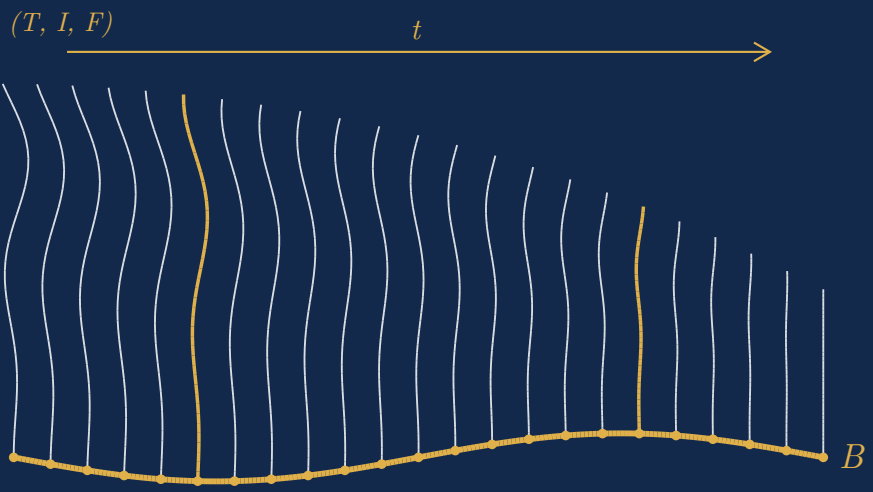
Fibrewise Neutrosophic Homotopy Theory

Foundations, Level Topologies, Multivalued and Ideal Methods



Saeid Jafari
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Abstract

This monograph develops fibrewise homotopy theory for spaces whose open sets are neutrosophic triples (T, I, F) of truth, indeterminacy and falsity degrees, over a fixed classical base B. The guiding equation is $p_Y H = p_X \mathrm{pr}_X$: a homotopy deforms a map while every point stays in its fibre. After fixing the lattice of triples, neutrosophic topologies and the crisp embedding of classical topology, we define fibrewise N-homotopies through a product-generated cylinder. We prove that homotopy is an equivalence relation under an explicit pasting convention and that homotopy equivalences satisfy two-out-of-three. We then develop separation conditions, pointed and relative homotopy, neutrosophic fibrations and cofibrations, mapping spaces, Lusternik–Schnirelmann category, local triviality and base change, path objects, mapping cylinders and homotopy pullbacks. Every hypothesis on products, pastings, pullbacks and exponentials is stated explicitly.

Two enlargements are treated in parallel: multivalued (upper semicontinuous) correspondences and their homotopies, and ideal modifications $\tau \mapsto \tau^*$ of topologies with their fibrewise traces. Explicit examples show that the three theories do not commute. A new final chapter attaches to each neutrosophic space three families of classical *level topologies*, one for each coordinate. It proves that fibrewise N-homotopies transfer to every level, so that the level profiles are invariants of the neutrosophic homotopy type. It also constructs bitopological neutrosophic spaces in which the indeterminacy coordinate alone obstructs a contraction. A conclusion lists open problems.

Keywords: fibrewise topology; fibrewise homotopy; neutrosophic topology; neutrosophic set; level topology; Hurewicz fibration; cofibration; upper semicontinuous correspondence; ideal topological space;

bitopological space.

MSC 2020: 55R70, 55P05, 55R05, 54A40, 03E72, 54C60, 54A10, 55M30.

Preface

This monograph gives a coherent account of fibrewise neutrosophic homotopy theory. It grew out of the authors' manuscript *Fibrewise Neutrosophic Homotopy Theory: Homotopies, Equivalences, and Fibrewise Topological Structure*, which has been rewritten here as a sequence of chapters. Notation has been corrected, hypotheses made explicit, and developments on multivalued and ideal fibrewise methods added. The book has a single chapter numbering and a single bibliography.

Throughout, the central equation is

$$p_Y H = p_X \mathrm{pr}_X,$$

which expresses deformation over a fixed base. We use one fixed lattice convention for neutrosophic membership triples, and we indicate how results transfer to the other standard convention (Remark 1.5); we do not claim validity for every variant of neutrosophic topology in the literature. The foundational literature includes James [9, 10], Hatcher [6], Smarandache [19, 20], Salama–Alblowi [17] and Jafari–Rajesh–Nordo [8].

The contributions are precise definitions, complete proofs, worked examples, and careful comparisons of different admissibility structures. Conditional statements keep their hypotheses. Chapters 1–10 develop the single-valued theory. Chapters 11–16 treat ideal and multivalued methods and compare the three theories. Chapters 17–20 collect examples, constructions and conventions. Chapter 21 isolates the phenomena that are genuinely neutrosophic, that is, invisible to crisp encodings. The Conclusion summarizes the results and lists open problems.

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Chapter 1

Classical Fibrewise Spaces and Neutrosophic Foundations

1.1 The topology of a projection

Fibrewise topology studies a space X together with a fixed continuous projection $p_X\colon X \to B$. For $b \in B$ the *fibre* is $X_b = p_X^{-1}(\{b\})$. Fibres are not required to be nonempty; this convention is classical [9, 10]. A map $f\colon X \to Y$ *over* B is a continuous map satisfying $p_Y f = p_X$. It induces restrictions $f_b\colon X_b \to Y_b$, each of which is continuous. The converse fails: continuity on each fibre does not in general give continuity on the total space.

Example 1.1 (A discontinuous assembly)**.** Take $B = [0,1]$ and $X = Y = B \times \mathbb{R}$ with the product topology. For $b > 0$ let $f_b(x) = x/b$, and put $f_0(x) = 0$. Every f_b is continuous, but $f(b,x) = (b, f_b(x))$ is not continuous at $(0,0)$: along $(b,x) = (1/n, 1/n)$ the second coordinate equals 1, whereas $f(0,0)$ has second coordinate 0. A fibrewise map must therefore be jointly continuous, not merely continuous on each fibre.

1.2 Membership triples as a complete lattice

For a set X, a *neutrosophic subset* is a triple $A = (T_A, I_A, F_A)$ of functions $X \to [0,1]$, in the single-valued sense of [19, 20, 22]. No sum constraint is imposed; in particular $0 \le T_A + I_A + F_A \le 3$. Throughout this book triples are ordered by

$$A \preceq C \iff T_A \le T_C, \quad I_A \ge I_C, \quad F_A \ge F_C.$$

Arbitrary joins and meets exist and are computed pointwise:

$$\bigvee_{\lambda} A_\lambda = \Big(\sup_\lambda T_{A_\lambda},\ \inf_\lambda I_{A_\lambda},\ \inf_\lambda F_{A_\lambda}\Big),$$
$$\bigwedge_{\lambda} A_\lambda = \Big(\inf_\lambda T_{A_\lambda},\ \sup_\lambda I_{A_\lambda},\ \sup_\lambda F_{A_\lambda}\Big).$$

The least and greatest elements are $\mathbf{0} = (0, 1, 1)$ and $\mathbf{1} = (1, 0, 0)$. This is one of the two standard inclusions of neutrosophic set theory [19, 17]; the other one orders the indeterminacy coordinate like the truth coordinate (Remark 1.5).

Proposition 1.2 (Distributive coordinate laws)**.** *For every triple A and every family (C_λ) on X,*

$$A \wedge \bigvee_{\lambda} C_\lambda = \bigvee_{\lambda} (A \wedge C_\lambda).$$

Proof. For the truth coordinate this is the identity $\min(a, \sup_\lambda c_\lambda) = \sup_\lambda \min(a, c_\lambda)$ in $[0, 1]$. For the indeterminacy and falsity coordinates it is the dual identity $\max(a, \inf_\lambda c_\lambda) = \inf_\lambda \max(a, c_\lambda)$. All equalities hold pointwise. □

Remark 1.3. Coordinatewise, the lattice of triples is L^X with $L = [0, 1] \times [0, 1]^{\mathrm{op}} \times [0, 1]^{\mathrm{op}}$. Neutrosophic topologies in the sense of the next section are therefore L-valued topologies in the sense of Goguen and Chang [5, 2] for this particular lattice. The specifically neutrosophic content lies in the interpretation of the three coordinates and in constructions that treat them differently; see Chapter 21.

1.3 Neutrosophic topologies and preimages

A *neutrosophic topology* $\mathcal{N}_X$ on X is a collection of triples containing $\mathbf{0}$ and $\mathbf{1}$ and closed under finite meets and arbitrary joins [17]. For a function $f\colon X \to Y$ and a triple V on Y put $f^{-1}(V) = (T_V \circ f,\ I_V \circ f,\ F_V \circ f)$. The function f is *N-continuous* when $f^{-1}(V) \in \mathcal{N}_X$ for all $V \in \mathcal{N}_Y$. Inverse images preserve all joins and all meets, because composition with f commutes with pointwise suprema and infima.

Theorem 1.4 (Composition)**.** *If $f\colon X \to Y$ and $g\colon Y \to Z$ are N-continuous, then $gf\colon X \to Z$ is N-continuous.*

Proof. For every $W \in \mathcal{N}_Z$ one has $(gf)^{-1}W = f^{-1}(g^{-1}W)$. The inner preimage belongs to $\mathcal{N}_Y$ and the outer one to $\mathcal{N}_X$. □

1.4 The crisp embedding and the choice of order

For a classical subset $U \subseteq X$, the characteristic triple adapted to the order of Section 1.2 is

$$\chi_U^N = (\chi_U,\, 1 - \chi_U,\, 1 - \chi_U).$$

In particular $\chi_\emptyset^N = \mathbf{0}$ and $\chi_X^N = \mathbf{1}$. Setting the indeterminacy coordinate to zero for every crisp subset, as in one formula of the source manuscript, would send the empty set to $(0, 0, 1)$, which is not the bottom element for the order fixed in Section 1.2. We therefore use χ_U^N throughout the book.

Remark 1.5 (The other standard order)**.** The choice above is convention-dependent, not a correction of an error. Under the other standard neutrosophic inclusion ($T_A \le T_C$, $I_A \le I_C$, $F_A \ge F_C$, see [19] and type 2 of [17]) the bottom element is $(0, 0, 1)$. The family $\{(\chi_U, 0, 1 - \chi_U) : U \in \tau\}$ is then closed under joins and finite meets, so it is a neutrosophic topology in the sense of [17], where $(1, 0, 0)$ is one of the admitted choices of 1_N. All results of this book can be transported to that convention by reversing the order of the second coordinate.

Theorem 1.6 (Crisp embedding theorem)**.** *For every ordinary topology τ on X, the family $\mathcal{N}_X^{\mathrm{cr}} = \{\chi_U^N : U \in \tau\}$ is a neutrosophic topology. Moreover, for classical topological spaces (X, τ_X) and (Y, τ_Y), a function $f\colon X \to Y$ is continuous if and only if it is N-continuous between these crisp encodings.*

Proof. The characteristic triples send arbitrary unions to joins, finite intersections to meets, the empty set to $\mathbf{0}$ and the full set to $\mathbf{1}$.

Furthermore $f^{-1}\chi^N_U = \chi^N_{f^{-1}U}$. Since $U \mapsto \chi^N_U$ is injective, $f^{-1}\chi^N_V \in \mathcal{N}^{\mathrm{cr}}_X$ holds exactly when $f^{-1}V \in \tau_X$. Thus the open-set preimage criterion is identical in both categories. □

1.5 Continuity over an ordinary base

Definition 1.7. Let (B, τ_B) be a classical space. A *fibrewise neutrosophic space* over B is a triple $(X, p_X, \mathcal{N}_X)$ such that $\chi^N_{p_X^{-1}(U)} \in \mathcal{N}_X$ for every $U \in \tau_B$. Morphisms are N-continuous maps over B.

The condition guarantees that the projection is compatible with the crisp open subsets of the base. Unlike an assignment of triples to the fibres individually, this definition keeps the global topology.

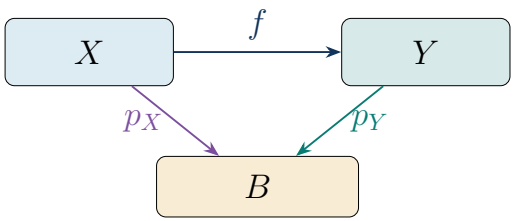


Figure 1.1: A fibrewise morphism satisfies $p_Y f = p_X$.

Chapter 2

Homotopies over a Base

2.1 The interval and its product topology

Let $I = [0,1]$ with its classical topology, encoded by the crisp neutrosophic topology of Theorem 1.6. The *neutrosophic product* on $X \times I$ is the neutrosophic topology generated by the rectangles

$$A \boxtimes \chi_U^N = \mathrm{pr}_X^{-1}(A) \wedge \mathrm{pr}_I^{-1}(\chi_U^N), \qquad A \in \mathcal{N}_X,\ U \subseteq I \text{ open}.$$

By Proposition 1.2 its members are $\mathbf{0}$, $\mathbf{1}$ and the joins of finite meets of rectangles. The parameter is topological; it is not a fourth membership coordinate. A *fibrewise N-homotopy* from f_0 to f_1 is an N-continuous map $H\colon X \times I \to Y$ such that

$$H(x,0) = f_0(x), \qquad H(x,1) = f_1(x), \qquad p_Y H(x,t) = p_X(x).$$

We write $f_0 \simeq_B^N f_1$.

Proposition 2.1 (Endpoint and fibre restrictions)**.** *Both endpoints of a fibrewise N-homotopy are N-continuous fibrewise maps. For every $b \in B$, the restriction $H_b\colon X_b \times I \to Y_b$ is an N-homotopy, each fibre carrying the induced subspace structure.*

Proof. The endpoint inclusions $i_j(x) = (x,j)$, $j = 0,1$, are N-continuous for the product structure: $i_j^{-1}(A \boxtimes \chi_U^N)$ equals A if $j \in U$ and $\mathbf{0}$ otherwise. Hence $f_j = H \circ i_j$ is N-continuous, and the projection equation gives $p_Y f_j = p_X$. If $x \in X_b$, then $p_Y H(x,t) = b$, so the restricted map lands in Y_b. Its continuity follows from the universal property of the induced (subspace) structure. $\square$

2.2 Reversal, concatenation and equivalence

For two N-homotopies $H\colon f \simeq_B^N g$ and $K\colon g \simeq_B^N h$ define

$$(H*K)(x,t) = \begin{cases} H(x,2t), & 0 \le t \le \frac{1}{2}, \\ K(x,2t-1), & \frac{1}{2} \le t \le 1, \end{cases} \qquad \overline{H}(x,t) = H(x,1-t).$$

For the product structure, ordinary reparameterizations of the interval are N-continuous, because they are continuous for the crisp interval. The concatenation $H * K$ is well defined since $H(x,1) = g(x) = K(x,0)$. Its N-continuity requires a pasting lemma for the two closed halves $X \times [0, \frac{1}{2}]$ and $X \times [\frac{1}{2}, 1]$.

Convention 2.2 (Pasting)**.** Whenever concatenation is used, we assume that the chosen neutrosophic structure satisfies the closed pasting lemma for these two halves. In the crisp model this is the classical pasting lemma.

Theorem 2.3 (Equivalence relation)**.** *Under Convention 2.2, $\simeq_B^N$ is an equivalence relation on the set of fibrewise N-maps $X \to Y$.*

Proof. The constant homotopy $H(x,t) = f(x)$ proves reflexivity, reversal proves symmetry and concatenation proves transitivity. Every construction preserves $p_Y H = p_X \mathrm{pr}_X$. □

2.3 A strong deformation retract over B

A fibrewise subspace $A \subseteq X$ is a *strong deformation retract* if there are a fibrewise N-retraction $r\colon X \to A$ and an N-homotopy $D\colon \mathrm{id}_X \simeq_B^N ir$ with $D(a,t) = a$ for every $a \in A$ and $t \in I$. For $X = B \times [0,1]$ with its crisp product topology and $A = B \times \{0\}$, the map $D((b,u),t) = (b,(1-t)u)$ has these properties, with $r(b,u) = (b,0)$.

2.4 Homotopy equivalences and their composition

An N-continuous fibrewise map $f\colon X \to Y$ is a *fibrewise N-homotopy equivalence* if there is a fibrewise N-map $g\colon Y \to X$ with $gf \simeq_B^N \mathrm{id}_X$

and $fg \simeq_B^N \mathrm{id}_Y$. In this and the following statements we assume that homotopies are compatible with composition on both sides: if $H\colon f \simeq_B^N f'$, then kH and $H \circ (h \times \mathrm{id}_I)$ are N-homotopies whenever k and h are fibrewise N-maps. In the crisp model this is automatic.

Theorem 2.4 (Composition of homotopy equivalences)**.** *If $f\colon X \to Y$ and $u\colon Y \to Z$ are fibrewise N-homotopy equivalences, then $uf\colon X \to Z$ is a fibrewise N-homotopy equivalence.*

Proof. Choose homotopy inverses $g\colon Y \to X$ of f and $v\colon Z \to Y$ of u. Then $(gv)(uf) = g(vu)f \simeq_B^N gf \simeq_B^N \mathrm{id}_X$ and $(uf)(gv) = u(fg)v \simeq_B^N uv \simeq_B^N \mathrm{id}_Z$. □

Corollary 2.5 (Two-out-of-three)**.** *Let $f\colon X \to Y$ and $u\colon Y \to Z$ be fibrewise N-maps. If two of f, u, uf are fibrewise N-homotopy equivalences, then so is the third.*

Proof. A map homotopic to a homotopy equivalence is itself one, with the same inverse. If u and uf are equivalences and v is an inverse of u, then $f \simeq_B^N (vu)f = v(uf)$ is a composite of equivalences, hence an equivalence by Theorem 2.4. If f and uf are equivalences and g is an inverse of f, then $u \simeq_B^N u(fg) = (uf)g$, and the same argument applies. □

Chapter 3

Separation Conditions in Fibrewise Neutrosophic Spaces

3.1 Point neighbourhoods and the role of a convention

Neutrosophic membership degrees do not by themselves determine a unique notion of open neighbourhood of a classical point. To obtain a well-defined separation theory, we use a specified *support functor*: for $A = (T_A, I_A, F_A)$ put $\operatorname{supp}(A) = \{x : T_A(x) > 0\}$. The supports of all members of $\mathcal{N}_X$ form an ordinary topology $\tau_T(X)$. Indeed, $\operatorname{supp}(\bigvee A_\lambda) = \bigcup \operatorname{supp}(A_\lambda)$ because a supremum is positive exactly when some term is, and $\operatorname{supp}(A \wedge C) = \operatorname{supp} A \cap \operatorname{supp} C$ because a minimum of two numbers is positive exactly when both are. For a crisp encoding, τ_T is the original topology. The support topology uses the truth coordinate only. It is the case $\alpha = 0$ of the level topologies of Chapter 21, where topologies built from the indeterminacy and falsity coordinates are also introduced. Other conventions require separate justification.

Definition 3.1 (Fibrewise separation hierarchy)**.** A fibrewise neutrosophic space is N-T_i for $i \in \{0, 1, 2\}$ if each fibre X_b satisfies the classical T_i condition for the support topology $\tau_T(X_b)$. It is N-R_0 if the specialization preorder of every fibre is symmetric.

Here the specialization preorder is the relation $x \sqsubseteq y$ if and only if $x \in \operatorname{cl}\{y\}$. We use $\sqsubseteq$ to avoid confusion with the order $\preceq$ of triples.

Proposition 3.2 (The implications)**.** *With the support convention,* N-$T_2 \Rightarrow N$-$T_1 \Rightarrow N$-T_0 *and* N-$T_1 \Rightarrow N$-R_0*. Moreover,* N-R_0 *together with* N-T_0 *implies* N-T_1*.*

Proof. All assertions reduce fibre by fibre to the corresponding classical statements. For the last one, let $x \neq y$ in a fibre and suppose that every open neighbourhood of x contains y. Then $x \in \mathrm{cl}\{y\}$, so by R_0 also $y \in \mathrm{cl}\{x\}$. Hence x and y have the same open neighbourhoods, contradicting T_0. Therefore some open neighbourhood of x omits y, and symmetrically some open neighbourhood of y omits x. □

3.2 Subspaces, products and the diagonal

For the support topology, the axioms T_0, T_1, T_2 are hereditary to subspaces. Their stability under finite products follows from the classical case, provided the chosen neutrosophic product yields the product support topology. For the product of Section 2.1 this is true; see Theorem 21.6. If a fibre is Hausdorff, its diagonal $\Delta_b = \{(x, x) : x \in X_b\}$ is closed in $X_b \times X_b$. These implications rest on the stated compatibility. They do not follow from the mere presence of three membership coordinates.

3.3 Homotopy is weaker than homeomorphism

A separation property cannot be inferred from the homotopy type: a non-Hausdorff, even non-T_1, space may be contractible. For a concrete instance, take the Sierpiński space $S = \{0, 1\}$ with open sets $\emptyset, \{1\}, S$. The map

$$H(x, t) = x \quad (t < 1), \qquad H(x, 1) = 0,$$

is continuous on $S \times I$, because $H^{-1}(\{1\}) = \{1\} \times [0, 1)$ is open. It is a homotopy from id_S to the constant map at 0, so S is contractible. Yet S is not T_1. With crisp triples and a one-point base, the same example is a fibrewise neutrosophic counterexample.

Chapter 4

Pointed and Relative Fibrewise Homotopy

4.1 Sections and pointed maps

A *section* is a map $s_X \colon B \to X$ with $p_X s_X = \mathrm{id}_B$. A *pointed fibrewise map* $f \colon (X, s_X) \to (Y, s_Y)$ satisfies $p_Y f = p_X$ and $f s_X = s_Y$. A *pointed N-homotopy* additionally satisfies $H(s_X(b), t) = s_Y(b)$ for all (b, t).

Proposition 4.1 (Pointed restriction)**.** *If H is a pointed fibrewise homotopy, then for every $b \in B$ the restriction $H_b \colon X_b \times I \to Y_b$ is a pointed homotopy based at $s_X(b)$ and $s_Y(b)$.*

Proof. The fibre condition restricts the codomain, and the pointed equation restricts to $H_b(s_X(b), t) = s_Y(b)$. □

4.2 Homotopies relative to a subspace

Let $A \hookrightarrow X$ be an admissible fibrewise inclusion. An N-homotopy H from f to g is *relative to* A if $H(a, t) = f(a) = g(a)$ for every $(a, t) \in A \times I$. This is stronger than equality of the two endpoint restrictions, because it controls all intermediate times. The restriction to A is constant in time, a fact that matters when relative deformations are composed or a map is descended to a quotient.

4.3 Pointed homotopy groups of fibres

For $n \geq 1$ and a section s, the pointed homotopy set $\pi_n^N(X_b, s(b))$ consists of admissible based maps $(S^n, *) \to (X_b, s(b))$ modulo based N-homotopy. The group law requires the standard pinch map to be N-continuous. In the crisp model this recovers the classical group $\pi_n(X_b, s(b))$, which is abelian for $n \geq 2$. Without the pinching and relative gluing axioms, the notation π_n^N alone does not guarantee that a group is defined.

Chapter 5

Neutrosophic Fibrations and Homotopy Lifting

5.1 The lifting square

Let $q\colon E \to Z$ be N-continuous. Given $u\colon W \to E$ and a homotopy $K\colon W \times I \to Z$ with $K(w,0) = q(u(w))$, a *lift* is a map $\widetilde{K}\colon W \times I \to E$ with $\widetilde{K}(w,0) = u(w)$ and $q\widetilde{K} = K$. The map q is a *neutrosophic Hurewicz fibration* if every such N-continuous lifting problem has an N-continuous lift. For spaces over B, all maps are also required to preserve the projection to B; this is the fibrewise formulation.

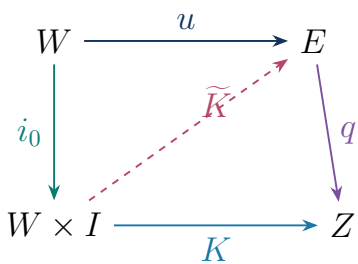


Figure 5.1: Homotopy lifting; the dashed map is the required lift.

Theorem 5.1 (Pullback stability)**.** *Let $q\colon E \to Z$ be a neutrosophic Hurewicz fibration and $g\colon A \to Z$ an N-continuous map. Equip $g^*E = \{(a,e) : g(a) = q(e)\}$ with the categorical pullback neutrosophic topology. Then $g^*E \to A$ is a neutrosophic Hurewicz fibration.*

Proof. Given a lifting problem for $g^*E \to A$, compose its homotopy with g and solve the resulting lifting problem for q. Pair the original

homotopy with the lift so obtained. The pullback equation holds by construction, and the pair is N-continuous by the universal property of the pullback. □

5.2 Path lifting and the classical model

Applying the lifting property to a one-point domain W lifts paths, so every N-fibration has path lifting. As in classical topology, the converse is false in general: individual path lifts need not assemble continuously into lifts of arbitrary parameterized families. For crisp neutrosophic spaces the lifting diagrams are exactly the classical Hurewicz diagrams, so the crisp N-fibrations are exactly the classical Hurewicz fibrations.

Chapter 6

Neutrosophic Cofibrations and Homotopy Extension

6.1 The extension property

An inclusion $i\colon A \hookrightarrow X$ is a *fibrewise neutrosophic cofibration* if the following holds for every space Y over B. Given an N-map $f\colon X \to Y$ over B and a fibrewise N-homotopy $h\colon A \times I \to Y$ with $h(a,0) = f(i(a))$, there is an N-homotopy $H\colon X \times I \to Y$ such that

$$H(x,0) = f(x), \qquad H(i(a),t) = h(a,t), \qquad p_Y H(x,t) = p_X(x).$$

This is the *fibrewise neutrosophic homotopy extension property* (NFHEP), the fibrewise version of the classical cofibration axiom [6, 9].

6.2 NDR pairs and explicit homotopies

A *fibrewise NDR pair* consists of an inclusion $A \subseteq X$, a continuous control map $u\colon X \to I$ and a fibrewise homotopy $D\colon X \times I \to X$ such that $u^{-1}(0) = A$, $D(x,0) = x$, $D(a,t) = a$ for $a \in A$, and $D(x,1) \in A$ whenever $u(x) < 1$ [21]. All maps must be admissible in the chosen neutrosophic category. An NDR pair yields an NFHEP inclusion whenever the classical NDR extension formula respects the chosen interval products and pasting operations. The claim cannot be deduced from an arbitrary three-coordinate control datum without first defining what its zero set is.

Example 6.1 (A product NDR pair)**.** For a classical space B, give $X = B \times I$ the crisp product structure and take $A = B \times \{0\}$. Set

$u(b, x) = x$ and $D((b, x), t) = (b, (1 - t)x)$. Then $u^{-1}(0) = A$, D fixes A pointwise and $D(x, 1) \in A$ for every x. The pair is a fibrewise NDR pair and hence a cofibration in the crisp category.

6.3 Retracts and relative deformations

If $A \subseteq X$ is a strong deformation retract over B, the terminal stage of the deformation is a fibrewise retraction. This alone does not make the inclusion a cofibration, because the homotopy extension property is a universal property quantified over all targets. The distinction between retractions and cofibrations persists in the neutrosophic setting. NDR data give a sufficient, but not a necessary, criterion for the cofibration property.

Chapter 7

Mapping Spaces and the Exponential Law

7.1 The compact-open construction

For topological spaces, the compact-open topology on $\mathrm{Map}(X, Y)$ has the subbasic sets $[K, U] = \{f : f(K) \subseteq U\}$ with $K \subseteq X$ compact and $U \subseteq Y$ open [12]. For fibrewise maps the relevant set is

$$\mathrm{Map}_B(X, Y) = \{f \colon X \to Y \text{ continuous} : p_Y f = p_X\},$$

with the subspace topology. A neutrosophic mapping-space topology must be specified separately; the set of functions alone does not determine one.

7.2 Parameterization by an interval

Given $H \colon X \times I \to Y$ over B, write $\widehat{H}(t)(x) = H(x, t)$. For the compact-open topology, continuity of H always implies continuity of $\widehat{H} \colon I \to \mathrm{Map}_B(X, Y)$; no assumption is needed. Conversely, recovering the continuity of H from that of $\widehat{H}$ requires the evaluation map $\mathrm{Map}_B(X, Y) \times X \to Y$ to be continuous, for instance when X is locally compact Hausdorff. In compactly generated categories these issues are handled by k-ification. An enriched neutrosophic exponential law must keep the same conditions and, in addition, require the N-continuity of evaluation.

Proposition 7.1 (A conditional adjunction)**.** *Suppose that the category of neutrosophic spaces over B has an internal hom $[X, Y]_B$ with evaluation and exponential adjunction. Then fibrewise N-homotopies*

$X \times I \to Y$ correspond bijectively to admissible paths $I \to [X, Y]_B$ with the same endpoints.

Proof. Apply the assumed natural bijection $\mathrm{Hom}_B(X \times I, Y) \cong \mathrm{Hom}(I, [X, Y]_B)$. The endpoint equations correspond under evaluation. □

Chapter 8

Fibrewise Homotopy Invariants

8.1 The pointwise family of invariants

A fibrewise homotopy equivalence $f\colon X \to Y$ restricts to a homotopy equivalence $f_b\colon X_b \to Y_b$ for every $b \in B$, so all homotopy invariants of the individual fibres agree. Agreement of these pointwise invariants, however, does not produce a global fibrewise map, let alone a fibrewise homotopy inverse. A bundle may have the same typical fibre everywhere and still be twisted over the base [21, 9]; the Möbius band and the cylinder over S^1 are the simplest example. For a given fibrewise map between fibrations, the situation is different (Section 18.8).

8.2 Lusternik–Schnirelmann category

For a nonempty neutrosophic space F, let $\mathrm{cat}_N(F)$ be the smallest $m \geq 1$ such that F has a cover by m admissible open subspaces whose inclusions into F are N-null-homotopic. For the empty space set $\mathrm{cat}_N(\emptyset) = 0$. This definition requires the chosen model to provide an open-cover convention and a homotopy theory of subspaces. It is the non-normalized convention, with $\mathrm{cat}(\mathrm{pt}) = 1$. The normalized convention now common in the literature [3] uses covers by $m+1$ sets, so that $\mathrm{cat}(\mathrm{pt}) = 0$; the two differ by one.

Theorem 8.1 (Homotopy invariance of LS category)**.** *Assume that admissible open covers pull back along N-maps, that homotopies restrict to preimages of open subspaces, and that Convention 2.2 holds. Then an N-homotopy equivalence $F \simeq^N G$ implies $\mathrm{cat}_N(F) = \mathrm{cat}_N(G)$.*

Proof. Choose $f\colon F \to G$ and $g\colon G \to F$ with $gf \simeq^N \mathrm{id}_F$, and let $G = \bigcup_{j=1}^{m} U_j$ be a categorical open cover. The sets $f^{-1}(U_j)$ cover F. Composing a null-homotopy of $U_j \hookrightarrow G$ with g, and precomposing with $f|_{f^{-1}(U_j)}$, gives a null-homotopy of $gf|_{f^{-1}(U_j)}$. Restricting a homotopy $\mathrm{id}_F \simeq^N gf$ to $f^{-1}(U_j)$ and concatenating it with this null-homotopy gives a null-homotopy of the inclusion $f^{-1}(U_j) \hookrightarrow F$. Therefore $\mathrm{cat}_N(F) \leq m$. Interchanging F and G gives the reverse inequality. □

8.3 The fundamental group and comparison homomorphisms

With a chosen section $s\colon B \to X$, the fibrewise family $\{\pi_1^N(X_b, s(b))\}_{b\in B}$ is invariant under pointed fibrewise N-homotopy equivalence whenever the pointed homotopy categories and group laws are defined. A forgetful functor U to ordinary spaces that preserves products and homotopies induces comparison homomorphisms

$$\Theta_{n,b}\colon \pi_n^N(X_b, s(b)) \longrightarrow \pi_n(UX_b, s(b)).$$

The maps $\Theta_{n,b}$ need not be isomorphisms without further hypotheses. For crisp encodings they are canonical isomorphisms.

Chapter 9

Local Triviality, Gluing and Base Change

9.1 Bundles and local products

A fibrewise space $p\colon X \to B$ is *locally trivial* with typical fibre F if each $b \in B$ has an open neighbourhood U with a fibrewise homeomorphism $\Phi_U\colon p^{-1}(U) \to U \times F$. For a neutrosophic bundle we require Φ_U and its inverse to be N-continuous for the declared product structure. Local triviality gives fibrewise identifications locally. It does not give a global product identification unless the transition functions are compatible (Section 18.5).

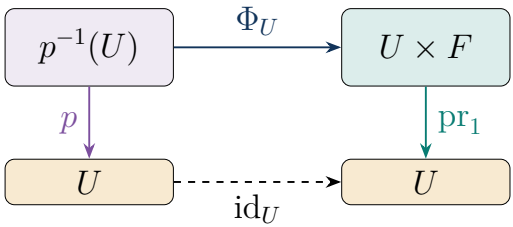


Figure 9.1: A local product chart and its compatibility with the base projection.

9.2 Change of base

For a continuous map $g\colon A \to B$ and a fibrewise space $p\colon X \to B$, the *pullback* is $g^*X = \{(a, x) : g(a) = p(x)\}$, with its projection to A. Give g^*X the categorical pullback neutrosophic structure. Then

all component maps are admissible, and pullback preserves identities and composites. To pull back homotopies one needs, in addition, that products with I commute with pullbacks:

$$(g^*X) \times I \cong g^*(X \times I) \quad \text{as neutrosophic spaces over } A. \tag{9.1}$$

This holds in the crisp model, but it is not automatic for the product-generated structure in general (compare Section 18.7).

Proposition 9.1. *Assume* (9.1). *If $f\colon X \to Y$ is a fibrewise N-homotopy equivalence over B, then $g^*f\colon g^*X \to g^*Y$ is a fibrewise N-homotopy equivalence over A.*

Proof. Pull back a homotopy inverse of f and the two homotopies; by (9.1) the pulled-back homotopies are defined on $(g^*X) \times I$ and $(g^*Y) \times I$. The endpoint equations and the base-preserving identities remain true after pullback. □

9.3 Gluing compatible local maps

Let (U_i) be an open cover of B and let $f_i\colon p_X^{-1}(U_i) \to p_Y^{-1}(U_i)$ be maps over U_i. If $f_i = f_j$ on each overlap, the set maps glue uniquely to a global $f\colon X \to Y$. The glued map is N-continuous if the chosen structure satisfies the *locality axiom*: a map is N-continuous whenever its restrictions to the members of an open cover are. This axiom must be checked. It does not follow from the existence of pointwise membership triples.

Chapter 10

Fibrewise Path Objects, Cylinders and Homotopy Pullbacks

10.1 Path spaces within a fibre

Let $p\colon X \to B$ be a fibrewise neutrosophic space in a category with an admissible exponential construction. The *fibrewise path object* is

$$P_B X = \{(b, \gamma) : b \in B,\ \gamma\colon I \to X_b \text{ admissible and continuous}\}.$$

The projection $(b, \gamma) \mapsto b$ makes it a space over B. Its endpoint maps $e_0(b, \gamma) = \gamma(0)$ and $e_1(b, \gamma) = \gamma(1)$ lie over the same point of B. If the path-object topology makes evaluation admissible, then $(e_0, e_1)\colon P_B X \to X \times_B X$ is an N-continuous fibrewise map. A path from x to y can exist only when $p(x) = p(y)$; this constraint comes before any topology on the path space.

Proposition 10.1 (Diagonal factorization)**.** *Assume the exponential law in the chosen neutrosophic category. Then the diagonal $\Delta_B\colon X \to X \times_B X$ factors as*

$$X \xrightarrow{c} P_B X \xrightarrow{(e_0, e_1)} X \times_B X,$$

where $c(x)$ is the constant path at x.

Proof. For every x the constant path $c(x)$ has both endpoints equal to x. The constant-path map is N-continuous by the exponential law, being adjoint to the projection $X \times I \to X$. Hence $(e_0, e_1)c(x) = (x, x) = \Delta_B(x)$. □

10.2 The fibrewise cylinder as a universal deformation domain

The *fibrewise cylinder* is $\mathrm{Cyl}_B(X) = X \times I$ with projection $p_X \mathrm{pr}_X$. Its endpoint maps are $i_j(x) = (x, j)$, $j = 0, 1$. A homotopy between f and g over B is precisely an admissible fibrewise map out of the cylinder with $Hi_0 = f$ and $Hi_1 = g$. This avoids any talk of "deforming" the three membership coordinates: what is deformed is a morphism, the cylinder supplies the parameter domain, and the triple-valued topology decides which candidate maps are admissible.

Lemma 10.2 (Functoriality of the cylinder)**.** *For every fibrewise N-map $f\colon X \to Y$, the formula $\mathrm{Cyl}_B(f)(x,t) = (f(x), t)$ defines an admissible map of cylinders, provided the product construction is functorial. This assignment preserves identities and composites.*

Proof. Functoriality of products gives the continuity of $f \times \mathrm{id}_I$. The base equation is $p_Y f \mathrm{pr}_X = p_X \mathrm{pr}_X$. The identity and composition laws hold pointwise. □

10.3 Mapping cylinders of fibrewise maps

For a map $f\colon X \to Y$ over B, form the quotient set

$$M_B(f) = (X \times I \amalg Y)/((x, 1) \sim f(x)).$$

The map to B sends cylinder points $[x, t]$ to $p_X(x)$ and points y to $p_Y(y)$. It is well defined on identified points because $p_Y f = p_X$. Give $M_B(f)$ the final neutrosophic structure induced by the two inclusions, when the adopted category has such final structures (Section 18.2). The inclusion of Y and the inclusion $x \mapsto [x, 0]$ of X are then fibrewise. The mapping cylinder separates an abstract homotopy class from a chosen representative map.

Proposition 10.3 (Mapping-cylinder projection)**.** *The map $M_B(f) \to B$ is well defined, and it is N-continuous whenever the*

final structure is compatible with both projections of the summands to the base.

Proof. The only nontrivial identification is $(x, 1) \sim f(x)$, whose two sides have the same image in B. Continuity follows from the final-structure criterion, by testing the projection on $X \times I$ and on Y. □

10.4 Fibrewise homotopy pullbacks

Given $f\colon X \to Z$ and $g\colon Y \to Z$ over B, their ordinary pullback is the set $\{(x, y) : f(x) = g(y)\}$. A *homotopy pullback* connects the two images by a path instead of requiring equality:

$$X \times_Z^h Y = \{(x, \gamma, y) : p_X(x) = p_Y(y),\ \gamma(0) = f(x),\\ \gamma(1) = g(y),\ \gamma(I) \subseteq Z_{p_X(x)}\}.$$

Its neutrosophic structure is induced from the product and path-object structures. When these constructions are available, the homotopy pullback has natural maps to X and Y over B. The strict pullback maps into the homotopy pullback through constant paths at $f(x) = g(y)$. This map is not an isomorphism in general, because nonconstant connecting paths contribute additional points.

Example 10.4 (Homotopy pullback over a point)**.** Let B be a point, $X = Y = \{*\}$, $Z = S^1$, and let both maps select $z_0 \in S^1$. The strict pullback is a point, whereas the homotopy pullback is the based loop space $\Omega(S^1, z_0)$. Its path components correspond to the winding numbers, and each component is contractible. The example lives in the crisp neutrosophic embedding. It shows that replacing the equation $f(x) = g(y)$ by a path is a substantial homotopical construction, not a notational variant.

10.5 Naturality of a path-object comparison

A fibrewise N-map $h\colon X \to Y$ takes a path γ in X_b to the path $h\gamma$ in Y_b. It therefore determines a map $P_B h\colon P_B X \to P_B Y$ whenever

composition is continuous on mapping spaces, and $e_j P_B h = h e_j$. If h is a fibrewise homeomorphism, so is $P_B h$, with inverse $P_B(h^{-1})$. For a fibrewise homotopy equivalence h, the map $P_B h$ is a homotopy equivalence under the usual hypotheses. It is not an isomorphism in general.

Chapter 11

Ideal Modifications and Their Homotopical Effect

11.1 The local function associated with an ideal

Let (X, τ) be a topological space. An *ideal* on X is a nonempty family $\mathcal{J}$ of subsets of X that is closed under subsets and under finite unions; in particular $\emptyset \in \mathcal{J}$. For $A \subseteq X$ its *local function* is

$$A^*_{\mathcal{J}} = \{x \in X : U \cap A \notin \mathcal{J} \text{ for every open } U \ni x\}$$

[14, 11]. The associated star closure $\mathrm{cl}^*(A) = A \cup A^*$ is a Kuratowski closure operator and defines a topology $\tau^* = \tau^*(\mathcal{J})$. A base for τ^* consists of the sets $U \setminus J$ with $U \in \tau$ and $J \in \mathcal{J}$ [11]. This description is convenient for explicit calculations. Since $U = U \setminus \emptyset$, always $\tau \subseteq \tau^*$.

Proposition 11.1 (Two extreme ideals)**.** *If $\mathcal{J} = \{\emptyset\}$, then $\tau^* = \tau$. If $\mathcal{J} = \mathcal{P}(X)$, then τ^* is the discrete topology.*

Proof. For the zero ideal the basic sets $U \setminus \emptyset = U$ are exactly the original open sets. For the full ideal, every subset $A \subseteq X$ equals $X \setminus (X \setminus A)$ with $X \in \tau$ and $X \setminus A \in \mathcal{J}$, so every subset is τ^*-open. □

11.2 An ideal on the total space and its trace on fibres

Let $p\colon X \to B$ be a continuous projection. An ideal $\mathcal{J}$ on X induces on each fibre the *trace ideal*

$$\mathcal{J}_b = \{J \cap X_b : J \in \mathcal{J}\}.$$

It is closed under subsets of X_b: if $K \subseteq J \cap X_b$, then $K \in \mathcal{J}$ by heredity, and $K = K \cap X_b$. It is closed under finite unions as well, so $\mathcal{J}_b$ is an ideal on X_b. For every open U and every $J \in \mathcal{J}$,

$$(U \setminus J) \cap X_b = (U \cap X_b) \setminus (J \cap X_b).$$

Theorem 11.2 (Fibrewise trace identity)**.** *Give each fibre the subspace topology and let τ^* be generated by the sets $U \setminus J$. Then*

$$(\tau^*)|_{X_b} = (\tau|_{X_b})^*_{\mathcal{J}_b}.$$

Proof. The restriction of a base is a base of the subspace topology, so the left-hand side has the base $(U \setminus J) \cap X_b$. By the displayed identity these are exactly the sets $(U \cap X_b) \setminus K$ with $K \in \mathcal{J}_b$, which form the base of the right-hand side. Arbitrary unions give equality of the two topologies. □

11.3 Ideal-continuous maps and pullback ideals

Let $f\colon X \to Y$ be continuous and let $\mathcal{J}_Y$ be an ideal on Y. The literal preimages $f^{-1}(J)$, $J \in \mathcal{J}_Y$, need not form a hereditary family, so the *inverse-image ideal* is defined as

$$f^!\mathcal{J}_Y = \{A \subseteq X : A \subseteq f^{-1}(J) \text{ for some } J \in \mathcal{J}_Y\}.$$

It is closed under subsets by definition, and under finite unions because $f^{-1}(J_1) \cup f^{-1}(J_2) = f^{-1}(J_1 \cup J_2)$.

Proposition 11.3 (Continuity for pulled-back ideals)**.** *Let $f\colon X \to Y$ be τ_X–τ_Y continuous and let $\mathcal{J}_X \supseteq f^!\mathcal{J}_Y$. Then f is continuous from (X, τ_X^*) to (Y, τ_Y^*).*

Proof. Since preimages commute with unions, it suffices to test a basic open set $V \setminus J$ of Y. Its preimage is $f^{-1}V \setminus f^{-1}J$, where $f^{-1}V$ is τ_X-open and $f^{-1}J \in \mathcal{J}_X$ by hypothesis. So the preimage is τ_X^*-open. □

11.4 A failure of preservation of contractibility

Take $X = [0,1]$ and the full ideal $\mathcal{J} = \mathcal{P}(X)$. The star topology is discrete, so its connected subsets are singletons and every continuous map $I \to (X, \tau^*)$ is constant. A homotopy from id_X to the constant map at c would give, for every x, a path $t \mapsto H(x,t)$ from x to c; for $x \neq c$ this path would be a nonconstant continuous map from the connected space I to a discrete space, which is impossible. Thus X is contractible, but it is not contractible after the full-ideal modification. Ideal homotopy is a theory with its own admissible paths, not a relabelling of ordinary homotopy. Sections 15.8, 17.6 and 20.1 use this example again.

11.5 The finite-set ideal on the real line

Let $X = \mathbb{R}$ with the Euclidean topology and let $\mathcal{J}_{\mathrm{fin}}$ consist of all finite subsets. Each $U \setminus F$ is Euclidean open, because finite subsets of a Hausdorff (indeed of any T_1) space are closed. Hence $\tau^* = \tau$: the ideal is nontrivial, but it does not change the admissible paths. Compared with the full ideal, this shows that the effect of an ideal cannot be predicted from its nontriviality alone.

11.6 Fibrewise ideal homotopy under pullback

Let $p_X \colon X \to B$ and $p_Y \colon Y \to B$ carry ideal topologies, and let $f \colon X \to Y$ be a fibrewise map continuous for them. An *ideal fibrewise homotopy* $H \colon X \times I \to Y$ is continuous from the chosen star-product topology into τ_Y^* and satisfies $p_Y H = p_X \mathrm{pr}_X$. Let $g \colon A \to B$ be a base map and choose the pullback ideals compatibly with inverse images, as in Proposition 11.3. Then the pullback of H is an ideal homotopy between the pulled-back spaces. The base equation is checked pointwise, and admissibility follows from the universal property of the pullback topology.

Chapter 12

Multivalued Homotopy and Continuous Selections

12.1 Relations as maps into a hyperspace

A *multivalued map* $F\colon X \rightrightarrows Y$ assigns a nonempty subset $F(x) \subseteq Y$ to each $x \in X$. It may be regarded as a single-valued map into a family of subsets of Y, but then the continuity notion depends on the topology chosen on that family. For topological spaces, F is *upper semicontinuous* if for every open $V \subseteq Y$ the set

$$F^{+}(V) = \{x : F(x) \subseteq V\}$$

is open. It is *lower semicontinuous* if for every open V the set $F^{-}(V) = \{x : F(x) \cap V \neq \emptyset\}$ is open [1, 15]. In a neutrosophic model one may replace open sets by specified triple-open sets, but the meaning of containment must then be stated explicitly (Definition 14.2).

Proposition 12.1 (Singleton correspondences)**.** *Let $f\colon X \to Y$ and $F_f(x) = \{f(x)\}$. Then F_f is upper semicontinuous if and only if f is continuous, and the same holds for lower semicontinuity.*

Proof. For every open $V \subseteq Y$, both $F_f^{+}(V)$ and $F_f^{-}(V)$ coincide with $f^{-1}(V)$. $\square$

12.2 Fibrewise compatibility of a correspondence

Let $p_X\colon X \to B$ and $p_Y\colon Y \to B$. A correspondence $F\colon X \rightrightarrows Y$ is *fibre-preserving* if

$$\emptyset \neq F(x) \subseteq Y_{p_X(x)} \qquad \text{for every } x \in X.$$

It then induces correspondences $F_b\colon X_b \rightrightarrows Y_b$. The nonempty-value assumption is essential for upper continuity: a correspondence with an empty value satisfies $F(x) \subseteq V$ vacuously for every V.

Lemma 12.2 (Restrictions to fibres)**.** *If F is fibre-preserving and upper semicontinuous, and the fibres carry subspace topologies, then every F_b is upper semicontinuous.*

Proof. Let $W = V \cap Y_b$ be relatively open. Since every value $F(x)$, $x \in X_b$, already lies in Y_b, we get $\{x \in X_b : F(x) \subseteq W\} = \{x \in X_b : F(x) \subseteq V\} = F^+(V) \cap X_b$, which is relatively open. □

12.3 Composing upper semicontinuous correspondences

For $F\colon X \rightrightarrows Y$ and $G\colon Y \rightrightarrows Z$, let $(G \circ F)(x) = \bigcup_{y \in F(x)} G(y)$. For every $W \subseteq Z$,

$$(G \circ F)^+(W) = F^+(G^+(W)),$$

since $G(F(x)) \subseteq W$ holds exactly when $F(x) \subseteq G^+(W)$. No compactness of values is needed. If a category also demands compact values, further conditions are needed for the unions $\bigcup_{y \in F(x)} G(y)$ to be compact; that requirement is separate from upper continuity.

Theorem 12.3 (Upper continuity of relation composition)**.** *Nonempty-valued upper semicontinuous correspondences are closed under composition. For fibre-preserving correspondences over B, the composite is fibre-preserving.*

Proof. The displayed identity proves continuity. If $y \in F(x)$ then $p_Y(y) = p_X(x)$, and if $z \in G(y)$ then $p_Z(z) = p_Y(y) = p_X(x)$. Hence every value of $G \circ F$ lies in the required fibre. □

12.4 Selections and their homotopies

A *selection* of F is a single-valued map $s\colon X \to Y$ with $s(x) \in F(x)$ for all x. The existence of a continuous selection requires hypotheses

on the spaces and on the geometry of the values; upper semicontinuity alone does not provide one. For a fibre-preserving correspondence, every selection respects the base: $p_Y s(x) = p_X(x)$. If (S_t) is a family of selections, then $H(x,t) = S_t(x)$ is a fibrewise homotopy as soon as the joint map $H\colon X \times I \to Y$ is continuous. Continuity of each S_t separately does not suffice.

Example 12.4 (An arc-valued deformation)**.** Identify S^1 with the unit complex numbers. For $0 \le a < \pi$ let $F_a(z) = \{ze^{i\theta} : -a \le \theta \le a\}$ be the closed arc of half-width a about z. Each F_a is an upper semicontinuous compact-valued correspondence. For fixed a, the family $t \mapsto F_{ta}(z)$ varies continuously in the Hausdorff metric, with $F_0(z) = \{z\}$. For the product bundle $B \times S^1$ this gives the fibrewise multivalued deformation $H((b,z),t) = \{b\} \times F_{ta}(z)$. The central selection $((b,z),t) \mapsto (b,z)$ is the constant homotopy. Other selections, such as $((b,z),t) \mapsto (b, ze^{ita})$, describe rotations; each must be checked for joint continuity.

12.5 Multivalued homotopy need not preserve points

The endpoint of a multivalued homotopy is a correspondence, not necessarily a single-valued map. If the first endpoint has singleton values and the second has nontrivial values, the deformation connects objects of different types. An endpoint comparison based on ordinary homotopy groups must therefore either restrict to singleton-valued endpoints or use a separately defined homotopy invariant of correspondences. In particular, replacing a point by a compact set is not automatically a homotopy equivalence of spaces.

12.6 Graph topology and fibrewise geometry

The *graph* of a correspondence is $\Gamma_F = \{(x,y) \in X \times Y : y \in F(x)\}$. If F is fibre-preserving, then $\Gamma_F \subseteq X \times_B Y = \{(x,y) : p_X(x) = p_Y(y)\}$, so correspondences can also be studied as subobjects of the fibre product. An upper semicontinuous correspondence with

compact values into a Hausdorff space always has a closed graph; more generally, closed values in a regular space suffice. For non-closed values the graph need not be closed. Conversely, a closed graph does not imply upper semicontinuity without additional compactness, for instance a compact range [1]. These classical facts persist in the crisp neutrosophic specialization.

Chapter 13

Categorical Formulation and Structural Theorems

13.1 Why the base matters

A family $\{X_b\}_{b\in B}$ can be organized in several nonequivalent ways: as a bare disjoint union, by a topology on the total set, by a bundle projection, or by a family of membership functions. None of these data determines the others. Classical fibrewise topology specifies a continuous map $p\colon X \to B$ with fibres $X_b = p^{-1}(\{b\})$ [9]. A map over B is a continuous map $f\colon X \to Y$ with $qf = p$. If f varies continuously with a parameter t, the identity $qH(x,t) = p(x)$ is the crucial constraint.

Definition 13.1 (A category of decorated spaces)**.** Fix a category $\mathcal{C}$ with finite products, a distinguished interval object $\mathbb{I}$ and a faithful underlying-set functor $|-|\colon \mathcal{C} \to \mathsf{Set}$. A *decorated space over* B is an arrow $p\colon X \to B$ of $\mathcal{C}$. A *fibrewise morphism* $f\colon (X,p) \to (Y,q)$ is a morphism of $\mathcal{C}$ with $qf = p$. The resulting slice category is denoted $\mathcal{C}/B$.

Proposition 13.2 (The slice is a category)**.** *Identities and composites of $\mathcal{C}$ define a category $\mathcal{C}/B$.*

Proof. If $qf = p$ and $rg = q$, then $r(gf) = (rg)f = qf = p$. The identity satisfies $p\,\mathrm{id}_X = p$, and the category laws are inherited from $\mathcal{C}$. □

13.2 The neutrosophic instance

In this book $\mathcal{C} = \mathsf{NTop}$ is the category of sets with a neutrosophic topology (Section 1.3) and N-continuous maps, with the order, joins and meets of Section 1.2. $L = [0,1] \times [0,1]^{\mathrm{op}} \times [0,1]^{\mathrm{op}}$ is a product of complete chains, hence complete, and so is its power L^X. Preimages preserve arbitrary joins, arbitrary meets and the constants $\mathbf{0}, \mathbf{1}$. The interval object is the crisp encoding of $[0,1]$ (Theorem 1.6), and products with the interval are the products of Section 2.1. By Theorem 1.6, the crisp encodings form a full subcategory of NTop isomorphic to Top. The order of Section 1.2 must never be silently mixed with the other standard order of Remark 1.5.

13.3 Product conventions and homotopy

Different product constructions are possible for generalized topologies. Here the product is generated by rectangles

$$A \boxtimes C = \big(\min(T_A, T_C),\ \max(I_A, I_C),\ \max(F_A, F_C)\big),$$

with A and C pulled back along the two projections. This convention makes both projections N-continuous, and on the crisp subcategory it gives the ordinary product topology: $\chi^N_U \boxtimes \chi^N_V = \chi^N_{U \times V}$, and joins of crisp triples are crisp. Concatenation of homotopies (Section 2.2) is then available under Convention 2.2.

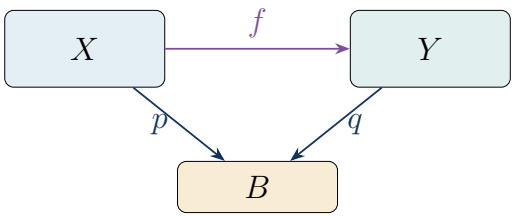


Figure 13.1: Morphisms over a fixed base; the triangle commutes.

13.4 A warning about standardization

The literature contains distinct notions of neutrosophic point membership, complement, closure, interior and topology. These are not merely different notations. A theorem formulated with open sets encoded as triples may fail for a threshold-open topology, a soft topology or an ideal-local modification. Every theorem of this book uses the conventions specified in its section.

Whether the slice categories NTop/B carry a model structure in the sense of Quillen [16, 7], with fibrewise N-homotopy equivalences as weak equivalences, is an open question. The fibrations and cofibrations of Chapters 5 and 6 are natural candidates. We use no model-categorical results in this book.

Chapter 14

Fibrewise Neutrosophic Multivalued Homotopy Theory

14.1 The reason for multivalued maps

An ordinary map assigns exactly one output to each input. A *multivalued map*, or *correspondence*, assigns a nonempty set of permitted outputs. This is useful for approximate measurements, unstable geometric reconstructions and uncertain trajectories. The multivalued theory must not be confused with the three membership coordinates of a neutrosophic subset: those coordinates decorate one subset, whereas a correspondence permits several output points. The two extensions can coexist, but they operate at different logical levels.

Let NTop be the category of neutrosophic spaces of Chapter 1. The arguments below use its inverse-image calculus. Where an operation needs extra separation or compactness, that assumption is stated. For background on multivalued analysis see [1], and for ordinary fibrewise theory see [9, 10].

Definition 14.1 (Nonempty correspondence)**.** A *correspondence* $R\colon X \rightrightarrows Y$ assigns a nonempty subset $R(x) \subseteq Y$ to each $x \in X$. It is *fibre-preserving over* B if $q(y) = p(x)$ for every $x \in X$ and $y \in R(x)$, that is, $R(x) \subseteq Y_{p(x)}$. We then write $R\colon X \rightrightarrows_B Y$.

Definition 14.2 (Universal inverse image of a triple)**.** For a triple $V = (T_V, I_V, F_V)$ on Y, define the triple $R^{\forall}V$ on X by

$$T_{R^{\forall}V}(x) = \inf_{y \in R(x)} T_V(y), \quad I_{R^{\forall}V}(x) = \sup_{y \in R(x)} I_V(y),$$

$$F_{R^{\forall}V}(x) = \sup_{y \in R(x)} F_V(y).$$

Thus $R^{\forall}V(x) = \bigwedge_{y \in R(x)} V(y)$ in the lattice of Section 1.2. R is *upper N-continuous* if $R^{\forall}V \in \mathcal{N}_X$ for every $V \in \mathcal{N}_Y$.

Remark 14.3. This is one precise convention; other multivalued continuities are possible. For a crisp open set V, the triple $R^{\forall}\chi_V^N$ is the crisp encoding of $\{x : R(x) \subseteq V\}$, so on crisp spaces the definition reduces to ordinary upper semicontinuity. A lower convention would use $\{x : R(x) \cap V \neq \emptyset\}$ instead; upper and lower continuity do not coincide in general (Section 14.10).

Proposition 14.4 (Identity correspondence)**.** *The singleton correspondence $\Delta_X(x) = \{x\}$ is upper N-continuous. A map $f\colon X \to Y$ is N-continuous if and only if the singleton correspondence $x \mapsto \{f(x)\}$ is upper N-continuous.*

Proof. For a singleton value, the infimum and suprema in the definition reduce to evaluation at the single point. Hence $\Delta_X^{\forall}V = V$ and $\{f\}^{\forall}V = f^{-1}V$. □

14.2 Composition

For $R\colon X \rightrightarrows Y$ and $S\colon Y \rightrightarrows Z$ define

$$(S \circ R)(x) = \bigcup_{y \in R(x)} S(y).$$

The union is nonempty. The formula defines a composite relation, not a function; no selection $y_x \in R(x)$ is involved.

Theorem 14.5 (Composition theorem)**.** *If $R\colon X \rightrightarrows_B Y$ and $S\colon Y \rightrightarrows_B Z$ are upper N-continuous, then $S \circ R$ is upper N-continuous and fibre-preserving.*

Proof. For $z \in S(y)$ and $y \in R(x)$ the projection equations give $r(z) = q(y) = p(x)$. For a triple W on Z, the truth coordinate of $(S \circ R)^{\forall}W$ at x is

$$\inf_{y \in R(x)} \inf_{z \in S(y)} T_W(z) = T_{R^{\forall}(S^{\forall}W)}(x),$$

and the other two coordinates satisfy the analogous identity with nested suprema. Hence $(S \circ R)^{\forall} W = R^{\forall}(S^{\forall} W)$. The inner triple is open because S is upper N-continuous, and its universal preimage under R is open because R is. $\square$

Corollary 14.6 (A correspondence category)**.** *Spaces over B, upper N-continuous fibre-preserving correspondences and the composition above form a category* NRel/B.

Proof. Composition is associative because unions are, and the singleton relations are two-sided identities. Closure under composition is Theorem 14.5. $\square$

14.3 A fully explicit two-fibre computation

Let $B = \{0, 1\}$ be discrete, and let $X = B \times \{a, b\}$ and $Y = B \times \{u, v, w\}$ carry the discrete crisp structures. Define

$$\begin{aligned} R(0,a) &= \{(0,u),(0,v)\}, \quad & R(0,b) &= \{(0,w)\}, \\ R(1,a) &= \{(1,u)\}, & R(1,b) &= \{(1,v),(1,w)\}. \end{aligned}$$

Every value of R lies in its prescribed fibre. Every correspondence between discrete spaces is upper continuous, since all universal inverse images are open. For $V = \{(0,u),(1,u)\}$, the universal inverse image is

$$\{x : R(x) \subseteq V\} = \{(1,a)\},$$

because $(0,v) \in R(0,a)$, and the values at $(0,b)$ and $(1,b)$ contain outputs outside V. The existential preimage $\{x : R(x) \cap V \neq \emptyset\} = \{(0,a),(1,a)\}$ is strictly larger: it also contains $(0,a)$, which has only one of its two outputs in V. (For $V' = \{(0,u),(0,v),(1,u)\}$ the two preimages coincide, both being $\{(0,a),(1,a)\}$, so V' would not show the difference.)

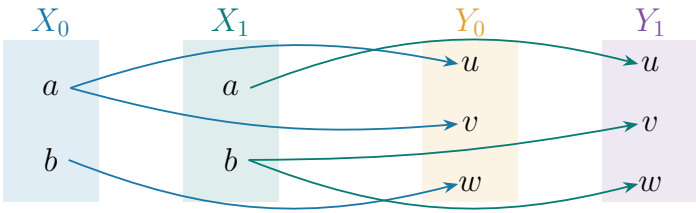


Figure 14.1: A fibre-preserving correspondence with several permitted outputs per input.

14.4 Homotopies of correspondences

The deformation parameter $I = [0, 1]$ carries its crisp structure, and $X \times I$ the product structure of Section 2.1.

Definition 14.7 (Upper fibrewise multivalued homotopy)**.** Let R_0 and R_1 be upper N-continuous correspondences $X \rightrightarrows_B Y$. An *upper fibrewise N-homotopy* from R_0 to R_1 is an upper N-continuous correspondence $H \colon X \times I \rightrightarrows Y$ with $H(x, 0) = R_0(x)$, $H(x, 1) = R_1(x)$ and $H(x, t) \subseteq Y_{p(x)}$ for all (x, t). We write $R_0 \simeq_B^{\mathrm{mv}} R_1$.

Lemma 14.8 (Endpoint restriction)**.** *The endpoint correspondences of an upper fibrewise multivalued homotopy are upper N-continuous.*

Proof. The inclusions $e_i(x) = (x, i)$ are N-continuous for the product structure, and the endpoint correspondence $x \mapsto H(x, i)$ is the composite $H \circ \{e_i\}$. Theorem 14.5 applies (with base $X \times I \to B$). □

Lemma 14.9 (Restriction to fibres)**.** *Assume that $X_b \times I$ carries the restriction of the product structure and that universal preimages commute with this restriction. Then an upper fibrewise homotopy H restricts to an upper multivalued homotopy $H_b \colon X_b \times I \rightrightarrows Y_b$.*

Proof. By the projection identity, every output of $H(x, t)$ lies in Y_b when $x \in X_b$. Under the stated hypotheses, restriction along subspace inclusions preserves upper continuity, and the endpoint identities restrict verbatim. □

14.5 Pasting: the indispensable extra hypothesis

A piecewise-defined correspondence may fail to be upper semicontinuous even when both pieces are. We therefore record an explicit sufficient property instead of treating concatenation as automatic.

Definition 14.10 (Multivalued interval pasting property)**.** A class of upper N-continuous correspondences has the *interval pasting property* if, whenever H and K agree as set-valued maps at the adjacent endpoints, the correspondence equal to $H(x, 2t)$ on $[0, \frac{1}{2}]$ and to $K(x, 2t-1)$ on $[\frac{1}{2}, 1]$ is again upper N-continuous.

Proposition 14.11 (Transitivity under pasting)**.** *In a class with the interval pasting property, $\simeq_B^{\mathrm{mv}}$ is transitive. It is reflexive whenever the projection $X \times I \to X$ is admissible, and symmetric whenever time reversal is admissible.*

Proof. For reflexivity use the constant correspondence $H(x,t) = R(x)$, for symmetry compose with $t \mapsto 1-t$, and for transitivity concatenate. In each case all values remain in the same fibre. □

14.6 A homotopy of uncertain intervals

Let B be any space and give $X = B \times [0,1]$ and $Y = B \times \mathbb{R}$ the crisp product topologies. For fixed $\varepsilon > 0$ define

$$H((b,x),t) = \{b\} \times [x+t,\ x+t+\varepsilon].$$

Each value is compact and nonempty, and the endpoints of the interval depend continuously on (b,x,t). Hence the upper inverse image of an open $V \subseteq Y$ is open. If $\{b\} \times [c,d] \subseteq V$ with V open in $B \times \mathbb{R}$, the tube lemma gives open $W \ni b$ and $\delta > 0$ with $W \times (c-\delta, d+\delta) \subseteq V$, and small changes of (b,x,t) keep the value inside this tube. Thus H is an upper continuous fibrewise multivalued homotopy from $R_0(b,x) = \{b\} \times [x, x+\varepsilon]$ to $R_1(b,x) = \{b\} \times [x+1, x+1+\varepsilon]$. No value is a singleton.

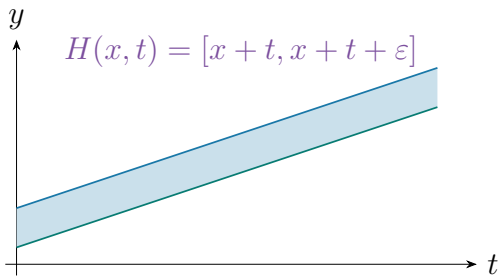


Figure 14.2: A moving interval of possible outputs in one fixed fibre.

14.7 Selection versus correspondence

A *selection* of R is a map $s\colon X \to Y$ with $s(x) \in R(x)$. Even when all values are nonempty, a continuous selection need not exist. In the crisp case, selection theorems require hypotheses such as lower semicontinuity, convex closed values in a Banach space and a paracompact domain [15, 1].

Proposition 14.12 (Selected homotopies)**.** *Suppose $H\colon X \times I \rightrightarrows_B Y$ has an N-continuous selection h with $h(x,0) = f(x)$ and $h(x,1) = g(x)$. Then h is a fibrewise N-homotopy from f to g.*

Proof. h is N-continuous by assumption. Membership in $H(x,t)$ forces $q(h(x,t)) = p(x)$, and the endpoint equalities hold by choice. □

Remark 14.13. An arbitrary selection need not respect prescribed endpoint maps. The endpoint conditions cannot be inferred from the mere existence of a continuous selection.

14.8 Graph methods

For $R\colon X \rightrightarrows Y$, the graph is $\Gamma_R = \{(x,y) : y \in R(x)\}$. When R is fibre-preserving, $\Gamma_R \subseteq X \times_B Y = \{(x,y) : p(x) = q(y)\}$. Correspondence theory can therefore also be approached through subobjects

of the fibre product, which makes comparisons with relations, spans and parameterized correspondences easy. The relation between closed graphs and upper semicontinuity is discussed in Section 12.6.

14.9 Homotopy equivalence of correspondences

One may define an equivalence of correspondences by requiring the composites to be homotopic to identity relations. This is in general different from ordinary homotopy equivalence, and it requires a homotopy relation that is a congruence: left and right composition must preserve the chosen homotopies.

Definition 14.14 (Correspondence homotopy equivalence)**.** In a class of upper N-continuous fibrewise correspondences in which homotopy is an equivalence relation compatible with composition, $R\colon X \rightrightarrows_B Y$ is a *correspondence homotopy equivalence* if there is $S\colon Y \rightrightarrows_B X$ with $S \circ R \simeq^{\mathrm{mv}}_B \Delta_X$ and $R \circ S \simeq^{\mathrm{mv}}_B \Delta_Y$.

Theorem 14.15 (Closure of equivalences)**.** *Under the assumptions of the preceding definition, composites of correspondence homotopy equivalences are correspondence homotopy equivalences.*

Proof. Let $R\colon X \rightrightarrows Y$ and $Q\colon Y \rightrightarrows Z$ have homotopy inverses S and P. Then $S \circ P$ is a homotopy inverse of $Q \circ R$: by associativity and compatibility with composition, $(S \circ P) \circ (Q \circ R) = S \circ (P \circ Q) \circ R \simeq S \circ R \simeq \Delta_X$, and symmetrically on the other side. $\square$

14.10 Upper and lower versions are distinct

Let $X = \{0\} \cup \{1/n : n \geq 1\} \subset \mathbb{R}$ and let $Y = \{0, 1\}$ be discrete. Put $R(0) = \{0, 1\}$ and $R(1/n) = \{0\}$. The existential preimage of $\{0\}$ is X, which is open. The existential preimage of $\{1\}$ is $\{0\}$, which is not open in X, so R is not lower semicontinuous at 0. On the other hand, R is upper semicontinuous at 0, since the only open set containing $R(0) = Y$ is Y itself, and it is trivially upper semicontinuous at the isolated points $1/n$. Upper continuity therefore does not imply lower

continuity, even on a compact metric domain. This must be kept in mind when results of the upper theory are translated to other notions of multivalued homotopy.

14.11 A fibrewise deformation with a parameter

For a product bundle $X = B \times F$, let $r_t \colon F \rightrightarrows F$ be a multivalued deformation whose total correspondence $(z, t) \mapsto r_t(z)$ is upper continuous. Define $H((b, z), t) = \{b\} \times r_t(z)$. Its values lie in fibres by construction. When the product conventions preserve upper continuity, H is a fibrewise multivalued homotopy that performs the same deformation in every fibre. For a nontrivial bundle one needs compatible local charts and transition maps, because fibrewise formulas in unrelated trivializations need not agree on overlaps.

Chapter 15

Ideal Fibrewise Neutrosophic Homotopy Theory

15.1 Why ideals change topology

An ideal of subsets identifies negligible or exceptional parts of a space, such as finite error sets, nowhere dense sets or a declared collection of defects. The ideal is not itself a neutrosophic open-set system; it modifies a topology through a separate operator. The local-function construction goes back to the theory of ideal topological spaces [14, 11]. Related neutrosophic ideal constructions are discussed by Salama and Smarandache [18]. We use the notation of Chapter 11: an ideal is denoted $\mathcal{J}$, its local function $A^*_{\mathcal{J}}$, and the associated topology τ^*, with base $\mathcal{B}^* = \{U \setminus J : U \in \tau, J \in \mathcal{J}\}$. An ideal is *proper* if $X \notin \mathcal{J}$.

Proposition 15.1 (Base for the ideal topology)**.** *The family $\mathcal{B}^*$ is a base for a topology on X, and $\tau \subseteq \tau^*$.*

Proof. $\mathcal{B}^*$ covers X, since $X = X \setminus \emptyset$. For basic sets, $(U \setminus J) \cap (V \setminus K) = (U \cap V) \setminus (J \cup K)$, which is basic because $\mathcal{J}$ is closed under finite unions. Every $U \in \tau$ equals $U \setminus \emptyset$. □

15.2 Simple ideals give genuinely different answers

Example 15.2 (Trivial ideal)**.** If $\mathcal{J} = \{\emptyset\}$, then $\tau^* = \tau$, and every notion of ideal continuity or ideal homotopy based on τ^* reduces to the original one.

Example 15.3 (Finite-subset ideal)**.** Let X be infinite and $\mathcal{J} = \mathrm{Fin}(X)$. Every set $U \setminus F$ with F finite is τ^*-open; in particular $X \setminus \{x\}$

is τ^*-open, so τ^* is T_1. Whether τ^* is strictly finer than τ depends on τ. If (X,τ) is T_1, finite sets are closed, every $U \setminus F$ is already τ-open, and $\tau^* = \tau$ (as for $\mathbb{R}$ in Section 11.5). If τ is the indiscrete topology, then τ^* is the cofinite topology together with $\emptyset$, which is strictly finer. In no case is τ^* automatically discrete: for τ indiscrete, a singleton $\{x\} = X \setminus (X \setminus \{x\})$ is not τ^*-open, because $X \setminus \{x\}$ is infinite.

Example 15.4 (Full power set)**.** If $\mathcal{J} = \mathcal{P}(X)$, then τ^* is discrete. This is why a properness condition is often imposed.

15.3 Fibrewise ideal systems

Definition 15.5 (Ideal-decorated space over a base)**.** Let $p\colon (X,\tau_X) \to (B,\tau_B)$ be continuous and let $\mathcal{J}_X$ be an ideal on X. The quadruple $(X,p,\tau_X,\mathcal{J}_X)$ is a *fibrewise ideal space*. Note that $p\colon (X,\tau_X^*) \to B$ is continuous, since $\tau_X \subseteq \tau_X^*$. When (X,τ_X) is the crisp reduction of a neutrosophic topology, we speak of an *ideal fibrewise neutrosophic space in the crisp ideal model*.

Remark 15.6. This construction is one precise model of ideal fibrewise neutrosophic homotopy. It is not a claim that all published notions of neutrosophic ideal topology coincide with it. A genuinely graded ideal on triples would require a separate lattice-ideal construction and a proof that the resulting topology supports interval products (Section 15.7).

Definition 15.7 (Ideal fibrewise map)**.** A map $f\colon X \to Y$ between ideal-decorated spaces over B is an *ideal fibrewise map* if $qf = p$ and $f\colon (X,\tau_X^*) \to (Y,\tau_Y^*)$ is continuous. Its crisp triple encoding is then N-continuous by Theorem 1.6.

Theorem 15.8 (Category of ideal fibrewise spaces)**.** *Ideal fibrewise spaces and ideal fibrewise maps, with identities and ordinary composition, form a category over B.*

Proof. Identities are continuous for each ideal topology. If f and g are ideal-continuous, the preimage of an ideal-open set V under

gf is $f^{-1}(g^{-1}V)$, which is ideal-open. The projection identities compose. □

15.4 Ideal fibrewise homotopies

We give $[0,1]$ its usual topology and equip the cylinder $X \times I$ with $\tau_X^* \times \tau_I$ and the projection $(x,t) \mapsto p(x)$. No separate ideal on the cylinder is used; without an explicit choice there would be no canonical product ideal.

Definition 15.9 (Ideal homotopy)**.** Ideal fibrewise maps $f, g\colon X \to Y$ are *ideal fibrewise homotopic*, written $f \simeq_{B,\mathcal{J}} g$, if there is a continuous map $H\colon (X,\tau_X^*) \times I \to (Y,\tau_Y^*)$ with $H(x,0) = f(x)$, $H(x,1) = g(x)$ and $qH(x,t) = p(x)$.

Proposition 15.10 (An equivalence relation)**.** *Ideal fibrewise homotopy is reflexive, symmetric and transitive.*

Proof. The constant homotopy proves reflexivity and time reversal proves symmetry. For transitivity, concatenate two homotopies using $t \mapsto 2t$ and $t \mapsto 2t-1$ on the closed halves of the interval. The classical pasting lemma applies because τ_X^* and τ_Y^* are ordinary topologies. Every construction preserves the projection equation. □

15.5 Transport of ideals and continuity criteria

Proposition 15.11 (Sufficient criterion)**.** *Let $f\colon (X,\tau_X) \to (Y,\tau_Y)$ be continuous with $f^{-1}(J) \in \mathcal{J}_X$ for every $J \in \mathcal{J}_Y$. Then $f\colon (X,\tau_X^*) \to (Y,\tau_Y^*)$ is continuous.*

Proof. This is Proposition 11.3, since the hypothesis means $f^!\mathcal{J}_Y \subseteq \mathcal{J}_X$. Explicitly, $f^{-1}(V \setminus J) = f^{-1}(V) \setminus f^{-1}(J)$ is basic τ_X^*-open, and preimages preserve unions. □

Remark 15.12. The criterion is sufficient but not necessary. It is a strong requirement for maps whose fibres over exceptional points are large, and failing it does not prove discontinuity.

Proposition 15.13 (Fibre restriction)**.** *If H is an ideal fibrewise homotopy and each fibre carries the subspace topology induced by τ_X^* (and likewise for Y), then $H_b\colon (X_b, \tau_X^*|_{X_b}) \times I \to (Y_b, \tau_Y^*|_{Y_b})$ is a homotopy between the restricted endpoint maps.*

Proof. For $x \in X_b$ the fibre equation gives $H(x,t) \in Y_b$. Continuity follows by restriction, using the canonical identification of products and subspaces for ordinary topologies. □

15.6 Restricted ideals on fibres

The trace ideal on a fibre is $\mathcal{J}_b = \{J \cap X_b : J \in \mathcal{J}_X\}$, and it defines a second topology $(\tau_X|_{X_b})^*_{\mathcal{J}_b}$ on X_b. By Theorem 11.2 it coincides with $(\tau_X^*)|_{X_b}$. Consequently fibre restriction commutes with ideal modification in the base model of this book, and the fibre restriction proposition above may be read with either topology on the fibres (Figure 15.1). For other ideal-modification procedures (for instance closure-defined ones without a base description, or graded ideals) the corresponding identity must be proved separately.

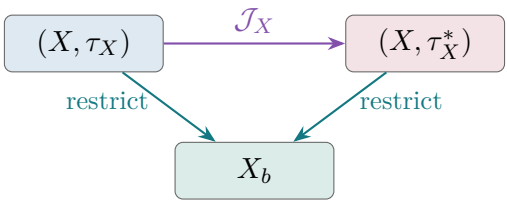


Figure 15.1: Ideal modification and fibre restriction commute in the trace-ideal base model (Theorem 11.2).

15.7 The graded neutrosophic extension: a proposal

A graded alternative starts from a *lattice ideal* $\mathcal{J}_X$ of triples: $\mathbf{0} \in \mathcal{J}_X$, and $\mathcal{J}_X$ is closed downward for $\preceq$ and under finite joins. Such

an object identifies negligible graded subsets rather than negligible ordinary subsets. A possible local operator is

$$A^{\star}_{\mathcal{J}}(x) = \bigwedge_{U \in \mathcal{N}_X,\ U(x) \succeq \lambda_x} \Phi(U \wedge A, \mathcal{J}),$$

where λ_x is a declared admissibility threshold and Φ an explicitly specified graded non-negligibility predicate. Without λ_x and Φ this expression is not a definition, and there is no canonical choice of these data. Its development is left as an open problem (see the Conclusion).

15.8 Counterexample: enlarging a topology changes homotopy

The example of Section 11.4 shows the effect of enlarging a topology: with the full ideal on $X = [0, 1]$, the identity of X is not homotopic to any constant map for τ^*. In particular the ordinary contraction $H(x, t) = (1 - t)x$ is not continuous from $X_{\mathrm{disc}} \times I$ to X_{disc}: for $x \neq 0$ the path $t \mapsto (1 - t)x$ would be a nonconstant continuous map from the connected space I to a discrete space, which is impossible. An ideal topology can change the homotopy type drastically.

15.9 Ideal fibrewise cofibrations and fibrations

In the crisp ideal model, ideal fibrewise cofibrations and fibrations are defined by the homotopy extension and lifting properties in the category of ideal-modified spaces over B. These definitions are meaningful because the cylinder is the ordinary product with I after modification. They are not equivalent, by definition, to the corresponding properties of the unmodified topologies.

Definition 15.14 (Ideal fibrewise HEP)**.** An ideal fibrewise inclusion $i\colon A \hookrightarrow X$ is an *ideal fibrewise cofibration* if, for every ideal fibrewise target Y, every ideal map $f\colon X \to Y$ and every ideal homotopy $h\colon A \times I \to Y$ with $h(a, 0) = f(a)$, there is an ideal fibrewise homotopy $H\colon X \times I \to Y$ with $H(x, 0) = f(x)$ and $H(a, t) = h(a, t)$.

Definition 15.15 (Ideal fibrewise HLP)**.** A map $r\colon E \to X$ over B is an *ideal fibrewise fibration* if every homotopy-lifting problem in the ideal-modified category over B has a solution. That is, for ideal fibrewise maps $g\colon Z \to E$ and ideal fibrewise homotopies $h\colon Z \times I \to X$ with $h(z,0) = r(g(z))$, all lying over B, there is an ideal fibrewise homotopy $\widetilde{h}$ with $\widetilde{h}(z,0) = g(z)$ and $r\widetilde{h} = h$.

15.10 Pullback stability

Theorem 15.16 (Ideal fibration pullback)**.** *Suppose the category of ideal-modified spaces has pullbacks and products with I, and that the pullback ideal topology agrees with the categorical pullback topology. If $r\colon E \to X$ is an ideal fibrewise fibration and $u\colon Y \to X$ an ideal fibrewise map, then the projection $Y \times_X E \to Y$ is an ideal fibrewise fibration.*

Proof. A lifting problem for the pullback consists of a homotopy in Y and a compatible initial map into $Y \times_X E$. Composing the homotopy with u gives a lifting problem for r; solve it. The pair of the original homotopy and the lift is a map into the pullback, because both have the same image in X. Continuity and the fibrewise identity follow from the universal property. □

15.11 Ideal deformation retracts

Definition 15.17. An *ideal fibrewise strong deformation retraction* of X onto A is an ideal fibrewise homotopy $H\colon X \times I \to X$ with $H(x,0) = x$, $H(x,1) \in A$ and $H(a,t) = a$ for $a \in A$.

Proposition 15.18. *If A is an ideal fibrewise strong deformation retract of X, then the inclusion $i\colon A \hookrightarrow X$ is an ideal fibrewise homotopy equivalence.*

Proof. Put $r(x) = H(x,1)$. Then r is ideal-continuous into A with the subspace topology, it is fibrewise, and $r|_A = \mathrm{id}_A$. The deformation H gives $ir \simeq_{B,\mathcal{J}} \mathrm{id}_X$, and $ri = \mathrm{id}_A$. □

15.12 Ideals of small sets in applications

Natural choices are the finite-set ideal, the ideal of subsets of a declared noise region, and the ideal of nowhere dense sets. The latter is an ideal: subsets of nowhere dense sets are nowhere dense, and so are finite unions of nowhere dense sets. It is not a σ-ideal in general, since countable unions of nowhere dense sets need not be nowhere dense ($\mathbb{Q} \subset \mathbb{R}$). For fibrewise data, a global ideal and its traces are automatically coherent under restriction to fibres. An arbitrary family $b \mapsto \mathcal{J}_b$ need not come from a global ideal and need not lead to a well-behaved topology on the total space.

Chapter 16

Comparing Single-Valued, Multivalued and Ideal Homotopies

16.1 Three distinct enlargements

We now compare the three constructions without conflating them. A neutrosophic topology enriches the open sets with degrees. A correspondence enriches the morphisms by permitting several outputs. An ideal modification changes which subsets are open, after declaring certain subsets negligible. These operations can be applied in different orders, and commutation is a theorem to be proved, not an identity of notation.

Theory	Extra datum	Continuity test
Neutrosophic	$A = (T, I, F)$	inverse images of open triples
Multivalued	$R(x) \subseteq Y$	universal inverse image $R^{\forall}V$
Ideal	$\mathcal{J} \subseteq \mathcal{P}(X)$	ordinary continuity after $\tau \mapsto \tau^{*}$

Table 16.1: The three enlargements of fibrewise homotopy theory.

16.2 Forgetful and comparison constructions

From a fibrewise N-map f one obtains the singleton relation $x \mapsto \{f(x)\}$. This preserves identities and composition on the nose, so it defines a faithful functor from the category of neutrosophic fibrewise spaces into NRel/B. By contrast, replacing topologies by ideal topologies may make a continuous map discontinuous, because source and target are both changed. Proposition 11.3 gives a sufficient criterion

for continuity to survive.

Proposition 16.1 (Crisp and singleton consistency)**.** *For crisp encodings of ordinary spaces, the composite of the crisp embedding and the singleton construction sends a fibrewise continuous map to a fibre-preserving upper semicontinuous singleton-valued correspondence. The induced notion of homotopy agrees with the usual fibrewise homotopy.*

Proof. Theorem 1.6 identifies continuity with N-continuity, and the identity-correspondence proposition of Chapter 14 identifies N-continuity with upper N-continuity of singleton correspondences. Applied to the cylinder $X \times I$, these identifications match the endpoint and projection conditions. □

16.3 Examples of noncommutation

Let $X = [0, 1]$ and $\mathcal{J} = \mathcal{P}(X)$. The interval is contractible, but it becomes discrete after modification. Passing to singleton correspondences does not change the set of outputs, whereas passing to τ^* radically changes the admissible paths. So a notion derived from ideals cannot be expected to coincide with the multivalued one merely because both contain the ordinary theory as a special case. Similarly, the example of Section 14.10 shows that universal and existential inverse-image conditions cannot be exchanged in a proof.

Example 16.2 (Constant-valued correspondence)**.** Let $R(x) = K$ for all x, where K is a fixed nonempty subset of one target fibre. For every open V, the universal inverse image is X or $\emptyset$, so R is upper continuous. If B has more than one point with nonempty fibres, however, a globally constant K is not fibre-preserving. A base-dependent choice $K_b \subseteq Y_b$ requires an additional continuity check.

16.4 Fibrewise invariants and naturality

A fibrewise equivalence in any of the three theories restricts to an equivalence in each fibre, provided restriction preserves the admissible homotopies. The converse fails in general: unrelated fibrewise equivalences may fail to assemble continuously across the base.

Proposition 16.3 (A necessary condition)**.** *Let $f\colon X \to Y$ be a fibrewise homotopy equivalence in the neutrosophic theory. If fibre subspaces and cylinders restrict compatibly, then each $f_b\colon X_b \to Y_b$ is a homotopy equivalence in the corresponding fibre category.*

Proof. Restrict a homotopy inverse g and the two homotopies to the fibre. The endpoint identities become $g_b f_b \simeq \mathrm{id}_{X_b}$ and $f_b g_b \simeq \mathrm{id}_{Y_b}$. □

16.5 A diagram of inclusions and modifications

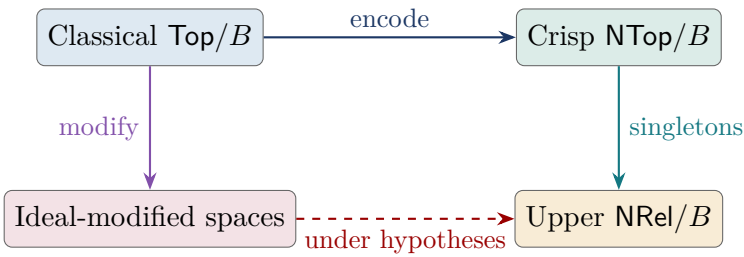


Figure 16.1: Routes through the theories; the dashed arrow needs explicit compatibility assumptions.

16.6 Comparison criteria

A reliable proof in this subject proceeds in a fixed order. First, declare the order on triples and the definitions of joins, meets, topology and continuity. Second, identify the category and the interval object. Third, state the base-preservation condition for every map and

homotopy. Fourth, make sure that products and restrictions carry the expected structures. Fifth, prove the continuity of the proposed constructions before declaring a homotopy invariant. Finally, test the crisp case as a consistency check, and test at least one non-crisp case (Chapter 21).

16.7 An invariant that records base dependence

Let $\mathcal{H}$ be a homotopy invariant of fibres in the chosen theory, and consider the assignment $b \mapsto \mathcal{H}(X_b)$. It is invariant under fibrewise homotopy equivalence when the fibre restriction theorem applies. Nevertheless, an assignment of bare isomorphism types does not determine the bundle: two bundles may have isomorphic fibres and differ by monodromy or gluing. Fibre transport, transition functions or other coherent data are needed for classification [21, 9].

Chapter 17

Examples, Calculations and Counterexamples

17.1 Finite models and admissibility checks

On finite models every continuity assertion can be checked mechanically. For a finite set X, the neutrosophic topology generated by a family of triples is obtained by closing the family, together with $\mathbf{0}$ and $\mathbf{1}$, under binary meets and then under joins. By Proposition 1.2 the result is closed under both operations. If X also carries an adjacency graph, graph continuity must be distinguished from neutrosophic continuity: the first is a statement about edges, the second about inverse images of open triples.

Example 17.1 (Three-valued membership on two points)**.** Let $X = \{a, b\}$, $A(a) = (.8, .1, .2)$, $A(b) = (.3, .4, .7)$, $C(a) = (.6, .2, .1)$ and $C(b) = (.9, .1, .4)$. Then

$$(A \vee C)(a) = (.8, .1, .1), \quad (A \vee C)(b) = (.9, .1, .4),$$
$$(A \wedge C)(a) = (.6, .2, .2), \quad (A \wedge C)(b) = (.3, .4, .7).$$

These values are computed coordinatewise from the lattice order. No normalization is imposed. For example, $(.8, .1, .1)$ happens to sum to one, but that is incidental and not part of the model. The family $\{\mathbf{0}, A \wedge C, A, C, A \vee C, \mathbf{1}\}$ is a neutrosophic topology on X that is not crisp. Its level topologies (Chapter 21) at $\alpha = 0.5$ are $\tau^{T}_{0.5} = \{\emptyset, \{a\}, X\}$ (from $T_{A\wedge C} = (.6, .3)$, $T_A = (.8, .3)$, $T_C = (.6, .9)$) and, at $\beta = 0.15$, $\tau^{I}_{0.15} = \{\emptyset, \{a\}, \{b\}, X\}$. So the truth and indeterminacy coordinates see different topologies on the same two points.

Example 17.2 (Pullback of a triple)**.** Let $f\colon \{0,1,2\} \to \{u,v\}$ be given by $f(0) = f(1) = u$ and $f(2) = v$. For $V(u) = (.7,.2,.3)$ and $V(v) = (.2,.6,.8)$, the pullback $f^{-1}V$ takes the first value at 0 and 1 and the second value at 2. To check continuity, it is not enough to inspect V alone unless V generates the target topology. Each generator must have an open pullback; preservation of joins and meets by preimages then extends the check to the whole topology.

17.2 A finite relation table

Let $X = \{a,b,c\}$ and $Y = \{u,v,w\}$ be discrete, and let $R(a) = \{u,v\}$, $R(b) = \{v\}$, $R(c) = \{v,w\}$. For a crisp open set $V \subseteq Y$, the upper inverse image consists of the inputs whose entire value lies in V. Some values are listed below; they form a simple unit test for an implementation of the universal pullback.

V	$\emptyset$	$\{v\}$	$\{u,v\}$	$\{v,w\}$	Y
$R^{\forall}(V)$	$\emptyset$	$\{b\}$	$\{a,b\}$	$\{b,c\}$	X

17.3 Two fibrewise circles

Let $B = \{0,1\}$ and $X = B \times S^1$ with the product topology and $p(b,z) = b$. The rotation by the angle θ in each fibre is $f_\theta(b,z) = (b, e^{i\theta}z)$. The homotopy $H((b,z),t) = (b, e^{it\theta}z)$ preserves the base and shows $f_\theta \simeq_B \mathrm{id}_X$ for every real θ. The construction persists under the crisp embedding. A multivalued variant replaces each rotated point by a short closed arc (Example 12.4); the upper continuity of that arc assignment must again be verified.

17.4 A family of discrete fibres

Let $B = [0,1]$ and $X = B \times \{0,1\}$, the second factor discrete. A continuous section of p has the form $s(b) = (b, \varepsilon(b))$ with $\varepsilon\colon B \to \{0,1\}$ continuous. Since B is connected, ε is constant, so this trivial

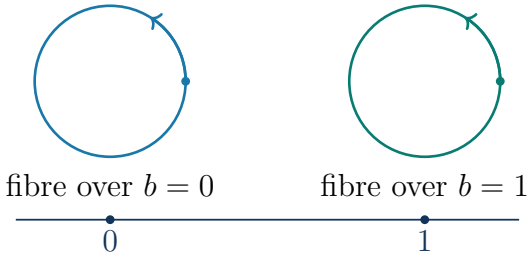


Figure 17.1: Rotations act separately within the two circular fibres.

two-sheeted family has exactly two continuous sections. The same holds for the crisp embedding. The example shows how the base imposes global coherence even though every individual fibre offers two choices.

17.5 An ordinary homotopy that fails to be fibrewise

Let $B = \mathbb{R}$, $X = Y = B \times \mathbb{R}$, with the first-coordinate projection. The maps $f(b, x) = (b, x)$ and $g(b, x) = (b, x + 1)$ are homotopic over B by $H((b, x), t) = (b, x + t)$. The similar-looking formula $K((b, x), t) = (b{+}t, x)$ is a continuous homotopy, but it is not fibrewise, since its base coordinate is $b + t$. Ordinary continuity never replaces the verification of $qK = p$.

17.6 An ideal destroys a path

Take $X = [0, 1]$ with $\mathcal{J} = \mathcal{P}(X)$, so that τ^* is discrete. For any continuous path $h\colon I \to (X, \tau^*)$, the image of the connected interval is a connected subset of a discrete space, hence a point, and h is constant. A homotopy from id_X to the constant map with value c would therefore produce a nonconstant path $t \mapsto H(x, t)$ for every $x \neq c$, a contradiction (compare Section 11.4). Ideal-modified homotopy cannot be inferred from ordinary homotopy without examining the modified topology.

17.7 A diagram for a homotopy cylinder

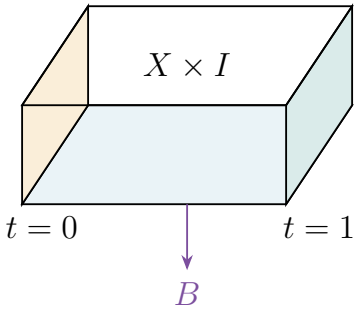


Figure 17.2: The cylinder records deformation time while the map to B stays fixed.

17.8 A fibrewise contraction of an interval bundle

On $X = B \times [0,1]$ let $r(b,x) = (b,0)$. The homotopy $H((b,x),t) = (b,(1-t)x)$ is a fibrewise strong deformation retraction of X onto $B \times \{0\}$, so $B \times \{0\}$ is a fibrewise strong deformation retract of X. Under the crisp embedding all maps remain admissible. If an ideal turns the interval fibre into a discrete space, the same formula no longer defines an ideal homotopy. The existence of a formula is not sufficient: its continuity must be checked in the chosen category.

17.9 Graded structures versus homotopies

Suppose that each point x carries three numbers $\langle T(x), I(x), F(x)\rangle$, and consider a family of transformations that keep the underlying points but change the three numbers. Such a family is not by itself a homotopy of maps $X \to Y$: the maps of underlying sets may be unchanged while the ambient neutrosophic topology varies with the parameter. Studying a varying topology requires a space of structures,

a topology on it and a continuity condition on the variation. This separates a homotopy inside one category from a deformation of the objects of the category.

Chapter 18

Constructive Methods: Cylinders, Pullbacks and Gluing

18.1 The mapping cylinder as a test object

Recall the fibrewise mapping cylinder $M_B(f)$ of Section 10.3. The inclusions of Y and of X at time 0 turn a map into a geometric space. Let $p\colon X \to B$ and $q\colon Y \to B$ be projections and f fibrewise, $qf = p$. The map from $X \times I \amalg Y$ to B given by $p(x)$ on the cylinder and by $q(y)$ on Y is constant on equivalence classes, so it descends to $M_B(f) \to B$. For the ordinary quotient topology, this map is continuous by the universal property of quotients.

Proposition 18.1 (Classical fibrewise mapping cylinder)**.** *For a continuous fibrewise map $f\colon X \to Y$, the mapping cylinder carries a canonical continuous projection to B, and the inclusion $i\colon Y \hookrightarrow M_B(f)$ is a fibrewise homotopy equivalence.*

Proof. Let $r\colon M_B(f) \to Y$ be the retraction $r([x,t]) = f(x)$, $r(y) = y$. Define $D([x,t],s) = [x, t + s(1-t)]$ and $D(y,s) = y$. The formula is compatible with the identification at $t = 1$, and it is continuous because I is locally compact Hausdorff, so that the product of the quotient map with id_I is again a quotient map. D fixes Y, satisfies $D(-,0) = \mathrm{id}$ and $D(-,1) = ir$, and is fibrewise, since $[x,t]$ and all its deformations project to $p(x)$. Finally $ri = \mathrm{id}_Y$. □

18.2 What must be checked for a neutrosophic cylinder

For a neutrosophic topology, writing the quotient set does not by itself specify a structure on it. One workable choice is the *final*

structure: a triple on $M_B(f)$ is open when its pullback along the quotient map is open in the sum of the product cylinder and Y. Since preimages preserve joins, meets and the constants, this is a neutrosophic topology, and it is the finest one making the quotient map N-continuous. The base projection is N-continuous whenever the map from the disjoint union to B is admissible. The deformation D, however, must still be checked in this quotient structure; a set-theoretic deformation is not automatically N-continuous. This is why the classical statement cannot be carried over to an arbitrary graded theory without qualification.

18.3 A constructive fibre product

Given $p\colon X \to B$ and $q\colon Y \to B$, the *fibre product* is

$$X \times_B Y = \{(x, y) \in X \times Y : p(x) = q(y)\},$$

with the subspace structure and projections $\pi_X(x, y) = x$, $\pi_Y(x, y) = y$. The common map to B is $p\pi_X = q\pi_Y$. If $u\colon Z \to X$ and $v\colon Z \to Y$ satisfy $pu = qv$, the unique induced map is $z \mapsto (u(z), v(z))$. This construction is the categorical pullback when the chosen product and subspace structures have the expected universal properties. For a topology of triples, this must be verified for the adopted product.

Proposition 18.2 (Fibre of a fibre product)**.** *Under the evident subspace compatibility, the fibre of $X \times_B Y$ over b is canonically identified with $X_b \times Y_b$.*

Proof. A pair (x, y) lies over b exactly when $p(x) = q(y) = b$, so the fibre is the set $X_b \times Y_b$. The topological identification requires that subspace and product structures agree on it; this is automatic for ordinary spaces and for the crisp product convention. □

18.4 Worked pullback calculation

Let $B = \mathbb{R}$, $X = B \times [0,1]$ and $Y = B \times S^1$ with the first-coordinate projections. Then

$$X \times_B Y \cong B \times [0,1] \times S^1, \qquad ((b,t),(b,z)) \mapsto (b,t,z),$$

with inverse $(b,t,z) \mapsto ((b,t),(b,z))$. Both maps are continuous for the product topologies and N-continuous for the crisp encodings. The fibre over b is the cylinder $[0,1] \times S^1$. In the ideal-modified setting this calculation can be repeated only after the ideal topology on the product has been specified: a product of two modified spaces need not be a modification of their product by any given ideal.

18.5 Local trivializations and gluing

A locally trivial family has charts $\varphi_i \colon p^{-1}(U_i) \to U_i \times F$ over a cover $\{U_i\}$ of B. On overlaps the transition maps have the form

$$\varphi_j \varphi_i^{-1}(b,z) = (b, g_{ji}(b)(z)),$$

and they satisfy the cocycle identities $g_{ii} = \mathrm{id}_F$ and $g_{ki} = g_{kj} g_{ji}$ on triple overlaps. The transition maps must be admissible in the chosen category. If they are merely bijections of sets, the glued family need not be well defined topologically. If they change the triples defining the open-set structure, a compatibility rule between neutrosophic charts must be imposed. This is a natural source of genuinely neutrosophic questions beyond the crisp embedding.

18.6 A compatibility criterion for glued fibrewise maps

Suppose given local fibrewise maps $f_i \colon p^{-1}(U_i) \to q^{-1}(U_i)$ that agree on overlaps. If each $p^{-1}(U_i)$ is open in X, the maps glue to a unique continuous $f \colon X \to Y$. Indeed, the maps agree pointwise, and the inverse image of an open set of Y is the union of its inverse images in the open pieces. Fibre preservation holds locally, hence globally. In

the neutrosophic setting the corresponding pasting theorem requires the locality axiom of Section 9.3, so that the local inverse-image triples assemble to an open triple. A family of local formulas is not a global construction until this compatibility is checked.

18.7 Homotopies and change of base

Let $u\colon C \to B$ be continuous and $p\colon X \to B$ a family. The pullback family over C is $u^{*}X = \{(c, x) : u(c) = p(x)\} \to C$. If $f\colon X \to Y$ is fibrewise over B, put $u^{*}f(c, x) = (c, f(x))$; this is well defined because $qf(x) = p(x) = u(c)$. In the classical category, continuity follows from the universal property of the pullback. If $H\colon X \times I \to Y$ is a fibrewise homotopy, then

$$(u^{*}H)((c, x), t) = (c, H(x, t))$$

is the pulled-back homotopy. It lies over C, and its endpoints are the pullbacks of the endpoints of H. Classical fibrewise homotopy equivalences are therefore preserved under change of base. The neutrosophic version holds under hypothesis (9.1) of Section 9.2, which makes pullbacks compatible with cylinders. Without this or a similar hypothesis it is an open problem, not an unconditional theorem.

18.8 From fibrewise to global equivalences

Suppose that maps $f_b\colon X_b \to Y_b$ are homotopy equivalences for every $b \in B$. This alone does not imply that the family $\{f_b\}$ forms a continuous map $X \to Y$, since the dependence on b may be discontinuous (Example 1.1). If a continuous fibrewise map f does exist, the answer depends on the projections. Dold's theorem [4] states the following. If p_X and p_Y are Hurewicz fibrations over a base admitting a numerable cover (for example a paracompact space or a CW complex), and $f\colon X \to Y$ is a map over B such that every f_b is a homotopy equivalence, then f is a fibrewise homotopy equivalence. Without

the fibration hypothesis, a fibrewise map that is a homotopy equivalence on every fibre need not admit fibrewise homotopy inverses. For neutrosophic fibrations, an analogue of Dold's theorem would require partitions of unity and the gluing properties of this chapter in the neutrosophic category. It is an open problem. In general, a fibrewise homotopy equivalence requires global inverse and homotopy data over the base, and this global coherence is what distinguishes parametrized homotopy theory from a family of independent homotopy problems.

18.9 A picture of a pullback

Figure 18.1 shows the fibre product $X \times_B Y$ with its two projections. The square commutes because $p\,\pi_X = q\,\pi_Y$, and it is universal among commuting squares with sides p and q.

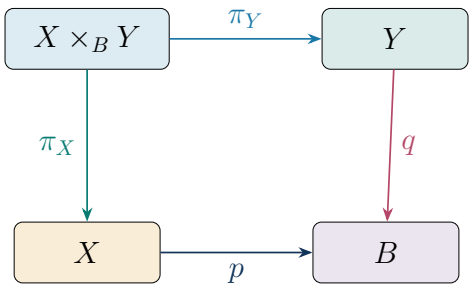


Figure 18.1: The fibre product is the pullback of the two projections.

Chapter 19

Terminology, Conventions and Structural Consistency

19.1 Foundational conventions

We summarize the conventions fixed throughout this monograph.

(1) A neutrosophic subset is a triple of $[0, 1]$-valued functions. The indeterminacy and falsity coordinates are ordered oppositely to the truth coordinate.

(2) A neutrosophic topology is closed under arbitrary joins and finite meets for that order.

(3) A continuous map is characterized by inverse images of open triples.

(4) A homotopy uses the product-generated cylinder; it is not merely a pointwise family of functions.

(5) A fibrewise homotopy preserves the base point $p(x)$ at every time.

The multivalued chapter uses universal inverse images. Its continuity is the upper version, not the lower one, and its composition is relational composition. The ideal chapters use ordinary set ideals and the base $U \setminus J$ for the modified topology. This is a crisp ideal model with an explicit neutrosophic embedding; the graded-ideal problem remains open. A proof combining the two must explain how the modified neutrosophic structure and the universal inverse-image operation interact.

19.2 Continuity and universal-property hypotheses

Before a new invariant is claimed, one should verify three things: that the proposed homotopy relation is an equivalence relation, that composition respects its classes, and that the invariant is constant on them. For a proposed fibration one must prove the lifting property for the specified class of test spaces, and for a cofibration the dual extension property. A mapping space requires an admissible evaluation map and a form of the exponential law. A change-of-base statement requires compatibility of pullbacks with cylinders. An ideal-local construction requires the heredity and finite-union axioms and a proof that the modified topology is well defined.

A crisp reduction is an important consistency test, but it does not prove the genuinely graded assertions. Conversely, a theorem that fails in the crisp subcategory cannot hold in full generality for a theory that extends that subcategory. Small classical counterexamples are therefore especially useful, and so are small non-crisp examples (Chapter 21).

19.3 A minimal glossary

Fibre
$X_b = p^{-1}(\{b\})$, the part of the total space over a base point b.

Slice category
The category of objects and arrows over a fixed base B.

Open triple
A truth–indeterminacy–falsity triple belonging to the chosen neutrosophic topology.

Crisp reduction
The specialization of the neutrosophic theory to ordinary topology through the characteristic triples χ_U^N.

Level topology
One of the classical topologies τ_α^T, τ_β^I, τ_γ^F obtained by thresholding one coordinate of the open triples (Chapter 21).

Upper correspondence
A nonempty-valued relation whose universal inverse images of open sets are open.

Selection
A single-valued map choosing a point in every value of a correspondence.

Ideal topology
The topology generated by the sets $U \setminus J$ with U open and J in the ideal.

Trace ideal
The ideal obtained from a global ideal by intersecting its members with a fibre.

Relative homotopy
A homotopy keeping a specified subspace fixed.

Strong deformation retract
A subspace onto which the space retracts by a homotopy that fixes the subspace pointwise at all times.

Chapter 20

A Worked Comparison of Three Homotopy Conventions

20.1 An interval family with three admissibility structures

Fix a nonempty base B, put $X = B \times [0,1]$ and $p(b,u) = b$. Under the crisp embedding, the fibrewise map $r(b,u) = (b,0)$ is homotopic to the identity by $H((b,u),t) = (b,(1-t)u)$. This homotopy is continuous for the product topology, and therefore N-continuous for the crisp encoding.

For the multivalued model, consider $F_t(b,u) = \{b\} \times [0,(1-t)u]$. At $t = 0$ its value is the interval $\{b\} \times [0,u]$, not the singleton $\{(b,u)\}$, so this formula cannot be a multivalued homotopy that starts at the identity. A singleton-valued initial correspondence is obtained from

$$G_t(b,u) = \{b\} \times [(1-t)u,\ u].$$

Then $G_0(b,u) = \{(b,u)\}$ and $G_1(b,u) = \{b\} \times [0,u]$. The values are compact intervals whose endpoints depend continuously on (b,u,t). Since the Hausdorff distance between two intervals is the larger of the distances between corresponding endpoints, G is a continuous hyperspace-valued deformation. It is fibre-preserving, because the first coordinate always equals b.

Now give every fibre the star topology of the full power-set ideal, so that each fibre is discrete. The map $H((b,u),t) = (b,(1-t)u)$ is then not an ideal homotopy, because for fixed $u > 0$ its time slice is a nonconstant map from the connected interval to a discrete fibre. The example illustrates three distinct statements: an ordinary

fibrewise contraction exists; a multivalued enlargement with compact values can be formed; and an ideal modification can eliminate all nonconstant paths. No implication among these statements holds without appropriate continuity and compatibility hypotheses.

20.2 The role of the base in the example

The base parameter is unchanged in all three constructions. This is more than a notational convenience. The identity $pH = p\,\mathrm{pr}_X$ is the defining equation of every fibrewise deformation, and $F_t(x) \subseteq X_{p(x)}$ is its multivalued analogue. A formula may be continuous in an underlying topology and still fail this equation. Conversely, satisfying the equation does not guarantee continuity in the neutrosophic, hyperspace or star-topology model. Both obligations must be checked independently in every example.

Chapter 21

Level Topologies and Genuinely Neutrosophic Phenomena

Most constructions of the previous chapters were tested on crisp encodings, where by Theorem 1.6 neutrosophic topology is ordinary topology. This chapter treats the non-crisp case. We first observe a symmetry between indeterminacy and falsity. We then attach to every neutrosophic space three families of classical *level topologies*, one for each coordinate, and prove that fibrewise N-homotopies induce classical fibrewise homotopies at every level. Finally we construct neutrosophic spaces in which the indeterminacy coordinate alone obstructs a contraction.

21.1 A symmetry between indeterminacy and falsity

Let $\sigma(T, I, F) = (T, F, I)$ on triples, and $\sigma_*\mathcal{N} = \{\sigma A : A \in \mathcal{N}\}$.

Proposition 21.1 (Exchange of indeterminacy and falsity)**.** *With the order of Section 1.2, σ is an order automorphism of the lattice of triples on X. It fixes $\mathbf{0}$ and $\mathbf{1}$ and commutes with arbitrary joins, arbitrary meets, preimages and the rectangles $A \boxtimes C$. Consequently $\sigma_*\mathcal{N}_X$ is a neutrosophic topology. Moreover, f is N-continuous for $(\mathcal{N}_X, \mathcal{N}_Y)$ if and only if it is N-continuous for $(\sigma_*\mathcal{N}_X, \sigma_*\mathcal{N}_Y)$, and the same holds for homotopies and fibrewise homotopies.*

Proof. The second and third coordinates are ordered in the same way, so exchanging them preserves $\preceq$; since $\sigma^2 = \mathrm{id}$, σ is an automorphism. Joins and meets are computed coordinatewise with the same operation in the second and third coordinates, preimages act coordinatewise,

and $\mathbf{0} = (0,1,1)$, $\mathbf{1} = (1,0,0)$ are fixed. The remaining assertions follow, since $f^{-1}(\sigma V) = \sigma f^{-1}(V)$ and $\sigma(A \boxtimes C) = \sigma A \boxtimes \sigma C$. □

Remark 21.2. Every definition and theorem of Chapters 1–20 is formulated through the lattice operations, preimages and products, so all of them are invariant under the exchange of I and F. In the order of Section 1.2, indeterminacy is *formally* a second falsity coordinate. Constructions that give indeterminacy a distinct role must break this symmetry. They can do so by treating the coordinates separately, as the level topologies below do. They can also use the other standard order of Remark 1.5, in which I is ordered like T; then σ is no longer an automorphism.

21.2 Level topologies

Definition 21.3 (Level topologies)**.** Let $\mathcal{N}_X$ be a neutrosophic topology on X. For $\alpha \in [0,1)$ and $\beta, \gamma \in (0,1]$ put

$$\tau^T_\alpha(\mathcal{N}_X) = \{\{x : T_A(x) > \alpha\} : A \in \mathcal{N}_X\},$$
$$\tau^I_\beta(\mathcal{N}_X) = \{\{x : I_A(x) < \beta\} : A \in \mathcal{N}_X\},$$
$$\tau^F_\gamma(\mathcal{N}_X) = \{\{x : F_A(x) < \gamma\} : A \in \mathcal{N}_X\}.$$

These are the *truth*, *indeterminacy* and *falsity level topologies*. The support topology of Chapter 3 is $\tau_T = \tau^T_0$.

Theorem 21.4. *Each $\tau^T_\alpha(\mathcal{N}_X)$, $\tau^I_\beta(\mathcal{N}_X)$ and $\tau^F_\gamma(\mathcal{N}_X)$ is a topology on X. For the crisp encoding of a topology τ, all of them equal τ.*

Proof. For the truth levels, write $[A]^T_\alpha = \{T_A > \alpha\}$. Then $[\mathbf{0}]^T_\alpha = \emptyset$ and $[\mathbf{1}]^T_\alpha = X$ because $0 \le \alpha < 1$. Since $\sup_\lambda t_\lambda > \alpha$ if and only if $t_\lambda > \alpha$ for some λ, we get $[\bigvee A_\lambda]^T_\alpha = \bigcup [A_\lambda]^T_\alpha$. Since $\min(s,t) > \alpha$ if and only if $s > \alpha$ and $t > \alpha$, we get $[A \wedge C]^T_\alpha = [A]^T_\alpha \cap [C]^T_\alpha$. As $\mathcal{N}_X$ is closed under joins and finite meets, τ^T_α is closed under unions and finite intersections.

For the indeterminacy levels, write $[A]^I_\beta = \{I_A < \beta\}$. Then $[\mathbf{0}]^I_\beta = \emptyset$ and $[\mathbf{1}]^I_\beta = X$ because $0 < \beta \le 1$. Joins take the infimum of

I, and $\inf_\lambda i_\lambda < \beta$ if and only if $i_\lambda < \beta$ for some λ. Meets take the maximum, and $\max(s,t) < \beta$ if and only if both $s < \beta$ and $t < \beta$. The falsity levels are treated identically.

For $\chi_U^N = (\chi_U, 1-\chi_U, 1-\chi_U)$ one has $\{\chi_U > \alpha\} = U$ for $\alpha < 1$ and $\{1-\chi_U < \beta\} = U$ for $\beta > 0$. □

Proposition 21.5 (Continuity at every level)**.** *If $f\colon X \to Y$ is N-continuous, then f is continuous from $\tau_\alpha^T(\mathcal{N}_X)$ to $\tau_\alpha^T(\mathcal{N}_Y)$ for every α, and likewise for every indeterminacy and falsity level.*

Proof. $f^{-1}\{T_V > \alpha\} = \{T_V \circ f > \alpha\} = [f^{-1}V]_\alpha^T$, and $f^{-1}V \in \mathcal{N}_X$. The other coordinates are analogous. □

Theorem 21.6 (Levels of products)**.** *For the product structure of Section 2.1 and every admissible level,*

$$\tau_\alpha^T(\mathcal{N}_X \boxtimes I) = \tau_\alpha^T(\mathcal{N}_X) \times \tau_I,$$
$$\tau_\beta^I(\mathcal{N}_X \boxtimes I) = \tau_\beta^I(\mathcal{N}_X) \times \tau_I,$$
$$\tau_\gamma^F(\mathcal{N}_X \boxtimes I) = \tau_\gamma^F(\mathcal{N}_X) \times \tau_I.$$

Proof. For a rectangle, $T_{A \boxtimes \chi_W^N}(x,t) = \min(T_A(x), \chi_W(t))$, which exceeds $\alpha < 1$ exactly when $T_A(x) > \alpha$ and $t \in W$. Hence $[A \boxtimes \chi_W^N]_\alpha^T = [A]_\alpha^T \times W$, and similarly $[A \boxtimes \chi_W^N]_\beta^I = [A]_\beta^I \times W$, since $\max(I_A(x), 1-\chi_W(t)) < \beta \le 1$ exactly when $I_A(x) < \beta$ and $t \in W$. By Proposition 1.2, the members of $\mathcal{N}_X \boxtimes I$ are $\mathbf{0}$, $\mathbf{1}$ and joins of finite meets of rectangles. By the proof of Theorem 21.4, levels turn joins into unions and finite meets into intersections. Therefore the level topology of the product consists of unions of finite intersections of the sets $[A] \times W$. This is exactly the product topology $\tau_\bullet(\mathcal{N}_X) \times \tau_I$, because $\tau_\bullet(\mathcal{N}_X)$ is a topology and the sets $[A] \times W$ form a base of the product. □

Theorem 21.7 (Homotopy transfer to levels)**.** *Let H be a fibrewise N-homotopy from f_0 to f_1 between fibrewise neutrosophic spaces over B. Then for every $\alpha \in [0,1)$ and $\beta, \gamma \in (0,1]$, H is a classical fibrewise homotopy from f_0 to f_1 between the spaces $(X, \tau_\alpha^T) \to B$ and $(Y, \tau_\alpha^T) \to B$, and likewise for τ_β^I and τ_γ^F.*

Proof. The projections are continuous at every level: by Definition 1.7, $\chi^{N}_{p_X^{-1}U} \in \mathcal{N}_X$ for $U \in \tau_B$, and its level sets equal $p_X^{-1}U$ (Theorem 21.4). By Proposition 21.5, H is continuous from the level topology of $\mathcal{N}_X \boxtimes I$, which by Theorem 21.6 is the product of the level topology of X with τ_I. The endpoint and base equations do not involve topology. □

Corollary 21.8 (Level profiles are invariants)**.** *If $f\colon X \to Y$ is a fibrewise N-homotopy equivalence, then for every level f is a classical fibrewise homotopy equivalence between the corresponding level spaces. In particular, the three* level profiles

$$\alpha \mapsto [(X, \tau_\alpha^T)], \qquad \beta \mapsto [(X, \tau_\beta^I)], \qquad \gamma \mapsto [(X, \tau_\gamma^F)],$$

with values in fibrewise homotopy types over B, are invariants of the fibrewise N-homotopy type. So is every classical homotopy invariant (number of path components of fibres, homotopy groups, LS category) evaluated along them.

Proof. Apply Theorem 21.7 to a homotopy inverse and to the two homotopies. □

For crisp spaces the three profiles are constant and equal to the classical type. The non-crisp information carried by a neutrosophic topology is thus visible in how the profiles vary and in how the three profiles differ from one another.

21.3 Bitopological neutrosophic spaces

The simplest non-crisp spaces carry two classical topologies, one read by the truth and falsity coordinates and one read by the indeterminacy coordinate.

Definition 21.9. Let τ and ρ be topologies on X. Put

$$\mathcal{N}(\tau, \rho) = \{(\chi_U,\ 1 - \chi_V,\ 1 - \chi_U) :\ U \in \tau,\ V \in \rho,\ U \subseteq V\}.$$

We write $\langle U, V \rangle$ for the triple $(\chi_U, 1 - \chi_V, 1 - \chi_U)$.

Proposition 21.10. *$\mathcal{N}(\tau,\rho)$ is a neutrosophic topology. Its level topologies are $\tau^T_\alpha = \tau^F_\gamma = \tau$ for all $\alpha \in [0,1)$, $\gamma \in (0,1]$, and $\tau^I_\beta = \rho$ for all $\beta \in (0,1]$. If $X \neq \emptyset$, it is never a crisp encoding: it always contains $\langle \emptyset, X\rangle = (0,0,1)$, which is not of the form $(\chi_U, 1-\chi_U, 1-\chi_U)$.*

Proof. $\mathbf{0} = \langle\emptyset,\emptyset\rangle$ and $\mathbf{1} = \langle X, X\rangle$. Coordinatewise, $\bigvee\langle U_\lambda, V_\lambda\rangle = \langle \bigcup U_\lambda, \bigcup V_\lambda\rangle$ and $\langle U,V\rangle \wedge \langle U',V'\rangle = \langle U\cap U', V\cap V'\rangle$, and these operations preserve the inclusion $U \subseteq V$. The level sets of $\langle U,V\rangle$ are U for the truth and falsity coordinates and V for the indeterminacy coordinate. Taking $U=\emptyset$ shows that every $V \in \rho$ occurs, and taking $V = X$ that every $U \in \tau$ occurs. Finally $\langle\emptyset, X\rangle$ is a member because $X \in \rho$. □

Theorem 21.11 (Bicontinuity criterion)**.** *A map $f\colon (X,\mathcal{N}(\tau,\rho)) \to (Y,\mathcal{N}(\tau',\rho'))$ is N-continuous if and only if f is τ–τ' continuous and ρ–ρ' continuous.*

Proof. One has $f^{-1}\langle U',V'\rangle = \langle f^{-1}U', f^{-1}V'\rangle$, and inclusion is preserved. If f is continuous for both pairs of topologies, this triple lies in $\mathcal{N}(\tau,\rho)$. Conversely, if f is N-continuous, then $f^{-1}\langle\emptyset,V'\rangle = \langle\emptyset, f^{-1}V'\rangle$ forces $f^{-1}V' \in \rho$, and $f^{-1}\langle U', Y\rangle = \langle f^{-1}U', X\rangle$ forces $f^{-1}U' \in \tau$. □

Theorem 21.12 (Products of bitopological spaces)**.** *If $\tau \subseteq \rho$, then $\mathcal{N}(\tau,\rho) \boxtimes I = \mathcal{N}(\tau\times\tau_I,\ \rho\times\tau_I)$.*

Proof. The crisp interval consists of the triples $\langle W,W\rangle$, $W \in \tau_I$, and $\langle U,V\rangle \boxtimes \langle W,W\rangle = \langle U\times W, V\times W\rangle$. Every rectangle therefore lies in $\mathcal{N}(\tau\times\tau_I, \rho\times\tau_I)$, which is a neutrosophic topology by Proposition 21.10; hence the generated product is contained in it. Conversely, let $\langle P,Q\rangle$ have $P \in \tau\times\tau_I$, $Q \in \rho\times\tau_I$ and $P \subseteq Q$. Write $P = \bigcup_i U_i \times W_i$ with $U_i \in \tau$ and $Q = \bigcup_j V_j \times W'_j$ with $V_j \in \rho$. Because $\tau \subseteq \rho$, each $\langle U_i, U_i\rangle$ belongs to $\mathcal{N}(\tau,\rho)$, and so does each $\langle\emptyset, V_j\rangle$. The join of the rectangles $\langle U_i\times W_i, U_i\times W_i\rangle$ and $\langle\emptyset, V_j\times W'_j\rangle$ is $\langle P, P\cup Q\rangle = \langle P, Q\rangle$. □

Corollary 21.13 (Neutrosophic homotopy as bitopological homotopy)**.** *Let $\tau \subseteq \rho$ on X and $\tau' \subseteq \rho'$ on Y. A map $H\colon X \times I \to Y$ is an N-homotopy between $(X, \mathcal{N}(\tau,\rho))$ and $(Y, \mathcal{N}(\tau',\rho'))$ if and only if it is simultaneously a homotopy for τ, τ' and for ρ, ρ'. The same holds fibrewise over B, provided p_X and p_Y are τ- and τ'-continuous.*

Proof. Combine Theorems 21.12 and 21.11, applied to $X \times I$. For the fibrewise statement, $\chi^N_{p_X^{-1}U} = \langle p_X^{-1}U, p_X^{-1}U\rangle$ lies in $\mathcal{N}(\tau,\rho)$ when $p_X^{-1}U \in \tau \subseteq \rho$. □

The neutrosophic homotopy theory of $\mathcal{N}(\tau,\rho)$ is thus the homotopy theory of the bitopological space (X,τ,ρ) in the sense of Kelly [13], with truth and falsity reading τ and indeterminacy reading ρ.

21.4 Indeterminacy obstructs a contraction

Example 21.14 (An interval that is not N-contractible)**.** Let $X = [0,1]$, let τ be the Euclidean topology and ρ the discrete topology, so that $\tau \subseteq \rho$, and consider $(X, \mathcal{N}(\tau,\rho))$. Its truth and falsity level topologies are Euclidean, hence contractible. Nevertheless id_X is not N-homotopic to any constant map. By Corollary 21.13, such a homotopy would be a homotopy for the discrete topology ρ, and for $x \neq c$ the path $t \mapsto H(x,t)$ would be a nonconstant continuous map from I to a discrete space. The obstruction lives entirely in the indeterminacy coordinate.

On the other hand, for $\tau = \{\emptyset, X\}$ and ρ Euclidean, the contraction $H(x,t) = (1-t)x$ is continuous for ρ, and for the indiscrete τ every map is continuous. By Corollary 21.13, $(X, \mathcal{N}(\{\emptyset, X\}, \rho))$ is N-contractible, although its truth level is indiscrete and its indeterminacy level is Euclidean.

Example 21.15 (A fibrewise version)**.** Let B be a space and $X = B \times [0,1]$ with $p(b,u) = b$. Let τ be the product topology and $\rho = \tau_B \times \tau_{\mathrm{disc}}$, so that $\tau \subseteq \rho$ and p is τ-continuous. In the crisp model, id_X is fibrewise homotopic to $r(b,u) = (b,0)$ (Section 2.3). For

$\mathcal{N}(\tau, \rho)$ it is not: a fibrewise N-homotopy would, by Corollary 21.13, be a fibrewise ρ-homotopy. For $u > 0$, the path $t \mapsto H((b, u), t)$ would then be a continuous path in the fibre $\{b\} \times [0, 1]$, whose ρ-subspace topology is discrete, joining (b, u) to $(b, 0)$. This is impossible.

Example 21.16 (Level profiles distinguish homotopy types)**.** Let $X_1 = ([0, 1], \chi^N_{\mathrm{Eucl}})$ and $X_2 = ([0, 1], \mathcal{N}(\tau_{\mathrm{Eucl}}, \rho_{\mathrm{disc}}))$. Their truth and falsity profiles coincide: both are constantly the contractible interval. Their indeterminacy profiles are the Euclidean and the discrete interval respectively, which are not homotopy equivalent (one is path connected, the other has uncountably many path components). By Corollary 21.8, X_1 and X_2 are not N-homotopy equivalent. No invariant computed from the truth coordinate alone, such as the support topology of Chapter 3, can detect this difference.

Remark 21.17 (The role of the exchange symmetry)**.** By Proposition 21.1, $\sigma_* \mathcal{N}(\tau, \rho)$ is a neutrosophic topology whose falsity coordinate reads ρ, and Examples 21.14 and 21.15 hold for it with "indeterminacy" replaced by "falsity". What the lattice structure cannot express is *which* of the two coordinates is the indeterminacy. The level profiles nevertheless record that the two coordinates carry different information, and that is the sense in which the examples above are genuinely neutrosophic. A theory in which indeterminacy plays a role of its own, not shared by falsity, requires one of the asymmetric devices of Remark 21.2.

Conclusion and Open Problems

Summary

This book has developed fibrewise homotopy theory over a classical base for spaces carrying neutrosophic topologies. The main points are as follows.

(1) *Foundations.* Triples are ordered by $T\uparrow$, $I\downarrow$, $F\downarrow$; neutrosophic topologies are closed under arbitrary joins and finite meets. The crisp embedding $U \mapsto (\chi_U, 1-\chi_U, 1-\chi_U)$ identifies classical topology with a full subcategory (Theorem 1.6). The other standard order is handled by Remark 1.5.

(2) *Homotopy over a base.* With the product-generated cylinder and an explicit pasting convention, fibrewise N-homotopy is an equivalence relation, and fibrewise N-homotopy equivalences are closed under composition and satisfy two-out-of-three (Theorem 2.4, Corollary 2.5).

(3) *Structural theory.* Separation conditions, pointed and relative homotopy, fibrations (with pullback stability), cofibrations and NDR pairs, mapping spaces, LS category (with homotopy invariance), local triviality, base change, path objects, mapping cylinders and homotopy pullbacks were developed. In each case the precise hypotheses needed in the neutrosophic category are recorded.

(4) *Multivalued and ideal methods.* Upper N-continuous correspondences form a category NRel/B (Theorem 14.5). Ideal topologies commute with restriction to fibres (Theorem 11.2). Explicit examples separate upper from lower continuity, and ideal homotopy from ordinary homotopy.

(5) *Genuinely neutrosophic phenomena.* The truth, indeterminacy and falsity level topologies are classical topologies (Theorem 21.4) compatible with products (Theorem 21.6). Every fibrewise N-homotopy is a classical fibrewise homotopy at every level (Theorem 21.7), so the level profiles are invariants of the neutrosophic homotopy type (Corollary 21.8). For the bitopological spaces $\mathcal{N}(\tau, \rho)$, neutrosophic homotopy coincides with simultaneous homotopy for the two topologies (Corollary 21.13), and the indeterminacy coordinate alone can obstruct a contraction (Examples 21.14 and 21.15).

A limitation should be stated plainly. In the order used throughout, indeterminacy and falsity enter all lattice-theoretic constructions symmetrically (Proposition 21.1). The theory therefore distinguishes the information carried by the two coordinates, but it cannot tell which of them is the indeterminacy without additional, asymmetric structure.

Open problems

(1) *Asymmetric indeterminacy.* Develop the theory for the order in which I is ordered like T, or with an operator on the "indeterminate zone" $\{x : I(x) > \max(T(x), F(x))\}$. Find a homotopy invariant that changes when I and F are exchanged.

(2) *Persistence of level profiles.* Study how the fibrewise homotopy type of (X, τ_α^T), (X, τ_β^I), (X, τ_γ^F) varies with the parameters, by analogy with persistent homology. Characterize the profiles that arise from neutrosophic spaces.

(3) *Products and pullbacks.* Determine when the product-generated cylinder satisfies the pasting lemma (Convention 2.2) and when products with I commute with pullbacks (condition (9.1)), beyond the crisp and bitopological cases. Without these conditions, change of base for homotopies (Section 18.7) remains open.

(4) *A neutrosophic Dold theorem.* Under which numerability conditions is a fibrewise map between neutrosophic fibrations that is an N-homotopy equivalence on every fibre a fibrewise N-homotopy equivalence (Section 18.8)?

(5) *Graded ideals.* Choose the threshold λ_x and the predicate Φ of Section 15.7 so that the graded local function yields a neutrosophic topology supporting interval products, and prove a trace theorem analogous to Theorem 11.2.

(6) *A neutrosophic interval.* Replace the crisp interval by a non-crisp neutrosophic interval, or by neutrosophic numbers $a + bI$, and compare the resulting homotopy relation with the one studied here.

(7) *Model structures.* Decide whether NTop/B, or the bitopological subcategory of Chapter 21, carries a Quillen model structure [16, 7] whose weak equivalences are the fibrewise N-homotopy equivalences.

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Fibrewise Neutrosophic Homotopy Theory

Foundations, Level Topologies, Multivalued and Ideal Methods

This monograph develops fibrewise homotopy theory for spaces whose open sets are neutrosophic triples of truth, indeterminacy and falsity degrees, over a fixed classical base. The guiding principle is deformation over the base: every point stays in its fibre throughout a homotopy.

Homotopies and equivalences, separation, fibrations and cofibrations, mapping spaces, Lusternik–Schnirelmann category, base change, path objects and homotopy pullbacks are treated with explicit hypotheses and complete proofs. Multivalued correspondences and ideal-modified topologies are developed in parallel and compared through explicit examples.

A final chapter introduces truth, indeterminacy and falsity level topologies, proves that fibrewise neutrosophic homotopies transfer to every level, and exhibits spaces in which indeterminacy alone obstructs a contraction. A conclusion collects open problems for further research.



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