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ABSOLUTES AND n-H-CLOSED SPACES

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ABSTRACT. In this paper the investigation of n-H-closed spaces that was started by Basile *et al.* (2019) is continued for every $n \in \omega$, $n \geq 2$. In particular, starting with the relationship between the absolute space PX of an arbitrary topological space X, reported by Ponomarev and Shapiro (1976) and introduced by Błaszczyk (1975, 1977), Ul'yanov (1975a,b) and Shapiro (1976), it is shown that the absolute PX is n-H-closed if and only if X is n-H-closed. For an arbitrary space X, a β-like extension (β for the Stone-Čech compactification) $\widehat{Y}$ is constructed for the semiregularization $PX(s)$ of the absolute PX such that $\widehat{Y}$ is a compact, extremally disconnected, completely regular (but not necessarily Hausdorff) extension of $PX(s)$, and $PX(s)$ is C^*-embedded in $\widehat{Y}$. The definition of the Fomin extension σX for a Hausdorff space X (Porter and Woods 1988) is extended to an arbitrary space X and $\sigma X \setminus X$ is shown to be homeomorphic to the remainder $\widehat{Y} \setminus PX(s)$. A similar result is established when X is an n-Hausdorff space defined by Basile *et al.* (2019). Further, we give a cardinality bound for any n-Hausdorff space X and show that the inequality $|X| \leq 2^{\chi(X)}$ for an H-closed space X proved by Dow and Porter (1982) can be extended to n-H-closed spaces.

1. Introduction

In 1963 Iliadis extended the concept of absolute spaces for Hausdorff spaces. For the past 50 years, the absolute spaces of Hausdorff spaces have played a major role in understanding Hausdorff spaces. For each Hausdorff space X, an extremally disconnected, Tychonoff space, denoted as EX and called the absolute space of X, is associated with X via a surjection (called the absolute map) from EX to X that is closed, irreducible, compact, and θ-continuous. The space EX is compact iff X is H-closed.

In recent years, the application of topology to areas outside of topology has expanded (see Rump 2009). Some of these areas include domain theory in computer science, spectrums in ring theory, lattice-ordered group theory, etc.. In 2007 Rump used category theory to extend the notion of absolutes to arbitrary spaces and proved how it can be used to obtain a well-known theorem by Conrad (1973). The expansion of the concept of absolute spaces to arbitrary spaces was paved in 1975–77 by Błaszczyk (1975, 1977), Shapiro (1976) and Ul'yanov (1975a,b).

In this paper we use the explicit description of an absolute space, developed by Shapiro (1976), that uses maximal filters of open sets on the space X (called open ultrafilters on X); this development is similar to the development of the Iliadis absolute space (Iliadis 1963). As noted by Błaszczyk (1975), Ul'yanov (1975a,b), Shapiro (1976), and Błaszczyk (1977), the absolute spaces are homeomorphic relative to the absolute map. In particular, this absolute space is the same absolute space as produced by Rump, some 30 years later.

The absolute space is not a generalization of the Iliadis absolute for a Hausdorff space but is very close (see Porter and Woods 1988, for details). However, this absolute space is an extension of the Banascheski absolute space described by Porter and Woods (1988) for a Hausdorff space. In a recent paper (Basile *et al.* 2019) the authors extended the concept of H-closed spaces to n-Hausdorff spaces defined by Bonanzinga (2013); the n-Hausdorff spaces is a class of spaces extending the property of Hausdorff. The authors realized that to expand the theory of n-H-closed spaces, a theory of absolutes spaces for non-Hausdorff spaces might be needed.

This paper starts, after a preliminary section, with a quick development of the absolute space PX for an arbitrary space X and connects n-Hausdorff and n-H-closed spaces to the absolute space PX. The middle third of the paper is a series of results applying the theory of absolutes spaces to show that the remainder of the n-Fomin extension of an arbitrary n-Hausdorff space is completely regular and is a subspace of an extremally disconnected space whose T_0-identification is Hausdorff. The final third of the paper shows that n-H-closed, semiregular spaces are minimal n-Hausdorff but the converse is false. The paper ends with results about the cardinality of n-H-closed spaces using absolute spaces.

2. Preliminaries

For a topological space X, we will use $\tau(X)$ (τ if there is no confusion) to denote the topology on X. For $x \in X$, we define $\tau(x) = \{U \in \tau(X) : x \in U\}$. An **open filter** $\mathcal{F}$ on X is a filter on the lattice $\tau(X)$ and an **open ultrafilter** on X is a maximal filter on $\tau(X)$. In particular, $\tau(x)$ is an open filter on X. Let $B(X)$ denote the set of all open ultrafilters on X. For an open filter $\mathcal{F}$ on X, $a_X\mathcal{F} = \bigcap_{F\in\mathcal{F}} cl(F)$ denotes the **adherence** of $\mathcal{F}$; when $a_X\mathcal{F}$ is nonempty, we say that $\mathcal{F}$ is fixed. The subscript in the notation $a_X\mathcal{F}$ is omitted when no confusion arises. For $U \in \tau(X)$, let $rU = int_X cl_X U$; it is easy to verify that $r(rU) = r(U)$ and $r(U \cap V) = rU \cap rV$. Sets of the form rU are called **regular open**. For a space X, let $X(s)$ denote the set X with the topology generated by $\{rU : U \in \tau(X)\}$; $X(s)$ is called the **semiregularization** of X. The identity function $X \to X(s)$ is a continuous bijection (see Sect. 2.2 of Porter and Woods 1988). A space X is **semiregular** if $X = X(s)$.

Our first preliminary result is a collection of known results about open ultrafilters, some are in Sect. 2 of Porter and Woods (1988) and others are straightforward to verify.

Proposition 1. *Let $\mathcal{U}$ be an open ultrafilter on a space X and let $\mathcal{U}_s$ be the open filter on $X(s)$ generated by the open ultrafilter base $r\mathcal{U} = \{rU : U \in \mathcal{U}\}$. Then:*
(a) *$\mathcal{U}_s$ is an open ultrafilter on $X(s)$ and the function $f_s : B(X) \to B(X(s)) : \mathcal{U} \mapsto \mathcal{U}_s$ is a bijection,*
(b) *for each $\mathcal{U} \in B(X)$, $a_X\mathcal{U} = a_{X(s)}\mathcal{U}_s = \bigcap_{U\in\mathcal{U}} cl_X U = \bigcap_{U\in\mathcal{U}} cl_X(rU)$,*
(c) *$\mathcal{U}$ is the unique open ultrafilter on X meeting $\{rU : U \in \mathcal{U}\}$,*

(d) $\mathcal{U}_s$ *is the unique open ultrafilter on* $X(s)$ *meeting* $\{rU : U \in \mathcal{U}\}$, *and*
(e) $\mathcal{U}_s = \mathcal{U} \cap \tau(X(s))$.

Throughout the paper, no additional hypotheses (for example, separation axioms) are included unless explicitly mentioned. A space X is **extremally disconnected** (Porter and Woods 1988) if disjoint open sets in X have disjoint closures (equivalently, the closure of an open set is also open); it is straightforward to show that a semiregular, extremally disconnected space is completely regular (not necessarily Hausdorff) and has a topological base of clopen sets. It is well-known (Porter and Woods 1988) that if Y is extremally disconnected, then $Y(s)$ is extremally disconnected and completely regular (but not necessarily Hausdorff).

For spaces X,Y, recall that a function $f : X \to Y$ is **separable** if for each $y \in Y$, distinct points in $f^{\leftarrow}(y)$ can be separated by disjoint open sets in X; **irreducible** if it is surjective and whenever A is a proper closed subset of X, then $f(A) \neq Y$; **compact** if $f^{\leftarrow}(y)$ is a compact subset of X for every $y \in Y$; and **perfect** if it is closed and compact.

We will need the following concepts (Bonanzinga 2013; Basile *et al.* 2019).

Definition 2. (Bonanzinga 2013) Let $n \in \omega$, $n \geq 2$. A space X is n**-Hausdorff** if for any distinct points $x_1,...,x_n \in X$, there are open subsets U_i of X containing x_i for every $i = 1,...,n$ such that $\bigcap_{i=1}^{n} U_i = \varnothing$. In particular, a space X is 2-Hausdorff iff X is Hausdorff.

Note that since $rU \cap rV = r(U \cap V)$ for open sets U,V in a space X, it follows that X is n-Hausdorff iff $X(s)$ is n-Hausdorff.

Definition 3. (Basile *et al.* 2019) Let $n \in \omega$, $n \geq 2$. An extension Y of X is said to be n**-Hausdorff except for** X if for points $p \in Y \backslash X$ and $q_1,...,q_{n-1} \in Y$, there are open sets $U,V_1,...,V_{n-1}$ in Y such that $p \in U$ and $q_i \in V_i$, $i = 1,...,n-1$, and $U \cap V_1 \cap \cdots \cap V_{n-1} = \emptyset$.

Theorem 4. (Basile *et al.* 2019) Let X be a space and $n \in \omega$, where $n \geq 2$. The following are equivalent:

(a) for every open filter $\mathcal{F}$ on X, $|a\mathcal{F}| \geq n-1$,
(b) for every open ultrafilter $\mathcal{U}$ on X, $|a\mathcal{U}| \geq n-1$,
(c) for every $A \in [X]^{<n-1}$ and for any family $\mathcal{U}$ of open subsets of X such that $X \setminus A \subseteq \bigcup \mathcal{U}$, there exists $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $X = \bigcup_{V \in \mathcal{V}} clV$, and
(d) X is closed in every extension of X that is n-Hausdorff except for X.

Comment: Part (b) of Theorem 4 is added to the result proven by Basile *et al.* (2019).

Definition 5. (Basile *et al.* 2019) Let $n \in \omega$, $n \geq 2$. A space is n**-qH-closed** if X satisfies one (and hence all) of the conditions of Theorem 4. A 2-qH-closed space is called **qH-closed**.

Notes: (a) If a space X is n-qH-closed then by Theorem 4 X is also qH-closed.
(b) By Proposition 1, a space X is n-qH-closed iff $X(s)$ is n-qH-closed.
(c) If Y is an extension of X then by Theorem 4 Y is qH-closed iff for every free open ultrafilter $\mathcal{U}$ on X, $a_Y\mathcal{U} \neq \varnothing$.

Definition 6. (Basile *et al.* 2019) Let $n \in \omega$, $n \geq 2$. An n-Hausdorff space X is called n**-H-closed** if X is closed in every n-Hausdorff space Y in which X is embedded.

Theorem 7. (Basile *et al.* 2019) An n-Hausdorff space X is n-H-closed iff it is n-qH-closed.

Recall the following definitions and concepts given by Porter and Woods (1988). Let Y be an extension of a space X. For $p \in Y$, let $O^p = \{U \cap X : p \in U \in \tau(Y)\}$ and for $U \in \tau(X)$, let $oU = \{p \in Y : U \in O^p\}$. Note that for $U, V \in \tau(X)$, $o(U \cap V) = oU \cap oV$, $o(\varnothing) = \varnothing$, and $oX = Y$. So, $\{oU : U \in \tau(X)\}$ forms a basis for a topology, denoted as $\tau^{\#}(Y)$, on Y. Denote by $\tau^{+}(Y)$ the topology on Y generated by the base $\mathcal{B} = \{U \cup \{p\} : U \in O^p$ and $p \in Y\}$. We have $\tau^{\#}(Y) \subseteq \tau(Y) \subseteq \tau^{+}(Y)$, and $Y^{\#}$(resp. Y^{+}) is used to denote the set Y with $\tau^{\#}(Y)$ (resp. $\tau^{+}(Y)$). Y^{+} is called a **simple extension** of the space X and $Y^{\#}$ is called a **strict extension** of the space X. Following Porter and Woods (1988, Sect. 7.2), if Y is an extension of X, then $Y(s)$ is a strict extension of $X(s)$.

If X is a space (not necessarily Hausdorff), then both the Katětov and Fomin H-closed extensions are defined as follows:

The **Katětov extension** for an arbitrary space is defined by $\kappa X = X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$ with the simple extension topology, where a set $U \subseteq \kappa X$ is open if $U \cap X$ is open in X and $U \cap X \in \mathcal{U}$ if $\mathcal{U} \in U$. In the non-Hausdorff setting there are simple extensions Y of X such that the function $Y \backslash X \to \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\} : p \mapsto \{U \cap X : p \in U \in \tau(Y)\}$ is a surjection but not one-to-one. The n-Katětov extension described by Basile *et al.* (2019) is such an extension. The **Fomin extension** σX of an arbitrary space X is defined as the strict extension of the Katětov extension; that is, an open basis for the topology of σX is $\{o(U) : U \in \tau(X)\}$, where $o(U) = U \cup \{\mathcal{U} \in \sigma X \backslash X : U \in \mathcal{U}\}$.

Let X be a space. A strict extension Y of X is said to be **σ-like** if for $p \in Y \backslash X$, O^p is an open ultrafilter on X. The Fomin extension of an arbitrary space is σ-like, and in the class of n-Hausdorff spaces, the Fomin extension n-σX of a space X defined by Basile *et al.* (2019) is also σ-like. If Y is a σ-like extension of X, there may be distinct points $p, q \in Y \backslash X$ such that $O^p = O^q$. As Y has the strict extension topology generated by the open base $\{o(U) : U \in \tau(X)\}$, for each $U \in \tau(Y)$, $p \in o(U)$ iff $q \in o(U)$. Thus, $\{o(U) : U \in O^p\}$ is an open base for both p and q; let $\hat{p} = \{q \in \sigma X \backslash X : O^p = O^q\}$. If $O^p \neq O^q$, then there are $U \in O^p$ and $V \in O^q$ such that $U \cap V = \varnothing$. The set $\{\hat{p} : p \in Y \backslash X\}$ is a partition into disjoint closed sets that can be separated by disjoint open sets.

Proposition 8. *Let Y be a σ-like extension of a space X, $T \in \tau(Y)$, and $U \in \tau(X)$. Then:*
(a) $T \subseteq o(T \cap X)$ and $o(U) \backslash X = o(rU) \backslash X$,
(b) $cl_Y o(U) = cl_X U \cup o(U) = cl_X U \cup o(rU)$ and
(c) $int_Y cl_Y o(U) = o(int_X cl_X U)$.

Proof. (a) Since $o(T \cap X) \cap X = T \cap X$, it suffices to show that if $\mathcal{U} \in T$, then $\mathcal{U} \in o(T \cap X)$. But $\mathcal{U} \in T$ implies that $T \cap X \in \mathcal{U}$ and hence, $\mathcal{U} \in o(T \cap X)$. By Proposition 1(c), $o(U) \backslash X = o(rU) \backslash X$.
(b) Clearly, $cl_X U \cup o(U) \subseteq cl_Y o(U) \subseteq cl_Y (o(U) \cap U) \cup o(U) = cl_Y U \cup o(U) = (cl_X U \backslash X) \cup cl_X U \cup o(U)$. Note that for $\mathcal{V} \in \sigma X \backslash X$, $\mathcal{V} \in cl_Y U$ iff $V \cap U \neq \varnothing$ for every $V \in \mathcal{V}$ iff $U \in \mathcal{V}$ iff $\mathcal{V} \in o(U)$. This shows that $cl_Y o(U) = cl_X U \cup o(U)$.
(c) To show $int_Y cl_Y o(U) = o(int_X cl_X U)$, we first apply a result from the proof of 2.2(i)(2) given by Porter and Woods (1988), to obtain that $(int_Y cl_Y o(U)) \cap X$
$= int_X cl_X (o(U) \cap X = int_X cl_X U$. By (a), $int_Y cl_Y o(U) \subseteq o(int_X cl_X U)$. To prove the converse, let $p \in o(int_X cl_X U) \backslash X$. Then, by (b), $p \in o(U) \subseteq int_{\sigma X} cl_{\sigma X} o(U)$ and hence, $o(int_X cl_X U) \subseteq int_Y cl_Y o(U)$. □

Proposition 9. *Let Y be a σ-like extension of a space X. Let $g : Y \to Y(s) : y \mapsto y$ denote the identity function (which is a continuous bijection).*
(a) For $p \in Y\backslash X$, $O^{g(p)} = O^p \cap \tau(X(s))$ is the unique open ultrafilter on $X(s)$ containing $\{rT : T \in O^p\}$;
(b) $g|_{Y\backslash X} : Y\backslash X \to Y(s)\backslash X(s)$ is a homeomorphism;
(c) $Y(s)$ is a σ-like extension of $X(s)$.

Proof. (a) The proof is straightforward. (b) For $p \in Y\backslash X, O^{g(p)} \in Y(s)\backslash X(s)$ by Proposition 1(b). Note that $\tau(Y\backslash X)$ is generated by $\{o(U)\backslash X : U \in \tau(X)\}$ and $\tau(Y(s)\backslash X(s))$ is generated by $\{o(int(cl(U)))\backslash X(s) : U \in \tau(X)\}$. This shows that the continuous bijection $id|_{Y\backslash X}$: $Y\backslash X \to Y(s)\backslash X(s)$ is a homeomorphism. (c) By (a), for $s(p) \in Y(s)\backslash X(s)$, $O^{s(p)}$ is an open ultrafilter on $X(s)$. By Sect. 7.2. of Porter and Woods (1988), $Y(s)$ is a strict extension of $X(s)$. □

For a subset A of a space X we will denote by $[A]^{<\lambda}$ ($[A]^{\leq\lambda}$) the family of all subsets of A of cardinality $< \lambda$ ($\leq \lambda$). We use the standard notation presented by Porter and Woods (1988) and Engelking (1989).

3. The absolute space for any space

Let X be a space and $B(X)$ denote the set $\{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on $X\}$ with the topology generated by the open basis $\{oU : U \in \tau(X)\}$ where $oU = \{\mathcal{U} \in B(X) : U \in \mathcal{U}\}$. $B(X)$ is the well-known Stone space generated by the Boolean algebra of regular open sets on X (see Porter and Woods 1988). $B(X)$ is a compact extremally disconnected Hausdorff space even though X may not be Hausdorff. Let EX be the dense subspace of $B(X)$ of fixed open ultrafilters on X (this is the absolute space of X, defined by Iliadis (1963), when X is Hausdorff). For each $x \in X$, let $k^{-1}(x)$ denote the subset $\{\mathcal{U} \in B(X) : x \in a\mathcal{U}\}$ of $B(X)$; the subspace $k^{-1}(x)$ of EX is compact Hausdorff. In a non-Hausdorff space X, it may happen that $k^{-1}(x) = k^{-1}(y)$ for distinct points $x, y \in X$.

We use the explicit construction of Shapiro (1976) to define the **absolute space** PX for an arbitrary space X.

In the product space $X \times EX$, we define the subset $PX = \{(x,\mathcal{U}) \in X \times EX : x \in a\mathcal{U}, \mathcal{U} \in B(X)\}$. For open sets $U,V \in \tau(X)$, the basic open set in PX is $(U \times oV) \cap PX$, which is denoted as $\langle U,V\rangle$. Also for an open set $U \in \tau(X)$, we let OU denote $\{(x,\mathcal{U}) \in PX : x \in a\mathcal{U}, \mathcal{U} \in oU\}$. The projection function $p : PX \to X : (x,\mathcal{U}) \mapsto x$ is the **absolute map**.

Here are some basic properties of the absolute space PX for an arbitrary space X and the absolute map $p : PX \to X$. The proofs are straightforward.

Proposition 10. *Let X be a space, $U,V,S,T \in \tau(X)$, and $\mathcal{U} \in EX$, then*
(a) $p : PX \to X$ is continuous and onto
(b) $p^{\leftarrow}[U] \cap OV = \langle U,V\rangle$, $OV = \langle X,V\rangle$, and $p^{\leftarrow}[U] = \langle U,X\rangle$
(c) $\langle U,V\rangle \cap \langle S,T\rangle = \langle U \cap S, V \cap T\rangle$ and $\langle U,V\rangle = \langle U, rV\rangle$
(d) $p^{\leftarrow}(U \cap V) \subseteq p^{\leftarrow}(U) \cap OV \subseteq O(U \cap V)$
(e) $p^{\leftarrow}[U] \cap OV \neq \varnothing$ iff $\langle U,V\rangle \neq \varnothing$ iff $U \cap V \neq \varnothing$.
(f) $cl\langle U,X\rangle = \langle X,U\rangle$
(g) $O(U \cap V) = OU \cap OV$ and $O(U \cup V) = OU \cup OV$

(h) If $cl(U) = cl(V)$, then $OU = OV$
(i) $PX \setminus OU = O(X \setminus clU)$ and $OU = O(rU)$
(j) $x \in a\mathcal{U}$ iff $\tau(x) \subseteq \mathcal{U}$
(k) $\tau(x) \subseteq \bigcap_{\tau(x)\subseteq \mathcal{U}} \mathcal{U}$ and $V \in \bigcap_{\tau(x)\subseteq \mathcal{U}} \mathcal{U}$ iff $rV \in \tau(x)$
(l) For $x \in X$, $p^{\leftarrow}(x) = \{x\} \times k^{\leftarrow}(x)$ and $\{x\} \times k^{\leftarrow}(x)$ is homeomorphic to $k^{\leftarrow}(x)$ and is a compact subset.
(m) Let $\{U_i : i \in I\}$ be a family of open subsets of X. Then $cl_{PX} \bigcup_I OU_i = O(\bigcup_I U_i)$.
(n) Let U, V be open subsets of X. Then $cl_{PX}\langle U, V\rangle = O(U \cap V)$.

Note that the set OU is closed by Proposition 10(i) (also by Proposition 10(b, f)); that is, OU is clopen.

Proposition 11. *Let X be a space. Then:*
(a) If W is open in PX, then $cl_{PX}W = OU$ for some U open in X.
(b) For $U \in \tau(X), p[OU] = cl_X U$.
(c) For $U \in \tau(X)$ and $x \in X$, $p^{\leftarrow}(x) \subseteq OU$ iff $x \in rU$.
(d) For $U, V \in \tau(X)$, $OU = OV$ iff $rU = rV$.

Proof. For (a), there is a family $\{\langle U_a, V_a\rangle : a \in A\}$ of basic open sets such that set $W = \bigcup_{a\in A}\langle U_a, V_a\rangle$. By Proposition 10(m,n), $clW \subseteq cl \bigcup_{a\in A} cl\langle U_a, V_a\rangle = cl \bigcup O(U_a \cap V_a) = clO(\bigcup(U_a \cap V_a)) = O(\bigcup(U_a \cap V_a))$. Also, by Proposition 10(m,n), $clW \supseteq \bigcup_{a\in A} cl\langle U_a, V_a\rangle \supseteq \bigcup_{a\in A} O(\langle U_a, V_a\rangle)$; thus, $clW = cl^2W \supseteq cl(\bigcup_{a\in A} O(\langle U_a, V_a\rangle)) = O(\bigcup_{a\in A}(U_a \cap V_a))$. To prove (b), note that since $p : PX \to X$ is a closed function, it suffices to show that $U \subseteq p[OU] \subseteq cl_X U$. If $x \in U$, let $\mathcal{U}$ be any open ultrafilter on X such that $U \in \mathcal{U}$. Then $(x, \mathcal{U}) \in OU$. So, $U \subseteq p[OU]$. Now, if $(x, \mathcal{U}) \in OU$, then $U \in \mathcal{U}$ and $p((x, \mathcal{U})) = x \in a\mathcal{U} \subseteq cl_X U$. Thus, $p[OU] \subseteq cl_X U$. To prove (c), suppose $p^{\leftarrow}(x) \subseteq OU$. Then $x \in a\mathcal{U}$ for every open ultrafilter $\mathcal{U}$ such that $\tau(x) \subseteq \mathcal{U}$, $U \in \mathcal{U}$. That is, $U \in \bigcap_{\tau(x)\subseteq\mathcal{U}} \mathcal{U}$. By Proposition 10(k), $rU \in \tau(x)$ and $x \in rU$. Conversely, suppose $x \in rU$. Then $\mathcal{U} \in p^{\leftarrow}(x)$ iff $x \in a\mathcal{U}$ iff $\tau(x) \subseteq \mathcal{U}$. Since $rU \in \tau(x)$, by Proposition 10(k), $U \in \bigcap_{\tau(x)\subseteq\mathcal{U}} \mathcal{U} = \bigcap_{\mathcal{U}\in p^{\leftarrow}(x)} \mathcal{U}$. Thus, for $\mathcal{U} \in p^{\leftarrow}(x)$, $x \in a\mathcal{U}$ and $\mathcal{U} \in o(U)$. So, $p^{\leftarrow}(x) \subseteq OU$. To prove (d), suppose $OU = OV$. By Proposition 11(b), $clU = clV$; thus, $rU = rV$. Conversely, suppose $rU = rV$. So, $O(rU) = O(rV)$. By Proposition 10(i), $O(U) = O(V)$. □

Theorem 12. (Ul'yanov 1975a; Shapiro 1976; Błaszczyk 1977) For a space X, the absolute space PX is extremally disconnected and the absolute map $PX \to X$ is continuous, onto, irreducible, separable, and perfect.

Proposition 13. *Let X be a space. Then:*
(a) *if $\mathcal{W}$ is an open ultrafilter on PX, there is an unique open ultrafilter $\mathcal{U}$ on X such that $\{rT : T \in \mathcal{W}\} = \{OU : U \in \mathcal{U}\}$ and $(x, \mathcal{U}) \in a\mathcal{W}$ iff $x \in a\mathcal{U}$,*
(b) *if $\mathcal{U}$ is an open ultrafilter on X, there is an unique open ultrafilter $\mathcal{W}$ on PX such that $\{rT : T \in \mathcal{W}\} = \{OU : U \in \mathcal{U}\}$,*
(c) *there is a bijection $g : B(PX) \to B(X)$ such that for each $\mathcal{W} \in B(PX)$, $\{rT : T \in \mathcal{W}\} = \{OU : U \in g(\mathcal{W})\}$, and*
(d) *for $\mathcal{W} \in B(PX)$, $a\mathcal{W} = \{(x, g(\mathcal{W})) \in PX : x \in a(g(\mathcal{W}))\}$.*

Proof. The proofs of (a) and (b) follow directly from the properties in Propositions 10 and 11. The proof of statement (c) is immediate from (a) and (b). The proof of (d) follows from the definition of $(x, g(\mathcal{W})) \in PX$. $\square$

Proposition 14. *Let $\mathcal{U}$ be an open ultrafilter on X and $\mathcal{V} = \{OU : U \in \mathcal{U}\}$ the corresponding open ultrafilter base on both PX and $PX(s)$. Then $a_{PX(s)}\mathcal{V} \neq \varnothing$ iff $a_{PX}\mathcal{V} \neq \varnothing$ iff $a_X\mathcal{U} \neq \varnothing$.*

Proof. This result follows from Propositions 1 and 13. $\square$

Another immediate consequence of Proposition 13(d) is the following result.

Proposition 15. *A space X is n-qH-closed iff PX is n-qH-closed.*

Proposition 16. *A space X is n-Hausdorff iff PX is n-Hausdorff.*

Proof. If X is n-Hausdorff, it follows quickly by Proposition 10(b,e) that PX is n-Hausdorff. Conversely suppose that PX is n-Hausdorff. The proof that X is n-Hausdorff is the same for any $n \geq 2$. Here is a proof by contradiction when $n = 2$. Assume there are two distinct points x_1, x_2 such that if $U_i \in \tau(x_i)$ for $i = 1,2$, $U_1 \cap U_2 \neq \varnothing$.

Let $\mathcal{U}$ be an open ultrafilter that contains the open filter base $\{U_1 \cap U_2 : U_i \in \tau(x_i), i = 1,2\}$. There are open sets $\langle V_i, W_i \rangle, i = 1,2$, such that $(x_i, \mathcal{U}) \in \langle V_i, W_i \rangle$ for $i = 1,2$ and $\langle V_1, W_1 \rangle \cap \langle V_2, W_2 \rangle = \varnothing$. By Proposition 10(c,e), $V_1 \cap W_1 \cap V_2 \cap W_2 = \varnothing$. As $V_i \in \tau(x_i)$ and $W_i \in \mathcal{U}$ for $i = 1,2$, it follows that $V_1 \cap V_2 \in \mathcal{U}$ and $W_1 \cap W_2 \in \mathcal{U}$. Hence, $V_1 \cap V_2 \cap W_1 \cap W_2 \neq \varnothing$, a contradiction. $\square$

Comment: By the above propositions, if X is n-qH-closed, then PX is also n-qH-closed. Now PX is extremally disconnected as the closure of an open set is clopen. It follows that its semiregularization $PX(s)$ is also extremally disconnected, qH-closed, and has a clopen basis. Then $PX(s)$ is completely regular (but not necessarily Hausdorff). By next proposition, $PX(s)$ is also compact.

Proposition 17. *If a space X is n-qH-closed and regular, then X is compact.*

Proof. Let X be a n-qH-closed and regular space and let $\mathcal{U}$ be an open cover of X. For every $x \in X$ there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. By regularity, there exists an open neighbourhood V_x of x such that $V_x \subseteq clV_x \subseteq U_x$. Put $\mathcal{V} = \{V_x : x \in X\}$. By hypothesis X is n-qH-closed, then there exists $\{V_x : x \in W\} \in [\mathcal{V}]^{<\omega}$ such that $X = \bigcup_{x \in W} clV_x$, hence $X = \bigcup_{x \in W} U_x$. $\square$

Corollary 18. *If a regular space X is n-H-closed, then X is compact and n-Hausdorff.*

Note that a compact n-Hausdorff space need not be regular. Indeed, in Example 5 of Bonanzinga *et al.* (2016), a compact 3-Hausdorff first countable non 3-normal space is constructed and since this space is not Hausdorff, it is not regular. We can notice that this space is not 3-H-closed. In fact, the open ultrafilter containing the neighborhoods of 1/4 plus the open set [0,1/4) has only one adherent point.

4. Applying the absolute PX to the Fomin extension

Our main goal in this section is to use the Fomin extension σX for a space X to construct a completely regular extension Z of $PX(s)$ such that $PX(s)$ is C^*-embedded in Z and the space $\sigma X \backslash X$ and $Z \backslash PX(s)$ are homeomorphic and Hausdorff. That is, an arbitrary space X

has an extension T such that $T\backslash X$ is Tychonoff and a subspace of an extremally disconnected space.

Construction. We start with a σ-like extension Y of X. The intermediate goal is to construct a σ-like extension $\widetilde{Y}$ of $PX(s)$ such that $Y\backslash X$ and $\widetilde{Y}\backslash PX(s)$ are homeomorphic. For each $p \in Y\backslash X$, O^p is an open ultrafilter on X, and by Proposition 13 there is an unique open ultrafilter $O^{h(p)}$ on PX such that $\{rS : S \in O^{h(p)}\} = \{OS : S \in O^p\}$. We denote the open ultrafilter on $PX(s)$ generated by the base $\{rS : S \in O^{h(p)}\} = \{OS : S \in O^p\}$ by $O_s^{h(p)}$. Let $\widetilde{Y} = \{h(p) : p \in Y\backslash X\} \cup PX(s)$ with the strict extension topology. Since $O_s^{h(p)}$ is an open ultrafilter on $PX(s)$ for each $p \in Y\backslash X$, $\widetilde{Y}$ is a σ-like extension of $PX(s)$. It follows that the function $h : Y\backslash X \to \widetilde{Y}\backslash PX(s) : p \mapsto h(p)$ is a bijection. More is true as noted in the next result.

Theorem 19. *Let Y be a σ-like extension of a space X. The bijection $g : Y\backslash X \to \widetilde{Y}\backslash PX(s) : p \mapsto h(p)$ is a homeomorphism.*

Proof. For $p \in Y\backslash X, \{oU\backslash X : U \in O^p\}$ is an open neighborhood base for p in $Y\backslash X$; note that $oU\backslash X = \{q \in Y\backslash X : U \in O^q\}$. Likewise, for $h(p) \in \widetilde{Y}\backslash PX(s)$, $\{o(OU)\backslash PX(s) : U \in O^p\}$ is an open neighborhood base for $h(p) \in \widetilde{Y}\backslash PX(s)$; note that $o(OU)\backslash PX(s) = \{h(q) : OU \in O^{g(q)}, q \in Y\backslash X\} = \{h(q) : U \in O^q, q \in Y\backslash X\}$. This shows that $g : Y\backslash X \to \widetilde{Y}\backslash PX(s) : p \mapsto h(p)$ is a homeomorphism. □

Lemma 20. *Let Y be a σ-like extension of a space X and T be a clopen set in X. Then $o(T)$ is also clopen in Y.*

Proof. As $X\backslash T$ is also clopen, then $o(T) \cap o(X\backslash T) = \varnothing$ and $Y = o(T) \cup o(X\backslash T)$. □

Corollary 21. *Let Y be a σ-like extension of a space X where X is extremally disconnected and semiregular. Then Y is extremally disconnected and completely regular.*

Proof. If $p \in Y\backslash X$, then O^p is an open ultrafilter on X and $rO^p = \{rU : U \in O^p\}$ is an open ultrafilter base on X that is contained in an unique open ultrafilter on X. Since X is semiregular, by Lemma 20, $\{o(rT) : T \in \tau(X)\}$ is a clopen basis for Y. So, Y is completely regular. Let $S \in \tau(Y)$, then $cl_Y S = cl_Y(S \cap X) = cl_Y r(S \cap X)$. Now $r(S \cap X)$ is clopen in X, and by Lemma 20, $o(r(S \cap X))$ is clopen in Y. So, $cl_Y S = cl_Y o(S) = o(S)$ is clopen. This shows that Y is extremally disconnected. □

Theorem 22. *Let Y be an extension of X such that O^p is an open ultrafilter for each $p \in Y\backslash X$. Then X is C^*-embedded in Y.*

Proof. Let $f : X \to \mathbb{R}$ be continuous and bounded. For $x \in X$, let $\mathcal{F}_x$ be the filter generated by $\{f[U] : x \in U \in \tau(X)\}$ and note that $f(x) \in c(\mathcal{F}_x)$ by the continuity of f. For $p \in Y\backslash X$, let $\mathcal{F}_p$ be the filter generated by $\{f[U] : U \in O^p\}$. Now, $f[X] \subseteq [a,b]$. As $[a,b]$ is compact, $a\mathcal{F}_p \neq \varnothing$. The next step is to show that $|a\mathcal{F}_p| = 1$. Assume that $c,d \in a\mathcal{F}_p$. There are open sets U,V in $\mathbb{R}$ such that $c \in U, d \in V$ and $U \cap V = \varnothing$.
Now $f^{\leftarrow}[U]$ meets O^p and, hence, $f^{\leftarrow}[U] \in O^p$; similarly, $f^{\leftarrow}[V] \in O^p$. But $f^{\leftarrow}[U] \cap f^{\leftarrow}[V] = \varnothing$, a contradiction. Thus, $|a\mathcal{F}_p| = 1$. By compactness of $[a,b]$, $\mathcal{F}_p$ converges to some point, denoted as $F(p)$, in $\mathbb{R}$. For $x \in X$, define $F(x) = f(x)$. The function $F : Y \to \mathbb{R}$ is an extension of f.

We will now show that F is continuous. Let $y \in Y$ and T be an open set in $\mathbb{R}$ such that $F(y) \in T$. There is an open set S in $\mathbb{R}$ such that $F(y) \in S \subseteq cl_{\mathbb{R}}S \subseteq T$. For $y = x \in X$, there is an open set U in Y such that $F[U \cap X] = f[U \cap X] \subseteq S$. Let $q \in U \backslash X$ and $F(q) \in V \in \tau(\mathbb{R})$. Then $U \cap X \in O^q$ and there is $R \in \tau(Y)$ such that $F[R \cap X] = f[R \cap X] \subseteq V$ and $R \cap X \in O^q$. Thus, $R \cap U \cap X \in O^q$. So, it follows that $\varnothing \neq F[R \cap U \cap X] \subseteq F[R \cap X] \cap F[U \cap X] \subseteq V \cap S$. This shows that $F(q) \in cl_{\mathbb{R}}S \subseteq T$. That is, $F[U] \subseteq T$. For $y = p \in Y \backslash X$, $F(p) \in c(\mathcal{F}_p)$ and there is an open set U in Y such that $F[U \cap X] \subseteq S$. By repeating the same proof as above, it follows that $F[U] \subseteq T$. □

Proposition 23. *Let Y be a σ-like extension of a space X and $\widetilde{Y}$ be the σ-like extension of $PX(s)$ constructed in Construction. Then Y is qH-closed iff $\widetilde{Y}$ is qH-closed.*

Proof. By Theorem 4, Y is qH-closed iff for every free open ultrafilter $\mathcal{U}$ on X, $a_Y\mathcal{U} \neq \varnothing$. Likewise, a σ-like extension $\widetilde{Y}$ of $PX(s)$ is qH-closed iff every free open ultrafilter $\mathcal{W}$ on $PX(s)$, $a_{\widetilde{Y}}\mathcal{W} \neq \varnothing$ on $PX(s)$.

Suppose Y is qH-closed, and $\mathcal{W}$ is a free open ultrafilter on $PX(s)$ such that $a_{PX(s)}\mathcal{W} = \varnothing$. Then $\{rT : T \in \mathcal{W}\}$ is an open ultrafilter base that generates the free open ultrafilter $\mathcal{W}$ on $PX(s)$; so, $a_{PX(s)}\mathcal{W}_s = a_{PX(s)}\mathcal{W} = \varnothing$. By Proposition 13, there is an unique open ultrafilter $\mathcal{U}$ on X such that $r\mathcal{W} = \{rT : T \in \mathcal{W}\} = \{OT : T \in \mathcal{U}\}$. By Proposition 14, as $a_{PX(s)}\mathcal{W} = \varnothing, a_X\mathcal{U} = \varnothing$. Thus there is a $p \in Y \backslash X$ such that $O^p = \mathcal{U}$ as Y is qH-closed. Hence, $p \in a_Y\mathcal{U} \neq \varnothing$. By Theorem 19 and the construction of $\widetilde{Y}$, $O^{h(p)} = \mathcal{W}$. Thus, $h(p) \in a_{\widetilde{Y}}\mathcal{W}$. This shows that $\widetilde{Y}$ is qH-closed. Conversely, suppose $\widetilde{Y}$ is qH-closed and $\mathcal{U}$ is a free open ultrafilter on X. By Proposition 13, there is an open ultrafilter $\mathcal{W}$ on $PX(s)$ such that $\{rW : W \in \mathcal{W}\} = \{OU : U \in \mathcal{U}\}$ is an open ultrafilter base on $PX(s)$. By Proposition 14, $a_{PX(s)}\mathcal{W} = \varnothing$ since $a_X\mathcal{U} = \varnothing$. As $\widetilde{Y}$ is qH-closed, by Theorem 19, there is $p \in Y \backslash X$ such that $O^{h(p)} = \mathcal{W} \in \widetilde{Y} \backslash PX(s)$. By the construction of $\widetilde{Y}$, $O^p = \mathcal{U}$ and $p \in a_Y\mathcal{U}$. This shows that Y is qH-closed. □

Let Y be a qH-closed, σ-like extension of a space X. By Theorem 12, PX is extremally disconnected, and, hence, the regular open subsets of PX are clopen. Thus, $(PX)(s)$ has a basis of clopen sets and is completely regular. Also, if U is open in $(PX)(s)$, then $cl_{PX}U = cl_{(PX)(s)}U$ is clopen in both PX and $(PX)(s)$. That is, $(PX)(s)$ is also extremally disconnected. As completely regular spaces are also semiregular, by Corollary 21, $\widetilde{Y}$ is extremally disconnected and completely regular. By Proposition 23, $\widetilde{Y}$ is qH-closed, thus using Proposition 17 we have that $\widetilde{Y}$ is compact. By Theorem 22, $(PX)(s)$ is C^*-embedded in $\widetilde{Y}$(defined in Theorem 19). Thus, for a qH-closed, σ-like extension Y of a space X, the extension $\tilde{Y}$ of $PX(s)$ is compact, extremally disconnected and $PX(s)$ is C*-embedded in $\tilde{X}$ and $Y \backslash X$ is homeomorphic to $\tilde{Y} \backslash PX(s)$. Two special cases of this theory are stated in the next result.

Theorem 24. (a) Let X be a space. The Fomin extension σX of X is qH-closed. The Fomin extension $\sigma PX(s)$ is compact, extremally disconnected, and completely regular; $PX(s)$ is C*-embedded in $\sigma PX(s)$; and $\sigma X \backslash X$ is homeomorphic to $\sigma PX(s) \backslash PX(s)$. Also, $\sigma X \backslash X$ is Hausdorff.
(b) Let X be an n-Hausdorff space X. The n-$\sigma PX(s)$ extension of $PX(s)$ is n-qH-closed. The

Fomin extension n-$\sigma PX(s)$ is compact, extremally disconnected, and completely regular; $PX(s)$ is C*-embedded in n-$\sigma PX(s)$; and n-$\sigma X\backslash X$ is homeomorphic to $\sigma PX(s)\backslash PX(s)$.

Proof. Note that both σX and n-σX are qH-closed, σ-like extensions of X and $\widetilde{Y} = \sigma(PX(s))$ and $\tilde{Y} = \mathrm{n}-\sigma(PX(s))$, respectively. By Theorem 19, $\sigma X\backslash X \cong \sigma(PX(s))\backslash(PX)(s)$ and n-$\sigma X\backslash X \cong \sigma(PX(s))\backslash(PX)(s)$. The conclusions follow. □

Final comment: The conclusions of Theorem 24 show that for an arbitrary space X, the extensions σX and n-σX have very nice remainders that are completely regular and subspaces of extremally disconnected spaces. More is true in the case of σX. As $\sigma X\backslash X = \{O^p : p \in \sigma X\backslash X\} = \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$ and when $O^p \neq O^q$ for distinct points p,q, $\sigma X\backslash X$ is Hausdorff. That is, $\sigma X\backslash X$ is Tychonoff and a subspace of an extremally disconnected space even though X may be far from being Hausdorff. When X is a Hausdorff space, the T_0 identification (sometimes called the Kolmogorov quotient) of n-$\sigma X\backslash X$ is Hausdorff. To apply the T_0 identification, first define the equivalence relation on n-$\sigma X\backslash X$ by saying that two points y,z are equivalent if $cl(\{p\}) = cl(\{q\})$. The induced quotient space of n-$\sigma X\backslash X$ is Hausdorff.

5. Functions and minimal n-Hausdorff spaces

Definition 25. A space X is called **minimal *n*-Hausdorff** if X has no strictly coarser n-Hausdorff topology.

We know (see Porter and Woods 1988) that a 2-Hausdorff space X is minimal 2-Hausdorff iff X is 2-H-closed and semiregular. We ask if this is true when 2 is replaced by an arbitrary n. The following shows this is true in only one direction.

Proposition 26. (a) If X is n-Hausdorff, then $X(s)$ is n-Hausdorff.
(b) If X is n-H-closed, then $X(s)$ is n-Hausdorff.

Proof. For (a), we can use that $r(\bigcap_n U_i) = \bigcap_n rU_i$. For (b), if $\mathcal{U}$ is an open ultrafilter on X, then $\mathcal{U}_s = \{rU : U \in \mathcal{U}\}$ is an ultrafilter base for $\mathcal{U}$ on X and for $\mathcal{U} \cap \tau(X(s))$ on $X(s)$. Thus, $a\mathcal{U} = a\mathcal{U}_s = a(\mathcal{U} \cap \tau(X(s))$. □

Example 1. A minimal 3-Hausdorff that is not 3-H-closed space.

Let $X = \{1,2,3\}$ with the topology $\{X, \varnothing, \{1\}, \{2,3\}\}$. Note that X is 3-Hausdorff and semiregular. The only coarser topology on X is the indiscrete topology and it is not 3-Hausdorff; so, X is minimal 3-Hausdorff. However, for the open ultrafilter $\mathcal{U} = \{X, \{1\}\}$, $a\mathcal{U}$ has only one adherent point, namely $\{1\}$. This shows that X is not 3-H-closed. □

We can show that n-H-closedness is preserved under continuous functions but first we prove the following stronger result.

Lemma 27. *Let $m,n \in \omega$, with $n \geq 2$ and $m \geq 1$. Let X be $(m \cdot n)$-H-closed, Y be n-Hausdorff, and $f : X \to Y$ be a continuous, m-to-1, surjective function. Then Y is n-H-closed.*

Proof. Let $A \subseteq Y$ such that $|A| < n-1$ and $\mathcal{C}$ be an open family in Y such that $Y\backslash A \subseteq \bigcup \mathcal{C}$. Since f is continuous, m-to-1 and onto, then $|f^{\leftarrow}[A]| < m(n-1)$ and $X\backslash f^{\leftarrow}[A] \subseteq \bigcup_{\mathcal{C}} f^{\leftarrow}[U]$. By Theorem 2 of Basile *et al.* (2019), there is a finite subfamily $\mathcal{D} \subseteq \mathcal{C}$ such that $X =$

$\bigcup_{\mathcal{D}} cl(f^{\leftarrow}[U])$. Then $Y = f[X] = \bigcup_{\mathcal{D}} f[cl(f^{\leftarrow}[U])] \subseteq \bigcup_{\mathcal{D}} clf(f^{\leftarrow}[U]) = \bigcup_{\mathcal{D}} clU$. Again, by Theorem 2 of Basile *et al.* (2019), this shows that Y is n-H-closed. □

Corollary 28. *Let X be n-H-closed, Y be n-Hausdorff, and $f : X \to Y$ be a continuous bijection. Then Y is also n-H-closed.*

Theorem 29. *A semiregular, n-H-closed space X is minimal n-Hausdorff.*

Proof. Let X be semiregular, n-H-closed, and σ be a coarser n-Hausdorff topology on X. Let f be the identity function on X. We will show that the continuous bijection $f : X \to (X, \sigma)$ is a homeomorphism. As the family of regular-closed subsets of X is a closed base for X, it suffices to show that $f[cl(U)]$ is closed in (X, σ) where U is open in X. By Proposition 3 of Basile *et al.* (2019), $cl(U)$ is n-H-closed. As $f|_{cl(U)} : cl(U) \to f[cl(U)]$ is a continuous bijection, by Corollary 28, $f[cl(U)]$ is n-H-closed. By the definition of n-H-closed, $f[cl(U)]$ is closed in (X, σ). □

Example 2. 3-H-closedness is not preserved under continuous, onto, compact, closed, irreducible functions.

We use the space in Remark 1 of Basile *et al.* (2019). Recall that $X = \omega \cup \{a, b\}$ where a set $U \subseteq X$ is defined to be open if $a \in U$ or $b \in U$, then $U \cap \omega$ is cofinite in ω, and each point $n \in \omega$ is isolated. Let $Y = X \cup \{c_n : n \in \omega\}$ where $c_n \notin X$ for $n \in \omega$. A set $U \subseteq Y$ is defined to be open if $a \in U$ or $b \in U$ when there is some $m \in \omega$ such that $\{n, c_n\} \subseteq U$ for $n \geq m$ and $c_n \in U$ iff $n \in U$. The space Y is 3-H-closed, and X is dense in Y. Let Z be the one-point compactification of ω where the point at infinity is denoted as p. Define $f : Y \to Z$ by $f(n) = f(c_n) = n$ for $n \in \mathbb{N}$ and $f(a) = f(b) = p$. Note that f is continuous, onto, compact and not separable (we can consider the point $p \in Z$). We now prove that f is closed. Let C be a closed subset of Y. We distinguish two cases. If the set C contains a or b, then $f(C)$ contains p and every subset of Z containing p is a closed subset of Z. If the set C does not contain a and b, then it is finite and also $f(C)$ is a finite, hence closed subset of Z. We can easily notice from the above comment that for every proper closed subset C of Y, we have $f(C) \neq Z$. This means f is also irreducible. As Z is Hausdorff, it is also n-Hausdorff for all $n \geq 2$. Then the compact space Z is H-closed. It was noted by Basile *et al.* (2019) that an H-closed space is not n-H-closed, for every $n > 2$; hence Z is not 3-H-closed. □

We now introduce the following definition which represents a generalization of the notion of separable function.

Definition 30. A function $f : X \to Y$ is called ***n*-separable** if for each $y \in Y$ and different points $x_1, \ldots, x_n \in f^{\leftarrow}(y)$, there exist open subsets U_i containing x_i, for every $i = 1, \ldots, n$, such that $\bigcap_{i=1}^{n} U_i = \emptyset$.

Remark 1. We can extend the space of Example 2. We consider $X = \omega \cup \{x_1, \ldots, x_n\}$ where $x_1, \ldots, x_n \notin \omega$ and U is defined to be open when if it contains one of x_i for $i = 1, \ldots, n$, then $U \cap \omega$ is cofinite in ω, and each point of ω is isolated. The space Y is now n-H-closed and the function $f : Y \to Z$ is continuous, onto, compact, closed, irreducible and not n-separable.

6. Cardinality bounds for n-H-closed spaces

Recall the following definitions given by Carlson and Porter (2018). The definition of a section is given after Proposition 2.1 of Carlson and Porter (2018), and also before Theorem 5.1 of Porter and Woods (1978).

Definition 31. (Carlson and Porter 2018) For a space X, define a *s*ection to be a one-to-one function

$$b : X \to \{\mathcal{U} : \mathcal{U} \text{ is a convergent open ultrafilter on X}\}$$

where $b(x)$ is an open ultrafilter on X converging to x, for any $x \in X$, i.e., $(x, b(x)) \in PX$.

For an open subset U of X, define $\widehat{U}_b = \{x \in X : U \in b(x)\}$. For what follows, the choice of a section will be inconsequential, so for convenience we will write $\widehat{U}$ instead of $\widehat{U}_b$.

Recall the following property:

Proposition 32. (Carlson and Porter 2018) Let X be a space, U an open set, and $\{U_i\}_{i=1}^n$ a family of open sets of X. Then:

(1) $U \subseteq \widehat{U}$,
(2) $\widehat{U} = \varnothing$ iff $U = \varnothing$,
(3) $\widehat{U} = X$ iff $U = X$,
(4) $\bigcap_{i=1}^n \widehat{U_i} = \widehat{\bigcap_{i=1}^n U_i}$,
(5) $\bigcup_{i=1}^n \widehat{U_i} = \widehat{\bigcup_{i=1}^n U_i}$,
(6) $X \backslash \widehat{U} = \widehat{X \backslash cl_X(U)}$,

Proof. For 1, notice that if $x \in U$ then $U \in b(x)$, thus $x \in \widehat{U}$. For 2, if $U = \varnothing$, then $U \notin b(x)$ for all $x \in X$. Thus $\widehat{U} = \varnothing$. If $\widehat{U} = \varnothing$, then $U = \varnothing$ by 1. For 3, suppose $\widehat{U} = X$. Then $U \in b(x)$ for every $x \in X$ and it follows that $U = X$. If $U = X$ then $\widehat{U} = X$ by 1. For 4, let $x \in \bigcap_{i=1}^n \widehat{U_i}$. Then $U_i \in b(x)$ for every $i = 1, \ldots, n$. Therefore $\bigcap_{i=1}^n U_i \in b(x)$ and $x \in \widehat{\bigcap_{i=1}^n U_i}$. The reverse inclusion follows from the superset property of a filter. For 5, let $x \in \bigcup_{i=1}^n \widehat{U_i}$. Then $x \in \widehat{U_i}$ for some i. Therefore $U_i \in b(x)$ and $\bigcup_{i=1}^n U_i \in b(x)$. It follows that $x \in \widehat{\bigcup_{i=1}^n U_i}$. Suppose $x \in \widehat{\bigcup_{i=1}^n U_i}$. Then $\bigcup_{i=1}^n U_i \in b(x)$. If $U_i \notin b(x)$ for all $i = 1, \ldots n$ then $X \backslash cl_X(U_i) \in b(x)$ for all i, contradicting that $\bigcup_{i=1}^n U_i \in b(x)$. Therefore $U_i \in b(x)$ for some i. It follows that $x \in \widehat{U_i}$ and thus $x \in \bigcup_{i=1}^n \widehat{U_i}$. For 6, let $x \in X \backslash \widehat{U}$. This is equivalent to $U \notin b(x)$, which in turn is equivalent to $X \backslash cl_X(U) \in b(x)$. Finally this is equivalent to $x \in \widehat{X \backslash cl_X(U)}$. □

Proposition 33. *X is n-Hausdorff iff for distinct points* $x_1, \ldots, x_n \in X$ *there exists open subsets* U_i *containing* $x_i \in U_i$ *for every* $i = 1, \ldots, n$ *such that* $\bigcap_{i=1}^n \widehat{U_i} = \varnothing$.

Proof. If X is n-Hausdorff, then for distinct points $x_1, ..., x_n \in X$ there exists open subsets U_i containing x_i for every $i = 1, ..., n$ such that $\bigcap_{i=1}^n U_i = \varnothing$, but then $\bigcap_{i=1}^n \widehat{U_i} = \widehat{\varnothing} = \varnothing$. The converse follows as $U \subseteq \widehat{U}$ every open subset U of X. □

The following represents a new characterization of n-qH-closed spaces.

Proposition 34. *X is n-qH-closed iff for every* $A \in [X]^{<n-1}$ *and a family* $\mathcal{U}$ *of open subsets of* X *such that* $X \setminus A \subseteq \bigcup \mathcal{U}$, *there exists* $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ *such that* $X = \bigcup_{V \in \mathcal{V}} \widehat{V}$.

Proof. Suppose X is n-qH-closed, then for every $A \in [X]^{<n-1}$ and a family $\mathcal{U}$ of open subsets of X such that $X \setminus A \subseteq \bigcup \mathcal{U}$, there exists $\mathcal{V} \in [\mathcal{U}]^{<\omega}$ such that $X = \bigcup_{V \in \mathcal{V}} V \subseteq \bigcup_{V \in \mathcal{V}} \widehat{V}$. Conversely, $X = \bigcup_{V \in \mathcal{V}} \widehat{V} = \widehat{\bigcup_{V \in \mathcal{V}} V} = \bigcup_{V \in \mathcal{V}} V$ by Proposition 32. □

Definition 35. (Carlson and Porter 2018) Let A be a subset of X. Define $\hat{c}(A) = \{x \in X :$ for every open subset U of X containing x, $\widehat{U} \cap A \neq \varnothing\}$. We say that a subset A is $\hat{c}$-dense in X if $\hat{c}(A) = X$. We define the $\hat{c}$-density of X, denoted by $d_{\hat{c}}(X)$, as the smallest cardinality of a $\hat{c}$-dense subset of X.

We observe that $cl(A) \subseteq \hat{c}(A)$ and therefore a dense set in a space is also $\hat{c}$-dense. Consequently, $d_{\hat{c}}(X) \leq d(X)$ for any space X.

We can prove the following for an n-Hausdorff space X, which is an improvement of Proposition 28 of (Bonanzinga 2013). We omit the proof because it follows step by step the proof of Proposition 28 of Bonanzinga (2013).

Proposition 36. *Let X be an n-Hausdorff space, then $|X| \leq d_{\hat{c}}(X)^{\chi(X)}$.*

Proof. Let D be a $\hat{c}$-dense set such that $|D| = d_{\hat{c}}(X)$ and let $\kappa = \chi(X)$. For all $x \in X$, let $\mathcal{N}_x$ be a neighborhood base at X such that $|\mathcal{N}_x| \leq \kappa$. For all $U \in \mathcal{N}_x$, let $x(U,D) \in \widehat{U} \cap D$ and $D(x) = \{x(U,D) : U \in \mathcal{N}_x\}$. Then $|D(x)| \leq |D|^{\kappa}$.

Define $\phi : X \to \left[[D]^{\leq\kappa}\right]^{\leq\kappa}$ by $\phi(x) = \{\widehat{U} \cap D(x) : U \in \mathcal{N}_x\}$. We show the cardinality of each fiber of ϕ is finite. By contradiction, assume there exists an infinite fiber $\phi^{-1}(\phi(x))$ of ϕ. Since X is n-Hausdorff, by Proposition 33 there exists $F = \{x_1, \cdots, x_n\} \subseteq \phi^{-1}(\phi(x))$ and neighborhoods $U_i \in \mathcal{N}_{x_i}$ of x_i for $i = 1, \ldots, n$ such that $\bigcap_{i=1}^{n} \widehat{U_i} = \varnothing$. Then $\phi(x_1) = \ldots = \phi(x_n) = \phi(x)$. For $i = 1, \ldots, n$, $\widehat{U_i} \cap D(x_i) \in \phi(x)$ and therefore $\widehat{U_i} \cap D(x_i) = \widehat{V_i} \cap D(x)$ where $V_i \in \mathcal{N}_x$. Let $V = \bigcap_{i=1}^{n} V_i$. Then $\widehat{V} \cap D(x) \neq \varnothing$. This contradicts that $\bigcap_{i=1}^{n} \widehat{U_i} = \varnothing$. As each fiber of ϕ is finite, we have that $|X| \leq (|D|^{\kappa})^{\kappa} = d_{\hat{c}}(X)^{\chi(X)}$. □

Recall the following

Theorem 37. (Bonanzinga *et al.* 2014, Theorem 3.1) Let $n \geq 2$ be finite, X be a set, $Y \subseteq X$ and for each $x \in X$, $\mathcal{H}(x) = \{V(\alpha, x) : \alpha < \kappa\}$ be a collection of subsets of X containing x which is closed under finite intersection. Assume the following:

- (n-**H**) if $x_1, \ldots, x_n \in X$ are distinct, then there exist $\alpha_1, \ldots, \alpha_n < \kappa$ such that $V(\alpha_1, x_1) \cap \ldots \cap V(\alpha_n, x_n) = \varnothing$ (n-Hausdorff condition)
- (**C**) for every function $f : X \to \kappa$, there exists $A \in [X]^{\leq\kappa}$ such that $Y \subseteq \bigcup_{x \in A} V(f(x), x)$ (cover condition)

Then $|Y| \leq 2^{\kappa}$.

Definition 38. (Carlson and Porter 2018) For a subset A of X, define $\widehat{L}(A, X)$ as the least cardinal κ such that for every cover $\mathcal{V}$ of A by sets open in X there exists $\mathcal{W} \in [\mathcal{V}]^{\leq\kappa}$ such that $A \subseteq \bigcup_{W \in \mathcal{W}} \widehat{W}$. Set $\widehat{L}(X) = \widehat{L}(X, X)$.

Of course, for every space X, $\widehat{L}(X) \leq L(X)$, where $L(X)$ is the Lindelöf number of X. We can prove the following:

Theorem 39. *If X is n-Hausdorff, then $|X| \leq 2^{\widehat{L}(X)\chi(X)}$.*

Proof. Let $\kappa = \widehat{L}(X)\chi(X)$ and $\mathcal{B}_x = \{B(\alpha,x) : \alpha < \kappa\}$ an open neighborhood system of x. Consider $\mathcal{H}(x) = \{V(\alpha,x) : \alpha < \kappa$ and $V(\alpha,x) = \widehat{B(\alpha,x)}\}$. Of course $\mathcal{H}(x)$ is closed under finite intersections and satisfies condition (n-**H**) of Theorem 37. We want to show that also condition (**C**) is satisfied. Let $f : X \to \kappa$ be a function. $\{B(f(x),x) : x \in X\}$ is an open cover of X and by $\widehat{L}(X) \leq \kappa$, there exists $A \in [X]^{\leq\kappa}$ such that $\bigcup_{x\in A} V(f(x),x) = X$. Applying Theorem 37 we have $|X| \leq 2^{\kappa}$. □

Corollary 40. *If X is n-Hausdorff, then $|X| \leq 2^{L(X)\chi(X)}$.*

The previous result improves the following one holding in the class of T_1 spaces.

Corollary 41. (Bonanzinga 2013) If X is a T_1 n-Hausdorff space, then $|X| \leq 2^{L(X)\chi(X)}$.

Corollary 42. *(*Arhangel'skii, 1969; see (Hodel 1984)) If X is Hausdorff, then $|X| \leq 2^{L(X)\chi(X)}$.

Corollary 43. *If X is n-H-closed, then $|X| \leq 2^{\chi(X)}$.*

Proof. It follows from Proposition 34 and Theorem 39. □

Corollary 44. (Dow and Porter 1982) If X is H-closed, then $|X| \leq 2^{\chi(X)}$.

In the papers by Bella and Cammaroto (1988) and Cammaroto *et al.* (2013) there are several open questions worthy of study in the area of n-Hausdorff spaces.

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