in \mathbb{R}$ satisfying $r_1 < r_2$ such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2),$$

where β and ρ are as in (2.6) and (2.7), and for each $\lambda \in \left]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the ${}^{[r_1]}(PS)^{[r_2]}$-condition. Then, for each $\lambda \in \left]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}\right[$, there exists $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that

$$I_\lambda(u_{0,\lambda}) = \inf_{u \in \Phi^{-1}(]r_1, r_2[)} I_\lambda(u).$$

with $u_{0,\lambda}$ being a critical point of I_λ.

Proof. Fix $\lambda \in \left]\dfrac{1}{\rho(r_1, r_2)}, \dfrac{1}{\beta(r_1, r_2)}\right[$. Then, since

$$\beta(r_1, r_2) < \frac{1}{\lambda} < \rho(r_1, r_2),$$

there exist $v_1, v_2 \in \Phi^{-1}(]r_1, r_2[)$ such that

$$\frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v_1)}{r_2 - \Phi(v_1)} < \frac{1}{\lambda} \quad \text{and} \quad \frac{1}{\lambda} < \frac{\Psi(v_2) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v_2) - r_1}.$$

Now, let $x_0 \in \Phi^{-1}(]r_1, r_2[)$ be such that

$$\Phi(x_0) - \lambda\Psi(x_0) = \min\{\Phi(v_1) - \lambda\Psi(v_1), \Phi(v_2) - \lambda\Psi(v_2)\}.$$

This implies the estimates:

$$\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \lambda\Psi(u) < r_2 - \Phi(x_0) + \lambda\Psi(x_0),$$

and

$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \lambda\Psi(u) < r_1 - \Phi(x_0) + \lambda\Psi(x_0).$$

Now, applying Theorem 2.2.1 to the functional $I_\lambda = \Phi - \lambda\Psi$, we obtain the assertion of the theorem. □

As a direct consequence of Theorem 2.2.3 we have the following result, which can be found in [21, Theorem 2.5].

Theorem 2.2.4. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions satisfying*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Suppose that there exist $r \in \mathbb{R}$ and $\hat{u} \in X$ with $0 < \Phi(\hat{u}) < r$ such that

$$\frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})} \tag{2.8}$$

and for each $\lambda \in \Lambda^{r,\hat{u}} := \left]\frac{\Psi(\hat{u})}{\Phi(\hat{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition. Then, for each $\lambda \in \Lambda^{r,\hat{u}}$, there exists $u_\lambda \in \Phi^{-1}(]0, r[)$ such that

$$I_\lambda(u_\lambda) \leq I_\lambda(u) \quad \textit{for all } u \in \Phi^{-1}(]0, r[),$$

with u_λ being a critical point of I_λ.

Proof. Our aim is to apply Theorem (2.2.3). To this end, let us choose $r_1 = 0$ and $r_2 = r$. Then, owing to (2.8), we obtain

$$\beta(0, r) \leq \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u) - \Psi(\hat{u})}{r - \Phi(\hat{u})} < \frac{r\frac{\Psi(\hat{u})}{\Phi(\hat{u})} - \Psi(\hat{u})}{r - \Phi(\hat{u})} = \frac{\Psi(\hat{u})}{\Phi(\hat{u})} = \rho(0, r).$$

Now, let $(v_n) \subseteq \Phi^{-1}(]0, r[)$ be such that $\lim_{n \to \infty} v_n = 0$. Then, for all $n \in \mathbb{N}$, it follows that

$$\beta(0, r) \leq \frac{\sup\limits_{u \in \Phi^{-1}(]0, r[)} \Psi(u) - \Psi(v_n)}{r - \Phi(v_n)}.$$

Since Φ and Ψ are continuous and $\Phi(0) = \Psi(0) = 0$, we obtain

$$\beta(0, r) \leq \frac{\sup\limits_{u \in \Phi^{-1}(]0, r[)} \Psi(u)}{r}.$$

This finally yields

$$\left]\frac{\Psi(\hat{u})}{\Phi(\hat{u})}, \frac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}\right[\subseteq \left]\frac{1}{\rho(0, r)}, \frac{1}{\beta(0, r)}\right[.$$

Since I_λ satisfies the $(PS)^{[r]}$-condition, it also fulfills the $^{[0]}(PS)^{[r]}$-condition. Hence, all the assumptions of theorem 2.2.3 are satisfied and the assertion of the theorem follows. □

Now, we recall the following important multiplicity result used in our study: a two critical points theorem established by Bonanno, D'Aguì and Winkert in 2019 [20, Theorem 2.10], which is a nontrivial consequence of the local minimum theorem presented in theorem 2.2.1 combined with the Ambrosetti-Rabinowitz theorem.

Theorem 2.2.5. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functions such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Suppose that there exist $r \in \mathbb{R}$ and $\hat{u} \in X$ with $0 < \Phi(\hat{u}) < r$ such that

$$\frac{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\hat{u})}{\Phi(\hat{u})} \tag{2.9}$$

and for each

$$\lambda \in \Lambda^{r,\hat{u}} = \left(\frac{\Phi(\hat{u})}{\Psi(\hat{u})}, \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)} \right)$$

the functional $I_\lambda = \Phi - \lambda\Psi$ fulfills the (PS)-condition and is unbounded from below. Then, for each $\lambda \in \Lambda^{r,\hat{u}}$, the functional I_λ admits at least two nontrivial critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

Proof. We fix λ as in the conclusion of the theorem. First, we mention that the (PS)-condition implies the $(\mathrm{PS})^{[r]}$-condition (see Bonanno [14]) and from Theorem 2.2.3 it follows the existence of $u_{\lambda,1} \in \Phi^{-1}(]0,r[)$ such that

$$I_\lambda(u_{\lambda,1}) \leq I_\lambda(u) \quad \text{for all } u \in \Phi^{-1}(]0,r[),$$

with $u_{\lambda,1}$ being a critical point of I_λ. In particular, $u_{\lambda,1} \neq 0$.

Claim 1. *$I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}(]-\infty,r])$ and $I_\lambda(u_{\lambda,1}) < 0$.*

Since $\lambda > \dfrac{\Psi(\hat{u})}{\Phi(\hat{u})}$, it follows that

$$\Phi(\hat{u}) - \lambda\Psi(\hat{u}) < 0 = \Phi(0) - \lambda\Psi(0),$$

that is

$$I_\lambda(u_{\lambda,1}) \leq I_\lambda(\hat{u}) < I_\lambda(0) = 0.$$

Thus, $I_\lambda(u_{\lambda,1}) < 0$ and $I_\lambda(u_{\lambda,1}) \leq I_\lambda(u)$ for all $u \in \Phi^{-1}([0,r[) = \Phi^{-1}(]-\infty,r[)$. In addition, since

$$\lambda < \frac{r}{\sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u)},$$

we have, for all $\tilde{u} \in X$ satisfying $\Phi(\tilde{u}) = r$, that

$$\Phi(\tilde{u}) - \lambda\Psi(\tilde{u}) \geq \Phi(\tilde{u}) - \lambda \sup_{u \in \Phi^{-1}(]-\infty,r])} \Psi(u) > \Phi(\tilde{u}) - r = 0,$$

that is,

$$I_\lambda(\tilde{u}) > I_\lambda(0) > I_\lambda(u_{\lambda,1}).$$

This proves Claim 1. Since the functional I_λ is unbounded from below, we find an element $\tilde{u}_{\lambda,2} \in X$ such that

$$I_\lambda(\tilde{u}_{\lambda,2}) < I_\lambda(u_{\lambda,1}).$$

Moreover, since $u_{\lambda,1}$ is a global minimum of I_λ on $\Phi^{-1}(]-\infty,r])$, it follows that $\Phi(\tilde{u}_{\lambda,2}) > r$. Now we are in the position to apply the nonsmooth Mountain Pass theorem; see Chang [34], which gives us an element $u_{\lambda,2} \in X$ being a critical point of I_λ with corresponding critical value

$$I_\lambda(u_{\lambda,2}) = c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_{\lambda,1}, \gamma(1) = \tilde{u}_{\lambda,2}\}.$$

Claim 2. $I_\lambda(u_{\lambda,2}) > 0$.

Let
$$L = r - \lambda \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u).$$

Since $\lambda < \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}$, we clearly have $L > 0$. Let $\gamma \in \Gamma$. As $\Phi(\gamma(0)) < r$ and $\Phi(\gamma(1)) > r$, we find $\tilde{t} \in]0,1[$ such that $\Phi(\gamma(\tilde{t})) = r$. For $\tilde{u} = \gamma(\tilde{t})$ we obtain
$$\Phi(\tilde{u}) - \lambda\Psi(\tilde{u}) \geq L,$$
which gives
$$I_\lambda(\gamma(\tilde{t})) \geq L.$$
We conclude that
$$\max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq L \quad \text{for each } \gamma \in \Gamma.$$
This finally yields
$$I_\lambda(u_{\lambda,2}) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\lambda(\gamma(t)) \geq L > 0.$$
This proves Claim 2, and the assertion of the theorem follows as well. □

The following theorem is a version of Bonanno [15, Theorem 2.1] for locally Lipschitz continuous functionals with some slightly different assumptions to obtain two different critical points, one is possibly zero.

Theorem 2.2.6. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz continuous functionals with Φ bounded from below and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$. Fix $r > 0$ such that*
$$\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) < +\infty,$$
and assume that, for each
$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$
the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and is unbounded from below. Then, for each
$$\lambda \in \left]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u)}\right[,$$
there exists $u_{\lambda,1} \in \Phi^{-1}(]-\infty, r[)$ such that
$$I_\lambda(u_{\lambda,1}) \leq I_\lambda(u) \quad \textit{for all } u \in \Phi^{-1}(]-\infty, r[),$$
and $u_{\lambda,1}$ is a critical point of I_λ. Moreover, there exists a second critical point $u_{\lambda,2}$ of I_λ.

Remark 2.2.1. *If we assume that*
$$\sup_{u\in\Phi^{-1}((-\infty,r))} \Psi(u) < +\infty \quad \textit{for all } r > \inf_X \Phi,$$
and if
$$\lambda^* = \sup_{r>\inf_X \Phi} \frac{1}{\beta(r)},$$

then the conclusion of Theorem 2.2.6 holds for all $\lambda \in]0, \lambda^*[$.

We now focus on the following theorem which ensures the existence of at least three critical points for locally Lipschitz continuous functionals satisfying the $(PS)_c$-condition under appropriate geometric assumptions. This result can be found in the paper by Bonanno and Marano [26, Theorem 3.5].

Theorem 2.2.7. *Assume X is a reflexive Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be locally Lipschitz continuous functions such that*

(a$_1$) *Φ is coercive and sequentially weakly lower semicontinuous;*

(a$_2$) *Ψ is sequentially weakly upper semicontinuous.*

Suppose that

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0,$$

and that there exist $r > 0$ and $\bar{v} \in X$, with $r < \Phi(\bar{v})$, such that:

(a$_3$) $\dfrac{\sup_{u \in \Phi^{-1}(]-\infty, r[)} \Psi(u)}{r} < \dfrac{\Psi(\bar{v})}{\Phi(\bar{v})}$;

(a$_4$) *for each* $\lambda \in \Lambda^{r,\bar{v}} := \left(\dfrac{\Phi(\bar{v})}{\Psi(\bar{v})}, \dfrac{r}{\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right)$, *the functional $I_\lambda = \Phi - \lambda\Psi$ is bounded below and satisfies the $(PS)_\mu$ for all $\mu \in \mathbb{R}$.*

Then, for each $\lambda \in \Lambda^{r,\bar{v}}$, the functional I_λ admits at least three distinct critical points in X.

Proof. Let $\lambda \in \Lambda^{r,\bar{v}}$ be fixed. Consider the functional $I_\lambda = \Phi - \lambda\Psi \colon X \to \mathbb{R}$, which is locally Lipschitz continuous, as both Φ and Ψ are.

By assumptions (a$_1$) and (a$_2$), I_λ is coercive and sequentially weakly lower semicontinuous. Hence, following the argument used in the proof of Theorem 3.1 in [17], we deduce the existence of two distinct critical points $u_1, u_2 \in X$.

Moreover, since I_λ satisfies the $(PS)_\mu$-condition, we can apply the nonsmooth Mountain Pass theorem 2.1.3 to obtain the existence of a third critical point $u_3 \in X$, as shown in [65, Corollary 2.1]. This concludes the proof. □

Finally, we recall an existence theorem for infinitely many critical points established by G. Bonanno and G. Molica Bisci in 2009 (see [28, Theorem 2.1]).This result is a more precise version of the infinitely many critical points theorem of S.A. Marano and D. Motreanu [66, Theorem 1.1], obtained by extending to the nonsmooth case an existence result of infinitely many critical points due to B. Ricceri (see [82]).

Theorem 2.2.8. *Let X be a reflexive real Banach space. Suppose $\Phi, \Upsilon \colon X \to \mathbb{R}$ are locally Lipschitz functionals, with Φ sequentially weakly lower semicontinuous, Υ sequentially weakly upper semicontinuous, and $j \colon X \to (-\infty, +\infty]$ is convex, proper, and lower semicontinuous. Let*

$$D(j) = \{u \in X \mid j(u) < +\infty\}, \quad \Psi = \Upsilon - j, \quad I_\lambda = \Phi - \lambda\Psi = (\Phi - \lambda\Upsilon) + \lambda j.$$

Assume Φ is coercive and that

$$D(j) \cap \Phi^{-1}((-\infty, r)) \neq \emptyset \quad \textit{for all } r > \inf_X \Phi.$$

For $r > \inf_X \Phi$, define

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty, r])} \frac{\sup_{v \in \Phi^{-1}((-\infty, r])} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$

and

$$\delta := \liminf_{r \to \infty} \varphi(r), \qquad \gamma := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *For every $r > \inf_X \Phi$ and $\lambda \in (0, 1/\varphi(r))$, the restriction of I_λ to $\Phi^{-1}((-\infty, r])$ has a global minimum, which is a critical point of I_λ on X.*

(b) *If $\gamma < \infty$, then for each $\lambda \in (0, 1/\gamma)$, either:*

(b_1) *I_λ admits a global minimum, or*

(b_2) *there exists a sequence of critical points $\{u_n\}$ such that $\Phi(u_n) \to \infty$.*

(c) *If $\delta < \infty$, then for each $\lambda \in (0, 1/\delta)$, either:*

(c_1) *a global minimum of Φ is a local minimum of I_λ, or*

(c_2) *there exists a sequence of pairwise distinct critical points $\{u_n\}$ of I_λ such that $\Phi(u_n) \to \inf_X \Phi$ and $u_n \rightharpoonup u^*$, where u^* is a global minimum of Φ.*

Proof. For the proof of (a), one argue as in the proof of [17, Theorem 3.1]. More precisely, let $1/\lambda > \varphi(r)$, then there exists $\bar{u} \in D(j)$ such that $\Phi(\bar{u}) < r$ and

$$\Phi(\bar{u}) - \lambda\Psi(\bar{u}) < r - \lambda \sup_{\Phi(x) < r} \Psi(x).$$

Moreover, put

$$M := \frac{r - \Phi(\bar{u})}{\lambda} + \Psi(\bar{u}).$$

Clearly,

$$\sup_{\Phi(x) < r} \Psi(x) < M.$$

Finally, put

$$\Psi_M(u) := \begin{cases} \Psi(u), & \text{if } \Psi(u) \le M, \\ M, & \text{if } \Psi(u) > M. \end{cases} \tag{2.10}$$

Since, owing to [30, Corollary III.8], j is sequentially weakly lower semicontinuous, a simple computation shows that Ψ_M is sequentially weakly upper semicontinuous. Put

$$J := \Phi - \lambda\Psi_M.$$

Clearly J is a sequentially weakly lower semicontinuous functional and, as it is easy to see, it is also coercive. Therefore (see [88, Theorem 1.2]), it admits a global minimum u_0. If $J(u_0) = J(\bar{u})$, then $\bar{u}$ satisfies the conclusion. Otherwise, assume $J(u_0) < J(\bar{u})$. In this case, we have $\Psi(u_0) < M$. In fact, from $J(u_0) < J(\bar{u})$ one has

$$\Phi(u_0) - \lambda\Psi_M(u_0) < \Phi(\bar{u}) - \lambda\Psi_M(baru).$$

Hence,

$$\Phi(u_0) < \lambda\Psi_M(u_0) + \Phi(\bar{u}) - \lambda\Psi(\bar{u}) \le \lambda M + \Phi(\bar{u}) - \lambda\Psi(\bar{u}) = r,$$

and from (2.10) one has $\Psi(u_0) < M$. Therefore,

$$\Phi(u_0) - \lambda\Psi(u_0) = \Phi(u_0) - \lambda\Psi_M(u_0) \le \Phi(u) - \lambda\Psi_M(u) \quad \text{for all } u \in X,$$

and, taking again (2.10) into account,

$$\Phi(u_0) - \lambda\Psi(u_0) \le \Phi(u) - \lambda\Psi(u) \quad \text{for all } u \in \Phi^{-1}(]-\infty, r[).$$

Hence, u_0 satisfies the conclusion.
Let us prove (b). Pick $\lambda \in (0, 1/\gamma)$ and assume that (b_1) is not true. We will show that when $\gamma < +\infty$ and I_λ does not possess a global minimum in X, then I_λ admits a sequence of critical points.
Let $a \in \mathbb{R}$ such that $\lambda < a < 1/\gamma$. From $\liminf_{r\to+\infty} \varphi(r) < 1/a$, there exists a sequence $\{r_n\}$ such that

$$\lim_{n\to+\infty} r_n = +\infty \,\text{and}\, \varphi(r_n) < 1/a \,\text{for all}\, n \in \mathbb{N}.$$

Put $r_{n_1} = r_1$. Owing to (a), one can find $u_1 \in \Phi^{-1}((-\infty, r_{n_1}))$ such that u_1 is a local minimum for I_λ. From our assumption, u_1 is not a global minimum for I_λ. Therefore, there exists $u_1' \in X$ such that $I_\lambda(u_1') < I_\lambda(u_1)$. Hence, $u_1' \notin \Phi^{-1}((-\infty, r_{n_1}))$. Let $r_{n_2} \in \{r_n\}$ such that $r_{n_2} > \Phi(u_1')$. Again, from (a), there exists $u_2 \in \Phi^{-1}((-\infty, r_{n_2}))$ such that u_2 is a local minimum for I_λ. Taking into account that u_2 is a global minimum in $\Phi^{-1}((-\infty, r_{n_2}))$, we have $I_\lambda(u_2) \le I_\lambda(u_1)$ and $I_\lambda(u_1') < I_\lambda(u_1)$. Hence, $\Phi(u_2) > r_{n_1}$. Reasoning inductively, we obtain a sequence $\{u_k\}$ of distinct critical points such that $\Phi(u_{k+1}) > r_{n_k}$ for all $k \in \mathbb{N}$. Hence, (b_2) holds.
Finally, we prove (c). Fix $\lambda \in (0, 1/\delta)$ and let $u \in X$ such that $\Phi(u) = \inf_X \Phi$. Assume that (c_1) does not hold, that is, u is not a local minimum for I_λ. Consider $a \in \mathbb{R}$ such that $\lambda < a < 1/\delta$. By our assumption, $\liminf_{r\to\inf_X \Phi} \varphi(r) = \delta < 1/a$, hence there exists a decreasing sequence $\{r_n\}$ such that $\lim_{n\to+\infty} r_n = \inf_X \Phi$ and $\varphi(r_n) < 1/a$ for all $n \in \mathbb{N}$. Put $r_{n_1} = r_1$. Owing to (a), there exists $u_1 \in \Phi^{-1}((-\infty, r_{n_1}))$, which is a local minimum for I_λ. Therefore, $u_1 \ne u$. Then, $\Phi(u) < \Phi(u_1)$. Let $r_{n_2} \in \{r_n\}$ such that $\Phi(u) < r_{n_2} < \Phi(u_1)$. Again from (a), there exists $u_2 \in \Phi^{-1}((-\infty, r_{n_2}))$ which is a local minimum for I_λ, with $\Phi(u) < \Phi(u_2) < \Phi(u_1)$. Reasoning inductively, we obtain a sequence $\{u_k\}$ of distinct local minima for I_λ such that $\Phi(u) < \Phi(u_k) < r_{n_k}$ for all $k \in \mathbb{N}$. Hence,

$$\lim_{k\to+\infty} \Phi(u_k) = \inf_X \Phi.$$

Moreover, since $\{u_k\} \subseteq \Phi^{-1}((-\infty, r_{n_1}))$ and Φ is coercive, then $\{u_k\}$ is bounded. Since X is reflexive, taking a subsequence if necessary, we may assume $\{u_k\}$ weakly converges to $u^* \in X$. From the weak sequential lower semicontinuity, one has

$$\Phi(u^*) \le \lim_{k\to+\infty} \Phi(u_k) = \inf_X \Phi,$$

that is, $\Phi(u^*) = \inf_X \Phi$. Hence, the conclusion is obtained. $\square$

Chapter 3

Existence results for differential inclusions

A differential inclusion is generally understood as a differential problem in which the reaction term, depending on the unknown function u but not on its derivatives, is replaced by a set-valued mapping $F(u)$, so that the nonlinear term is not given by a single-valued function but is allowed to take values within a prescribed set, providing a natural generalization of ordinary and partial differential equations. This framework, as discussed by Iannizzotto [53], is particularly well-suited for modeling systems characterized by irregularities, discontinuities, or multiple possible responses, which naturally arise in contexts such as control theory, friction dynamics, game theory, as well as in applications across physics, mechanics, economics, biology, and the social sciences.The theory of differential inclusions underwent rapid development in the 1980s, with comprehensive treatments provided by authors such as Aubin and Cellina [7, 33], as well as Hu and Papageorgiou [52]. The analysis of such problems is strongly influenced by the properties of the multifunction that governs its dynamics. In particular, regularity (upper or lower semicontinuity), compactness, and convexity of the values of F play a decisive role in the choice of the mathematical tools to be employed in proving the existence of solutions. When F is lower semicontinuous, it is often possible to construct a continuous single-valued selection, that is, a continuous map $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$. This approach is guaranteed by Michael's selection theorem, which ensures the existence of such a selection when F has nonempty closed convex values and X is a paracompact space. In this way, the inclusion reduces to a classical differential equation, making available well-established methods such as the method of upper and lower solutions or, when an appropriate functional framework is present, variational techniques.

If F is upper semicontinuous, an effective strategy for establishing the existence of solutions relies on fixed point theorems for multivalued maps, such as those of Kakutani or Ky Fan. These results guarantee the existence of solutions even in the presence of discontinuities. In this framework, we recall the seminal contribution of S.A. Marano [64], where the existence of at least one solution for a class of elliptic differential inclusions is proved.

Variational methods have proved to be particularly effective tools for establishing existence and multiplicity results. In this setting, the inclusion is associated with a locally Lipschitz functional, and solutions are characterized as generalized critical points in the sense of Clarke's theory. This approach makes it possible to deal with inclusions generated by discontinuous or non-differentiable nonlinearities, for which classical methods are no longer applicable. A first variational framework for second order elliptic inclusions with set-valued mappings of the type

$$F(u) = [f^-(u), f^+(u)],$$

that is, pointwise the interval between the lower and upper limits of a discontinuous single-valued mapping f, was developed in [34], based on the nonsmooth critical point theory for locally Lipschitz functionals introduced in [39] (see also [49]). Further developments and significant contributions in this direction can be found in [25],[40],[54],[55], [56],[57], [77].

3.1 Dirichlet elliptic differential inclusion with Laplace operator

In this section we establish the existence of two distinct positive solutions for an elliptic differential inclusion driven by the Laplace operator, under the assumption that the nonlinear term is given by an upper semicontinuous set-valued mapping with compact convex values and subcritical growth. As a direct consequence of our approach, we also present a multiplicity result for elliptic Dirichlet problems, particularly those involving discontinuous nonlinearities. The results presented in this section are obtained in [29], in collaboration with G. Bonanno, D. O'Regan, and B. Vassallo.

3.1.1 Introduction to the problem and variational setting

Consider the following elliptic differential inclusion:

$$\begin{cases} -\Delta u \in \lambda G(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D_λ}$$

where $\Omega \subseteq \mathbb{R}^N$, with $N \geq 3$, is a bounded open set with smooth boundary $\partial\Omega$, $\lambda > 0$ is a real parameter, and $G : \mathbb{R} \to 2^{\mathbb{R}}$ is an upper semicontinuous set-valued mapping with nonempty, compact, and convex values. We define $f_G(t) = \min G(t)$ for all $t \in \mathbb{R}$, and recall that, as stated in Proposition 1.2.3, the function f_G is lower semicontinuous. We assume the following growth condition:

(f_D) There exist exponents $s \in [1, 2[$ and $q \in]2, \frac{2N}{N-2}[$, and constants $a_s, a_q > 0$ such that
$$0 \leq f_G(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1} \quad \text{for all } t \in \mathbb{R}.$$

The variational framework for our problem is the following. We consider as Banach space X the Sobolev space $W_0^{1,2}(\Omega)$, endowed with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2} \quad \text{for all } u \in W_0^{1,2}(\Omega).$$

It is well known that a function $u \in W_0^{1,2}(\Omega)$ is a generalized weak solution of (D_λ) if there is a function $w \in L^2(\Omega)$ such that

(i) $w(x) \in G(u(x))$ *a.e.* in Ω

(ii) u is a weak solution of the problem

$$\begin{cases} -\Delta u = \lambda w & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

that is, one has

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega wv \, dx \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Put $F_G(\xi) = \int_0^\xi f_G(t)\,dt$ for all $\xi \in \mathbb{R}$. Clearly, F_G is a locally Lipschitz function. Moreover, we define

$$f_G^-(t) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|h|<\delta} f_G(t+h) \quad \text{and} \quad f_G^+(t) = \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|h|<\delta} f_G(t+h). \tag{3.1}$$

Finally, we introduce the functionals $\Phi, \Psi, I_\lambda \colon W_0^{1,2}(\Omega) \to \mathbb{R}$, defined for all $u \in W_0^{1,2}(\Omega)$ by

$$\Phi(u) = \frac{1}{2}\|u\|^2, \tag{3.2}$$

$$\Psi(u) = \int_\Omega F_G(u(x))\,dx \tag{3.3}$$

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{3.4}$$

Denoting by $2^* = \frac{2N}{N-2}$ the critical Sobolev exponent, it is well known that there exists a continuous embedding (see Theorem 1.1.1):

$$W_0^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega),$$

from which one has

$$\|u\|_{2^*} \le C\|u\| \quad \text{for all } u \in W_0^{1,2}(\Omega), \tag{3.5}$$

where the constant C, given by

$$C = \frac{1}{\sqrt{N(N-2)\pi}} \left(\frac{N!}{2\,\Gamma\left(1+\frac{N}{2}\right)} \right)^{\frac{1}{N}},$$

is the best constant, see Talenti [89], and Γ stands for the Gamma function. By Hölder's inequality and (3.5) we obtain

$$\|u\|_p = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}} \le |\Omega|_N^{\frac{2^*-p}{p2^*}} \|u\|_{2^*} \le |\Omega|_N^{\frac{2^*-p}{p2^*}} C\|u\| \tag{3.6}$$

for all $u \in W_0^{1,2}(\Omega)$ and for all $p \in [1, 2^*[$, where $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. In addition, we set

$$R(x) = \sup\{\delta : B(x,\delta) \subseteq \Omega\}$$

for all $x \in \Omega$ and $R = \sup_{x\in\Omega} R(x)$ for which there exists $x_0 \in \Omega$ such that $B(x_0, R) \subseteq \Omega$. We point out the following proposition, which is based on Corollary 1.4.1.

Proposition 3.1.1. *If $u \in W_0^{1,2}(\Omega)$ is a critical point of the functional I_λ defined by (3.4), then u is a generalized weak solution for problem (D_λ).*

Proof. Let $u \in W_0^{1,2}(\Omega)$ be a critical point of I_λ. Then

$$0 \in \partial I_\lambda(u), \quad \text{which means } I_\lambda^\circ(u)(v) \geq 0 \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Therefore,

$$\begin{aligned} 0 &\leq (\Phi - \lambda\Psi)^\circ(u;v) \\ &\leq \Phi'(u;v) + \lambda(-\Psi)^\circ(u;v) \\ &= \Phi'(u;v) + \lambda(\Psi)^\circ(u;-v) \quad \text{for all } v \in W_0^{1,2}(\Omega). \end{aligned}$$

Hence,

$$\Phi'(u;-v) \leq \lambda(\Psi)^\circ(u;-v) \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

which is equivalent to

$$\Phi'(u;s) \leq \lambda(\Psi)^\circ(u;s) \quad \text{for all } s \in W_0^{1,2}(\Omega).$$

Therefore,

$$\Phi'(u) \in \lambda\partial\Psi(u). \tag{3.7}$$

Now, by Corollary 1.4.1, Ψ is locally Lipschitz on $W_0^{1,2}(\Omega)$ and

$$\partial\Psi(u) \subseteq [f_G^-(u(x)), f_G^+(u(x))] \quad \text{a.e. in } \Omega,$$

where the inclusion means: for all $L \in \partial\Psi(u) \subseteq (W_0^{1,2}(\Omega))^*$ there exists $w \in L^2(\Omega)$ such that

$$L(v) = \lambda \int_\Omega wv \, \mathrm{d}x \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

and

$$w(x) \in [f_G^-(u(x)), f_G^+(u(x))] \quad \text{a.e. in } \Omega.$$

Then, from (3.7), there exists $w \in L^2(\Omega)$ such that

$$\Phi'(u)(v) = \int_\Omega wv \, \mathrm{d}x \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

with

$$w(x) \in [f_G^-(u(x)), f_G^+(u(x))] \quad \text{a.e. in } \Omega.$$

In other words, u is a weak solution of

$$\begin{cases} -\Delta u = \lambda w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, we claim

$$[f_G^-(u(x)), f_G^+(u(x))] \subseteq G(u(x)) \quad \text{a.e. in } \Omega.$$

Indeed, for all $\delta > 0$,

$$\operatorname*{ess\,inf}_{|h|<\delta} f_G(t+h) \geq \inf_{|h|<\delta} \min G(t+h),$$

and since $\min G$ is upper semicontinuous, it follows

$$f_G^-(t) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|h|<\delta} f_G(t+h) \geq \min G(t).$$

Similarly, one obtains $f_G^+(t) \leq \max G(t)$, hence the claim is proved. Therefore, we conclude that $w(x) \in G(u(x))$ for a.a. $x \in \Omega$. $\square$

Remark 3.1.1. *Thanks to Proposition 1.2.3* $\min G$ *and* $\max G$ *are measurable selections and any convex combination of these functions is still a measurable selection. However, we choose as a particular selection the minimum of G since its growth behaviour does not impose restrictions on the growth of the set-valued mapping G, which can also have supercritical growth. Therefore, our choice is motivated by the aim of considering a more general case that is possible.*

3.1.2 Positive generalized weak solutions

In this section, we present our main result on the existence of two nontrivial positive solutions for the Dirichlet elliptic differential inclusion given in (D_λ). Put

$$K = \frac{R^2}{2}\cdot\frac{1}{2^N-1}\cdot\frac{1}{2\left(C|\Omega|_N^{\frac{1}{N}}\right)^2},$$

and

$$\begin{aligned} K_\delta &= \frac{1}{K}\cdot\frac{1}{2\left(C|\Omega|_N^{\frac{1}{N}}\right)^2}\cdot\frac{\delta^p}{F_G(\delta)},\\ K_\gamma &= \frac{1}{2\left(C|\Omega|_N^{\frac{1}{N}}\right)^2}\cdot\frac{1}{\frac{a_s}{s}\gamma^{s-p}+\frac{a_q}{q}\gamma^{q-p}}, \end{aligned} \tag{3.8}$$

for two positive constants δ and γ, respectively. The main result reads as follow.

Theorem 3.1.1. *Let* $G : \mathbb{R} \to 2^{\mathbb{R}}$ *be an upper semicontinuous set-valued mapping with compact convex values satisfying condition* (f_D). *Assume that there exist two positive constants* δ, γ, *with* $\delta < \gamma$, *such that*

$$\frac{a_s}{s}\gamma^{s-2}+\frac{a_q}{q}\gamma^{q-2} < K\frac{F_G(\delta)}{\delta^2} \tag{3.9}$$

and suppose that there exist two constants $m > 2$ *and* $l > 0$ *such that*

$$0 < mF_G(\xi) \leq \xi f_G(\xi) \quad \text{for all } \xi \geq l. \tag{3.10}$$

Then, for each $\lambda \in]K_\delta, K_\gamma[$, *problem* ($D_\lambda$) *admits at least two distinct positive generalized weak solutions.*

Proof. Let $\lambda \in]K_\delta, K_\gamma[$ be fixed. From (3.8) and (3.9) we easily see that the interval $]K_\delta, K_\gamma[$ is nonempty. We want to apply Theorem 2.2.5. First, we mention that condition (3.10), known as the Ambrosetti-Rabinowitz condition, implies that the functional I_λ is unbounded from below and satisfies the Palais-Smale condition (see for example [81]). So, we only need to show that inequality (2.9) is satisfied. To this end, put

$$r = \frac{|\Omega|_N^{\frac{2}{2^*}}}{2C^2}\gamma^2 \tag{3.11}$$

and note that the growth condition in (f_D) implies

$$F_G(t) \leq \frac{a_s}{s}|t|^s + \frac{a_q}{q}|t|^q \quad \text{for all } t \in \mathbb{R}. \tag{3.12}$$

Taking into account (3.6), (3.11) and (3.12), we have

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\left(\frac{a_s}{s}\|u\|_s^s + \frac{a_q}{q}\|u\|_q^q\right)}{r} \\
&\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\left(\frac{a_s}{s}C^s|\Omega|_N^{\frac{2^*-s}{2^*}}\|u\|^s + \frac{a_q}{q}C^q|\Omega|_N^{\frac{2^*-q}{2^*}}\|u\|^q\right)}{r} \\
&\leq \frac{\left(\frac{a_s}{s}C^s|\Omega|_N^{\frac{2^*-s}{2^*}}(2r)^{\frac{s}{2}} + \frac{a_q}{q}C^q|\Omega|_N^{\frac{2^*-q}{2^*}}(2r)^{\frac{q}{2}}\right)}{r} \\
&= 2C^2|\Omega|_N^{\frac{2^*-2}{2^*}}\left(\frac{a_s}{s}\left(\frac{2C^2r}{|\Omega|_N^{\frac{2}{2^*}}}\right)^{\frac{s-2}{2}} + \frac{a_q}{q}\left(\frac{2C^2r}{|\Omega|_N^{\frac{2}{2^*}}}\right)^{\frac{q-2}{2}}\right) \\
&= 2C^2|\Omega|_N^{\frac{2}{N}}\left(\frac{a_s}{s}\gamma^{s-2} + \frac{a_q}{q}\gamma^{q-2}\right) \\
&= \frac{1}{K_\gamma},
\end{aligned}$$

where

$$\gamma = \left(\frac{2C^2r}{|\Omega|_N^{\frac{2}{2^*}}}\right)^{\frac{1}{2}}.$$

This implies that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{1}{\lambda}. \tag{3.13}$$

In order to prove the other inequality, let

$$v_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0,R), \\ \frac{2\delta}{R}(R-|x-x_0|) & \text{if } x \in B(x_0,R) \setminus B\left(x_0,\frac{R}{2}\right), \\ \delta & \text{if } x \in B\left(x_0,\frac{R}{2}\right). \end{cases}$$

We easily see that $v_\delta \in W_0^{1,2}(\Omega)$. Moreover, one has

$$\begin{aligned}\Phi(v_\delta) &= \frac{1}{2}\int_{B(x_0,R)\setminus B\left(x_0,\frac{1}{2}R\right)} \frac{(2\delta)^2}{R^2}\mathrm{d}x \\ &= \frac{1}{2}\frac{(2\delta)^2}{R^2}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\left(R^N-\left(\frac{1}{2}R\right)^N\right) \\ &= \frac{2\left(2^N-1\right)}{2^N}R^{N-2}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\delta^2\end{aligned} \tag{3.14}$$

and

$$\Psi(v_\delta) \geq \int_{B\left(x_0,\frac{R}{2}\right)} F_G(\delta)dx = F_G(\delta)\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{R^N}{2^N}.$$

Combining these estimates yields

$$\begin{aligned}\frac{\Psi(v_\delta)}{\Phi(v_\delta)} &\geq \frac{R^2}{2}\frac{1}{2^N-1}\frac{F_G(\delta)}{\delta^2} \\ &= 2C^2|\Omega|_N^{\frac{2}{N}}K\frac{F_G(\delta)}{\delta^2} = \frac{1}{K_\delta} > \frac{1}{\lambda}.\end{aligned} \tag{3.15}$$

From (3.13) and (3.15) we obtain

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(v_\delta)}{\Phi(v_\delta)}. \tag{3.16}$$

We only need to show that $\Phi(v_\delta) < r$. Setting

$$\hat{k} = \left(\frac{2\left(2^N-1\right)}{2^N}R^{N-2}\frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)}\frac{2C^2}{|\Omega|_N^{\frac{2}{2^*}}}\right)^{\frac{1}{2}},$$

we get from (3.14) that

$$\Phi(v_\delta) = \hat{k}^2\frac{|\Omega|_N^{\frac{2}{2^*}}}{2C^2}\delta^2.$$

Recall that $\delta < \gamma$ and we are going to show that $\hat{k}\delta < \gamma$.
First, we observe that

$$\begin{aligned}\frac{1}{\hat{k}^2} &= \frac{2^N}{2\left(2^N-1\right)}\frac{1}{R^{N-2}}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\frac{|\Omega|_N^{\frac{2}{2^*}}}{2C^2} \\ &= \frac{R^2}{2}\frac{1}{2^N-1}\frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}}\frac{\Gamma\left(1+\frac{N}{2}\right)}{\pi^{\frac{N}{2}}}\frac{2^N}{R^N}|\Omega|_N \\ &= \frac{|\Omega|_N}{\left|B\left(x_0,\frac{R}{2}\right)\right|}K \geq K.\end{aligned} \tag{3.17}$$

Now, we apply (3.9) in combination with the growth condition in (3.12) to obtain

$$\frac{\frac{a_s}{s}\gamma^s+\frac{a_q}{q}\gamma^q}{\gamma^2}<K\frac{\frac{a_s}{s}\delta^s+\frac{a_q}{q}\delta^q}{\delta^2}. \tag{3.18}$$

Arguing by contradiction, we assume that $\hat{k}\delta \geq \gamma$. Then, this fact together with $\delta < \gamma$ and (3.17) gives

$$\frac{\frac{a_s}{s}\gamma^s+\frac{a_q}{q}\gamma^q}{\gamma^2}\geq\frac{\frac{a_s}{s}\gamma^s+\frac{a_q}{q}\gamma^q}{\hat{k}^2\delta^2}\geq\frac{1}{\hat{k}^2}\frac{\frac{a_s}{s}\delta^s+\frac{a_q}{q}\delta^q}{\delta^2}\geq K\frac{\frac{a_s}{s}\delta^s+\frac{a_q}{q}\delta^q}{\delta^2},$$

which is a contradiction to (3.18). Hence, $\hat{k}\delta<\gamma$ and this implies $\Phi(v_\delta)<r$.

Now, taking also (3.16) into account, we are in the position to apply Theorem 2.2.5 which says that I_λ admits two distinct non-zero critical points u_1, u_2. Therefore, Proposition 3.1.1 ensures that they are solutions of problem (D_λ). So, in particular, there are $w_i \in L^2(\Omega)$, $i=1,2$, such that u_i are weak solutions of

$$\begin{cases} -\Delta u=\lambda w_i & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega,\end{cases}$$

where $w_i(x)\in G(u_i(x))\subseteq[0,+\infty[$ a.e in Ω, $i=1,2$, owing to (f_D). Hence, the strong maximum principle guarantees that u_i, $i=1,2$, are positive in Ω and the conclusion is achieved. □

Now put

$$\hat{\lambda}=\frac{1}{2C^2|\Omega|^{\frac{2}{N}}_N}\left(\frac{s}{a_s}\right)^{\frac{q-2}{q-s}}\left(\frac{q}{a_q}\right)^{\frac{2-s}{q-s}}\left(\frac{2-s}{q-2}\right)^{\frac{2-s}{q-s}}\frac{q-2}{q-s}.$$

As a consequence, we consider the following case when the nonlinear term has a particular behaviour near 0.

Theorem 3.1.2. *Let $G:\mathbb{R}\to 2^{\mathbb{R}}$ be an upper semicontinuous set-valued mapping with compact convex values satisfying conditions* (f_D) *and* (3.10)*. Assume that*

$$\limsup_{t\to0^+}\frac{F_G(t)}{t^2}=+\infty. \tag{3.19}$$

Then, for each $\lambda\in(0,\hat{\lambda})$, problem (D_λ) admits at least two distinct positive generalized weak solutions.

Proof. Let $\lambda\in(0,\hat{\lambda})$ be fixed. We easily see that $\hat{\lambda}=\sup_\gamma K_\gamma$ and so there is $\gamma>0$ such that $\lambda<K_\gamma$. On the other side, since

$$\limsup_{t\to0^+}2C^2K|\Omega|^{\frac{2}{N}}_N\frac{F_G(t)}{t^2}=+\infty,$$

there exists $\delta<\gamma$ such that

$$2C^2K|\Omega|^{\frac{2}{N}}_N\frac{F_G(\delta)}{\delta^2}>\frac{1}{\lambda}.$$

Hence, $\lambda \in (K_\delta, K_\gamma)$ and so condition (3.9) is fulfilled. Therefore, the statement of the theorem follows from Theorem 3.1.1. □

Now, we give an application of previuos results to Dirichlet problems with discontinuous nonlinearities.

Theorem 3.1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be an almost every continuous function satisfying the following condition:*

($\tilde{f}_D$) *there are $s \in [1,2[$ and $q \in]2,2^*[$ and two positive constants a_s and a_q such that*

$$0 \leq f(t) \leq a_s|t|^{s-1} + a_q|t|^{q-1} \quad \textit{for all } t \in \mathbb{R}.$$

Put $F(t) = \int_0^t f(s)\,\mathrm{d}s$, $t \in \mathbb{R}$, and assume that there exist two constants $m > 2$ and $l > 0$ such that

$$0 < mF(\xi) \leq \xi f(\xi) \quad \textit{for all } \xi \geq l \tag{3.20}$$

and

$$\limsup_{t\to 0^+} \frac{F(t)}{t^2} = +\infty.$$

We denote by

$$\tilde{D}_f = \{t \in \mathbb{R} : f \text{ is not continuous at t}\}.$$

and we require

$$\inf_{\tilde{D}_f} f > 0. \tag{3.21}$$

Then, for each $\lambda \in]0, \hat{\lambda}[$, problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \Omega, \end{cases} \tag{$\tilde{D}_\lambda$}$$

admits at least two distinct positive weak solutions.

Proof. First we observe that the assumption "f is a.e. continuous in $\mathbb{R}$" signifies $|\tilde{D}_f| = 0$. Moreover, assumption ($\tilde{f}_D$) implies that f is a locally bounded function. So, F is well defined and it is a locally Lipschitz function. Moreover, we can define f^-, f^+ as in (3.1). Consider $\partial F(t)$, $t \in \mathbb{R}$. Moreover, put $L(t) = \{a \in \mathbb{R} : \langle a, t\rangle \in \partial F(t)\}$. Since $L(t)$ and ∂F are homeomorphic sets, from [39, Propositions 2.1.5 (d) and 2.1.2 (a)] L is an upper semicontinuous set-valued mapping with convex and compact values. Moreover, from [34, Example 1] one has $L(t) \subseteq [f^-(t), f^+(t)]$ a.e. in $\mathbb{R}$. So, in particular, one has $L(t) = f(t)$ for a.a. $t \in \mathbb{R}$ since $|\tilde{D}_f| = 0$. Moreover, taking into account that $\min L(t) = f(t)$ for a.a. $t \in \mathbb{R}$, simple computations show that L satisfies the assumptions (f_D), (3.10) and (3.19) of Theorem 3.1.2. Hence, problem

$$\begin{cases} -\Delta u \in \lambda L(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive generalized weak solutions. Let u be one of them. One has that there is a function $w \in L^2(\Omega)$ such that

(j) $w(x) \in L(u(x))$ *a.e.* in Ω

(jj)

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \lambda \int_\Omega wv\mathrm{d}x \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

Since $\partial\Omega$ is smooth enough, from usual regularity results it follows that $u \in W^{2,2}(\Omega)$ and from (jj) one has

$$-\Delta u(x) = \lambda w(x)$$

for a.a. $x \in \Omega$. Now, we claim that $|u^{-1}(\tilde{D}_f)|_N = 0$. Arguing by a contradiction, assume $|u^{-1}(\tilde{D}_f)|_N > 0$. From one side, Lemma 1 by De Giorgi, Buttazzo and Dal Maso [44] ensures that $-\Delta u(x) = 0$ for a.a. $x \in u^{-1}(\tilde{D}_f)$. On the other side, (j) ensures $w(x) \in L(u(x))$ for a.a. $x \in u^{-1}(\tilde{D}_f)$, for which $w(x) = f(u(x))$ for a.a. $x \in u^{-1}(\tilde{D}_f)$. Therefore, from (3.21) we obtain $w(x) > 0$ for a.a. $x \in u^{-1}(\tilde{D}_f)$. Hence, one has $0 = -\Delta u(x) = \lambda w(x) > 0$ for a.a. $x \in u^{-1}(\tilde{D}_f)$ and this is absurd, so our claim is proved. Hence, one has

$$w(x) = f(u(x))$$

for a.a. $x \in \Omega$ and from (jj) follows

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \lambda \int_\Omega f(u(x))v \,\mathrm{d}x \quad \text{for all } v \in W_0^{1,2}(\Omega).$$

This completes the proof.

□

We now give two examples of applications of Theorem 3.1.3. The first has a nonlinearity with only one discontinuity point, while the nonlinearity on the second one has an uncountable set of discontinuity points.

Example 3.1.1. The following problem has been introduced in [9]. Put

$$g(s) = \begin{cases} 1 & \text{if} \quad s \leq 0 \\ 0 & \text{if} \quad s > 0 \end{cases}$$

and consider the problem

$$\begin{cases} -\Delta u = u^{p-1} + bg(u-a) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.22}$$

with $2 < p < 2^*$ and $a, b > 0$. Theorem 3.1.3 ensures that there is

$$\hat{b} = \left(\frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}}\right)^{\frac{p-1}{p-2}} \left(\frac{p}{p-2}\right)^{\frac{1}{p-2}} \left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}$$

such that for each $a > 0$ and for each $b \in]0, \hat{b}[$, problem (3.22) admits two positive weak solutions.
Indeed, by choosing $p = q$ and $s = 1$, all assumptions of Theorem 3.1.3 are verified.
In particular, one has

$$0 \leq |u|^{q-1} + bg(u-a) \leq |u|^{q-1} + b$$

for all $u \in \mathbb{R}$. Hence, Theorem 3.1.3 ensures that the problem

$$\begin{cases} -\Delta u = \lambda\Big(u^{p-1} + bg(u-a)\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions for each $\lambda \in]0, \hat{\lambda}[$ where

$$\hat{\lambda} = \frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}} \left(\frac{1}{b}\right)^{\frac{p-2}{p-1}} (p)^{\frac{1}{p-1}} \left(\frac{1}{p-2}\right)^{\frac{1}{p-1}} \frac{p-2}{p-1},$$

so that $\hat{\lambda} > 1$ if and only if $b < \hat{b}$.

Example 3.1.2. Put

$$g(s) = \begin{cases} 0 & \text{if } \; s \in C \\ 1 & \text{if } \; s \in \mathbb{R} \setminus C, \end{cases}$$

where C is the Cantor set, and consider the problem

$$\begin{cases} -\Delta u = u^{p-1} + bg(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.23}$$

with $2 < p < 2^*$ and $b > 0$. Theorem 3.1.3 ensures that there is

$$\hat{b} = (2C^2|\Omega|_N^{\frac{2}{N}})^{\frac{p-1}{p-2}} \left(\frac{p}{p-2}\right)^{\frac{1}{p-2}} \left(\frac{p-2}{p-1}\right)^{\frac{p-1}{p-2}}$$

such that for each $b \in]0, \hat{b}[$, problem (3.23) admits two positive weak solutions. Indeed, by choosing $p = q$ and $s = 1$, all assumptions of Theorem 3.1.3 are verified. Hence, Theorem 3.1.3 ensures that the problem

$$\begin{cases} -\Delta u = \lambda\Big(u^{p-1} + bg(u)\Big) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits two positive weak solutions for each $\lambda \in]0, \hat{\lambda}[$, where

$$\hat{\lambda} = \frac{1}{2C^2|\Omega|_N^{\frac{2}{N}}} \left(\frac{1}{b}\right)^{\frac{p-2}{p-1}} (p)^{\frac{1}{p-1}} \left(\frac{1}{p-2}\right)^{\frac{1}{p-1}} \frac{p-2}{p-1},$$

so that $\hat{\lambda} > 1$ if and only if $b < \hat{b}$.

Remark 3.1.2. Consider the following problem involving a hemivariational inequality

$$\begin{cases} -\displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx + \lambda \int_\Omega F^\circ(u(x); v(x)) \, dx \geq 0 & \text{for all } v \in W_0^{1,2}(\Omega) \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where F° stands for Clarke's generalized directional derivative of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ given by $F(\xi) = \int_0^\xi f(z)dz$, $z \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is a locally essentially bounded function. If we assume ($\tilde{\mathrm{f}}_D$), (3.19) and (3.20) and then for each $\lambda \in]0, \hat{\lambda}[$, problem ($\tilde{D}_\lambda$) admits at least two positive solutions. Indeed, arguing as in the first part of the proof of Theorem 3.1.3, condition (i) and (ii) are obtained. Hence, taking into account that $w(x) \in L(u(x))$ if and only if $w(x)v(x) \leq F^\circ(u(x); v(x))$, one has

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \lambda \int_\Omega wv \,\mathrm{d}x \leq \lambda \int_\Omega F^\circ(u(x); v(x))\mathrm{d}x \quad \text{for all } v \in W_0^{1,2}(\Omega),$$

which is the conclusion.

3.2 Periodic boundary differential inclusions

This section is dedicated to the study of an ordinary differential inclusion involving an upper semicontinuous set-valued mapping with compact convex values, subject to periodic conditions. Under weak assumptions on the reaction term, we prove the existence of two nontrivial weak solutions. We then apply this result to analyze a periodic ordinary differential equation. The main results presented in this section were obtained in [4] in collaboration with E. Amoroso, G. Bonanno, and G. Klun.

3.2.1 Foundations of the problem and variational structure

We consider the following second order ordinary differential inclusion coupled with periodic boundary conditions:

$$\begin{cases} -u'' + u \in \lambda G(u) & \text{in }]0, T[, \\ u(0) = u(T), \quad u'(0) = u'(T). \end{cases} \tag{P_λ}$$

Assume that

(H$_G$) $G \colon \mathbb{R} \to 2^{\mathbb{R}}$ is an u.s.c. set-valued mapping with compact convex values. We denote by

$$f_G(t) = \min G(t) \qquad \text{for all } t \in \mathbb{R}$$

and we define

$$f_G^-(t) = \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|h|<\delta} f_G(t+h) \quad \text{and} \quad f_G^+(t) = \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|h|<\delta} f_G(t+h).$$

We denote by C_T^∞ the space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$, i.e., functions $u \in C^\infty(\mathbb{R})$ such that $u(t+T) = u(t)$ for all $t \in \mathbb{R}$. Using the notation from the book of Mawhin and Willem [67], given $u \in L^1([0,T])$ we say that $\varphi \in L^1([0,T])$ is the weak derivative of u if

$$\int_0^T u(t)v'(t)\,\mathrm{d}t = -\int_0^T \varphi(t)v(t)\,\mathrm{d}t, \qquad \text{for all } v \in C_T^\infty,$$

and we denote $u' = \varphi$. We take as Banach space X the periodic Sobolev space $W_T^{1,2}$ defined as follows

$$W_T^{1,2} = \left\{ u \in L^2([0,T]) : u' \in L^2([0,T]) \right\},$$

endowed with the norm $\|u\|_{1,2} = (\|u\|_2 + \|u'\|_2)^{\frac{1}{2}}$. It is well known that $W_T^{1,2}$ is a reflexive Banach space, $C_T^\infty \subset W_T^{1,2}$ and for all $u \in W_T^{1,2}$

$$\|u\|_\infty \leq k\|u\|_{1,2}, \quad \text{for some } k > 0.$$

In particular, from [67, Proposition 1.1] we have the following estimate for the embedding constant

$$k = \max\left\{\frac{1}{\sqrt{T}}, \sqrt{T}\right\}.$$

Here we give the definition of generalized weak solution for problem (P_λ).

Definition 3.2.1. *We say that a function $u \in W_T^{1,2}$ is a generalized weak solution of problem (P_λ) if there exists a function $w \in L^2(0,T)$ such that:*

(i) $w(x) \in G(u(x))$ for almost every $x \in [0,T]$;

(ii) u is a weak solution of the problem

$$\begin{cases} -u''(x) + u(x) = \lambda w(x) & \text{for } x \in]0,T[, \\ u(0) = u(T), \quad u'(0) = u'(T), \end{cases}$$

that is

$$\int_0^T u'(x)v'(x)\,\mathrm{d}x + \int_0^T u(x)v(x)\,\mathrm{d}x = \lambda \int_0^T w(x)v(x)\,\mathrm{d}x \quad \text{for all } v \in C_T^\infty. \tag{3.24}$$

When G is a single-valued mapping, the previous definition coincides with the usual definition of weak solution. Moreover, we observe that as a consequence of Fundamental Lemma in [67], $u(0) = u(T)$ and from the definition of weak derivative and (3.24) follows that u' admits weak derivative and so $u'(0) = u'(T)$. For all $u \in W_T^{1,2}$, we consider the functionals $\Phi, \Psi \colon W_T^{1,2} \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\|u\|_{1,2}^2, \qquad \Psi(u) = \int_0^T F_G(u(x))\,\mathrm{d}x,$$

where

$$F_G(t) = \int_0^t f_G(\tau)\,\mathrm{d}\tau, \quad \text{for all } t \in \mathbb{R},$$

Since f_G is a bounded function on every compact interval (as a measurable selection of an upper semicontinuous multifunction with compact values), it follows that for any bounded interval $[a,b] \subset \mathbb{R}$, there exists a constant $M > 0$ such that

$$|F_G(t) - F_G(s)| \leq M|t-s| \quad \text{for all } t, s \in [a,b],$$

i.e., F_G is Lipschitz continuous on bounded subsets of $\mathbb{R}$. Hence, F_G is locally Lipschitz continuous on $\mathbb{R}$. The so-called energy functional associated with problem (P_λ) is given by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \quad \text{for all } u \in W_T^{1,2}. \tag{3.25}$$

Observe that $\Phi \in C^1(W_T^{1,2}, \mathbb{R})$ and

$$\Phi'(u)(v) = \int_0^T u'(x)v'(x)\,\mathrm{d}x + \int_0^T u(x)v(x)\,\mathrm{d}x, \quad \text{for all } u, v \in W_T^{1,2}. \tag{3.26}$$

Moreover, standard computations show that the functional Ψ is well defined and sequentially weakly continuous on $W_T^{1,2}$. The next proposition, based on a result of Chang [34], give the relationship between critical points and generalized weak solutions.

Proposition 3.2.1. *Let $u \in W_T^{1,2}$ be a critical point of functional I_λ defined in (3.25). Then u is a generalized weak solution for problem (P_λ).*

Proof. Fix $\lambda > 0$ and let $u \in W_T^{1,2}$ be a critical point of I_λ. One has $0 \in \partial I_\lambda(u)$, which means

$$I_\lambda^\circ(u;v) \geq 0 \quad \text{for all } v \in W_T^{1,2}.$$

Therefore, taking into account Proposition 1.3.1, we obtain

$$0 \leq (\Phi - \lambda\Psi)^\circ(u;v) \ \leq \ \Phi'(u)(v) + \lambda(-\Psi)^\circ(u;v) \ \text{ for all } v \in W_T^{1,2}.$$

From (3.26), it follows that

$$-\left(\int_0^T u'(x)v(x)\mathrm{d}x + \int_0^T u(x)v(x)\mathrm{d}x\right) \leq \lambda(-\Psi)^\circ(u,v) \quad \text{for all } v \in W_T^{1,2},$$

that is

$$\Phi'(u)(-v) \leq \lambda\Psi^\circ(u,-v) \text{ for all } v \in W_T^{1,2},$$

i.e.

$$\Phi'(u)(s) \leq \lambda\Psi^\circ(u,s) \text{ for all } s \in W_T^{1,2}.$$

Hence, one has

$$\Phi'(u) \in \lambda\partial\Psi(u).$$

Clearly, $\Phi'(u)$ is a linear and continuous operator on $W_T^{1,2}$. We also observe that Ψ is a locally Lipschitz functional on $L^2([0,T])$. Indeed, if we consider $w \in L^2([0,T])$ and $\bar{\delta} > 0$, Mean Value theorem and Hölder inequality ensure that

$$\begin{aligned}
|\Psi(u) - \Psi(v)| &\leq \int_0^T \int_{v(t)}^{u(t)} |f(\tau)|\mathrm{d}\tau)\,\mathrm{d}t \\
&\leq \int_0^T |g(t)||u(t) - v(t)|\mathrm{d}t \\
&\leq \|g\|_2\|u - v\|_2 \qquad\qquad \text{for all } u, v \in B(w,\delta),
\end{aligned}$$

where $g(t) \in [\min\{u(t),v(t)\},\max\{u(t),v(t)\}]$ for all $t \in [0,T]$ with $g \in B(w,2\bar{\delta})$. Now, taking into account that $W_T^{1,2}$ is dense in $L^2([0,T])$, from Theorem 1.4.2 one has

$$\Phi'(u) \in \lambda\partial(\Psi)\big|_{W_T^{1,2}}(u) \subset \lambda\partial(\Psi)\big|_{L^2([0,T])}(u) \subseteq (L^2([0,T]))^*. \tag{3.27}$$

Being $\Phi'(u)$ a linear and continuous operator on $L^2([0,T])$, from Riesz representation theorem 1.1.4 there exists a unique $w \in L^2([0,T])$ such that

$$\Phi'(u)(v) = \lambda\int_0^T w(x)v(x)\,\mathrm{d}x \quad \text{for all } v \in L^2([0,T]).$$

Therefore, from Theorem 1.4.1 one has

$$w(x) \in [f_G^-(u), f_G^+(u))] \quad \text{for a.a. } x \in [0,T],$$

that we can write
$$\Phi'(u) \subseteq [f_G^-(u(x)), f_G^+(u(x))].$$
To reach the conclusion, we verify that
$$[f_G^-(u(x)), f_G^+(u(x))] \subseteq G(u(x)),$$
that is
$$\min G(u(x)) \leq f_G^-(u(x)) \leq f_G^+(u(x)) \leq \max G(u(x)).$$
We observe that, for every $\delta > 0$, we have
$$\operatorname*{ess\,inf}_{|h|<\delta} f_G(t+h) \geq \inf_{|h|<\delta} \min G(t+h).$$
Since f_G is lower semicontinuous, for $\delta \to 0^+$, it follows that
$$f_G^-(t) \geq \min G(t).$$
Similarly, we find that $f_G^+(t) \leq \max G(t)$ and this concludes the proof. $\square$

3.2.2 Existence of two nontrivial solutions

This subsection is devoted to obtaining the existence of two critical points for the energy functional related to our problem, as defined in (3.25). These critical points, as guaranteed by Proposition 3.2.1, will represent generalized weak solutions to the problem (P_λ). The main result of this investigation can be stated as follows.

Theorem 3.2.1. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued mapping satisfying (H_G). Assume that there exist two positive constants c, d with $d < c$, such that*
$$\frac{\max_{|\xi|\leq c} F_G(\xi)}{c^2} < \frac{1}{Tk^2}\frac{F_G(d)}{d^2} \tag{3.28}$$
and there exist two constants $\eta > 2$ and $l > 0$ such that
$$0 < \eta F_G(t) \leq t f_G(t) \quad \textit{for all } t \geq l. \tag{AR}$$
Then, for each $\lambda \in \Lambda_{c,d} = \left]\frac{d^2}{2TF_G(d)}, \frac{c^2}{2Tk^2 \max_{|\xi|<c} F_G(\xi)}\right[$, problem ($P_\lambda$) admits at least two distinct generalized weak solutions with opposite energy sign.

Proof. First, we prove that I_λ fulfils the (PS)-condition for each $\lambda > 0$.
Fix $\lambda > 0$ and let $\{u_n\} \subseteq W_T^{1,2}$ be such that (P_1) and (P_2) hold. We observe that
$$\eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) = \left(\frac{1}{2}\eta - 1\right)\|u_n\|_{1,2}^2 - \lambda \int_0^T \eta F_G(u_n(x))\,\mathrm{d}x + \lambda \Psi^\circ(u_n, u_n)$$
for all $n \in \mathbb{N}$. Furthermore, we note that
$$\Psi^\circ(u_n; u_n) = \int_0^T \max_{\zeta \in \partial F_G(u_n(x))} \zeta \cdot u_n(x)\,\mathrm{d}x \geq \int_0^T f_G(u_n(x))(u_n(x))\,\mathrm{d}x$$
where
$$f_G(u_n(x)) \in \partial F_G(u_n(x)) \qquad \text{for a.a. } x \in [0, T].$$

Then,

$$\begin{aligned}\eta I_\lambda(u_n) - I_\lambda^\circ(u_n)(u_n) &= \left(\frac{1}{2}\eta - 1\right)\|u_n\|_{1,2}^2 \\ &\quad - \lambda\int_0^T (\eta F_G(u_n(x)) - f_G(u_n(x))(u_n(x)))\mathrm{d}x.\end{aligned}$$

Note that since F_G is locally Lipschitz continuous, it is in particular locally bounded on every compact interval. Moreover, F_G is differentiable almost everywhere with derivative f_G, which is therefore locally essentially bounded. Hence, both F_G and f_G are locally essentially bounded functions, that is, for every compact interval $I \subseteq [0, T]$ there exist $C_1, C_2 > 0$ such that

$$|f_G(u(x))| \leq C_1 \quad \text{e} \quad |F_G(u(x))| \leq C_2, \quad \text{for a.a. } x \in I. \tag{3.29}$$

Using (AR)-condition and taking into account (3.29), we get

$$\left(\frac{1}{2}\eta - 1\right)\|u_n\|_{1,2}^2 \leq \eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) + \lambda T\,(\eta C_1 + lC_2)\,. \tag{3.30}$$

From (P_1) we have that there exists $M > 0$ such that

$$I_\lambda(u_n) \leq M \quad \text{for all } n \in \mathbb{N}. \tag{3.31}$$

Therefore, by combining (P_1) and (P_2) with (3.30), one has

$$\left(\frac{1}{2}\eta - 1\right)\|u_n\|_{1,2}^2 \leq \eta M + \varepsilon_n\|u_n\|_{1,2} + \lambda T\,(\eta C_1 + lC_2) \qquad \text{for all } n \in \mathbb{N},$$

and follows that the sequence $\{u_n\}$ is bounded in $W_T^{1,2}$. Since $W_T^{1,2}$ is a reflexive Banach space, there exists a subsequence, still denoted by $\{u_n\}$, which converges weakly in $W_T^{1,2}$. Moreover, due to the compact embedding of $W_T^{1,2}$ into $C^0([0, T])$, this subsequence converges strongly in $C^0([0, T])$. Hence, we have

$$u_n \rightharpoonup u \quad \text{in } W_T^{1,2} \quad \text{and} \quad u_n \to u \quad \text{in } C^0([0, T]).$$

Since Φ' is a linear and continuous operator, from the weak convergence it follows that

$$\langle \Phi'(u), u_n - u\rangle \to 0.$$

Furthermore, standard arguments yield

$$\langle \Phi'(u_n), u_n - u\rangle \to 0.$$

Therefore,

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0. \tag{3.32}$$

On the other hand, by using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle &= \langle \Phi'(u_n), u_n\rangle - \langle \Phi'(u_n), u\rangle - \langle \Phi'(u), u_n\rangle + \langle \Phi'(u), u\rangle \\ &= \|u_n\|_{1,2}^2 - \langle \Phi'(u_n), u\rangle - \langle \Phi'(u), u_n\rangle + \|u\|_{1,2}^2 \\ &\leq \|u_n\|_{1,2}^2 + \|u\|_{1,2}^2 - 2\|u_n\|_{1,2}\|u\|_{1,2} \\ &= \left(\|u_n\|_{1,2} - \|u\|_{1,2}\right)^2.\end{aligned}$$

Hence, by (3.32), we deduce that

$$\lim_{n\to+\infty}\|u_n\|_{1,2} = \|u\|_{1,2}.$$

Since $W_T^{1,2}$ is uniformly convex, it follows from Proposition III.30 in [31] that

$$u_n \to u \quad \text{strongly in } W_T^{1,2}.$$

Next, we prove that the energy functional I_λ is unbounded from below. To this end, classical computations show that if f_G satisfies the (AR)-condition, then there exist two constants A, B, with $A > 0$ and $B \geq 0$ such that

$$F_G(t) \geq At^\eta - B \quad \text{for all } t \geq 0.$$

Fix $u \in W_T^{1,2}$ with $u \geq 0$ and $\|u\|_{1,2} \neq 0$, and let $h > 0$. Then, one has

$$\begin{aligned} I_\lambda(hu) &= \Phi(hu) - \lambda\Psi(hu) \\ &\leq \frac{1}{2}\|hu\|_{1,2}^2 - \lambda\int_0^T F_G(hu(x))\,\mathrm{d}x \\ &\leq \frac{h^2}{2}\|u\|_{1,2}^2 - \lambda\int_0^T \big(A(hu(x))^\eta - B\big)\,\mathrm{d}x \\ &= \frac{h^2}{2}\|u\|_{1,2}^2 - \lambda h^\eta\int_0^T Au(x)^\eta\,\mathrm{d}x + \lambda\int_0^T B\,\mathrm{d}x \\ &\leq A_1h^2 - \lambda A_2h^\eta + \lambda A_3, \end{aligned}$$

for some constants $A_1, A_2, A_3 \geq 0$. Since $\eta > 2$, letting $h \to +\infty$ yields

$$I_\lambda(hu) \to -\infty,$$

showing that the functional I_λ is unbounded from below.

Now, we need to show that inequality (2.9) is satisfied. To this end, fix $\lambda \in \Lambda_{c,d}$ and put

$$r = \frac{c^2}{2k^2}, \quad \hat{u} = d.$$

Clearly, $\hat{u} \in W_T^{1,2}$ and

$$\Phi(\hat{u}) = \frac{1}{2}\|\hat{u}\|_{1,2}^2 = \frac{1}{2}Td^2.$$

Hence,

$$\frac{\Psi(\hat{u})}{\Phi(\hat{u})} = \frac{2}{T}\frac{\int_0^T F_G(d)\mathrm{d}x}{d^2} = \frac{2F_G(d)}{d^2}.$$

Note that, from the embedding of $W_T^{1,2}$ in C_T^∞, for all $u \in W_T^{1,2}$ such that $\Phi(u) < r$ one has

$$\max_{x\in[0,T]}|u(x)| \leq k\|u\|_{1,2} < k\sqrt{2r} = c.$$

Hence,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{r} = 2k^2\frac{\sup_{\|u\|_{1,2}\leq\sqrt{2r}}\int_0^T F_G(u(x))\mathrm{d}x}{c^2}$$
$$\leq 2Tk^2\frac{\max_{|t|\leq c}F_G(t)}{c^2} < \frac{2F_G(d)}{d^2} = \frac{\Psi(\hat{u})}{\Phi(\hat{u})}.$$

It remains to prove that $0 < \Phi(\hat{u}) < r$. Since $\Phi(\hat{u}) = \frac{1}{2}Td^2$, we have to verify that $d^2 < \frac{c^2}{Tk^2}$. Arguing by contradiction, suppose that $d^2 \geq \frac{c^2}{Tk^2}$. This, together with the assumption $d < c$, provides

$$\frac{\max_{|t|\leq c}F_G(t)}{c^2} \geq \frac{F_G(d)}{c^2} \geq \frac{F_G(d)}{Tk^2d^2}$$

that is in contradiction with hypothesis (3.28). Thus, condition (2.9) holds and $\Lambda_{c,d}$ is nonempty from hypothesis (3.28). Therefore, Theorem 2.2.5 ensures that the energy functional I_λ admits two nontrivial critical points $u_{\lambda_1}, u_{\lambda_2}$ such that $I_\lambda(u_{\lambda_1}) < 0 < I_\lambda(u_{\lambda_2})$ and taking into account Proposition 3.2.1 they are nontrivial generalized weak solutions for problem (P_λ). □

We point out the following consequence of Theorem 3.2.1.

Corollary 3.2.1. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be a set-valued mapping with compact convex values. Assume that f_G fulfills (AR) and*

$$\limsup_{t\to 0^+}\frac{F_G(t)}{t^2} = +\infty. \tag{3.33}$$

Then, for each $\lambda \in]0,\hat{\lambda}[$, where

$$\hat{\lambda} = \sup_{c>0}\frac{c^2}{2Tk^2\max_{|\xi|\leq c}F_G(\xi)} \tag{3.34}$$

problem (P_λ) admits at least two distinct generalized weak solutions.

Proof. Fix $\lambda \in]0,\hat{\lambda}[$. Then , there exists $c > 0$ such that $\lambda < \frac{c^2}{2Tk^2\max_{|\xi|<c}F_G(\xi)}$. Moreover, from (3.33), one has

$$\limsup_{d\to 0^+}\frac{F_G(d)}{2Tk^2d^2} = \frac{1}{2Tk^2}\limsup_{d\to 0^+}\frac{F_G(d)}{d^2} + \infty.$$

Therefore, there is $d < c$ small enough such that

$$\frac{F_G(d)}{Tk^2d^2} > \frac{1}{\lambda}.$$

Hence, $\lambda \in \left]\frac{d^2}{2TF_G(d)}, \frac{c^2}{2Tk^2\max_{|\xi|<c}F_G(\xi)}\right[$ which implies (3.28). Therefore, the statement of the corollary follows from Theorem 3.2.1. □

As a consequence, we give an application of the previous result to the following periodic differential problem with discontinuous reaction term

$$\begin{cases} -u'' + u = \lambda f(u) & \text{in }]0,T[\\ u(0) - u(T) = u'(0) - u'(T) = 0. \end{cases} \tag{$\tilde{P}_\lambda$}$$

We state the following assumptions:

(f_1^P) $f\colon \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function,

(f_2^P) either $\operatorname*{ess\,inf}_{t \in D_f^P} f(t) - t > 0$ or $\operatorname*{ess\,sup}_{t \in D_f^P} f(t) - t < 0$ for all $t \in D_f^P$, where

$$D_f^P = \{z \in [0,T] : f \text{ is discontinuous at } z\}.$$

We also put $F(t) = \int_0^t f(\tau)d\tau$ for all $t \in \mathbb{R}$.

Theorem 3.2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ satisfying (f_1^P), (f_2^P), (AR)-condition and*

$$\frac{\max_{|\xi| \le c} F(\xi)}{c^2} < \frac{1}{Tk^2} \frac{F(d)}{d^2}. \tag{3.35}$$

Then, for each $\lambda \in \Lambda_{c,d}$, problem ($\tilde{P}_\lambda$) admits at least two distinct weak solutions.

Proof. Note that condition (f_1^P) ensures that f is a locally bounded function, F is well defined and it is a locally Lipschitz function. We define f^-, f^+ as in (H_G). Consider $\partial F(t)$, $t \in \mathbb{R}$ and put $L(t) = \{a \in \mathbb{R} : \langle a, t \rangle \in \partial F(t)\}$. Arguing as in the proof of Proposition 3.2.1 one has $L(t) \subseteq [f^-(t), f^+(t)]$ a.e. in $\mathbb{R}$. In particular, being f almost every continuous, we know that $|D_f^P| = 0$. Moreover, $L(t) = f(t)$ for a.a. $t \in \mathbb{R}$ and it satisfies all the assumptions of Theorem 3.2.1. Hence, problem

$$\begin{cases} -u'' + u \in \lambda L(u) & \text{in } [0,T] \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases}$$

admits two distinct generalized weak solutions. Let u be one of them. It means that there exists a unique $w \in L^2([0,T])$ such that

(h_1) $w(x) \in L(u(x))$ *a.e.* in $[0,T]$

(h_2) $\displaystyle\int_0^T u'(x)v'(x)dx + \int_0^T u(x)v(x)dx = \lambda \int_0^T w(x)v(x)dx$ for all $v \in C_T^\infty$.

From usual regularity results it follows that $u \in W^{2,2}([0,T]) \cap W_T^{1,2}$ and from (h_1), taking into account that $L(u(x)) \subseteq [f^-(u(x)), f^+(u(x))]$ and $\lambda > 0$, one has

$$-u''(x) \in [f^-(u(x)) - u(x), f^+(u(x)) - u(x)]. \tag{3.36}$$

In order to prove that it is a weak solution of problem ($\tilde{P}_\lambda$), we show that $|u^{-1}(D_f^P)| = 0$. Arguing by contradiction, assume $|u^{-1}(D_f^P)| > 0$. From one side, [44, Lemma 1] ensures that

$$-u''(x) = 0 \text{ for a.a. } x \in u^{-1}(D_f^P). \tag{3.37}$$

On the other side, from (f_2^P) we can assume that $\operatorname*{ess\,inf}_{t \in D_f^P} f(t) - t > 0$. So, exploiting this condition in (3.36), we obtain $-u''(x) > 0$, leading to a contradiction. The same reasoning applies if $\operatorname*{ess\,sup}_{t\in D_f^p} f(t) - t < 0$.

Therefore, $w(x) = \lambda f(u(x))$ for almost all $x \in [0, T]$ and from (h$_2$)

$$\int_0^T u'(x)v'(x)\mathrm{d}x + \int_0^T u(x)v(x)\mathrm{d}x = \lambda \int_0^T f(u(x)v\,\mathrm{d}x \quad \text{for all } v \in W_T^{1,2},$$

which completes the proof. □

We conclude by providing an example.

Example 3.2.1. *Let $T = 1$ and*

$$g(u) = \begin{cases} 2, & u < \frac{1}{2} \\ 3, & u = \frac{1}{2} \\ 3u^2 - 3u + \dfrac{15}{4}, & u > \frac{1}{2}. \end{cases}$$

Clearly, g is an almost everywhere continuous function satisfying (AR)-condition and such that

$$\operatorname*{ess\,inf}_{t\in D_g} g(t) - \frac{1}{2} = g\left(\frac{1}{2}\right) - \frac{1}{2} > 0.$$

Moreover,

$$\limsup_{t\to 0^+} \frac{G(t)}{t^2} = +\infty.$$

where $G = \int_0^t g(\tau)\mathrm{d}\tau$ for all $t \in \mathbb{R}$. Therefore, from the combined use of Theorem 3.2.2 and Corollary 3.2.1, for each $\lambda \in \left]0, \frac{4}{21}\right[$, problem

$$\begin{cases} -u'' + u = \lambda g(u) & \text{in }]0,1[\\ u(0) - u(1) = u'(0) - u'(1) = 0 \end{cases}$$

admits at least two distinct weak solutions.

Chapter 4

Differential problems with highly discontinuous nonlinearities

In this chapter we analyze two classes of differential problems characterized by the presence of highly discontinuous nonlinear terms. Such problems pose a significant challenge in mathematical analysis due to the technical difficulties arising from the discontinuities.
By highly discontinuous functions we refer to the class $\mathcal{H}$, introduced below:

Definition 4.0.1. *We denote by $\mathcal{H}$ the family of highly discontinuous functions, that is the family of all functions $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that*

($\mathcal{H}$) (h$_1$) *the function $x \mapsto f(x,t)$ is measurable for all $t \in \mathbb{R}$;*

(h$_2$) *there exists a set $\Omega_0 \subset \Omega$ with $|\Omega_0| = 0$ such that the set*

$$D_f := \cup_{x \in \Omega \setminus \Omega_0} \{t \in \mathbb{R} \colon f(x,\cdot) \text{ is discontinuous at } t\}$$

has measure zero;

(h$_3$) *the function $t \mapsto f(x,t)$ is locally essentially bounded for a.a. $x \in \Omega$;*

(h$_4$) *the functions*

$$f^-(x,t) := \lim_{\xi \to 0^+} \operatorname*{ess\,inf}_{|t-z|<\xi} f(x,z), \qquad f^+(x,t) := \lim_{\xi \to 0^+} \operatorname*{ess\,sup}_{|t-z|<\xi} f(x,z)$$

are superpositionally measurable, that is, $f^-(x,u(x))$ and $f^+(x,u(x))$ are measurable for all measurable functions $u\colon \Omega \to \mathbb{R}$, according to Definition 1.4.2.

Functions belonging to $\mathcal{H}$ allow for the effective modelling of phenomena characterized by significant discontinuities, which arise in various applied contexts. Examples include models of phase transitions in physics, population dynamics, fluid mechanics problems or economic and social models with threshold effects or stepwise responses.

In the presence of such discontinuous nonlinearities, traditional analytical methods based on continuity and differentiability are no longer applicable, making it necessary to employ advanced techniques such as nonsmooth variational methods and approaches based on generalized critical points.

4.1 Complete Sturm Liouville differential equations under mixed boundary conditions

This section is devoted to the study of a Sturm Liouville problem with weight functions that are not necessarily sign-definite, coupled with mixed boundary conditions. Such problems naturally arise in various applications and present difficulties due to the sign changes in the weights and the mixed nature of the boundary conditions. We establish the existence of at least three distinct generalized solutions for a range of positive parameters, and we also consider, as particular cases, the autonomous setting and the case in which the nonlinearity is nonnegative on a suitable interval. The results presented in this chapter were obtained in [69].

4.1.1 Weak formulation of the problem

We investigate the following mixed boundary value problem for a complete nonlinear ordinary differential equation of Sturm Liouville type, characterized by sign-changing coefficients and a possibly discontinuous nonlinearity:

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda f(x,u), & \text{in } (a,b), \\ u(a) = 0, \quad u'(b) = 0, \end{cases} \tag{M_λ}$$

where $\lambda \in \mathbb{R}^+$ is a positive parameter. The weight functions $\gamma, \delta \in L^\infty([a,b])$ are allowed to change sign and in particular δ fulfills the condition

$$\operatorname*{ess\,inf}_{x\in[a,b]} \delta(x) > -\left(\frac{\pi}{2(b-a)}\right)^2. \tag{4.1}$$

We impose the following assumptions:

(f_M) the function $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ is assumed to satisfy:

- (m_1) f belongs to the class $\mathcal{H}$;
- (m_2) there exists a function $C \in L^\infty([a,b])$ such that
$$|f(x,t)| \le C(x)(1+|t|) \quad \text{for all } (x,t)\in[a,b]\times\mathbb{R};$$
- (m_3) for each $\lambda > 0$, for almost every $x \in [a,b]$, and for all $t \in D_f^M$ (with D_f^M defined below), the following implication holds:
$$\lambda f^-(x,t) - \delta(x)z \le 0 \le \lambda f^+(x,t) - \delta(x)z \;\Rightarrow\; \lambda f(x,t) - \delta(x)z = 0.$$

The assumptions $f \in \mathcal{H}$ need to be adapted to our framework, and in particular hypothesis (h_2) becomes:

there exists a set $I \subset [a,b]$ with $|I| = 0$ such that the set

$$D_f^M = \bigcup_{x\in[a,b]\setminus I} \{t \in \mathbb{R} : f(x,\cdot) \text{ is discontinuous at } t\}$$

has measure zero.

We consider as Banach space X the Sobolev space

$$W_a^{1,2}([a,b]) = \{u \in W^{1,2}([a,b]) : u(a) = 0\}.$$

equipped with the norm

$$\|u\|_{1,2} = \left(\int_a^b |u(x)|^2\,dx\right)^{\frac{1}{2}} + \left(\int_a^b |u'(x)|^2\,dx\right)^{\frac{1}{2}}.$$

In Bonanno, D'Aguì and Morabito [19], we proved that the norm $\|\cdot\|_{1,2}$ is equivalent to the classical gradient norm in L^2, denoted by $\|\cdot\|_0$. Moreover, we established two Poincaré-type inequalities with exact estimates of the involved constants. These constants are optimal, meaning they cannot be improved, and their optimality is rigorously demonstrated in the same paper.

For the reader's convenience, we recall here the statements of these results together with their proofs.

Proposition 4.1.1 ([19], Proposition 2.1). *On the space $W_a^{1,2}([a,b])$, the norm $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{1,2}$.*

Proof. For all $u \in W_a^{1,2}([a,b])$, one has

$$\|u\| = \|u\|_2 + \|u'\|_2 \geq \|u'\|_2 = \|u\|_0.$$

Further, taking into account that u is absolutely continuous and $u' \in L^1([a,b])$, from the fundamental theorem of integral calculus, one has

$$u(x) = \int_a^x u'(t)\,\mathrm{d}t,$$

then

$$|u(x)| \leq \int_a^x |u'(t)|\,\mathrm{d}t \leq \int_a^b |u'(x)|\,\mathrm{d}x.$$

Applying Holder's inequality, we obtain

$$|u(x)| \leq (b-a)^{\frac{1}{2}}\|u'\|_{L^2}. \tag{4.2}$$

By squaring and integrating between a and b, one has

$$\|u\|_2 \leq (b-a)\,\|u'\|_2. \tag{4.3}$$

Therefore, taking (4.3) into account

$$\|u\| = \|u\|_2 + \|u'\|_2 \leq \big(1 + (b-a)\big)\|u\|_0. \tag{4.4}$$

Thus, the proof is complete.

□

Proposition 4.1.2 ([19], Proposition 2.2). *(Poincaré inequalities) For all $u \in W_a^{1,2}([a,b]$, the following inequalities hold:*

(a_1) $\max_{x\in[a,b]} |u(x)| \leq (b-a)^{\frac{1}{2}}\|u'\|_2$;

(a_2) $\|u\|_2 \leq \frac{2(b-a)}{\pi}\|u'\|_2$.

Proof.

(j) This Poincaré inequality is obtained as in formula (4.2). We observe that $(b-a)^{\frac{1}{2}}$ is the best constant $\hat{k}$ among those for which $\|u\|_\infty \leq \hat{k}\|u\|_0$ for all $u \in W_a^{1,2}([a,b]$. Indeed, arguing by a contradiction, assume that there is a positive constant $\hat{k} < (b-a)^{\frac{1}{2}}$ for which $\|u\|_\infty \leq k\|u\|_0$ for all $u \in W_a^{1,2}([a,b])$. In particular, chosing $\overline{u}(x) \in W_a^{1,2}([a,b])$ so defined $\overline{u}(x) = x - a$, one has $\|\overline{u}\|_\infty = (b-a) \leq \hat{k}\|\overline{u}\|_0$, that is $(b-a) \leq \hat{k}(b-a)^{\frac{1}{2}}$ and this is an absurd, then our claim is proved.

(jj) From (4.3), one has

$$\|u\|_2 \leq (b-a)\|u'\|_2.$$

However, $\tilde{k} = (b-a)$ is not the best constant for Poincaré inequality in $W_a^{1,2}([a,b])$. In order to obtain the best constant, we reason as done in [13, Lemmma 8.1] for Dirichlet problems. We consider the following problem

$$\begin{cases} -u'' = \lambda u & \text{in }]a,b[, \\ u(a) = u'(b) = 0. \end{cases} \tag{4.5}$$

Standard computations show that $\lambda_0 = \dfrac{\pi^2}{4(b-a)^2}$ is the first eigenvalue of the above problem. Let $A\colon W_a^{1,2}([a,b]) \to \mathbb{R}^+$ be defined by:

$$A(u) = \left\{ \frac{\|u'\|_2^2}{\|u\|_2^2} : \|u\|_2 \neq 0 \right\}.$$

We prove that A has a minimizer in $W_a^{1,2}([a,b])$, $\inf A = \lambda_0$ and the value of the best constant in (jj) is $\dfrac{1}{\sqrt{\lambda_0}}$. First we note that by (4.3) A is bounded from below.
Let u_n be a minimizing sequence, that is $A(u_n) \to \inf A$, as $n \to +\infty$. Note that

$$z_n = \frac{u_n}{\|u_n'\|_2}$$

satisfies $\|z_n'\|_2 = 1$. Up to a subsequence, $z_n \to z$ in $L^2([a,b])$ (being the embedding compact, see for instance [30, Theorem IX.16]) and weakly in $W_a^{1,2}([a,b])$.
First, we need to prove that $\|z\|_2 \neq 0$. Since A is homogeneous of degree 0, z_n is a minimizing sequence and $A(z_n)$ is a bounded sequence, i.e. there exists a positive constant $\tilde{C}$ such that

$$1 = \|z_n'\|_2^2 \leq \tilde{C}\|z_n\|_2^2.$$

From $\|z_n\|_2 \geq \sqrt{\frac{1}{\tilde{C}}}$, taking into account that z_n converges strongly to z in L^2, passing at the limit as $n \to +\infty$ we get $\|z\|_2 > 0$.
Moreover, we observe that

$$0 \leq \int_a^b (z_n'(x) - z'(x))^2 \, \mathrm{d}x = \int_a^b (z_n'(x))^2 \, \mathrm{d}x + \int_a^b (z'(x))^2 \, \mathrm{d}x - 2\int_a^b z_n'(x) z'(x) \, \mathrm{d}x.$$

Therefore,

$$\int_a^b (z_n'(x))^2 \, \mathrm{d}x \geq 2\int_a^b z_n'(x) z'(x) \, \mathrm{d}x - \int_a^b (z'(x))^2 \, \mathrm{d}x.$$

If we pass to the limit in the above inequality, we have

$$\liminf_{n\to+\infty} \int_a^b (z_n'(x))^2 \, \mathrm{d}x \geq \int_a^b (z'(x))^2 \, \mathrm{d}x.$$

Since z_n is a minimizing sequence, one has

$$\inf A = \lim_{n\to+\infty} A(z_n) = \liminf_{n\to+\infty} \frac{\displaystyle\int_a^b (z_n'(x))^2 \, \mathrm{d}x}{\displaystyle\int_a^b (z_n(x))^2 \, \mathrm{d}x} \geq \frac{\displaystyle\int_a^b (z'(x))^2 \, \mathrm{d}x}{\displaystyle\int_a^b z(x)^2 \, \mathrm{d}x} = A(z).$$

Consequently z is a minimizer for A.
Now, we prove that $\lambda_0 = \min_{u\in W_a^{1,2}([a,b]),u\neq 0} A(u)$.
Let $u \in W_a^{1,2}([a,b])$ be a minimizer for A. It follows that $A'(u) = 0$, that is

$$\frac{2\int_a^b u'(x)v'(x)\,\mathrm{d}x \cdot \int_a^b u(x)^2\,\mathrm{d}x - 2\int_a^b (u'(x))^2\,\mathrm{d}x \cdot \int_a^b u(x)v(x)\,\mathrm{d}x}{\left(\int_a^b u(x)^2\,\mathrm{d}x\right)^2} = 0.$$

Hence,

$$\int_a^b u'(x)v'(x)\,\mathrm{d}x = \frac{\int_a^b (u'(x))^2\,\mathrm{d}x}{\int_a^b u(x)^2\,\mathrm{d}x} \cdot \int_a^b u(x)v(x)\,\mathrm{d}x \quad \text{for all } v \in W_a^{1,2}([a,b]).$$

Therefore, if u minimizes A, u is an eigenfunction of (4.5) and $\frac{\int_a^b u'(x)^2\,\mathrm{d}x}{\int_a^b u(x)^2\,\mathrm{d}x} = A(u) = \inf(A)$ is the corresponding eigenvalue.
Let us prove that $\inf A = \lambda_0$. Since λ_0 is the smallest eigenvalue of (4.5), one has

$$\lambda_0 \le \inf A.$$

We prove the opposite inequality. Let w be the eigenfunction corresponding to λ_0. One has

$$\int_a^b w'(x)u'(x)\,\mathrm{d}x = \lambda_0 \int_a^b w(x)u(x)\,\mathrm{d}x \quad \text{for all } u \in W_a^{1,2}([a,b]).$$

Consequently, choosing $u = w$ we obtain

$$\int_a^b w'(x)^2\,\mathrm{d}x = \lambda_0 \int_a^b w(x)^2\,\mathrm{d}x \quad \text{for all } u \in W_a^{1,2}([a,b]).$$

Therefore, one has

$$\inf A = \inf \frac{\int_a^b u'(x)^2\,\mathrm{d}x}{\int_a^b u(x)^2\,\mathrm{d}x} \le \frac{\int_a^b w'(x)^2\,\mathrm{d}x}{\int_a^b w(x)^2\,\mathrm{d}x} = \lambda_0 \frac{\int_a^b w(x)^2\,\mathrm{d}x}{\int_a^b w(x)^2\,\mathrm{d}x} = \lambda_0.$$

□

Now, we equippe the space $W_a^{1,2}([a,b])$ with the following norm

$$\|u\|_a = \left(\int_a^b e^{-\Gamma(x)}|u'(x)|^2\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)|u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

where

$$\Gamma(x) = \int_a^x \gamma(\xi)\,\mathrm{d}\xi \qquad \text{for all } x \in [a,b]\,.$$

The following result holds.

Proposition 4.1.3 ([19], Proposition 2.3). *Assume that condition (4.1) holds. Then $\|\cdot\|_a$ is a norm on $W^{1,2}_a([a,b])$ and it is equivalent to $\|(\cdot)'\|_2$. In particular, one has*

$$m\|u'\|_2 \le \|u\|_a \le M\|u'\|_2\,, \tag{4.6}$$

for all $u \in W^{1,2}_a([a,b])$, where m, M, with $M \ge m > 0$, are given by

$$m = \begin{cases} \left(\min\limits_{x\in[a,b]} e^{-\Gamma(x)}\right)^{\frac{1}{2}} & \text{if } \operatorname*{ess\,inf}\limits_{x\in[a,b]} \delta(x) \ge 0 \\ \left[\min\limits_{x\in[a,b]} e^{-\Gamma(x)}\left(1+\operatorname*{ess\,inf}\limits_{x\in[a,b]}\delta(x)\left(\dfrac{2(b-a)}{\pi}\right)^2\right)\right]^{\frac{1}{2}} & \text{if } \operatorname*{ess\,inf}\limits_{x\in[a,b]} \delta(x) < 0 \end{cases},$$
$$M = \begin{cases} \left[\max\limits_{x\in[a,b]} e^{-\Gamma(x)}\left(1+\operatorname*{ess\,sup}\limits_{x\in[a,b]}\delta(x)\left(\dfrac{2(b-a)}{\pi}\right)^2\right)\right]^{\frac{1}{2}} & \text{if } \operatorname*{ess\,sup}\limits_{x\in[a,b]} \delta(x) \ge 0 \\ \left(\max\limits_{x\in[a,b]} e^{-\Gamma(x)}\right)^{\frac{1}{2}} & \text{if } \operatorname*{ess\,sup}\limits_{t\in[a,b]} \delta(x) < 0 \end{cases}. \tag{4.7}$$

Proof. Let's prove (4.7). One has

$$\|u\|_a^2 = \int_a^b e^{-\Gamma(x)}|u'(x)|^2 \mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)|u(x)|^2 \mathrm{d}x$$
$$\ge \min_{x\in[a,b]} e^{-\Gamma(x)}\left(\int_a^b |u'(x)|^2\mathrm{d}x + \operatorname*{ess\,inf}_{x\in[a,b]}\delta(x)\int_a^b |u(x)|^2\mathrm{d}x\right).$$

We distinguish two cases: If $\operatorname{ess\,inf}_{x\in[a,b]} \delta(x) \ge 0$, then

$$\|u\|_a^2 \ge \min_{x\in[a,b]} e^{-\Gamma(x)} \int_a^b |u'(x)|^2\mathrm{d}x = \min_{x\in[a,b]} e^{-\Gamma(x)}\|u'\|_2^2.$$

If instead $\operatorname{ess\,inf}_{x\in[a,b]} \delta(x) < 0$, then using (4.1) and inequality (a_2) in Proposition 4.1.2, we obtain

$$\|u\|_a^2 \ge \min_{x\in[a,b]} e^{-\Gamma(x)}\left(\|u'\|_{L^2}^2 + \operatorname*{ess\,inf}_{x\in[a,b]}\delta(x)\int_a^b |u(x)|^2\mathrm{d}x\right)$$
$$\ge \min_{x\in[a,b]} e^{-\Gamma(x)}\left(\|u'\|_{L^2}^2 + \operatorname*{ess\,inf}_{x\in[a,b]}\delta(x)\left(\frac{2(b-a)}{\pi}\right)^2\|u'\|_2^2\right)$$
$$= \min_{x\in[a,b]} e^{-\Gamma(x)}\left(1 + \operatorname*{ess\,inf}_{x\in[a,b]}\delta(x)\left(\frac{2(b-a)}{\pi}\right)^2\right)\|u'\|_2^2,$$

with

$$\min_{x\in[a,b]} e^{-\Gamma(x)}\left(1 + \operatorname*{ess\,inf}_{x\in[a,b]}\delta(x)\left(\frac{2(b-a)}{\pi}\right)^2\right) > 0.$$

To prove the other inequality, one has

$$\|u\|_a^2 \leq \max_{x\in[a,b]} e^{-\Gamma(x)} \left(\int_a^b |u'(x)|^2 dx + \operatorname*{ess\,sup}_{x\in[a,b]} \delta(x) \int_a^b |u(x)|^2 dx \right).$$

If $\operatorname{ess\,sup}_{x\in[a,b]} \delta(x) \geq 0$, then again by ($a_2$) in Proposition 4.1.2 we get

$$\begin{aligned}\|u\|_a^2 &\leq \max_{x\in[a,b]} e^{-\Gamma(x)} \left(\|u'\|_{L^2}^2 + \operatorname*{ess\,sup}_{x\in[a,b]} \delta(x) \left(\frac{2(b-a)}{\pi}\right)^2 \|u'\|_2^2 \right) \\ &= \max_{x\in[a,b]} e^{-\Gamma(x)} \left(1 + \operatorname*{ess\,sup}_{x\in[a,b]} \delta(x) \left(\frac{2(b-a)}{\pi}\right)^2 \right) \|u'\|_{L^2}^2.\end{aligned}$$

If instead $\operatorname{ess\,sup}_{x\in[a,b]} \delta(x) < 0$, then

$$\|u\|_X^2 \leq \max_{x\in[a,b]} e^{-\Gamma(x)} \|u'\|_2^2.$$

Hence, (4.7) is proved. Moreover, observe that $W_a^{1,2}([a,b])$ is an Hilbert space with the dot product

$$\langle u, v\rangle = \int_a^b e^{-\Gamma(x)} u'(x)v'(x)\, dx + \int_a^b e^{-\Gamma(x)} \delta(x)u(x)v(x)\, dx,$$

that clearly induces the norm $\|\cdot\|_a$. □

Remark 4.1.1. *Combining* (4.6) *and Poincarè inequality* (a_1), *one has*

$$\max_{x\in[a,b]} |u(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{m} \|u\|_a \quad \textit{for all } u \in W_a^{1,2}([a,b]) \tag{4.8}$$

where m is given in (4.7) *and which will be useful later on.*

Now, we recall the definition of generalized solution for the problem (M_λ).

Definition 4.1.1. *We say that a function $u \in W_a^{1,2}([a,b])$ is a generalized solution of* (M_λ) *if $u' \in AC([a,b])$ and*

$$-u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x, u(x)) \quad \textit{a.e. in } [a,b].$$

Now, put

$$F(x,t) = \int_0^t f(x,\xi)\, d\xi \qquad \text{for all } (x,t) \in [a,b] \times \mathbb{R}.$$

Thanks to the hypotheses on the function f, one has

(f'$_1$) $x \to F(x,t)$ is measurable for all $x \in \mathbb{R}$;

(f'$_2$) $|F(x,t)| \leq \operatorname*{ess\,sup}_{|\xi|\leq|t|} |f(x,\xi)|t|$ for almost every $x \in [a,b]$, for all $t \in \mathbb{R}$.

Furthermore, consider $\Phi, \Psi : W_a^{1,2}([a,b]) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\|u\|_a^2,$$

$$\Psi(u) = \int_a^b e^{-\Gamma(x)} F(x, u(x))\, \mathrm{d}x,$$

for all $u \in W_a^{1,2}([a,b])$. The functional Φ is Gâteaux differentiable and its derivative is given by

$$\Phi'(u,v) = \int e^{-}\Gamma(x) u'(x) v'(x)\, \mathrm{d}x + \int e^{-\Gamma}(x) \delta(x) u(x) v(x)\, \mathrm{d}x.$$

The functional $\Psi\colon W_a^{1,2}([a,b]) \to \mathbb{R}$ is well defined, due to embedding $W_a^{1,2}([a,b]) \subset C([a,b])$ and the assumptions (f'_1), (f'_2) and (f_3). Indeed,

$$\begin{aligned} e^{-\Gamma(x)} |F(x,u(x))| &\leq \|e^{-\Gamma}\|_\infty \operatorname*{ess\,sup}_{|\xi| \leq |u(x)|} |f(x,\xi)| |u(x)| \\ &\leq \|e^{-\Gamma}\|_\infty \|u\|_\infty \operatorname*{ess\,sup}_{|\xi| \leq \|u\|_\infty} |f(x,\xi)| \\ &\leq \|e^{-\Gamma}\|_\infty \|u\|_\infty |C(x)| (1 + \|u\|_\infty) \in L^1([a,b]). \end{aligned}$$

In order to apply variational methods within a nonsmooth framework, it is essential to verify some basic regularity properties of the associated energy functionals. The following lemma ensures that the functionals Φ and Ψ defined on the Banach space $W_a^{1,2}([a,b])$ are locally Lipschitz continuous. This property is a prerequisite for the use of generalized gradient techniques in the sense of Clarke.

Lemma 4.1.1. *The functionals Φ and Ψ are locally Lipschitz continuous in $W_a^{1,2}([a,b])$.*

Proof. First, observe that the function $F(x,\cdot)$ is locally Lipschitz continuous for each fixed x, as it is the integral of a locally essentially bounded function. To prove that Ψ is locally Lipschitz continuous in $W_a^{1,2}([a,b])$, let $w \in W_a^{1,2}([a,b])$ and let $\rho > 0$ be arbitrary. For any function $h \in B(w,\rho)$, where $B(w,\rho)$ denotes the ball of center w and radius ρ, follows that

$$\|h\|_\infty \leq \|h - w\|_\infty + \|w\|_\infty \leq \rho \frac{(b-a)^{\frac{1}{2}}}{m} + \|w\|_\infty = K.$$

Moreover, since f is locally essentially bounded, and the set $[a,b] \times [-K,K]$ is compact in $\mathbb{R}^2$, there exists a constant $l \geq 0$ such that

$$\operatorname*{ess\,sup}_{(x,t) \in [a,b] \times [-K,K]} |f(x,t)| = l.$$

Then, for any $u, v \in B(w,\rho)$, one has

$$\begin{aligned} |\Psi(u) - \Psi(v)| &\leq \int_a^b e^{-\Gamma(x)} \left| \int_{v(x)}^{u(x)} f(x,t)\, \mathrm{d}t \right| \mathrm{d}x \\ &\leq \|e^{-\Gamma}\|_\infty \int_a^b l \|u - v\|_\infty\, \mathrm{d}x \\ &\leq \|e^{-\Gamma}\|_\infty \frac{l(b-a)^{3/2}}{m} \|u - v\|_a, \end{aligned}$$

which shows that Ψ is locally Lipschitz continuous in $W_a^{1,2}([a,b])$.The same conclusion holds for Φ since it corresponds to a rescaling of the squared norm. □

For all $u \in W_a^{1,2}([a,b])$, let

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

be the energy functional associated with the problem (M_λ). From Lemma 4.1.1, it follows that I_λ is a locally Lipschitz functional. Therefore, we can apply the theory of generalized differential calculus to seek generalized solutions to the problem (M_λ), which will correspond to the critical points of I_λ. Indeed, the following theorem holds.

Theorem 4.1.1. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$. Assume that f satisfies (f_M). Then, each critical point of I_λ is a generalized solution to problem (M_λ) and belongs to $W^{2,2}([a,b])$.*

Proof. Let $u_0 \in W_a^{1,2}([a,b])$ be a critical point of the functional I_λ, that is,

$$0 \in \partial(\Phi - \lambda\Psi)(u_0),$$

By [39, Proposition 2.1.5], this condition is equivalent to

$$(\Phi - \lambda\Psi)^\circ(u_0; w) \geq 0 \quad \text{for all } w \in W_a^{1,2}([a,b]),$$

In our case, this becomes

$$\Phi'(u_0)(w) + \lambda\left(-\Psi\right)^\circ(u_0; w) \geq 0 \quad \text{for all } w \in W_a^{1,2}([a,b]).$$

which also means that

$$-\left(\int_a^b e^{-\Gamma(x)} u_0'(x) w'(x)\, \mathrm{d}x + \int_a^b e^{-\Gamma(x)} \delta(x) u_0(x) w(x)\, \mathrm{d}x\right) \leqslant \lambda(-\Psi)^\circ(u_0; w) \tag{4.9}$$

for all $w \in W_a^{1,2}([a,b])$. We proceed in three steps.

Claim 1. *The functional Ψ is locally Lipschitz in $L^2([a,b])$.*

Let $w \in L^2([a,b])$ and fix $\rho > 0$. Consider $u, v \in B_{L^2}(w,\rho)$. Using the growth condition (m_2) and Hölder's inequality, we estimate:

$$\begin{aligned}
|\Psi(u) - \Psi(v)| &= \int_a^b e^{-\Gamma(x)} \left(F(x,u(x)) - F(x,v(x))\right) \mathrm{d}x \\
&\leq \|e^{-\Gamma}\|_\infty \int_a^b \left|\int_{v(x)}^{u(x)} f(x,t)\, \mathrm{d}t\right| \mathrm{d}x \\
&\leq \|e^{-\Gamma}\|_\infty \int_a^b \left|\int_{v(x)}^{u(x)} C(x)(1+|t|)\, \mathrm{d}t\right| \mathrm{d}x \\
&\leq \|e^{-\Gamma}\|_\infty \|C\|_\infty \|u-v\|_2 \cdot (b-a)^{1/2} + \int_a^b \left|\int_{v(x)}^{u(x)} |t|\, \mathrm{d}t\right| \mathrm{d}x
\end{aligned} \tag{4.10}$$

for any $u, v \in B_{L^2}(w,\rho)$. By the Mean Value theorem assuming without loss of generality that $v(x) < u(x)$ for each $x \in [a,b]$, there exists $g(x) \in [v(x), u(x)]$ such that

$$\left|\int_{v(x)}^{u(x)} |x|\, \mathrm{d}x\right| = |g(x)|\,|u(x) - v(x)|.$$

Then, applying Hölder's inequality, one has

$$\int_a^b |g(x)|\,|u(x)-v(x)|\,\mathrm{d}x \le \int_a^b |u(x)|\,|u(x)-v(x)|\,\mathrm{d}x \le \|u\|_2\,\|u-v\|_2. \qquad (4.11)$$

Combining (4.10) and (4.11), we obtain

$$|\Psi(u)-\Psi(v)| \le \|e^{-\Gamma}\|_\infty \|C\|_\infty \left(\sqrt{b-a}+\|u\|_2\right)\|u-v\|_2,$$

which proves that Ψ is locally Lipschitz continuous in $L^2([a,b])$.

Claim 2. *u_0' belongs to $AC([a,b]) \cap W^{2,2}([a,b])$.*

For all $w \in W_a^{1,2}([a,b])$, put

$$L(w) = -\left(\int_a^b e^{-\Gamma(x)} u_0'(x) w'(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)} \delta(x) u_0(x) w(x)\,\mathrm{d}x\right). \qquad (4.12)$$

L is a linear and continuous functional on $W_a^{1,2}([a,b])$ and from (4.9) we have $L \in \partial(-\lambda\Psi_{|W_a^{1,2}([a,b])})(u_0) = \lambda\partial(-\Psi_{|W_a^{1,2}([a,b])})(u_0)$. By Theorem 1.4.2 since $W_a^{1,2}([a,b])$ is dense in $L^2([a,b])$ we obtain

$$\partial(-\Psi_{|W_a^{1,2}([a,b])})\,(u_0) \subseteq \partial(-\Psi_{|L^2([a,b])})\,(u_0)\,.$$

Then, we have

$$L \in \lambda\,\partial(-\Psi_{|W_a^{1,2}([a,b])})(u_0) \subseteq \lambda\,\partial(-\Psi_{|L^2([a,b])})(u_0) \subseteq L^2([a,b])^*,$$

which implies that L is a continuous linear functional on $L^2([a,b])$. Therefore, by the Riesz representation theorem, there exists a unique element $\bar{w} \in L^2([a,b])$ such that

$$L(w) = \langle w, \bar{w}\rangle_{L^2} = \int_a^b w(x)\bar{w}(x)\,\mathrm{d}x \quad \text{for all } w \in L^2([a,b]). \qquad (4.13)$$

Hence

$$-\left(\int_a^b e^{-\Gamma(x)} u_0'(x) w'(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)} \delta(x) u_0(x) w(x)\,\mathrm{d}x\right) = \int_a^b w(x)\bar{w}(x)\,\mathrm{d}x$$

for all $w \in W_a^{1,2}([a,b])$, i.e.

$$-\int_a^b e^{-\Gamma(x)} u_0'(x) w'(x)\,\mathrm{d}x = -\int_a^b \left(e^{-\Gamma(x)}\delta(x)u_0(x) + \bar{w}(x)\right) w(x)\,\mathrm{d}x.$$

Hence, we have $e^{-\Gamma}u_0' \in AC([a,b])$, whose weak derivative is $e^{-\Gamma}\delta u_0 + \bar{w}$. Since $e^{\Gamma} \in AC([a,b])$ as well, it follows that their product also belongs to $AC([a,b])$. In particular, we obtain

$$u_0'' = \gamma u_0' + \delta u_0 + e^{\Gamma}\bar{w} \in L^2([a,b]),$$

and therefore $u_0 \in W^{2,2}([a,b])$.

Claim 3. *$-u_0''(x) + \gamma(x)u_0'(x) + \delta(x)u_0(x) = \lambda f(x, u_0(x))$ a.e. in $[a,b]$.*

Since u_0 is a critical point of I_λ integrating by part in (4.12), we obtain

$$-e^{-\Gamma(b)}u_0'(b)w(b)+\int_a^b \left(e^{-\Gamma(x)}u_0'(x)\right)' w(x)\,\mathrm{d}x-\int_a^b e^{-\Gamma(x)}\delta(x)u_0(x)w(x)\,\mathrm{d}x=L(w).$$

Taking into account (4.13), necessarily $u_0'(b)=0$. Therefore,

$$\begin{aligned}\int_a^b e^{-\Gamma(x)}u_0''(x)w(x)\,\mathrm{d}x-\int_a^b e^{-\Gamma(x)}\gamma(x)u_0'(x)w(x)\,\mathrm{d}x&\\ -\int_a^b e^{-\Gamma(x)}\delta(x)u_0(x)w(x)\,\mathrm{d}x=L(w).&\end{aligned}\tag{4.14}$$

From Theorem 1.4.1 we also know that, for almost every $x\in[a,b]$,

$$\partial(-\Psi_{|L^2([a,b])})(u_0)\subseteq[(-\lambda e^{-\Gamma}f)^-(x,u_0(x)),(-\lambda e^{-\Gamma}f)^+(x,u_0(x))]\tag{4.15}$$

Combining equations (4.14) and (4.15), and considering that $\lambda>0$ as well as the positivity and continuity of $e^{-\Gamma}$, we deduce that

$$e^{-\Gamma}\left(-u_0''(x)+\gamma(x)u_0'(x)+\delta(x)u_0(x)\right)\in[\lambda e^{-\Gamma}f^-(x,u_0(x)),\lambda e^{-\Gamma}f^+(x,u_0(x))],$$

that is,

$$\begin{aligned}\lambda\int_a^b e^{-\Gamma(x)}f^-(x,u_0(x))w(x)\,\mathrm{d}x\le&-\int_a^b e^{-\Gamma(x)}u_0''(x)w(x)\,\mathrm{d}x\\ &+\int_a^b e^{-\Gamma(x)}\gamma(x)u_0'(x)w(x)\,\mathrm{d}x\\ &+\int_a^b e^{-\Gamma(x)}\sigma(x)u_0(x)w(x)\,\mathrm{d}x\\ \le&\,\lambda\int_a^b e^{-\Gamma(x)}f^+(x,u_0(x))w(x)\,\mathrm{d}x,\end{aligned}$$

for all $w\in W_a^{1,2}([a,b])$.
Since every $w\in W_a^{1,2}([a,b])$ can be expressed as $w=e^\Gamma v$, with $v\in W_a^{1,2}([a,b])$, one has

$$-u_0''(x)+\gamma(x)u_0'(x)\in\left[\lambda f^-(x,u_0(x))-\delta(x)u_0(x),\ \lambda f^+(x,u_0(x))-\delta(x)u_0(x)\right]$$

for almost every $x\in[a,b]$. Noting that f belongs to $\mathcal{H}$ we consider two cases: if $x\in[a,b]\setminus u_0^{-1}(D_f^M)$, then $f(x,u_0(x))=f^-(x,u_0(x))=f^+(x,u_0(x))$. So, the previous inclusion can be written as

$$-u_0''(x)+\gamma(x)u_0'(x)=\lambda f(x,u_0(x))-\sigma(x)u_0(x)\quad\text{a.e. in }[a,b]\setminus u_0^{-1}(D_f^M).$$

If $x\in u_0^{-1}(D_f^M)$, we can observe that, using [86, Theorem 1] one has $u_0'=0$ a.e. in $u_0^{-1}(D_f^M)$, hence also $u''(0)=0$ a.e. in $u_0^{-1}(D_f^M)$. On the other hand, for a.a. $u_0^{-1}(D_f^M)$

$$0=-u_0''(x)+\gamma(x)u_0'(x)\in[\lambda f^-(x,u_0(x))-\delta(x)u_0(x),\lambda f^+(x,u_0(x))-\delta(x)u_0(x)],$$

then, using assumption (m$_3$), one has

$$\lambda f(x, u_0(x)) - \delta(x)u_0(x) = 0, \quad \text{a.e. in } u_0^{-1}(D_f^M)$$

which gives us $-u_0''(x) + \gamma(x)u_0'(x) = 0 = \lambda f(x, u_0(x)) - \delta(x)u_0(x)$ almost everywhere in $u_0^{-1}(D_f^M)$. Finally $-u_0''(x) + \gamma(x)u_0'(x) + \delta(x)u_0(x) = \lambda f(x, u_0(x))$ a.e. in $[a, b]$, which proves the claim. □

4.1.2 Existence of three generalized solutions

Here, we aim to investigate the existence of three generalized solutions to problem (M_λ). For the sake of clarity and reader convenience, we put

$$\bar{K} = \frac{1}{2}\frac{m^2}{M^2}\frac{\min_{x\in[a,b]} e^{-\Gamma(x)}}{\max_{x\in[a,b]} e^{-\Gamma(x)}} \tag{4.16}$$

where m, M are defined in (4.7). The following lemma presents key properties of the functionals introduced in Section 4.1.1, which are required in order to apply the abstract critical points theorem 2.2.7.

Lemma 4.1.2. *Let X, Φ, Ψ and I_λ be defined as in Section 4.1.1. Then, the following properties hold:*

(a) Φ is coercive and weakly lower semicontinuous;

(b) Ψ is weak upper semicontinuous;

(c) $I_\lambda = \Phi - \lambda\Psi$ is bounded from below and satisfies the $(PS)_\mu$- condition.

Proof. The statement (a) is a direct consequence of the fact that, since Φ is a norm, is coercive and weakly lower semi-continuous with respect to sequences. We now turn to verify (b). Let us consider a sequence $\{u_n\} \subset W_a^{1,2}([a, b])$ such that $u_n \rightharpoonup u$ weakly in $W_a^{1,2}([a, b])$, for some $u \in W_a^{1,2}([a, b])$. By the Rellich–Kondrachov compactness theorem 1.1.2, the compact embedding $W^{1,2}([a, b]) \hookrightarrow L^1([a, b])$ implies that $u_n \to u$ strongly in $L^1([a, b])$. Taking into account assumption (m$_2$), we estimate:

$$\begin{aligned}\left|\int_a^b (F(x, u(x)) - F(x, u_n(x)))\, dx\right| &\leq \int_a^b \left(\int_{u_n(x)}^{u(x)} |f(x,t)|\,\mathrm{d}t\right) \mathrm{d}x \\ &\leq \int_a^b \left(\int_{u_n(x)}^{u(x)} C(x)(1 + |t|)|\,\mathrm{d}t\right) \mathrm{d}x \\ &\leq \|C\|_\infty \left(\|u - u_n\|_1 + \int_a^b \left(\int_{u_n(x)}^{u(x)} |t|\mathrm{d}t\right) \mathrm{d}x\right).\end{aligned}$$

Using the Mean Value theorem, one has that

$$\int_a^b \left(\int_{u_n(x)}^{u(x)} |t|\,\mathrm{d}t\right) \mathrm{d}x = \int_a^b |g_n(x)||u(x) - u_n(x)|\,\mathrm{d}x,$$

where the function $g_n(x) \in [\min\{u_n(x), u(x)\}, \max\{u_n(x), u(x)\}]$ for almost every $x \in [a, b]$. Moreover, thanks to the uniform boundedness principle ([30, Theorem 2.2]) and taking into account that the embedding $W_a^{1,2}([a, b]) \hookrightarrow L^\infty([a, b])$ is continuous, i.e., for any $u \in W_a^{1,2}([a, b])$,

$$\|u\|_\infty < k\|u\|_a, \tag{4.17}$$

being k the best constants for which such embedding is satisfied, the sequences $\{\|u_n\|\}$ and so $\{\|u_n\|_\infty\}$ are bounded. Thus, there exists $M \in [0,+\infty)$ such that

$$\|u_n\|_\infty, \|u\|_\infty < M$$

and

$$\int_a^b \left(\int_{u_n(x)}^{u(x)} |t| \mathrm{d}t \right) \mathrm{d}x = \int_a^b |g_n(x)||u(x) - u_n(x)| \mathrm{d}x \leq M\|u - u_n\|_1 \to 0.$$

We thus obtain

$$\Psi(u) = \int_a^b F(x, u(x))\, \mathrm{d}x = \lim_{n\to\infty} \int_a^b F(x, u_n(x))\, \mathrm{d}x = \limsup_{n\to\infty} \int_a^b F(x, u_n(x))\, \mathrm{d}x.$$

Hence, for almost every $x \in [a, b]$, we have

$$|F(x, u_n(x))| = \left| \int_0^{u_n(x)} f(x,t)\, \mathrm{d}t \right| \leq \int_{-M}^{M} |f(x,t)|\, \mathrm{d}t \leq 2MA,$$

where $A = \operatorname{ess\,sup}_{(x,t)\in[a,b]\times[-M,M]} |f(x,t)| < +\infty$, and the last inequality holds since f is locally essentially bounded and $[0, T] \times [-M, M]$ is a compact set of $\mathbb{R}^2$.

Clearly, the constant function $2MA \in L^1([a, b])$, so we are in position to apply the reverse Fatou's lemma 1.1.2, which yields:

$$\Psi(u) = \int_a^b F(x, u(x))\, \mathrm{d}x \geq \limsup_{n\to\infty} \int_a^b F(x, u_n(x))\, \mathrm{d}x$$

and proves that Ψ is sequentially weakly upper semi-continuous. Now, in order to verify that I_λ is bounded below, we show that it is coercive. According to assumption (m_2), we have the lower bound

$$(\Phi - \lambda\Psi)(u) \geq \frac{1}{2}\|u\|_X^2 - \lambda \int_a^b \mu(x)(1 + |u(x)|^s)\, \mathrm{d}x, \tag{4.18}$$

for some function $\mu \in L^1([a, b])$ and an exponent s such that $s < 2$. Moreover, using (4.17), one has

$$\begin{aligned}\int_a^b \mu(x)(1 + |u(x)|^s)\, \mathrm{d}x &\leq \|\mu\|_{L^1} + \int_a^b \mu(x)|u(x)|^s\, dx \\ &\leq \|\mu\|_{L^1} + \|\mu\|_{L^1}\|u\|_\infty^s.\end{aligned}$$

Substituting in (4.18), we conclude

$$(\Phi - \lambda\Psi)(u) \geq \frac{1}{2}\|u\|^2 - \lambda\|\mu\|_1 - \lambda k^s \|\mu\|_1 \|u\|^s_{W_a^{1,2}([a,b])}.$$

Since $s < 2$, it follows that $\Phi - \lambda\Psi$ is coercive.
Let us now prove that $\Phi - \lambda\Psi$ satisfies the $(PS)_\mu$ condition for any $\mu \in \mathbb{R}$.
Let $\{u_n\} \subset X$ be a sequence such that $(\Phi - \lambda\Psi)(u_n) \to \mu$ and

$$(\Phi - \lambda\Psi)^\circ(u_n; v - u_n) \geq -\epsilon_n\|v - u_n\|_X, \quad \text{for all } v \in W_a^{1,2}([a,b]) \tag{4.19}$$

with $\epsilon_n \to 0^+$. Since the functional $\Phi - \lambda\Psi$ is coercive, it follows that the sequence $\{u_n\}$ is bounded in norm. Hence, by the Banach–Alaoglu theorem 1.1.5,

there exists a subsequence (still denoted by $\{u_n\}$) that converges weakly to some limit $u \in W_a^{1,2}([a,b])$.
In addition, thanks to the compact embedding $W_a^{1,2}([a,b]) \hookrightarrow L^2([a,b])$, we can extract a further subsequence (again denoted by $\{u_n\}$) such that $u_n \to u$ strongly in $L^2([a,b])$. Choosing $v = u$ in (4.19) yields

$$\Phi'(u_n)(u-u_n) + \lambda\,(-\Psi)^\circ(u_n; u-u_n) \geq -\epsilon_n \|u-u_n\|_a. \tag{4.20}$$

Observe that

$$\begin{aligned}\Phi'(u_n)(u-u_n)\,\mathrm{d}x &= \int_a^b e^{-\Gamma(x)} u_n'(x)(u'(x)-u_n'(x))\,\mathrm{d}x \\ &\quad + \int_a^b e^{-\Gamma(x)}\delta(x)u_n(x)(u(x)-u_n(x))\,\mathrm{d}x \\ &= \int_a^b e^{-\Gamma(x)}u_n'(x)u'(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)u_n(x)u(x)\,\mathrm{d}x \\ &\quad - \int_a^b e^{-\Gamma(x)}\left(u_n'^2(x)+\delta(x)u_n^2(x)\right)\mathrm{d}x.\end{aligned}$$

Now, applying Young's inequality, we estimate:

$$\begin{aligned}\Phi'(u_n)(u-u_n)\,\mathrm{d}x \leq &\int_a^b e^{-\Gamma(x)}u_n'^2(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)u'^2(x)\,\mathrm{d}x \\ &+ \int_a^b e^{-\Gamma(x)}\delta(x)u_n^2(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)u^2(x)\,\mathrm{d}x - \|u_n\|_a^2.\end{aligned}$$

This can be written compactly as

$$\Phi'(u_n)(u-u_n)\,\mathrm{d}x \leq \frac{1}{2}\|u_n\|_a^2 + \frac{1}{2}\|u\|_a^2 - \|u_n\|_a^2 = -\frac{1}{2}\|u_n\|_a^2 + \frac{1}{2}\|u\|_a^2.$$

Therefore, from the inequality (4.20)

$$\Phi'(u_n)(u-u_n) + \lambda(-\Psi)^\circ(u_n; u-u_n) \geq -\epsilon_n\|u-u_n\|_a,$$

we deduce

$$-\,\epsilon_n\|u-u_n\|_a + \frac{1}{2}\|u\|_X^2 - \frac{1}{2}\|u_n\|_a^2 \leq \lambda(-\Psi)^\circ(u_n; u-u_n). \tag{4.21}$$

Recall that, as proved in Lemma 4.1.1,Ψ is locally Lipschitz in $L^2([a,b])$ and by Theorem 1.4.2 we have

$$(-\Psi|_X)^\circ(u;v) \leq (-\Psi)^\circ|_{L^2}(u;v), \quad \text{for all } u,v \in W_a^{1,2}([a,b]).$$

Moreover, by the upper semi-continuity of $(-\Psi)^\circ$ on $L^2([a,b])^*$, which follows from [39, Proposition 2.1.1], we obtain

$$\limsup_{n\to\infty}(-\Psi)^\circ\,|_{L^2([a,b])}\,(u_n; u-u_n) \leq 0.$$

Hence, from (4.21)

$$-\epsilon_n\|u-u_n\|_a + \frac{1}{2}\|u_n\|_a^2 \leq \frac{1}{2}\|u\|_a^2 + \lambda(-\Psi)^\circ(u_n; u-u_n),$$

we obtain

$$\limsup_{n\to\infty} \|u_n\|_a^2 \leq \|u\|_a^2.$$

as $n \to \infty$ and $\epsilon_n \to 0^+$. Thus, the sequence $\{u_n\}$ is weakly convergent and norm bounded in X, which is a Hilbert space and hence uniformly convex. Therefore, by Proposition III.30 in [30], we conclude that $u_n \to u$ strongly in $W_a^{1,2}([a,b])$ and part (c) is proved. $\square$

Finally, we can present our main result.

Theorem 4.1.2. *Let $f\colon [a,b] \to \mathbb{R}$ be an almost every continuous function satisfying assumption (f_M). Assume that there exist two positive costants c,d with $c < d$ such that*

(j_1) $F(x,t) \geq 0$ *for all* $x \in \left[a, \dfrac{a+b}{2}\right[$, *for all* $t \in [0,d]$

(j_2) $\dfrac{\int_a^b \max_{|\xi|\leq c} F(x,\xi)\,d\xi}{c^2} < \dfrac{K}{2} \dfrac{\int_{\frac{a+b}{2}}^b F(x,d)}{d^2}$,

(j_3) *there exist $\mu \in L^1([a,b])$ and $s \in \mathbb{R}$ with $s < 2$ such that*

$$F(x,\xi) \leq \mu(x)(1+|\xi|^s).$$

Then, for each $\lambda \in \Lambda^{c,d}$, where

$$\Lambda^{c,d} = \left] \frac{m^2}{(b-a)\max\limits_{x\in[a,b]} e^{-\Gamma(x)}} \frac{1}{K} \frac{d^2}{\int_{\frac{a+b}{2}}^b F(x,d)\,dx}, \frac{m^2}{2(b-a)\max\limits_{x\in[a,b]} e^{-\Gamma(x)}} \frac{c^2}{\int_a^b \max\limits_{|\xi|<c} F(x,\xi)\,d\xi} \right[$$

problem (M_λ) admits at least three generalized solutions.

Proof. Our goal is to apply Theorem 2.2.7. Let us consider the functionals Φ, Ψ and I_λ as defined in Section 4.1.1, and assume that the conditions of Lemma 4.1.2 are satisfied. It is immediate to verify that

$$\inf_{u\in} \Phi(u) = \Phi(0) = 0 = \Psi(0),$$

so the assumptions of Theorem 2.2.7 are fulfilled.

Put $r = \dfrac{m^2}{2(b-a)}c^2$. Taking (4.8) into account, for each $u \in W_a^{1,2}([a,b]) : \Phi(u) = \dfrac{1}{2}\|u_n\|_a^2 < r$, one has

$$\max_{x\in[a,b]} |u(x)| \leq \left(\frac{2(b-a)}{m^2} r\right)^{\frac{1}{2}} = c,$$

for all $x \in [a,b]$. So,

$$\begin{aligned}
\frac{\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u)}{r} &\leq \frac{\max_{x\in[a,b]} e^{-\Gamma(x)} \int_a^b \max_{|\xi|<c} F(x,\xi)\,d\xi}{r} \\
&\leq \frac{2(b-a)}{m^2} \frac{\max_{x\in[a,b]} e^{-\Gamma(x)} \int_a^b \max_{|\xi|<c} F(x,\xi)\,d\xi}{c^2}.
\end{aligned}$$

Next, choose as function test

$$\bar{v} = \begin{cases} \dfrac{2d}{b-a}(x-a), & \text{if } x \in \left[a, \dfrac{a+b}{2}\right[\\ \\ d, & \text{if } x \in \left[\dfrac{a+b}{2}, b\right]. \end{cases}$$

Clearly, $\bar{v} \in W_a^{1,2}([a,b])$. Moreover,

$$\|\bar{v}\|_2^2 = \frac{4d^2}{b-a}.$$

Then,

$$\Phi(\bar{v}) = \frac{1}{2}\|\bar{v}\|_a^2 \leq \frac{1}{2}M^2\|\bar{v}'\|_2^2 = \frac{2M^2d^2}{b-a}. \tag{4.22}$$

Furthermore, since $c < d$, holds that

$$\Phi(\bar{v}) \geq \frac{2m^2d^2}{b-a} \geq \frac{2m^2c^2}{b-a} > \frac{m^2c^2}{2(b-a)} = r.$$

On the other hand,

$$\begin{aligned} \Psi(\bar{v}) = \int_a^b e^{-\Gamma(x)}F(x,\bar{v})\,\mathrm{d}x &\geq \min_{x\in[a,b]} e^{-\Gamma(x)} \int_a^b F(x,\bar{v}(x))\,\mathrm{d}x \\ &\geq \min_{x\in[a,b]} e^{-\Gamma(x)} \int_{\frac{a+b}{2}}^b F(x,d)\,\mathrm{d}x. \end{aligned} \tag{4.23}$$

Combining (4.22),(4.23) and taking into account (4.16), one has

$$\begin{aligned} \frac{\Psi(\bar{v})}{\Phi(\bar{v})} &\geq \frac{b-a}{2M^2} \min_{x\in[a,b]} e^{-\Gamma(x)} \frac{\int_{\frac{a+b}{2}}^b F(x,d)\,\mathrm{d}x}{d^2} \\ &= \frac{b-a}{m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} K \frac{\int_{\frac{a+b}{2}}^b F(x,d)\,\mathrm{d}x}{d^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}((-\infty,r])}\Psi(u)}{r} &\leq \frac{2(b-a)}{m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} \frac{\int_a^b \max_{|\xi|<c} F(x,\xi)\,\mathrm{d}\xi}{c^2} \\ &< \frac{b-a}{m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} \bar{K} \frac{\int_{\frac{a+b}{2}}^b F(x,d)\,\mathrm{d}x}{d^2} \leq \frac{\Psi(\bar{v})}{\Phi(\bar{v})} \end{aligned}$$

and

$$\Lambda^{c,d} \subseteq \left(\frac{\Phi(\bar{v})}{\Psi(\bar{v})}, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}\right). \tag{4.24}$$

Therefore, for every $\lambda \in \Lambda^{c,d}$, the functional I_λ has at least three critical points in $W_a^{1,2}([a,b])$. According to Theorem 4.1.1 each of these critical points corresponds to a solution of (M_λ) thus completing the proof. □

4.1.3 Application to the autonomous problem

This subsection is devoted to the analysis of a boundary value problem in which the nonlinear term f does not exhibit explicit dependence on the variable $x \in [a, b]$. Specifically, we focus on the following boundary value problem:

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda g(u), & \text{in } (a, b), \\ u(a) = u'(b) = 0, \end{cases} \tag{AM_λ}$$

where $\gamma, \delta \in L^\infty([a, b])$ and

$$\operatorname*{ess\,inf}_{x\in[a,b]} \delta(x) > -\left(\frac{\pi}{2(b-a)}\right)^2, \quad \lambda > 0.$$

We assume that the function $g \colon \mathbb{R} \to \mathbb{R}$ is locally essentially bounded and almost everywhere continuous. Define

$$G(s) = \int_0^s g(t)\,\mathrm{d}t.$$

We have the following result.

Theorem 4.1.3. *Suppose that there exist two positive constants c_1, d_1 with $c_1 < d_1$, such that:*

(g_1)

$$\frac{\max_{|\xi|\le c_1} G(\xi)}{c_1^2} < \frac{K}{4} \cdot \frac{G(d_1)}{d_1^2};$$

(g_2) *there exists $0 < s < 1$ such that*

$$\lim_{t\to\pm\infty} \frac{g(t)}{|t|^s} = 0;$$

(g_3) *for almost every $x \in [a, b]$, for all $z \in D_g = \{t \in \mathbb{R} \colon g \text{ is discontinuous at } t\}$ and for all $\lambda \in \Lambda^{c_1,d_1}$, with*

$$\Lambda^{c_1,d_1} = \left] \frac{2m^2}{K(b-a)^2 \max\limits_{x\in[a,b]} e^{-\Gamma(x)}} \cdot \frac{d_1^2}{G(d_1)}, \frac{m^2}{2(b-a)^2 \max\limits_{x\in[a,b]} e^{-\Gamma(x)}} \cdot \frac{c_1^2}{\max\limits_{|\xi|\le c_1} G(\xi)} \right[,$$

one has

$$\lambda g^-(z) - \delta(x)z \le 0 \le \lambda g^+(z) - \delta(x)z \quad \Rightarrow \quad \lambda g(z) - \delta(x)z = 0,$$

where g^- and g^+ are defined as in (h_4).

Then, for each $\lambda \in \Lambda^{c_1,d_1}$, problem ($AM_\lambda$) admits at least three generalized solutions.

The following is a special case when g is nonnegative on a suitable interval.

Theorem 4.1.4. *Suppose that there exist two positive constants c_2, d_2 with $c_2 < d_2$, such that:*

(g'_1) *g is nonnegative in $[0, d_2]$ and* $\dfrac{G(c_2)}{c_2^2} < \dfrac{K}{4}\dfrac{G(d_2)}{d_2^2}$,

(g'_2) *there exists* $0 < s < 1$ *such that*

$$\lim_{t\to\pm\infty} \frac{g(t)}{|t|^s} = 0;$$

(g'_3) *for almost every* $x \in [a,b]$, *for all* $z \in D_g = \{t \in \mathbb{R} \colon g$ *is discontinuous at* $t\}$ *and for all* $\lambda \in \Lambda^{c_2,d_2}$, *with*

$$\Lambda^{c_2,d_2} = \left] \frac{2m^2}{K(b-a)^2 \max\limits_{x\in[a,b]} e^{-\Gamma(x)}} \cdot \frac{d_2^2}{G(d_2)}, \frac{m^2}{2(b-a)^2 \max\limits_{x\in[a,b]} e^{-\Gamma(x)}} \cdot \frac{c_2^2}{G(c_2)} \right[,$$

one has

$$\lambda g^-(z) - \delta(x)z \le 0 \le \lambda g^+(z) - \delta(x)z \quad \Rightarrow \quad \lambda g(z) - \delta(x)z = 0.$$

Then, for each $\lambda \in \Lambda^{c_2,d_2}$, *problem* ($AM_\lambda$) *admits at least three generalized solutions.*

4.2 Neumann elliptic differential problems with p-Laplace operator

This section investigates a nonlinear differential problem driven by the p-Laplace operator and subject to Neumann boundary value conditions. In particular, we focus on cases where the nonlinear term on the right-hand side is highly discontinuous proving the existence of at least two nontrivial weak solutions. The results presented here correspond to those obtained in [43], in collaboration with G. D'Aguì and P. Winkert.

4.2.1 Setting of the problem and variational context

Let $\Omega \subseteq \mathbb{R}^N$, $N \ge 3$ be a bounded domain with a C^1-boundary $\partial\Omega$ and $\nu(x)$ is the outer unit normal of Ω at $x \in \partial\Omega$. We study the following highly discontinuous elliptic PDE involving the p-Laplace operator under Neumann boundary conditions:

$$\begin{cases} -\Delta_p u + \delta(x)|u|^{p-2}u = \lambda f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{N_λ}$$

where $p < N$ and we assume that $\delta \in L^\infty(\Omega)$ such that

$$\operatorname*{ess\,inf}_{x\in\Omega} \delta(x) > 0 \quad \text{for a.a. } x \in \Omega. \tag{4.25}$$

Within this context, we introduce the precise hypotheses concerning the nonlinearity. We consider a function $f \colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying the following conditions:

(f_N) (f_1) f belongs to $\mathcal{H}$;

(f_2) there exist constants $s \in [1, p[$ and $q \in]p, p^*[$ such that

$$|f(x,t)| \le a_s|t|^{s-1} + a_q|t|^{q-1}$$

for almost every $x \in \Omega$ and all $t \in \mathbb{R}$, where a_s and a_q are positive constants;

(f$_3$) for each $\lambda > 0$, for almost every $x \in \Omega$ and all $t \in D_f$, it holds that

$$\lambda f^-(x,t) \le \delta(x)|t|^{p-2}t \le \lambda f^+(x,t) \quad \text{implies} \quad \lambda f(x,t) = \delta(x)|t|^{p-2}t,$$

(AR) there exist constants $\eta > p$ and $m > 0$ such that

$$0 < \eta F(x,t) \le t f(x,t)$$

for almost every $x \in \Omega$ and all $|t| \ge m$.

Since we are interested in two nontrivial weak solutions for the Neumann elliptic problem (N_λ), the natural functional setting is the Sobolev space $W^{1,p}(\Omega)$, equipped with its standard norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p.$$

Nevertheless, due to assumption (4.25), it is more suitable for our purposes to consider the equivalent norm given by

$$\|u\| = \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_\Omega \delta(x)|u|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Recalling that the space $W^{1,p}(\Omega)$ is compactly embedded into suitable $L^r(\Omega)$-spaces, as stated in Theorem 1.1.2 (we can also refer to [76, Theorem 4.5.25 (a)]), we denote by h_q the best constant for which we have

$$\|u\|_q \le h_q \|u\| \quad \text{for all } 1 \le q < p^*.$$

We define two functionals $\Phi, \Psi \colon W^{1,p}(\Omega) \to \mathbb{R}$ given by

$$\Phi(u) = \frac{\|u\|^p}{p}, \quad \Psi(u) = \int_\Omega F(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p}(\Omega).$$

Standard arguments show that Φ is a C^1 functional on $W^{1,p}(\Omega)$ with derivative

$$\langle \Phi'(u), v\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla v + \delta(x)|u|^{p-2}uv \right) \mathrm{d}x.$$

In accordance with assumptions (f$_N$) , in particular due to the growth condition (f$_2$), the functional Ψ is locally Lipschitz on $W^{1,p}(\Omega)$. Consequently, the energy functional associated with the problem (N_λ), given by

$$I_\lambda = \Phi - \lambda\Psi$$

is also locally Lipschitz. In this context a weak solution of problem (N_λ) is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla v + \delta(x)|u|^{p-2}uv \right) \mathrm{d}x = \lambda \int_\Omega f(x,u)v\,\mathrm{d}x \tag{4.26}$$

is satisfied for all $v \in W^{1,p}(\Omega)$. The next lemma shows that any function u satisfying (4.26) is a critical point of the energy functional I_λ.

Lemma 4.2.1. *Let f be a function which satisfies hypotheses (f$_1$),(f$_2$) and (f$_3$).Then, for each $\lambda > 0$, the critical points of the functional I_λ are weak solutions of problem (N_λ).*

Proof. Fix $\lambda > 0$ and let $u_0 \in W^{1,p}(\Omega)$ be a critical point of the energy functional I_λ, namely

$$(\Phi - \lambda\Psi)^\circ(u_0; v) \geq 0 \quad \text{for all } v \in W^{1,p}(\Omega).$$

In particular, by using (p_2) of Proposition 1.3.5, one has

$$0 \leq \Phi^\circ(u_0; v) + (-\lambda\Psi)^\circ(u_0; v) = \Phi'(u_0, v) + (-\lambda\Psi)^\circ(u_0; v).$$

Replacing the derivative of Φ, we obtain

$$-\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla v + \delta(x)|u|^{p-2}uv)\,\mathrm{d}x \leq (-\lambda\Psi)^\circ(u_0; v),$$

that is,

$$\Phi'(u_0) \in \partial(\lambda\Psi)(u_0). \tag{4.27}$$

Since $W^{1,p}(\Omega)$ is embedded and dense in $L^p(\Omega)$, taking Theorem 1.4.2 into account, one has

$$\partial(-\lambda\Psi)(u_0) \subseteq \partial(-\lambda\Psi)|_{L^p(\Omega)}(u_0).$$

Therefore, if we define

$$T^*(w) = -\left[\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla w\,\mathrm{d}x + \int_\Omega \delta(x)|u_0|^{p-2}u_0 w\,\mathrm{d}x\right]$$

for all $w \in W^{1,p}(\Omega)$, one has that T^* is a linear and continuous operator on $W^{1,p}(\Omega)$ such that $T^* \in \lambda\partial(-\Psi)(u_0)$ and, additionally, it is a linear and continuous operator on $L^p(\Omega)$. Moreover, since f satisfies hypothesis (f_2) and $-\lambda f \in \mathcal{H}$, we can apply Theorem 1.4.1 and obtain

$$\partial(\lambda\Psi)(u_0)_{L^p(\Omega)} \subseteq \lambda[f^-(x, u_0(x)), f^+(x, u_0(x))]|_{L^q(\Omega)}$$

where p and q are conjugated exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and

$$[f^-(x, u_0(x)), f^+(x, u_0(x))]|_{L^q(\Omega)} = \{w \in L^q(\Omega) \colon w(x) \in [f^-(x, u_0(x)), f^+(x, u_0(x))]\}.$$

From (4.27), one has

$$\Phi'(u_0) \in \lambda[f^-(x, u_0(x)), f^+(x, u_0(x))]|_{L^q(\Omega)},$$

then

$$-\Delta_p u \in [-\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^-(\cdot, u_0(\cdot)), -\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^+(\cdot, u_0(\cdot))]|_{L^q(\Omega)}.$$

Thus, there exists a unique

$$w_0 \in [-\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^-(\cdot, u_0(\cdot)), -\delta(\cdot)|u_0(\cdot)|^{p-2}u_0(\cdot) + \lambda f^+(\cdot, u_0(\cdot))]|_{L^q(\Omega)},$$

such that $-\Delta_p u_0 = w_0$, that is

$$\int_\Omega |\nabla u_0|^{p-2}\nabla u_0 \cdot \nabla v\,\mathrm{d}x = \int_\Omega w_0 v\,\mathrm{d}x \tag{4.28}$$

for all $v \in L^p(\Omega)$. Now, we show that

$$w_0(x) = -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f(x, u_0(x)) \quad \text{a.e. in } \Omega. \tag{4.29}$$

Let $\Omega_1 \subseteq \Omega$ be a set such that $|\Omega_1| = 0$ and for all $x \in \Omega \setminus \Omega_1$, one has

$$w_0(x) \in [-\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^-(x, u_0(x)), -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^+(x, u_0(x))].$$

We denote by $\Omega_f = \{x \in \Omega \colon u_0(x) \in D_f\}$. Clearly $\Omega_f = u_0^{-1}(D_f)$. Formula (4.28) ensures that u_0 is a weak solution of the problem

$$-\Delta_p u = w_0(x) \quad \text{in } \Omega, \qquad |\nabla u|^{p-2}\nabla u \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

Using Lemma 1 by De Giorgi, Buttazzo and Dal Maso [44], one has $-\Delta_p u_0(x) = 0$ for a.a. $x \in \Omega_f$, so that $w_0 = 0$ for a.a. $x \in \Omega_f$.

Let $\Omega_3 \subseteq \Omega_f$ be a set of measure zero such that $w_0 = 0$ for all $x \in \Omega_f \setminus \Omega_3$. Taking into account assumption (f_3), there exists $\Omega_2 \subseteq \Omega$ with $|\Omega_2| = 0$ such that for all $x \in \Omega \setminus \Omega_2$ and for all $t \in D_f$ one has

$$\lambda f(x, t) = \delta(x)|t|^{p-2}t. \tag{4.30}$$

We put $\Omega^* = \bigcup_{i=0}^{3} \Omega_i$ and prove (4.29) for each $x \in \Omega \setminus \Omega^*$. We observe that $|\Omega^*| = 0$. Fix $x \in \Omega \setminus \Omega^*$. There are two possible cases: if $u_0(x) \notin D_f$, in particular $x \in \Omega \setminus \Omega_0$ and the definition of D_f implies the continuity of the function $f(\cdot, u_0(\cdot))$ at the point x. Then

$$w_0(x) = -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f(x, u_0(x)).$$

If, on the other hand $u_0(x) \in D_f$, one has that $x \in (\Omega \setminus \Omega_1) \cap (\Omega \setminus \Omega_3)$ and

$$0 = w_0(x) \in [-\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^-(x, u_0(x)), -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f^+(x, u_0(x))],$$

that is

$$\lambda f^-(x, u_0(x)) \leq \delta(x)|u_0(x)|^{p-2}u_0(x) < x \leq \lambda f^+(x, u_0(x)).$$

Note that $x \in \Omega \setminus \Omega_2$, from (4.30) one has

$$\lambda f(x, u_0(x)) = \delta(x)|u_0(x)|^{p-2}u_0(x).$$

Therefore

$$w_0(x) = 0 = -\delta(x)|u_0(x)|^{p-2}u_0(x) + \lambda f(x, u_0(x))$$

and this completes the proof. □

4.2.2 Weak solutions with opposite energy sign

In the following we prove the existence of at least two nontrivial weak solutions to problem (N_λ) by applying the abstract theorem 2.2.5, under suitable assumptions on the nonlinearity f that guarantee the required geometric structure and compactness. As a consequence, we deduce a corollary establishing the multiplicity of solutions for all sufficiently small values of λ, assuming a superlinearity condition on $F(x, t)$ near the origin.

Theorem 4.2.1. *Let f be a function satisfying assumptions (f_N). Suppose that there exist $r, d > 0$ such that*

(k_1) $d < \left(\frac{pr}{\|\delta\|_1}\right)^{\frac{1}{p}}$,

(k_2) $\frac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \frac{a_q}{q} h_q^q (pr)^{\frac{q}{p}} < \int_\Omega F(x, d)\, \mathrm{d}x$,

then for each $\lambda \in \Lambda$, with

$$\Lambda := \left] \frac{d^p \|\delta\|_1}{p \int_\Omega F(x, d)\, \mathrm{d}x}, \frac{r}{\frac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \frac{a_q}{q} h_q^q (pr)^{\frac{q}{p}}} \right[,$$

problem (N_λ) admits at least two nontrivial weak solutions with opposite energy sign.

Proof. Our aim is to apply Theorem 2.2.5. Note that the Ambrosetti-Rabinowitz condition stated in (AR) implies that the functional I_λ is unbounded from below and satisfies the Palais-Smale condition. So, we only have to verify that condition (2.9) is satisfied. Fix $\lambda \in \Lambda$ and let $\tilde{u} \in W^{1,p}(\Omega)$ be such that $\tilde{u}(x) = d$ for all $x \in \Omega$. It easy to verify that $0 < \phi(\tilde{u}) < r$. Indeed

$$0 < \Phi(\tilde{u}) = \frac{1}{p}\|\tilde{u}\|^p = \frac{1}{p}\int_\Omega \delta(x) d^p\, \mathrm{d}x = \frac{d^p}{p}\|\delta\|_1 < r, \tag{4.31}$$

from hypothesis (k_1). Moreover,

$$\Psi(\tilde{u}) = \int_\Omega F(x, d)\, \mathrm{d}x. \tag{4.32}$$

Then, we consider $u \in X$ such that $\Phi(u) < r$. We observe that

$$\Phi^{-1}(]-\infty, r]) = \left\{u \in W^{1,p}(\Omega) \colon \frac{1}{p}\|u\|^p < r\right\} = \left\{u \in W^{1,p}(\Omega) \colon \|u\| < (pr)^{\frac{1}{p}}\right\}$$

Therefore, exploiting the Sobolev embedding provided in Proposition 1.1.2 and the growth condition (f_2), which implies that

$$F(x, t) \le \frac{a_s}{s}|t|^s + \frac{a_q}{q}|t|^q \quad \text{for a.a. } x \in \Omega \text{ and for all } t \in \mathbb{R},$$

we obtain

$$\begin{aligned}
\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) &= \sup_{u \in \Phi^{-1}(]-\infty, r])} \int_\Omega F(x, u(x))\, \mathrm{d}x \\
&\le \sup_{u \in \Phi^{-1}(]-\infty, r])} \int_\Omega \left(\frac{a_s}{s}|u|^s + \frac{a_q}{q}|u|^q\right) \mathrm{d}x \\
&= \sup_{u \in \Phi^{-1}(]-\infty, r])} \left(\frac{a_s}{s}\|u\|_s^s + \frac{a_q}{q}\|u\|_q^q\right) \\
&\le \sup_{u \in \Phi^{-1}(]-\infty, r])} \left(\frac{a_s}{s} h_s^s \|u\|^s + \frac{a_q}{q} h_q^q \|u\|^q\right) \\
&\le \frac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \frac{a_q}{q} h_q^q (pr)^{\frac{q}{p}}.
\end{aligned}$$

Hence, combining the previous inequality, (4.31), (4.32) and taking into account the choice of λ one has

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})}.$$

Then, hypothesis (2.9) is verified and Theorem 2.2.5 ensures the existence of two nontrivial weak solutions $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$. □

Corollary 4.2.1. *Let hypothesis* (f_N) *be satisfied and suppose that*

$$\limsup_{t\to 0^+} \frac{F(x,t)}{t^p} = +\infty \quad \textit{uniformly for a.a. } x \in \Omega. \tag{4.33}$$

Then, for each $\lambda \in]0, \lambda^*[$, *where*

$$\lambda^* = \sup_{r>0} \frac{r}{\frac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \frac{a_q}{q} h_q^q (pr)^{\frac{q}{p}}},$$

problem (N_λ) *admits at least two nontrivial weak solutions.*

Proof. Fix $\lambda \in]0, \lambda^*[$. Then, there exists $r > 0$ such that

$$\lambda < \frac{r}{\frac{a_s}{s} h_s^s (pr)^{\frac{s}{p}} + \frac{a_q}{q} h_q^q (pr)^{\frac{q}{p}}}. \tag{4.34}$$

On the other hand, from (4.33), it follows that

$$\limsup_{d\to 0^+} \frac{\|\delta\|_1}{p} \frac{d^p}{F(x,d)} = 0,$$

so there exist $d > 0$ small enough such that

$$\frac{\|\delta\|_1}{p} \frac{d^p}{F(x,d)} < \lambda,$$

which together with (4.34) implies (k_2). Then, the assertion follows. □

Chapter 5

Hemivariational inequalities

Hemivariational inequalities, introduced by Panagiotopoulos (see [74]), extend the classical theory of variational inequalities to the nonsmooth setting, useful for the treatment of nonconvex and nondifferentiable potentials. The central analytical tool is the Clarke's generalized gradient, which replaces the classical derivative and makes it possible to address problems with nonmonotone or discontinuous constitutive laws, as encountered in solid mechanics, unilateral contact with complex friction, plasticity, and systems exhibiting hysteresis effects.

From a variational standpoint, such problems correspond to stationary points of locally Lipschitz functionals subject to convex constraints, where the stationarity condition is expressed as an inequality involving Clarke's derivative.

In this chapter, we study a fourth-order hemivariational inequality, which combines the nonsmooth structure described above with the analytical difficulties inherent to higher-order differential operators. Problems of this nature arise, for example, in the modeling of elastic and viscoelastic beams or plates subject to unilateral constraints and nonmonotone boundary interactions. The presence of fourth-order derivatives requires an appropriate functional framework together with suitable compactness arguments in order to apply variational methods in the nonsmooth context.

5.1 Fourth-order hemivariational problem

This section is devoted to the study of a fourth-order hemivariational inequality. The main result, obtained in collaboration with G. Bonanno and B. Di Bella in [22], establishes the existence of at least three distinct solutions for a parameter-dependent problem involving Clarke's generalized directional derivative. By imposing suitable assumptions on the nonlinear term, we identify a precise interval of the parameter where multiple solutions occur. As a consequence, in the autonomous case we obtain a multiplicity result for a fourth-order differential inclusion, together with an application to a fourth-order differential equation.

5.1.1 Preliminaries on the problem and functional setting

Consider the following fourth-order hemivariational problem:

find $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ *such that*

$$\begin{aligned} &-\int_0^1 \left[p(t)u''(t)v''(t) - q(t)u'(t)v'(t) + r(t)u(t)v(t)\right] \mathrm{d}t \\ &+ \lambda \int_0^1 F^\circ(t, u(t); v(t))\, \mathrm{d}t \geq 0 \end{aligned} \tag{E_λ}$$

for all $v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, where

- λ is a real positive parameter,
- p, q, r are suitable regular functions,
- F° stands for Clarke's generalized directional derivative of a locally Lipschitz function $F\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ given by $F(t,\xi) = \int_0^\xi f(t,x)\mathrm{d}x$,
- $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ locally essentially bounded function.

Regarding the weight functions p, q and r we have the following hypotheses:

let $p \in C^2([0,1]), q \in C^1([0,1]), r \in C([0,1])$ be real functions and put

$$p^- = \min_{[0,1]} p, \quad q^- = \min_{[0,1]} q, \quad r^- = \min_{[0,1]} r.$$

Moreover, consider the following set of conditions according to their signs:

(L_1) $p^- > 0, q^- \geq 0, r^- \geq 0;$

(L_2) $p^- > 0, q^- < 0, r^- \geq 0$ *and* $p^- + q^- k > 0;$

(L_3) $p^- > 0, q^- \geq 0, r^- < 0$ *and* $p^- + r^- k^2 > 0.$

(L_4) $p^- > 0, q^- < 0, r^- < 0$ *and* $p^- + q^- k + r^- k^2 > 0,$

where

$$k = \frac{1}{\pi^2}.$$

Further, consider the following condition:

(L) $\min\{p^- + q^- k,\ p^- + r^- k^2,\ p^- + q^- k + r^- k^2\} > 0.$

Put

$$\sigma := \min\{p^-, p^- + q^- k,\ p^- + r^- k^2,\ p^- + q^- k + r^- k^2\}. \tag{5.1}$$

Clearly, condition (L) implies that $\sigma > 0$.
When (L) holds, we set

$$\delta := \sqrt{\sigma} = \left(\min\{p^-,\ p^- + q^- k,\ p^- + r^- k^2,\ p^- + q^- k + r^- k^2\}\right)^{1/2}. \tag{5.2}$$

Note that δ is well-defined due to $\sigma > 0$ ensured by condition (L). Moreover, condition (L) is equivalent to assuming one of the conditions (L_1)–(L_4), as stated in the following [23, Proposition 2.2]. On the nonlinearity f we state the following assumptions

(E_f) Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a function such that:

(e_1) $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is measurable for a.a. $t \in [0,1]$;

(e_2) There exist a constant $a > 0$ and $s \in [0,1)$ such that

$$|f(t,x)| \leq a(1 + |x|^s) \quad \text{for all } (t,x) \in [0,1] \times \mathbb{R}.$$

We take as Banach space X the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ endowed with the norm

$$\|u\| = \left(\int_0^1 (p(t)|u''(t)|^2 + q(t)|u'(t)|^2 + r(t)|u(t)|^2) \mathrm{d}t \right)^{1/2}$$

for all $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, which is equivalent to the usual one, provided that $\sigma > 0$ (see [23, Proposition 2.5]).

We recall the following result, which provides an estimate of the uniform norm in terms of the norm in $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ (see [23, Proposition 2.5]).

Proposition 5.1.1. *For every $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, the following inequality holds:*

$$\|u\|_\infty \leq \frac{1}{2\pi\delta}\|u\|. \tag{5.3}$$

Next, for all $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, we define the functionals $\Phi, \Psi, I_\lambda : X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{2}\|u\|^2,$$

$$\Psi(u) = \int_0^1 F(t, u(t))\,\mathrm{d}t, \tag{5.4}$$

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u). \tag{5.5}$$

where I_λ is the so-called energy functional associated to our problem (E_λ). The functional Φ is of class C^1, and its derivative is given by

$$\Phi'(u)(v) = \int_0^1 \left(p(t)u''(t)v''(t) + q(t)u'(t)v'(t) + r(t)u(t)v(t)\right) \mathrm{d}t,$$

for all $u, v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. In addition, the functional Ψ, as defined in (5.4), is well defined, locally Lipschitz and sequentially weakly continuous on $u, v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. We point out the following result.

Proposition 5.1.2. *If $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ is a critical point of functional I_λ defined by (5.5), then u is a solution for problem (E_λ).*

Proof. Let $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ be a critical point of I_λ. One has $0 \in \partial I(u)$, which means $I^\circ(u)(v) \geq 0$ for all $v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. Hence,

$$0 \leq (\Phi - \lambda\Psi)^\circ(u; v) \;\leq\; \Phi'(u; v) + \lambda(-\Psi)^\circ(u; v) = \Phi'(u; v) + \lambda(\Psi)^\circ(u; -v)$$

for all $v \in X$. Therefore, taking into account that

$$(\Psi)^\circ(u; w) \leq \int_0^1 F^\circ(t, u(t); w(t))\,\mathrm{d}t$$

for all $w \in X$ (see [39, formula 2 p.77]) one has

$$\Phi'(u; v) + \lambda \int_0^1 F^\circ(t, u(t); -v(t))\,\mathrm{d}t \geq 0, \quad \text{for all } v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1]),$$

that is

$$-\Phi'(u; -v) + \lambda \int_0^1 F^\circ(t, u(t); -v(t))\,\mathrm{d}t \geq 0, \quad \text{for all } v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1]),$$

that we can write as

$$-\Phi'(u;w)+\lambda\int_0^1 F^\circ(t,u(t);w(t))\,\mathrm{d}t\geq 0,\quad \text{for all } w\in W^{2,2}([0,1])\cap W_0^{1,2}([0,1]),$$

so u turns out a solution to (E_λ). □

5.1.2 Existence of solutions in $W^{2,2}([0,1])\cap W_0^{1,2}([0,1])$

Here, we establish our main result and highlight some of its consequences and applications. The principal tool employed in the analysis is the abstract critical point theorem 2.2.7 presented in Chapter 3, which is due to Bonanno and Marano [26]. First, put

$$K=\frac{2\delta^2\pi^2}{\frac{2048}{27}p^+ +\frac{32}{9}q^+ +\frac{13}{40}r^+},$$

where p^+,q^+,r^+ are the max in $[0,1]$ of the functions p,q,r respectively. Moreover, for all $x>0$, we put

$$A(x)=\int_0^1 \max_{|\xi|\leq x} F(t,\xi)\mathrm{d}t,$$

and

$$B(x)=\int_{3/8}^{5/8} F(t,x)\mathrm{d}t+\inf_{\xi\in[0,x]}\left(\int_0^{3/8} F(t,\xi)\mathrm{d}t+\int_{5/8}^1 F(t,\xi)\mathrm{d}t\right),$$

Theorem 5.1.1. *Let (E_f) be satisfied and assume that there exist two positive constants c,d, with $c<d$ such that*

$$\frac{A(c)}{c^2}<K\frac{B(d)}{d^2}. \tag{5.6}$$

Then, for each $\lambda\in\left]\frac{2\delta^2\pi^2}{K}\frac{d^2}{B(d)},2\delta^2\pi^2\frac{c^2}{A(c)}\right[$, problem ($E_\lambda$) admits at least three distinct solutions.

Proof. Fix λ as in the conclusion, and let Φ, Ψ, and I_λ be as in Section 5.1.1. Our aim is to apply Theorem 2.2.7. Since Φ is sequentially weakly lower semicontinuous on the space $W^{2,2}([0,1])\cap W_0^{1,2}([0,1])$, as already observed, assumptions (a_1) and (a_2) of Theorem 2.2.7 are satisfied. We now verify condition (a_3). To this end, define

$$r=2\delta^2\pi^2c^2,$$

where the existence of the constant $c>0$ is ensured by our assumptions. By Proposition 5.1.1, it follows that

$$|u(t)|\leq c\quad \text{for all } u\in W^{2,2}([0,1])\cap W_0^{1,2}([0,1]) \text{ with } \|u\|^2<2r.$$

Hence, for all such u, we have

$$\Psi(u)=\int_0^1 F(t,u(t))\,\mathrm{d}t\leq\int_0^1\max_{|s|\leq c}F(t,s)\,\mathrm{d}t=A(c).$$

Therefore, one has

$$\frac{\sup_{\Phi(u)<r}\Psi(u)}{r}\leq\frac{A(c)}{2\pi^2\delta^2c^2}. \tag{5.7}$$

Next, put

$$v_d(t) = \begin{cases} -\frac{64}{9}d\left(t^2 - \frac{3}{4}t\right) & \text{if } t \in \left[0, \frac{3}{8}\right[\\ d & \text{if } t \in \left[\frac{3}{8}, \frac{5}{8}\right] \\ -\frac{64}{9}d\left(t^2 - \frac{5}{4}t + \frac{1}{4}\right) & \text{if } t \in \left]\frac{5}{8}, 1\right] \end{cases}$$

Clearly, $v_d \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ and

$$\|v_d\|^2 \leq \frac{4\delta^2\pi^2}{K}d^2.$$

Hence,

$$\Phi(v_d) \leq \frac{2\delta^2\pi^2}{K}d^2. \tag{5.8}$$

Now, taking into accont that $c < d$ one has

$$r = 2\delta^2\pi^2c^2 < 2\delta^2\pi^2d^2 = 2\delta^2\pi^2\|v\|_\infty^2 \leq \frac{1}{2}\|v_d\|^2 = \Phi(v_d)\,,$$

that is,

$$\Phi(v_d) > r.$$

Moreover, simply computations show that

$$\begin{aligned} \Psi(v_d) &= \int_0^{3/8} F(t, v_d(t))dt + \int_{3/8}^{5/8} F(t,d)dt + \int_{5/8}^1 F(t, v_d(t))dt \\ &\geq \int_{3/8}^{5/8} F(t,d)dt + \inf_{\xi\in[0,d]}\left(\int_0^{3/8} F(t,\xi)dt + \int_{5/8}^1 F(t,\xi)dt\right) \\ &= B(d). \end{aligned}$$

So, by (5.8) one has

$$\frac{\Psi(v_d)}{\Phi(v_d)} \geq \frac{1}{2\delta^2\pi^2}K\frac{B(d)}{d^2}\,. \tag{5.9}$$

Combining (5.6), (5.7), (5.8) and (5.9) our claim is proved.
Next, we prove that I_λ is coercive, which, in view of the weakly lower semicontinuity of I_λ, implies the first part of (a_4). For each $u \in X$, from (e_2) and (5.3) one has

$$\begin{aligned} \Psi(u) &= \int_0^1 F(t, u(t))\,\mathrm{d}t \\ &\leq \int_0^1 \left(a|u(t)| + \frac{a}{s+1}|u(t)|^{s+1}\right)\mathrm{d}t \\ &\leq a\|u\|_\infty + \frac{a}{s+1}\|u\|_\infty^{s+1} \\ &\leq \frac{a}{2\pi\delta}\left(\|u\| + \frac{\|u\|^{s+1}}{(s+1)(2\pi\delta)^s}\right). \end{aligned}$$

This leads to

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{\lambda a}{2\pi\delta}\left(\|u\| + \frac{\|u\|^{s+1}}{(s+1)(2\pi\delta)^s}\right).$$

We infer that $I_\lambda(u) \to +\infty$, as $\|u\| \to \infty$, meaning that I_λ is coercive on $W^{2,2}([0,1]) \cap$

$W_0^{1,2}([0,1])$. To show that I_λ satisfies the $(PS)_\mu$-condition, fix $\mu \in \mathbb{R}$ and let $\{u_n\} \subset W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ be a sequence such that (P_1^μ) and (P_2^μ) hold. By the coercivity of I_λ, the sequence $\{u_n\}$ is bounded in $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. Since $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ is reflexive, there exists a subsequence, still denoted $\{u_n\}$, and an element $\bar{u} \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ such that

$$u_n \rightharpoonup \bar{u} \quad \text{weakly in } W^{2,2}([0,1]) \cap W_0^{1,2}([0,1]).$$

Therefore, by evaluating (P_2^μ) at $\bar{u}$ instead of v, we obtain

$$\Phi'(u_n; \bar{u} - u_n) + \lambda(-\Psi)^\circ(u_n; \bar{u} - u_n) \geq -\varepsilon_n \|\bar{u} - u_n\|.$$

Since

$$\begin{aligned}
\Phi'(u_n; \bar{u} - u_n) &= \Phi'(u_n; \bar{u}) - \|u_n\|^2 \\
&\leq \|u_n\| \|\bar{u}\| - \|u_n\|^2 \\
&\leq \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|\bar{u}\|^2 - \|u_n\|^2 \\
&= -\frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|\bar{u}\|^2,
\end{aligned}$$

we get

$$-\varepsilon_n \|\bar{u} - u_n\| \leq \lambda(-\Psi)^\circ(u_n; \bar{u} - u_n) - \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|\bar{u}\|^2. \tag{5.10}$$

On the other hand, taking into account that $\lambda(-\Psi)^\circ$ is upper semicontinuous on $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$, one has

$$\limsup_{n \to +\infty} \lambda\,(-\Psi)^\circ(u_n; \bar{u} - u_n) \leq \lambda\,(-\Psi)^\circ(\bar{u}; 0) = 0.$$

At this point, passing to the upper limit, from (5.10) we get

$$\limsup_{n \to \infty} \|u_n\| \leq \|\bar{u}\|.$$

Thanks to [30, Proposition III.30] we conclude that $\{u_n\}$ converges to $\bar{u}$ in $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ and assumption (a_4) of Theorem 2.2.7 holds. Finally, conditions (5.7) and (5.9) furnish

$$\Lambda = \left] \frac{2\delta^2\pi^2}{K} \frac{d^2}{B(d)}, 2\delta^2\pi^2 \frac{c^2}{A(c)} \right[\subseteq \left] \frac{\Phi(v_d)}{\Psi(v_d)}, \frac{r}{\sup_{\Phi(u) \leq r} \Psi(u)} \right[.$$

Hence, Theorem 2.2.7 ensures that, for each $\lambda \in \Lambda$, the functional I_λ admits at least three distinct critical points in $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ which are, as seen in Proposition 5.1.2, three solutions for the problem (E_λ) and the proof is complete. □

Remark 5.1.1. *If $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ is a critical point of I_λ, as proved in Proposition 5.1.2*

$$\Phi'(u; w) \leq \lambda(\Psi)^\circ(u; w) \quad \textit{for all } w \in X.$$

Therefore, taking [39, formula 1, p.76] into account, one has

$$\Phi'(u) \in \lambda \partial \Psi(u) = \lambda \partial \int_0^1 F(t, u(t))\, dt \subset \lambda \int_0^1 \partial F(t, u(t))\, dt.$$

So, there exists a function $v\colon [0,1] \to \mathbb{R}$ *such that*

$$\begin{aligned} &v(t) \in \partial F(t, u(t)) \quad \text{a.e. } t \in [0,1],\\ &v(\cdot)\, w(\cdot) \in L^1([0,1]) \quad \text{for every } w \in X,\\ &\Phi'(u; w) = \lambda \int_0^1 v(t)\, w(t)\, \mathrm{d}t \quad \text{for all } w \in X. \end{aligned}$$

That means that u *is a solution of the problem*

$$\begin{cases} (p(t)u'')'' - (q(t)u')' + r(t)u = \lambda v, & \text{in } (0,1),\\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases}$$

and then it is a solution of the following differential inclusion

$$\begin{cases} (p(t)u'')'' - (q(t)u')' + r(t)u \in \lambda\, \partial F(t,u), & \text{in } (0,1),\\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0. \end{cases} \tag{Q_λ}$$

As a consequence, we obtain that, under the assumptions of Theorem 5.1.1, the differential inclusion (Q_λ) *admits three distinct solutions for each*

$$\lambda \in \left] \frac{2\delta^2\pi^2}{K} \frac{d^2}{B(d)}, 2\delta^2\pi^2 \frac{c^2}{A(c)} \right[.$$

Remark 5.1.2. *We point out that if the function* f *is nonnegative in* $[0,1] \times \mathbb{R}$*, condition (5.6) is verified when there are two positive constants* c, d*, with* $c < d$*, such that*

$$\frac{\int_0^1 F(t,c)\, dt}{c^2} < K \frac{\int_{3/8}^{5/8} F(t,d)}{d^2}.$$

In this case, the interval of parameters for which problem (E_λ) *admits three distinct solutions is*

$$\bar{\Lambda} = \left] \frac{2\delta^2\pi^2}{K} \frac{d^2}{\int_{3/8}^{5/8} F(t,d)dt}, 2\delta^2\pi^2 \frac{c^2}{\int_0^1 F(t,c)dt} \right[.$$

Remark 5.1.3. *As seen in the proof of Theorem 5.1.1, the assumption* (f_2) *guarantees that* I_λ *is coercive for every* $\lambda > 0$*. It occurs, as a special case, when*

$$\lim_{x \to +\infty} \frac{|F(t,x)|}{|x|^{s+1}} = 0, \text{with } 0 \le s < 1$$

uniformly with respect to $t \in [0,1]$ *and for some* $s < 1$*. With similar calculations, one can see that the following condition*

$$\lim_{x \to +\infty} \frac{|F(t,x)|}{|x|^2} = 0$$

is even more general than the previous one (but not more general than (e_2)*) and it implies that* I_λ *is coercive for all* $\lambda > 0$*.*

5.1.3 Autonomous case and application to a fourth-order differential equation

We now focus on the autonomous case of the problem. Precisely, let $g\colon \mathbb{R} \to \mathbb{R}$ be a nonnegative function such that

(g_1) g is measurable;

(g_2) there exist constants $a > 0$ and $s \in [0, 1)$ such that

$$|g(x)| \le a(1 + |x|^s) \quad \text{for all } x \in \mathbb{R}.$$

Under assumption (g_1), the function $G : \mathbb{R} \to \mathbb{R}$, defined by

$$G(\xi) = \int_0^\xi g(x)\,\mathrm{d}x, \quad \text{for all } \xi \in \mathbb{R},$$

is locally Lipschitz continuous. Hence, we can consider its Clarke's generalized directional derivative, denoted by G°. In the autonomous framework, we introduce the positive constant

$$T = \frac{1}{4}K = \frac{\delta^2\pi^2}{\frac{4096}{27}p^+ + \frac{64}{9}q^+ + \frac{13}{20}r^+}.$$

We point out the following result.

Theorem 5.1.2. *Assume that there exist two positive constants c, d, with $c < d$, such that*

$$\frac{G(c)}{c^2} < T\frac{G(d)}{d^2}. \tag{5.11}$$

Then, for each

$$\lambda \in \Lambda_G := \left]\frac{2\delta^2\pi^2}{T} \cdot \frac{d^2}{G(d)}, 2\delta^2\pi^2 \cdot \frac{c^2}{G(c)}\right[,$$

there exist three distinct functions $u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ satisfying the following inequality:

$$\begin{aligned} &-\int_0^1 \left[p(t)u''(t)v''(t) - q(t)u'(t)v'(t) + r(t)u(t)v(t)\right]\,\mathrm{d}t \\ &\quad + \lambda \int_0^1 G^\circ(u(t); v(t))\,\mathrm{d}t \ge 0 \end{aligned} \tag{A_λ}$$

for all $v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$.

Proof. The claim follows as a direct application of Theorem 5.1.1, in the special case where the nonlinearity is of the form

$$f(t,x) = \alpha(t)g(x), \quad \text{for all } (t,x) \in [0,1] \times \mathbb{R},$$

with $\alpha(t) \equiv 1.$, taking Remark 5.1.2 into account. □

As a way of example, here we point out one of the possible consequences of previous results.
Take a measurable function $g : \mathbb{R} \to \mathbb{R}$ and denote D_g the set of discontinuity points of g, that is,

$$D_g = \{x \in \mathbb{R} : g \text{ is discontinuous at } x\}.$$

Moreover, consider the problem

$$\begin{cases} (p(t)u'')'' - (q(t)u')' = \lambda g(u), & t \in [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0. \end{cases} \tag{R_λ}$$

From Theorem 5.1.2 we obtain the following multiplicity result for the previous fourth-order nonlinear differential equation with discontinuous data. For our purpose, put

$$\underline{g}(x) = \lim_{\delta \to 0^+} \operatorname*{ess\,inf}_{|\xi - x| < \delta} g(\xi); \qquad \overline{g}(x) = \lim_{\delta \to 0^+} \operatorname*{ess\,sup}_{|\xi - x| < \delta} g(\xi)$$

for all $x \in \mathbb{R}$.

Theorem 5.1.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be an essentially locally bounded and almost everywhere continuous function such that*

$$\inf_{D_g} \underline{g} > 0.$$

Assume that

$$\lim_{x \to 0^+} \frac{G(x)}{x^2} = \lim_{x \to +\infty} \frac{|G(x)|}{x^2} = 0 \tag{5.12}$$

and put

$$\overline{\lambda} = \frac{2\delta^2 \pi^2}{T} \inf_{d > 0} \frac{d^2}{G(d)}.$$

Then, for each $\lambda > \overline{\lambda}$, problem ($R_\lambda$) admits two distinct non-zero weak solutions.

Proof. First, we observe that conditions (g_1) and (g_2) are satisfied (see also Remark 5.1.3). Next, fix $\lambda > \overline{\lambda}$. Then, there exists $d > 0$ such that

$$\lambda > \frac{2\delta^2 \pi^2}{T} \cdot \frac{d^2}{G(d)}.$$

The first condition in (5.12) allows us to choose $c < d$ such that

$$\frac{T}{2\delta^2 \pi^2} \cdot \frac{G(c)}{c^2} < \frac{1}{\lambda},$$

from which we derive the strict inequality

$$\frac{T}{2\delta^2 \pi^2} \cdot \frac{G(c)}{c^2} < \frac{1}{\lambda} < \frac{T}{2\delta^2 \pi^2} \cdot \frac{G(d)}{d^2}.$$

Hence, condition (5.11) is satisfied and $\lambda \in \Lambda_G$. Therefore, by Theorem 5.1.2, there exist three distinct functions

$$u \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$$

satisfying the following variational inequality:

$$-\int_0^1 \left[p(t)u''(t)v''(t) - q(t)u'(t)v'(t) \right] \, \mathrm{d}t + \lambda \int_0^1 G^\circ(u(t); v(t)) \, \mathrm{d}t \geq 0$$

for all $v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. Clearly, two of them are non-zero functions. Let u be one of them. It follows that (see Remark 5.1.1) there exists a function $w \colon [0,1] \to$

$\mathbb{R}$ such that

$$w(t) \in \partial G(u(t)) \quad \text{for a.a. } t \in [0,1],$$

and

$$-\int_0^1 \left[p(t)u''(t)v''(t) - q(t)u'(t)v'(t)\right] \, dt = \lambda \int_0^1 w(t)v(t)\,\mathrm{d}t$$

for all $v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. In particular, one has

$$(p(t)u''(t))'' - (q(t)u'(t))' = \lambda w(t), \quad \text{a.e. in } [0,1]. \tag{5.13}$$

We claim that $u(t) \in \mathbb{R} \setminus D_g$ for a.e. $t \in [0,1]$. To this end, define $\Omega_0 = u^{-1}(D_g)$ and assume by contradiction that $|\Omega_0| > 0$. Since $|D_g| = 0$, it follows from [44] that

$$u'(t) = 0 \quad \text{and} \quad u''(t) = 0 \quad \text{for all } t \in \Omega_0.$$

Therefore, in particular, $u'(t) = 0$ and $u''(t) = 0$ for all $t \in \Omega_0$. Then, from equation (5.13), we deduce that

$$w(t) = 0 \quad \text{for a.a. } t \in \Omega_0.$$

On the other hand, since $w(t) \in \partial G(u(t))$, from Corollary 1.4.1 we obtain

$$w(t) \in [\underline{g}(u(t)), \overline{g}(u(t))] \quad \text{a.e. in } [0,1]. \tag{5.14}$$

So, in particular, one has $w(t) \geq \underline{g}(u(t))$ a.e in Ω_0. Hence, owing to our assumption, one has

$$w(t) > 0$$

for a.a. $t \in \Omega_0$. This is an absurd and our claim is proved.
At this point, since $|\Omega_0| = 0$, from (5.14) we obtain $w(t) = g(u(t))$ for a.a. $t \in [0,1]$. It follows that

$$-\int_0^1 \left[p(t)u''(t)v''(t) - q(t)u'(t)v'(t)\right] \, \mathrm{d}t = \lambda \int_0^1 g(u(t))v(t)\,\mathrm{d}t$$

for all $v \in W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$. This means that u is a weak solution of problem (R_λ) and the conclusion follows. □

Example 5.1.1. *Consider the following problem:*

$$\begin{cases} u^{(iv)} + 2u''' + 2u'' + u' = 10^4\, g(u) & t \in [0,1], \\ u(0) = u(1) = 0, \\ u''(0) = u''(1) = 0, \end{cases} \tag{S}$$

where $g : \mathbb{R} \to \mathbb{R}$ is defined as follows

$$g(x) = \begin{cases} x^2 & \text{if } x \in [-1,1], \\ \frac{1}{2}\sqrt{|x|} & \text{if } |x| > 1. \end{cases}$$

Hence, Theorem 5.1.1, arguing as in the first part of the proof of Theorem 5.1.3, ensures that problem (S) admits two non-zero weak solutions. Indeed, it is enough to observe that equation in (S) is equivalent to $(e^t u'')'' - (e^t u')' = 10^4 e^t g(u)$ and verify that $\dfrac{2\delta\pi^2}{K}\dfrac{1}{e^{5/8} - e^{3/8}} \inf_{d>0} \dfrac{d^2}{G(d)} = \dfrac{2144}{9}\dfrac{e}{e^{5/8} - e^{3/8}} < 10^4$.

List of Symbols

a.e.	almost everywhere
a.a.	almost all
u.s.c	upper semicontinuous
l.s.c	lower semicontinuous
$\mathbb{R}, \mathbb{K}$	real field and a generic numerical field
$C(X)$	space of continuous real functions on X
$C(X, Y)$	space of continuous functions from X to Y
$C^k(X)$	space of k-times continuously differentiable real functions on X
C_T^∞	space of indefinitely differentiable T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$
X^*, X^{**}	dual and bidual spaces of Banach space X
2^Y	power set of Y
$M(\Omega)$	space of measurable functions in Ω
$L^r(\Omega)$	Lebesgue space with constant exponent r
$W^{1,r}(\Omega)$	Sobolev space with constant exponent r
$W_0^{1,r}(\Omega)$	completion of $C_c^\infty(\Omega)$ in $W^{1,r}(\Omega)$
$\Vert \cdot \Vert_r$	usual norm in $L^r(\Omega)$
$\Vert \cdot \Vert_{1,r}$	usual norm in $W^{1,r}(\Omega)$
$\Vert \cdot \Vert_{1,r,0}$	equivalent norm in $W_0^{1,r}(\Omega)$
$\langle \cdot, \cdot \rangle$	duality pairing between X^* and X
$\Vert \cdot \Vert_X$	norm in the Banach space X
f°	Clarke generalized derivative
∂f	Clarke generalized gradient of f
$\mathrm{Dom}(f)$	domain of the function f
∇, Δ	gradient and Laplacian operators
$A^\circ, \overline{A}, \partial A$	interior, closure, and boundary of the set A
$B(x, r)$	open ball centered at x with radius r
$\overline{B}(x, r)$	closed ball centered at x with radius r
$\to, \rightharpoonup$	strong and weak convergence

$\hookrightarrow, \hookrightarrow\!\!\!\!\rightarrow$	continuous and compact embeddings
$\lvert A \rvert$	Lebesgue measure of the set A

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List of publications

The thesis presents a unified exposition of results obtained in the author's papers:

1. E. Amoroso, G. Bonanno, G. Klun, and V. Morabito. "Two solutions for ordinary differential inclusions with periodic boundary conditions". Submitted. 2025

2. G. Bonanno, B. Di Bella, and V. Morabito. "Multiple solutions for fourth-order ordinary differential inclusions". In: *Discrete and Continuous Dynamical Systems - Series S* 18.6 (2025), pp. 1550–1560. DOI: 10.3934/dcdss.2024108

3. G. Bonanno, V. Morabito, D. O'Regan, and B. Vassallo. "Two positive solutions for elliptic differential inclusions". In: *Applied Math.* 4.4 (2024), pp. 1404–1417. DOI: 10.3390/appliedmath4040074

4. G. D'Aguì, V. Morabito, and P. Winkert. "Elliptic Neumann problems with highly discontinuous nonlinearities". In: *Applied Mathematics Letters* 163 (2025), p. 109455. DOI: 10.1016/j.aml.2025.109455

5. V. Morabito. "Multiple solutions to a nonlinear parametric Sturm-Liouville problem with mixed boundary conditions". Preprint. 2025

In addition, there is a list of other author's papers on the study of existence and multiplicity of solutions for different type of problems through critical point theory:

1. E. Amoroso, G. D'Aguì, and V. Morabito. "On a complete parametric Sturm–Liouville problem with sign changing coefficients". In: *AIMS Mathematics* 9.3 (2024), pp. 6499–6512. DOI: 10.3934/math.2024316.

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4. G. D'Aguì and V. Morabito. "Multiplicity results for mixed boundary value problems with complete Sturm–Liouville differential equations". Preprint. 2025.

5. Valeria Morabito and Patrick Winkert. "Multiple solutions to logarithmic double phase problems involving superlinear nonlinearities". In: *Zeitschrift für Analysis und ihre Anwendungen* (2026). DOI: 10.4171/ZAA/1812.