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Three Solutions for a Discrete Fourth-Order Boundary Value Problem with Three Parameters

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ABSTRACT: This paper presents several sufficient conditions for the existence of at least three classical solutions of a boundary value problem for a fourth-order difference equation. Fourth-order boundary value problems act as models for the bending or deforming of elastic beams. In different fields of research, such as computer science, mechanical engineering, control systems, artificial or biological neural networks, economics and many others, the mathematical modelling of important questions leads naturally to the consideration of nonlinear difference equations. Our technical approach is based on variational methods. An example is included in the paper. Numerical computations of the example confirm our theoretical results.

Key Words: Discrete boundary value problem, fourth-order boundary value problem, three solutions, variational methods, critical point theory.

Contents

1. Introduction

Let $T \in \mathbb{N}$, $T > 2$ and $[2,T]_{\mathbb{N}}$ be the discrete interval given by $\{2,3,4,\dots,T\}$. In this paper, we will examine a three-parameter boundary value problem (BVP) for a nonlinear fourth-order difference equation, with the intention of proving the existence of three solutions. The equation to be studied can be viewed as a discrete version of the generalized beam equation. Consider the fourth-order BVP

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(k,u(k)), & k \in [2,T]_{\mathbb{N}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0, \end{cases} \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$, $\Delta^{i+1}u(k) = \Delta(\Delta^i u(k))$, $\lambda \geq 0$, $f : [2,T]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and α, β are real parameters satisfying

$$1 + (T-1)T\alpha_- + T(T-1)^3\beta_- > 0,$$

where

$$\gamma_- = \min\{\gamma, 0\} \quad \text{for any} \quad \gamma \in \mathbb{R}.$$

Fourth-order boundary value problems act as models for the bending or deforming of elastic beams, and, therefore, have important applications in engineering and physical sciences. Boundary value problems for fourth-order problems have been of great concern in recent years; see [5,26,30] and the references therein.

A considerable number of problems, which are strictly connected both with boundary value differential problems and numerical simulations of some mathematical models arising from many research areas (e.g., biological, physical and computer science), can be formulated as special cases of nonlinear algebraic systems (see, for instance, [31]). Recently, there is a trend to study difference equations by using fixed point theory, lower and upper solutions method, variational methods and critical point theory, Morse theory, and the mountain-pass theorem. Many interesting results are obtained, see, for example, [7,10,22,25,27,3,4] and the references therein. See also [8,9,23,24] for related studies.

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In recent years, discrete nonlinear fourth-order boundary value problems have been widely investigated. We refer the reader to [19,12,1,11,13,14,17,21,28,31] and the references therein. For example, Zhang et al. in [31], established the existence of positive solution to the fourth-order BVP

$$\begin{cases} \Delta^4 x(t-2) = \lambda a(t) f(t, x(t)), \quad t \in \mathbb{N},\ 2 \le t \le T, \\ x(0) = x(T+2) = 0, \\ \Delta^2 x(0) = \Delta^2 x(T) = 0 \end{cases}$$

by using the method of upper and lower solutions. Graef et al., in [14], by employing variational methods and critical point theory, obtained the existence of multiple solutions to a periodic boundary value problem for the fourth-order nonlinear difference equation

$$\Delta^4 u(t-2) - \Delta(p(t-1)\Delta u(t-1)) + q(t)u(t) = f(t, u(t)), \quad t \in [1, N]_{\mathbb{N}}$$

under the periodic boundary condition (BC)

$$\Delta^i u(-1) = \Delta^i u(T-1), \quad i = 0, 1, 2, 3$$

where $N \in \mathbb{N}$ and $f \in \mathrm{C}(\in [1, N]_{\mathbb{N}} \times \mathbb{R}, \mathbb{R})$. In [11], using the variational method and the classic mountain-pass lemma of Ambrosetti and Rabinowitz, the existence of at least two nontrivial solutions to a discrete fourth-order boundary value problem was discussed. Applications of the results to a related eigenvalue problem were also presented. In [17], by using a consequence of the local minimum theorem due to Bonanno, the existence of at least one solution under algebraic conditions on the nonlinear terms and two solutions for the discrete nonlinear fourth-order boundary value problem

$$\begin{cases} \Delta^4 u(t-2) + \delta \Delta^2 u(t-1) - \xi u(t) \\ \quad = \lambda f(t, u(t)) + \mu g(t, u(t)) + h(u(t)), \quad t \in [a+1, b+1]_{\mathbb{N}}, \\ u(a) = \Delta^2 u(a-1) = 0,\ u(b+2) = \Delta^2 u(b+1) = 0 \end{cases}$$

where $f, g : [a+1, b+1]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ are two continuous functions and $h : \mathbb{R} \to \mathbb{R}$ is a strictly monotone Lipschitz continuous function, under algebraic conditions with the classical Ambrosetti–Rabinowitz (AR) condition on the nonlinear terms, was discussed. Furthermore, by employing two critical point theorems, one due to Averna and Bonanno, and another one due to Bonanno, the existence of two and three solutions for the above problem in the case $\mu = 0$, were guaranteed. Ousbika and El Allali in [28] based on the critical point theory, proved the existence of three solutions for the discrete nonlinear fourth-order boundary value problems with four parameters

$$\begin{cases} \Delta^4 u(k-2) - \alpha \Delta^2 u(k-1) + \beta u(k) = \lambda f(k, u(k)) + \mu g(k, u(k)), \quad k \in [2, T]_{\mathbb{N}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0. \end{cases}$$

Motivated by the above facts, in the present paper, we study the existence of at least one nontrivial classical solution for (1.1) under an asymptotical behaviour of the nonlinear datum at zero, see Theorem 3.1. In Theorem 3.3, we present an application of Theorem 3.1. We present Example 3.4, in which the hypotheses of Theorem 3.3 are fulfilled. In Theorem 3.5, we present a simple consequence of Theorem 3.3, in which the function f has separated variables. In Theorem 3.6, we offer a consequence of Theorem 3.3 in the case when f does not depend upon k. In Theorem 3.7, we obtain the existence of at least two positive solutions under suitable conditions on the nonlinear term at zero and at infinity, while, finally, in Theorem 3.8, we ensure the existence of at least four nonnegative solutions.

2. Preliminaries

In the present paper, X denotes a finite-dimensional real Banach space and $I_\lambda : X \to \mathbb{R}$ is a functional satisfying the following structure hypothesis:

$I_\lambda(u) := \Phi(u) - \lambda \Psi(u)$ for all $u \in X$, where $\Phi, \Psi : X \to \mathbb{R}$ are two functions of class C^1 on X such that Φ is coercive, i.e., $\lim_{\|u\| \to \infty} \Phi(u) = \infty$, and λ is a positive real parameter.

In this framework, a finite-dimensional variant of [2, Theorem 3.3] (see also [2, Corollary 3.1 and Remark 3.9]) is as follows.

For all r, r_1, r_2 with $r_2 > r_1$ and $r_2 > \inf_X \Phi$, and all $r_3 > 0$, we define

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)},$$
$$\beta(r_1,r_2) := \inf_{u\in\Phi^{-1}(-\infty,r_1)} \sup_{v\in\Phi^{-1}[r_1,r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)},$$
$$\gamma(r_2,r_3) := \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \Psi(u)}{r_3},$$
$$\alpha(r_1,r_2,r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2,r_3)\}.$$

Theorem 2.1 (See [2, Theorem 3.3])**.** *Assume that*

(a_1) Φ *is convex and* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$,

(a_2) *for every* $u_1, u_2 \in X$ *such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$, *one has*

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 *with* $r_1 < r_2$ *such that*

(a_3) $\varphi(r_1) < \beta(r_1,r_2)$,

(a_4) $\varphi(r_2) < \beta(r_1,r_2)$,

(a_5) $\gamma(r_2,r_3) < \beta(r_1,r_2)$.

Then, for each $\lambda \in \left(\frac{1}{\beta(r_1,r_2)}, \frac{1}{\alpha(r_1,r_2,r_3)}\right)$, *the functional* $\Phi - \lambda\Psi$ *admits three distinct critical points* u_1, u_2, u_3 *such that*

$$u_1 \in \Phi^{-1}(-\infty,r_1), \quad u_2 \in \Phi^{-1}[r_1,r_2), \quad u_3 \in \Phi^{-1}(-\infty,r_2+r_3).$$

We refer the interested reader to the papers [5,15,16,18,20], in which Theorem 2.1 has been successfully employed to obtain the existence of at least three solutions for boundary value problems.

We define the real vector space E

$$E = \left\{u : [0,T+2]_{\mathbb{N}} \to \mathbb{R} : \ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0\right\},$$

which is a $(T-1)$-dimensional Hilbert space (see [29]) with the inner product

$$(u,v) = \sum_{k=2}^{T} u(k)v(k).$$

The associated norm is defined by

$$\|u\| = \left(\sum_{k=2}^{T} |u(k)|^2\right)^{\frac{1}{2}}.$$

Lemma 2.2 (See [28, Lemma 2.5])**.** *For any* $u, v \in E$, *we have*

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) = \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2),$$
$$\sum_{k=2}^{T} \Delta u(k-1)\Delta v(k-1) = -\sum_{k=2}^{T} \Delta^2 u(k-1)v(k).$$

Put

$$F(k,t) := \int_0^t f(k,\xi)\mathrm{d}\xi \quad \text{for all} \quad (k,t) \in [2,T]_{\mathbb{N}} \times \mathbb{R}.$$

We consider the functionals Φ, Ψ, I_λ defined by

$$\Phi(u) = \frac{1}{2}\left(\sum_{k=2}^{T+1} |\Delta^2 u(k-1)|^2 + \alpha \sum_{k=2}^{T} |\Delta u(k-1)|^2 + \beta \sum_{k=2}^{T} |u(k)|^2\right), \tag{2.1}$$

$$\Psi(u) = \sum_{k=1}^{T} F(k, u(k)), \tag{2.2}$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for every $u \in E$. Note that the solutions of (1.1) are exactly the critical points of $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$.

Definition 2.3. *We say that $u \in E$ is a solution of $I'_\lambda(u)v = 0$ for all $v \in E$ if, for any $v \in E$, we have*

$$\begin{aligned} I'_\lambda(u)(v) =& \sum_{k=2}^{T} \Delta^4 u(k-2)v(k) - \alpha \sum_{k=2}^{T} \Delta^2 u(k-1)v(k) \\ &+ \beta \sum_{k=2}^{T} u(k)v(k) - \lambda \sum_{k=2}^{T} f(k,u(k))v(k) \\ =& 0. \end{aligned}$$

We say that $u \in E$ is a solution of (1.1) if, for any $v \in E$, we have

$$\sum_{k=2}^{T} \Delta^4 u(k-2)v(k) - \alpha \sum_{k=2}^{T} \Delta^2 u(k-1)v(k) + \beta \sum_{k=2}^{T} u(k)v(k) = \lambda \sum_{k=2}^{T} f(k,u(k))v(k).$$

Lemma 2.4 (See [28, Lemma 2.6])**.** *For any $u \in E$, we have*

$$\Phi(u) \geq \frac{1}{2}\rho\|u\|^2,$$

where

$$\rho = \left(1 + (T-1)T\alpha_- + T(T-1)^3\beta_-\right) T^{-1}(T-1)^3.$$

Proposition 2.5 (See [6])**.** *Let E be a real reflexive Banach space and E^* be the dual space of E. Suppose that $T : E \to E^*$ is a continuous operator and there exists $\omega > 0$ such that*

$$\langle Tv - Tv, u - v\rangle \geq \omega\|u - v\|^2 \quad \text{for all} \quad u, v \in E.$$

Then $T : E \to E^*$ is a homeomorphism between E and E^*.

3. Main Results

For convenience, we use the notation

$$\mathcal{F}(\theta) = \sum_{k=2}^{T} \max_{|t|\leq\theta} F(k,t) \quad \text{for} \quad \theta > 0.$$

We present our main result as follows.

Theorem 3.1. *Assume that there exist $\theta_1, \theta_2, \theta_3, \sigma > 0$ with*

$$\theta_1 < \sigma\sqrt{T-1} \quad \textit{and} \quad \sqrt{\frac{2+\alpha+(T-1)\beta}{\rho}}\sigma < \theta_2 < \theta_3$$

such that

(A_1) $f(k,t) \geq 0$ *for each* $(k,t) \in [2,T]_{\mathbb{N}} \times [-\theta_3, \theta_3]$,

(A_2)

$$\max\left\{\frac{\mathcal{F}(\theta_1)}{\theta_1^2}, \frac{\mathcal{F}(\theta_2)}{\theta_2^2}, \frac{\mathcal{F}(\theta_3)}{\theta_3^2-\theta_2^2}\right\} < \frac{\rho}{2+\alpha+(T-1)\beta} \frac{\sum_{k=2}^{T} F(k,\sigma) - \mathcal{F}(\theta_1)}{\sigma^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\sigma^2}{\sum_{k=2}^{T} F(k,\sigma) - \mathcal{F}(\theta_1)}, \frac{\rho}{2}\min\left\{\frac{\theta_1^2}{\mathcal{F}(\theta_1)}, \frac{\theta_2^2}{\mathcal{F}(\theta_2)}, \frac{\theta_3^2-\theta_2^2}{\mathcal{F}(\theta_3)}\right\}\right),$$

(1.1) *possesses at least three nonnegative solutions u_1, u_2, u_3 such that*

$$\max_{k\in[2,T]_{\mathbb{N}}} |u_1(k)| < \theta_1, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_2(k)| < \theta_2, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_3(k)| < \theta_3.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. We consider the auxiliary problem

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda \hat{f}(k, u(k)), & k \in [2,T]_{\mathbb{N}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0, \end{cases} \tag{3.1}$$

where $\hat{f} : [2,T]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ is a continuous functions defined by

$$\hat{f}(k,\xi) = \begin{cases} f(k,-\theta_3) & \text{if} \quad \xi < -\theta_3, \\ f(k,\xi) & \text{if} \quad -\theta_3 \leq \xi \leq \theta_3, \\ f(k,\theta_3) & \text{if} \quad \xi > \theta_3. \end{cases}$$

If any solution of (1.1) satisfies the condition

$$-\theta_3 \leq u(k) \leq \theta_3 \quad \text{for every} \quad k \in [2,T]_{\mathbb{N}},$$

then any solution of (3.1) clearly turns to be also a solution of (1.1). Therefore, for our goal, it is enough to show that our conclusion holds for (1.1). Fix λ as in the conclusion. In order to apply Theorem 2.1 to our problem, we consider Φ, Ψ as given in (2.1) and (2.2), respectively. Note that the solutions of (1.1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. By Lemma 2.4, we see that Φ is coercive. On the other hand, Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous, and its Gâteaux derivative is the functional $\Phi'(u) \in E^*$ given by

$$\begin{aligned} \Phi'(u)(v) &= \sum_{k=2}^{T+1} \Delta^2 u(k-2)\Delta^2 v(k-2) + \alpha\sum_{k=2}^{T} \Delta u(k-1)\Delta v(k-1) \\ &\quad +\beta\sum_{k=2}^{T} u(k)v(k) \end{aligned}$$

for every $u, v \in E$. We prove that Φ' admits a continuous inverse on E^*. We have

$$\langle \Phi'(u), u \rangle = 2\Phi(u) \quad \text{for every} \quad u \in E.$$

Then

$$\langle \Phi'(u) - \Phi'(v), u - v\rangle = 2\Phi(u - v) \geq \rho\|u - v\|^2 \quad \text{for each} \quad u, v \in E.$$

Hence, by Proposition 2.5, $(\Phi')^{-1} : E^* \to E$ exists and is continuous. Secondly, we show that Ψ is compact. Suppose that $u_n \to u \in E$. Then, since f is continuous and from (2.2), we deduce that $\Psi'(u_n) \to \Psi(u_n)$, and thus Ψ' is compact. It is well known that Ψ is a differentiable functional whose differential at the point $u \in E$ is

$$\Psi'(u)(v) = \sum_{k=2}^{T} f(k, u(k))v(k) \quad \text{for any} \quad v \in E$$

as well as it is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : E \to E^*$ is a compact operator. Now put

$$r_1 := \frac{\rho}{2}\theta_1^2, \quad r_2 := \frac{\rho}{2}\theta_2^2, \quad \text{and} \quad r_3 := \frac{\rho}{2}(\theta_3^2 - \theta_2^2).$$

We define $w_\sigma(k) = \sigma$ for every $k \in [2, T]_{\mathbb{N}}$. Clearly, $w_\sigma \in E$. It is easy to verify that

$$\Phi(w_\sigma) = \frac{1}{2}\left(2 + \alpha + (T - 1)\beta\right)\sigma^2.$$

On the other hand, we have

$$\Psi(w_\sigma) = \sum_{k=2}^{T} F(k, \sigma).$$

From the conditions

$$\theta_3 > \theta_2, \quad \theta_1 < \sigma\sqrt{T-1}, \quad \text{and} \quad \sqrt{\frac{2 + \alpha + (T-1)\beta}{\rho}}\sigma < \theta_2,$$

we get $r_3 > 0$ and $r_1 < \Phi(w) < r_2$. Taking into account the fact that

$$|u(k)| \leq \|u\| \leq \sqrt{\frac{2\Phi(u)}{\rho}} \quad \text{for any} \quad k \in [2, T]_{\mathbb{N}},$$

we get from the definition of r_1 that

$$\Phi^{-1}(-\infty, r_1) = \{u \in E : \ \Phi(u) < r_1\} \subseteq \{u \in E : \ |u| \leq \theta_1\},$$

and by the same argument as above,

$$\Phi^{-1}(-\infty, r_2) \subseteq \{u \in E : \ |u| \leq \theta_2\}.$$

Hence, we obtain

$$\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u) \leq \sum_{k=2}^{T} \max_{|t| \leq \theta_1} F(k, t) = \mathcal{F}(\theta_1).$$

In a similar way, we have

$$\sup_{u \in \Phi^{-1}(-\infty, r_2)} \Psi(u) \leq \mathcal{F}(\theta_2)$$

and

$$\sup_{\Phi(u) < r_2 + r_3} \Psi(u) \leq \mathcal{F}(\theta_3).$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_1)$ and $\Phi(0) = \Psi(0) = 0$, we obtain

$$\begin{aligned}
\varphi(r_1) &= \inf_{u \in \Phi^{-1}(-\infty, r_1)} \frac{\left(\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)\right) - \Psi(u)}{r_1 - \Phi(u)} \\
&\leq \frac{\sup_{u \in \Phi^{-1}(-\infty, r_1)} \Psi(u)}{r_1} \leq \frac{\mathcal{F}(\theta_1)}{r_1} = \frac{2}{\rho}\frac{\mathcal{F}(\theta_1)}{\theta_1^2},
\end{aligned}$$

$$\varphi(r_2) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} \le \frac{\mathcal{F}(\theta_2)}{r_2} = \frac{2}{\rho}\frac{\mathcal{F}(\theta_2)}{\theta_2^2},$$

and

$$\theta(r_2,r_3) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3} \le \frac{\mathcal{F}(\theta_3)}{r_3} = \frac{2}{\rho}\frac{\mathcal{F}(\theta_3)}{\theta_3^2-\theta_2^2}.$$

For each $u \in \Phi^{-1}(-\infty, r_1)$, one has

$$\beta(r_1,r_2) \ge \frac{\sum_{k=2}^{T} F(k,\sigma) - \mathcal{F}(\theta_1)}{\frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\sigma^2}.$$

Due to (A_2), we get

$$\alpha(r_1,r_2,r_3) < \beta(r_1,r_2).$$

Now we show that the functional I_λ satisfies the assumption (a_2) of Theorem 2.1. Let u_1 and u_2 be two local minima of I_λ. Then u_1 and u_2 are critical points of I_λ, and so, they are solutions of (1.1). We want to prove that they are nonnegative. Let u_* be a (nontrivial) solution of (1.1). Arguing by a contradiction, assume that

$$\mathcal{A} := \{k \in [2,T]_{\mathbb{N}} :\ u_*(k) < 0\} \neq \emptyset.$$

Put $\bar{v}(k) = \min\{u_*(k), 0\}$ for $k \in [2,T]_{\mathbb{N}}$. Clearly, $\bar{v} \in E$, and one has

$$\sum_{k=2}^{T+1} \Delta^2 u_*(k-2)\Delta^2\bar{v}(k-2) + \alpha\sum_{k=2}^{T}\Delta u_*(k-1)\Delta\bar{v}(k-1) \\ + \beta\sum_{k=2}^{T} u_*(k)\bar{v}(k) - \lambda\sum_{k=2}^{T} f(k,u_*(k))\bar{v}(k) = 0.$$

By choosing $\bar{v} = u_*$ and since f is nonnegative, we have

$$0 \le \rho\|u_*\|_{\mathcal{A}}^2 \le \sum_{k\in\mathcal{A}} |\Delta^2 u_*(k-2)|^2 + \alpha\sum_{k\in\mathcal{A}}|\Delta u_*(k-1)|^2 + \beta\sum_{k\in\mathcal{A}}|u_*(k)|^2 \le 0,$$

and thus

$$\|u_*\|_{\mathcal{A}}^2 = 0,$$

which contradicts the fact that u_* is a nontrivial solution. Hence, u_* is positive. Thus, our claim is proved. Then, we observe $u_1(k) \ge 0$ and $u_2(k) \ge 0$ for every $k \in [2,T]_{\mathbb{N}}$. Thus, it follows that

$$\lambda f(k, su_1 + (1-s)u_2) \ge 0 \quad \text{for all} \quad k \in [2,T]_{\mathbb{N}} \quad \text{and all} \quad s \in [0,1],$$

and, consequently,

$$\Psi(su_1 + (1-s)u_2) \ge 0 \quad \text{for every} \quad s \in [0,1].$$

Hence, Theorem 2.1 implies that for every

$$\lambda \in \left(\frac{\frac{1}{2}\left(2+\alpha+(T-1)\beta\right)\sigma^2}{\sum_{k=2}^{T} F(k,\sigma) - \mathcal{F}(\theta_1)}, \frac{\rho}{2}\min\left\{\frac{\theta_1^2}{\mathcal{F}(\theta_1)}, \frac{\theta_2^2}{\mathcal{F}(\theta_2)}, \frac{\theta_3^2-\theta_2^2}{\mathcal{F}(\theta_3)}\right\}\right),$$

the functional I_λ has three critical points $u_1, u_2, u_3 \in E$, such that $\Phi(u_1) < r_1$, $\Phi(u_2) < r_2$, and $\Phi(u_3) < r_2 + r_3$, that is,

$$\max_{k\in[2,T]_{\mathbb{N}}} |u_1(k)| < \theta_1, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_2(k)| < \theta_2, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_3(k)| < \theta_3.$$

Then, taking into account the fact that the solutions of (1.1) are exactly the critical points of the functional I_λ, we have the desired conclusion. □

Remark 3.2. *We observe that, in Theorem 3.1, no asymptotic conditions on f are needed, and only algebraic conditions on f are imposed to guarantee the existence of the solutions.*

Now, we deduce the following straightforward consequence of Theorem 3.1.

Theorem 3.3. *Assume that there exist $\theta_1, \theta_4, \sigma > 0$ with*

$$\theta_1 < \min\left\{\sigma, \sigma\sqrt{T-1}\right\} \quad \textit{and} \quad \max\left\{\sigma, \sqrt{\frac{2+\alpha+(T-1)\beta}{\rho}}\sigma\right\} < \theta_4$$

such that

(A_3) $f(k,t) \geq 0$ *for each* $(k,t) \in [2,T]_{\mathbb{N}} \times [-\theta_4, \theta_4]$,

(A_4)

$$\max\left\{\frac{\mathcal{F}(\theta_1)}{\theta_1^2}, \frac{2\mathcal{F}(\theta_4)}{\theta_4^2}\right\} < \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{1}{2}\left(\rho+2+\alpha+(T-1)\beta\right)\sigma^2}{\sum_{k=2}^T F(k,\sigma)}, \frac{\rho}{2}\min\left\{\frac{\theta_1^2}{\mathcal{F}(\theta_1)}, \frac{\theta_4^2}{2\mathcal{F}(\theta_4)}\right\}\right),$$

(1.1) *possesses at least three nonnegative solutions u_1, u_2, u_3 such that*

$$\max_{k\in[2,T]_{\mathbb{N}}} |u_1(k)| < \theta_1, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_2(k)| < \frac{\theta_4}{\sqrt{2}}, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_3(k)| < \theta_4.$$

Proof. Choose $\theta_2 = \theta_4/\sqrt{2}$ and $\theta_3 = \theta_4$. So, from (A_4), one has

$$\begin{aligned}\frac{\mathcal{F}(\theta_2)}{\theta_2^2} &= \frac{2\mathcal{F}(\theta_4/\sqrt{2})}{\theta_4^2} \leq \frac{2\mathcal{F}(\theta_4)}{\theta_4^2} \\ &< \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2}\end{aligned} \tag{3.2}$$

and

$$\frac{\mathcal{F}(\theta_3)}{\theta_3^2-\theta_2^2} = \frac{2\mathcal{F}(\theta_4)}{\theta_4^2} < \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2}. \tag{3.3}$$

Moreover, taking into account that $\theta_1 < \sigma$, by using (A_4), we have

$$\begin{aligned}&\frac{\rho}{2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma) - \mathcal{F}(\theta_1)}{\sigma^2} \\ &> \frac{\rho}{2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2} - \frac{\rho}{2+\alpha+(T-1)\beta}\frac{\mathcal{F}(\theta_1)}{\theta_1^2} \\ &> \frac{\rho}{2+\alpha+(T-1)\beta}\left(\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2} - \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2}\right) \\ &= \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{\sum_{k=2}^T F(k,\sigma)}{\sigma^2}.\end{aligned}$$

Hence, from (A_4), (3.2), and (3.3), it is easy to see that the assumption (A_2) of Theorem 3.1 is satisfied, and thus follows the conclusion. □

We now present the following example in order to illustrate Theorem 3.3.

Example 3.4. *Let $T = 4$. We consider the problem*

$$\begin{cases} \Delta^4 u(k-2) - \Delta^2 u(k-1) + u(k) = \lambda f(u), \quad k \in [2,4]_{\mathbb{N}}, \\ u(1) = \Delta u(0) = \Delta u(4) = \Delta^3 u(0) = \Delta^3 u(3) = 0, \end{cases} \tag{3.4}$$

where

$$f(\xi) = \begin{cases} 4\xi^3 & \text{if} \quad \xi \le 1, \\ 4\xi + \dfrac{\cos(\ln(\xi))}{\xi} & \text{if} \quad \xi > 1. \end{cases}$$

By the expression of f, we have

$$F(\xi) = \begin{cases} \xi^4 & \text{if} \quad \xi \le 1, \\ 2\xi^2 + \sin(\ln(\xi)) - 1 & \text{if} \quad \xi > 1. \end{cases}$$

By simple calculations, we obtain $\rho = \frac{64}{5}$. Taking $\theta_1 = \frac{1}{10}$, $\theta_4 = 10^4$, and $\sigma = 1$, all conditions in Theorem 3.3 are satisfied. Therefore, it follows that for each

$$\lambda \in (47, 6400),$$

(3.4) *possesses at least three nonnegative solutions u_1, u_2, u_3 such that*

$$\max_{k\in[2,4]_{\mathbb{N}}} |u_1(k)| < \frac{1}{10}, \quad \max_{k\in[2,4]_{\mathbb{N}}} |u_2(k)| < \frac{10^4}{\sqrt{2}}, \quad \max_{k\in[2,4]_{\mathbb{N}}} |u_3(k)| < 10^4.$$

We next point out a simple consequence of Theorem 3.3, in which the function f has separated variables.

Theorem 3.5. *Let $f_1 : [2,T]_{\mathbb{N}} \to \mathbb{R}$ and $f_2 \in \mathrm{C}(\mathbb{R},\mathbb{R})$ be two functions. Put*

$$\tilde{F}(t) = \int_0^t f_2(\xi)\mathrm{d}\xi \quad \text{for all} \quad t \in \mathbb{R}$$

and assume that there exist $\theta_1, \theta_4, \sigma > 0$ with

$$\theta_1 < \min\left\{\sigma, \sigma\sqrt{T-1}\right\} \quad \text{and} \quad \max\left\{\sigma, \sqrt{\frac{2+\alpha+(T-1)\beta}{\rho}}\sigma\right\} < \theta_4$$

such that

(A_5) $f_1(k) \ge 0$ *for each* $k \in [2,T]_{\mathbb{N}}$ *and* $f_2(\xi) \ge 0$ *for each* $\xi \in [-\theta_4, \theta_4]$,

(A_6)

$$\max\left\{\frac{\max_{|t|\le\theta_1} \tilde{F}(t)}{\theta_1^2}, \frac{2\max_{|t|\le\theta_4} \tilde{F}(t)}{\theta_4^2}\right\} < \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{\tilde{F}(\sigma)}{\sigma^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{1}{2}(\rho+2+\alpha+(T-1)\beta)\sigma^2}{\tilde{F}(\sigma)\sum_{k=2}^T f_1(k)}, \right.$$

$$\left. \frac{\rho}{2\sum_{k=2}^T f_1(k)} \min\left\{\frac{\theta_1^2}{\max_{|t|\le\theta_1} \tilde{F}(t)}, \frac{\theta_4^2}{2\max_{|t|\le\theta_4} \tilde{F}(t)}\right\}\right),$$

the problem

$$\begin{cases} \Delta^4 u(k-1) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f_1(k) f_2(u(k)), \quad k \in [2,T]_{\mathbb{N}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0 \end{cases}$$

possesses at least three nonnegative solutions u_1, u_2, u_3 such that

$$\max_{k\in[2,T]_{\mathbb{N}}} |u_1(k)| < \theta_1, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_2(k)| < \frac{1}{\sqrt{2}}\theta_4 \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_3(k)| < \theta_4.$$

Proof. Set $f(k,u) = f_1(k)f_2(u)$ for each $(k,u) \in [2,T]_{\mathbb{N}} \times \mathbb{R}$. Since

$$F(k,\xi) = f_1(k)\tilde{F}(\xi),$$

from (A_5) and (A_6), we obtain (A_3) and (A_4), respectively. □

Now, we present a simple consequence of Theorem 3.3 in the case when f does not depend upon k.

Theorem 3.6. *Assume that there exist $\theta_1, \theta_4, \sigma > 0$ with*

$$\theta_1 < \min\left\{\sigma, \sigma\sqrt{T-1}\right\} \quad \textit{and} \quad \max\left\{\sigma, \sqrt{\frac{2+\alpha+(T-1)\beta}{\rho}}\sigma\right\} < \theta_4$$

such that

(A_7) $f(\xi) \geq 0$ for each $\xi \in [-\theta_4, \theta_4]$,

(A_8)

$$\max\left\{\frac{\max_{|t|\leq\theta_1} F(t)}{\theta_1^2}, \frac{2\max_{|t|\leq\theta_4} F(t)}{\theta_4^2}\right\} < \frac{\rho}{\rho+2+\alpha+(T-1)\beta}\frac{F(\sigma)}{\sigma^2}.$$

Then, for every

$$\lambda \in \left(\frac{\frac{1}{2}(\rho+2+\alpha+(T-1)\beta)\sigma^2}{(T-1)F(\sigma)}, \right.$$
$$\left. \frac{\rho}{2(T-1)} \min\left\{\frac{\theta_1^2}{\max_{|t|\leq\theta_1} F(t)}, \frac{\theta_4^2}{2\max_{|t|\leq\theta_4} F(t)}\right\}\right),$$

the problem

$$\begin{cases} \Delta^4 u(k-2) - \alpha\Delta^2 u(k-1) + \beta u(k) = \lambda f(u(k)), & k \in [2,T]_{\mathbb{N}}, \\ u(1) = \Delta u(0) = \Delta u(T) = \Delta^3 u(0) = \Delta^3 u(T-1) = 0 \end{cases}$$

possesses at least three nonnegative solutions u_1, u_2, u_3 such that

$$\max_{k\in[2,T]_{\mathbb{N}}} |u_1(k)| < \theta_1, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_2(k)| < \frac{1}{\sqrt{2}}\theta_4, \quad \max_{k\in[2,T]_{\mathbb{N}}} |u_3(k)| < \theta_4.$$

The following result is a consequence of Theorem 3.3.

Theorem 3.7. *Let $f : [2,T]_{\mathbb{N}} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\xi f(k,\xi) > 0 \quad \textit{for all} \quad (k,\xi) \in [2,T]_{\mathbb{N}} \times (\mathbb{R} \setminus \{0\}).$$

Assume that

(A_9) $\lim_{\xi\to 0} \frac{f(k,\xi)}{|\xi|} = \lim_{|\xi|\to+\infty} \frac{f(k,\xi)}{|\xi|} = 0.$

Then, for every

$$\lambda > \overline{\lambda} := \frac{\rho+2+\alpha+(T-1)\beta}{2} \max\left\{\inf_{\sigma>0} \frac{\sigma^2}{\sum_{k=2}^T F_1(k,\sigma)}, \inf_{\sigma<0} \frac{(-\sigma)^2}{\sum_{k=2}^T F(k,\sigma)}\right\},$$

(1.1) *possesses at least four distinct nontrivial solutions.*

Proof. Set

$$f_1(k,\xi) = \begin{cases} f(k,\xi) & \text{if} \quad (k,\xi) \in [2,T]_{\mathbb{N}} \times [0,+\infty), \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_2(k,\xi) = \begin{cases} -f(k,-\xi) & \text{if} \quad (k,\xi) \in [2,T]_{\mathbb{N}} \times [0,+\infty), \\ 0 & \text{otherwise} \quad , \end{cases}$$

and define

$$F_1(k,\xi) := \int_0^\xi f_1(k,x)\mathrm{d}x \quad \text{for every} \quad (k,\xi) \in [2,T]_{\mathbb{N}} \times \mathbb{R}.$$

Fix $\lambda > \lambda^*$ and let $\sigma > 0$ be such that

$$\lambda > \frac{\frac{1}{2}(\rho + 2 + \alpha + (T-1)\beta)\sigma^2}{\sum_{k=2}^{T} F_1(k,\sigma)}.$$

From

$$\lim_{\xi \to 0} \frac{f_1(k,\xi)}{|\xi|} = \lim_{|\xi| \to +\infty} \frac{f_1(k,\xi)}{|\xi|} = 0,$$

there exists $\theta_1 > 0$ such that

$$\theta_1 < \min\left\{\sigma, \sigma\sqrt{T-1}\right\} \quad \text{and} \quad \frac{\sum_{k=2}^{T} \max_{|t| \le \theta_1} F_1(k,t)}{\theta_1^2} < \frac{\rho}{2\lambda},$$

and there exists $\theta_4 > 0$ such that

$$\max\left\{\sigma, \sqrt{\frac{2+\alpha+(T-1)\beta}{\rho}}\sigma\right\} < \theta_4 \quad \text{and} \quad \frac{\sum_{k=2}^{T} \max_{|t| \le \theta_4} F_1(k,t)}{\theta_4^2} < \frac{\rho}{4\lambda}.$$

Then, (A_4) in Theorem 3.3 is satisfied, and

$$\lambda \in \left(\frac{\frac{1}{2}(\rho + 2 + \alpha + (T-1)\beta)\sigma^2}{\sum_{k=2}^{T} F_1(k,\sigma)}, \right.$$
$$\left. \frac{\rho}{2} \min\left\{ \frac{\theta_1^2}{\sum_{k=2}^{T} \max_{|t| \le \theta_1} F_1(k,t)}, \frac{\theta_4^2}{2\sum_{k=2}^{T} \max_{|t| \le \theta_4} F_1(k,t)} \right\} \right).$$

Hence, the problem $(P_\lambda^{f_1})$ admits two positive solutions u_1 and u_2, which are positive solutions of (1.1). Next, arguing in the same way, from

$$\lim_{\xi \to 0} \frac{f_2(k,\xi)}{|\xi|} = \lim_{|\xi| \to +\infty} \frac{f_2(k,\xi)}{|\xi|} = 0,$$

we ensure the existence of two positive solutions u_3 and u_4 for the problem $(P_\lambda^{f_2})$. Clearly, $-u_3$ and $-u_4$ are negative solutions of (1.1), and the conclusion is achieved. □

Remark 3.8. *We explicitly observe that in Theorem 3.7, no symmetric condition on f is assumed. However, whenever f is an odd continuous nonzero function such that*

$$f(k,\xi) \ge 0 \quad \textit{for all} \quad (k,\xi) \in [2,T]_{\mathbb{N}} \times [0,+\infty),$$

(A_9) can be replaced by

(A_{10}) $\lim\limits_{\xi \to 0^+} \frac{f(k,\xi)}{\xi} = \lim\limits_{\xi \to +\infty} \frac{f(k,\xi)}{\xi} = 0,$

ensuring the existence of at least four distinct nontrivial solutions of (1.1) for every

$$\lambda > \lambda^* := \inf_{\sigma > 0} \frac{\frac{1}{2}(\rho + 2 + \alpha + (T-1)\beta)\sigma^2}{\sum_{k=2}^{T} F(k,\sigma)}.$$

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