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SELECTIVELY STRONGLY STAR-MENGER SPACES AND RELATED PROPERTIES

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ABSTRACT. A space X is *selectively strongly star-Menger* (briefly, *selSSM*) if for each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X and each sequence $(D_n : n \in \mathbb{N})$ of dense subspaces of X, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ is an open cover of X. This property is between *absolute countable compactness* [M. V. Matveev, *Topol. Appl.* **58**, 81 (1994)] and *selective absolute star-Lindelöfness* [M. Bonanzinga *et al.*, *Topol. Appl.* **221**, 517 (2017)] and represents a "selective version" of the selection principle *strongly star Menger* [L. D. R. Kočinac, *Publ. Math. Debrecen* **55**, 421 (1999)]. In this paper, we study some properties of selectively strongly star-Menger spaces, the relation with related properties and give some example distinguishing the properties considered.

1. Introduction and definitions

In this paper we use standard notation and terminology following Hodel (1984) and Engelking (1989). Let $\mathscr{U}$ be a cover of a space X and M be a subset of X; the star of M with respect to $\mathscr{U}$ is the set $st(M, \mathscr{U}) = \bigcup\{U : U \in \mathscr{U} \text{ and } U \cap M \neq \emptyset\}$. The star of a one-point set $\{x\}$ with respect to a cover $\mathscr{U}$ is denoted by $st(x, \mathscr{U})$. Recall that a space X is said to be *star-compact (star-Lindelöf)* if for every open cover $\mathscr{U}$ of X there exists a finite (countable) subset $F \subset X$ such that $st(F, \mathscr{U}) = X$; X is *absolutely countably compact* (briefly, *acc*) (Matveev 1994) if for every open cover $\mathscr{U}$ of X and every dense subspace $D \subset X$ there exists a finite subset $F \subset D$ such that $st(F, \mathscr{U}) = X$; X is *absolutely star-Lindelöf* (briefly, *a-star-Lindelöf*) (Bonanzinga 1998) if for every open cover $\mathscr{U}$ of X and every dense subspace $D \subset X$ there exists a countable subset $C \subset D$ such that $st(C, \mathscr{U}) = X$. See also Matveev (1998) for a survey on star covering properties.

Bonanzinga *et al.* (2017) considered the following selective version of *a-star-Lindelöfness* (see also Bhowmik 2011; Bal *et al.* 2018), where another terminology is used; note that this notion is strictly related to selective separability (Bella *et al.* 2008, 2009).

Definition 1.1. (Bonanzinga *et al.* 2017) A space X has the *selective absolutely star-Lindelöf* property (briefly, *sel-a-star-Lindelöf*) if for every open cover $\mathscr{U}$ of X and every sequence $(D_n : n \in \mathbb{N})$ of dense subspaces of X, there exists a sequence $(F_n : n \in \mathbb{N})$, of finite subsets $F_n \subset D_n$, such that $st(\bigcup_{n\in\mathbb{N}} F_n, \mathscr{U}) = X$.

The previous property is between acc and sel-a-star-Lindelöf properties. In particular

$$acc \Rightarrow sel\text{-}a\text{-}star\text{-}Lindel\ddot{o}f \Rightarrow a\text{-}star\text{-}Lindel\ddot{o}f.$$

In a similar way we can introduce the following property (see also Cuzzupé 2017).

Definition 1.2. A space X is *selectively strongly star-Menger* (briefly, *selSSM*) if for each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X and each sequence $(D_n : n \in \mathbb{N})$ of dense subspaces of X, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ is an open cover of X.

Note that *selSSM* property is between *acc* and *sel-a-star-Lindelöf* properties. In particular

$$acc \Rightarrow selSSM \Rightarrow sel\text{-}a\text{-}star\text{-}Lindel\ddot{o}f.$$

The selSSM property is a "selective version" of the following selection principle introduced by Kočinac (1999) (see also Bonanzinga *et al.* 2009; Sakai 2014):

Definition 1.3. (Kočinac 1999; Bonanzinga *et al.* 2009; Sakai 2014) X is *strongly star-Menger* (briefly, *SSM*) if for every sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ is an open cover of X.

Recall the definition of $\mathfrak{b}$ and $\mathfrak{d}$. For $f, g \in \mathbb{N}^{\mathbb{N}}$ put $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. A subset B of $\mathbb{N}^{\mathbb{N}}$ is *bounded* if there is $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leq^* g$ for each $f \in B$. $D \subset \mathbb{N}^{\mathbb{N}}$ is *dominating* if for each $g \in \mathbb{N}^{\mathbb{N}}$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of an unbounded subset of $\mathbb{N}^{\mathbb{N}}$ is denoted by $\mathfrak{b}$, and the minimal cardinality of a dominating subset of $\mathbb{N}^{\mathbb{N}}$ is denoted by $\mathfrak{d}$.

In this paper we study the selSSM and related properties. Even if for completeness we introduce also Rothberger-type properties, we study only Menger-type and Hurewicz-type properties. In Section 2 we consider some properties related to selSSM property and prove that for spaces having cardinality less than $\mathfrak{d}$ all the considered properties are equivalent. In particular, we prove that for Isbell-Mrwóka spaces $\Psi(\mathscr{A})$, all the considered Menger-type properties are equivalent to the condition $|\mathscr{A}| < \mathfrak{d}$. Also similar results are obtained for the corresponding Hurewicz-type properties. In Section 3 we consider some relative versions of the considered properties (recall Bonanzinga and Pansera 2007 as one of the first papers on relative star selection principles).

The selSSM property is in fact a "star-selection principle".

Let X be a space and Y a subspace of X. We use the symbol:

$\mathscr{O}_X$: the collection of open covers of X;

$\mathscr{O}_{YX}$: the collection of open covers of Y by sets open in X;

Ω_X: the collection of open ω-covers of X. An open cover $\mathscr{U}$ of X is an ω-cover (Gerlits and Nagy 1982) if X does not belong to $\mathscr{U}$ and every finite subset of X is contained in an element of $\mathscr{U}$;

Ω_{YX}: the collection of open ω-covers of Y by sets open in X;

Γ_X: the collection of open γ-covers of X. An open cover $\mathscr{U}$ of X is an γ-cover (Gerlits and Nagy 1982) if it is infinite and each x belongs to all but finitely many elements of $\mathscr{U}$;

Γ_{YX}: the collection of open γ-covers of Y by sets open in X.

We shall drop the subscript X or Y in the indication of the family of covers when it is clear from the context which space we are referring to.

Definition 1.4. (Kočinac 1999) Let $\mathscr{A}, \mathscr{B}$ be collections of subsets of an infinite space X. Then

$SS_1(\mathscr{A}, \mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there exists a sequence $(x_n : n \in \mathbb{N})$ of points of X such that $\{st(x_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathscr{B}$;

$SS_{fin}(\mathscr{A}, \mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of X such that $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathscr{B}$.

Kočinac (1999) used the following terminology. Let X be a space.

- X is SSR (strongly star-Rothberger) if it satisfies $SS_1(\mathscr{O}, \mathscr{O})$;
- X is SSM (strongly star-Menger) if it satisfies $SS_{fin}(\mathscr{O}, \mathscr{O})$;
- X is SSH (strongly star-Hurewicz) if it satisfies $SS_{fin}(\mathscr{O}, \Gamma)$.

Bonanzinga *et al.* (2009), using a different terminology, considered the following weaker star versions of the properties in Definition **1.4**:

Definition 1.5. (Bonanzinga *et al.* 2009) Let $\mathscr{A}, \mathscr{B}$ be collections of subsets of an infinite space X. Then

$NSS_1(\mathscr{A}, \mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there exists a sequence $(x_n : n \in \mathbb{N})$ of points of X such that for every open $O_n \ni x_n$, $n \in \mathbb{N}$, $\{st(O_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathscr{B}$;

$NSS_{fin}(\mathscr{A}, \mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ there exists a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for every open $O_n \supset A_n$, $n \in \mathbb{N}$, $\{st(O_n, , \mathscr{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathscr{B}$.

Bonanzinga *et al.* (2009) used the following terminology. Let X be a space.

- X is NSR (neighbourhood strongly star-Rothberger) if it satisfies $NSS_1(\mathscr{O}, \mathscr{O})$;
- X is NSM (neighbourhood strongly star-Menger) if it satisfies $NSS_{fin}(\mathscr{O}, \mathscr{O})$;
- X is NSH (neighbourhood strongly star-Hurewicz) if it satisfies $NSS_{fin}(\mathscr{O}, \Gamma)$.

Recently, De la Rosa and Garcia-Balan (2021) used the following terminology for selective versions of properties in Definitions **1.4** and **1.7**.

Definition 1.6. (De la Rosa and Garcia-Balan 2021; see also Kočinac 2021) Let $\mathscr{A}, \mathscr{B}$ be collections of subsets of an infinite space X. Then

$selSS_1(\mathscr{A}, \mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ and each sequence $(D_n : n \in \mathbb{N})$ of dense set of X there exists a sequence $(x_n : n \in \mathbb{N})$ of points $x_n \in D_n$, $n \in \mathbb{N}$, such that $\{st(x_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathscr{B}$;

$selSS_{fin}(\mathscr{A}, \mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of elements of $\mathscr{A}$ and each sequence $(D_n : n \in \mathbb{N})$ of dense set of X there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ belongs to $\mathscr{B}$.

De la Rosa and Garcia-Balan (2021) used the following terminology. Let X be a space.

- X is selSSR (selectively strongly star-Rothberger) if it satisfies $selSS_1(\mathscr{O},\mathscr{O})$;
- X is selSSM (selectively strongly star-Menger) if it satisfies $selSS_{fin}(\mathscr{O},\mathscr{O})$;
- X is selSSH (selectively strongly star-Hurewicz) if it satisfies $selSS_{fin}(\mathscr{O},\Gamma)$.

Definition 1.7. (De la Rosa and Garcia-Balan 2021) Let $\mathscr{A},\mathscr{B}$ be collections of subsets of an infinite space X. Then

$selNSS_1(\mathscr{A},\mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n\in\mathbb{N})$ of elements of $\mathscr{A}$ and each sequence $(D_n : n\in\mathbb{N})$ of dense set of X there exists a sequence $(x_n : n\in\mathbb{N})$ of points $x_n\in D_n$, $n\in\mathbb{N}$, such that for every open $O_n\ni x_n$, $n\in\mathbb{N}$, $\{st(O_n,\mathscr{U}_n) : n\in\mathbb{N}\}$ belongs to $\mathscr{B}$;

$selNSS_{fin}(\mathscr{A},\mathscr{B})$ denote the selection hypothesis: For each sequence $(\mathscr{U}_n : n\in\mathbb{N})$ of elements of $\mathscr{A}$ and each sequence $(D_n : n\in\mathbb{N})$ of dense set of X there exists a sequence $(F_n : n\in\mathbb{N})$ of finite subsets $F_n\subset D_n$, $n\in\mathbb{N}$, such that for every open $O_n\supset F_n$, $n\in\mathbb{N}$, $\{st(O_n,,\mathscr{U}_n) :\ n\in\mathbb{N}\}$ belongs to $\mathscr{B}$.

De la Rosa and Garcia-Balan (2021) used the following terminology. Let X be a space.

- X is selNSR (neighbourhood selectively strongly star-Rothberger) if it satisfies $selNSS_1(\mathscr{O},\mathscr{O})$;
- X is selNSM (neighbourhood selectively strongly star-Menger) if it satisfies $selNSS_{fin}(\mathscr{O},\mathscr{O})$;
- X is selNSH (neighbourhood selectively strongly star-Hurewicz) if it satisfies $selNSS_{fin}(\mathscr{O},\Gamma)$.

See also Kočinac (2015) for a survey on star selection principles.

Recall that a family of sets is almost disjoint (a.d., for short) if the intersection of any two distinct elements is finite. Let $\mathscr{A}$ be an a.d. family of infinite subsets of $\mathbb{N}$. Put $\Psi(\mathscr{A})=\mathbb{N}\cup\mathscr{A}$ and topologize $\Psi(\mathscr{A})$ as follows: the points of $\mathbb{N}$ are isolated and a basic neighbourhood of a point $a\in\mathscr{A}$ takes the form $\{a\}\cup(A\setminus F)$, where F is a finite set. $\Psi(\mathscr{A})$ is called a Ψ-space or a Isbell-Mrówka space (see Engelking 1989).

2. SelSSM and related properties.

Proposition 2.1. A space X is selSSM iff for each sequence $(\mathscr{U}_n : n\in\mathbb{N})$ of open covers of X and each sequence $(D_n : n\in\mathbb{N})$ of dense subspaces of X, there exists a sequence $(F_n : n\in\mathbb{N})$ of finite subsets $F_n\subset D_n$, $n\in\mathbb{N}$, such that for every $x\in X$ there exists $n\in\mathbb{N}$ such that $st(x,\mathscr{U}_n)\cap F_n\neq\emptyset$.

Proof. It it enough to note that if $x\in X$ then

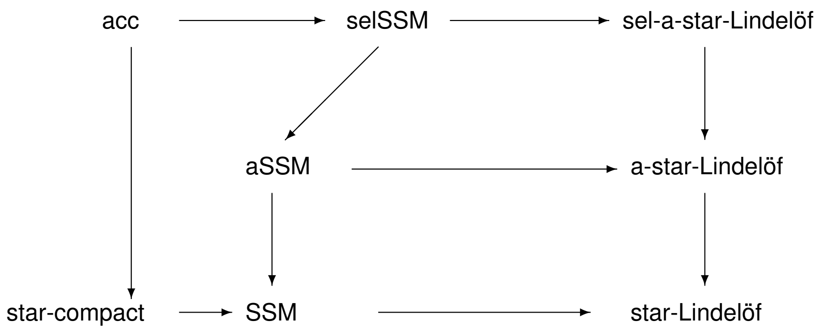
$$x\in st(F_n,\mathscr{U}_n)\Leftrightarrow st(x,\mathscr{U}_n)\cap F_n\neq\emptyset.$$

□

Caserta *et al.* (2011) gave the following selective version of strongly star-Menger property.

Definition 2.1. (Caserta *et al.* 2011) A space X is *absolutely strongly star-Menger* (briefly, *aSSM*) if for each sequence $(\mathscr{U}_n : n\in\mathbb{N})$ of open covers of X and each dense subspace D of X, there exists a sequence $(F_n : n\in\mathbb{N})$ of finite subsets of D such that $\{st(F_n,\mathscr{U}_n) :\ n\in\mathbb{N}\}$ is an open cover of X.

The implications in the following diagram are easy to see:



Note that any star-compat non a-star-Lindelöf space is an example of a SSM space which is not aSSM: consider, for example, the space $X = \omega_1 \times (\omega_1 + 1)$ (see Bonanzinga 1998).

Bonanzinga *et al.* (2017, Example 8) gave an *a-star-Lindelöf* not *sel-a-star-Lindelöf* space. Of course every countable discrete space is a *selSSM* not countably compact, hence not *acc*, space.

The following questions are open:

Question 2.1. Does exist an *aSSM* which is not *sel-a-star-Lindelöf*?

Question 2.2. Does exist an *aSSM* not *selSSM* space?

Bonanzinga and Matveev proved the following characterization:

Theorem 2.1. (Bonanzinga and Matveev 2009) The following properties are equivalent:

(i) $\Psi(\mathscr{A})$ is SSM
(ii) $|\mathscr{A}| < \mathfrak{d}$.

Now we have the following characterization.

Proposition 2.2. Let X be a topological space. Suppose $|X| < \mathfrak{d}$. Then all the following properties are equivalent in X:

(i) selSSM
(ii) sel-a-star-Lindelöf
(iii) a-star-Lindelöf
(iv) aSSM

Proof. Since (i)⇒(ii)⇒(iii), (i)⇒(iv) and (iv)⇒(iii), we only prove that (iii) implies (i). Let $(\mathscr{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X and $(D_n : n \in \mathbb{N})$ a sequence of dense subsets of X. Fixed $n \in \mathbb{N}$, since X is an a-star-Lindelöf space, we can find a countable subset $C_n \subseteq D_n$ such that $X = st(C_n, \mathscr{U}_n)$. We enumerate $C_n = \{c_{n,k}\}_{k\in\mathbb{N}}$ for all $n \in \mathbb{N}$. For each $x \in X$ and $n \in \mathbb{N}$ we can find $f_x(n) \in \mathbb{N}$ such that $c_{n,f_x(n)} \in st(x, \mathscr{U}_n)$. Since the set $\{f_x : x \in X\}$ is not cofinal in $(\mathbb{N}^{\mathbb{N}}, \leq)$, there are $g \in \mathbb{N}^{\mathbb{N}}$ and $n_x \in \mathbb{N}$ such that $f_x(n_x) < g(n_x)$ for all $x \in X$. Let $F_n = \{c_{n,j} : j \leq g(n)\}$. Then F_n is a finite subset of D_n for all $n \in \mathbb{N}$. Let $x \in X$. Then obviously $x \in st(F_{n_x}, \mathscr{U}_{n_x})$. This prove that $X = \bigcup\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$. □

Corollary 2.1. (Bonanzinga *et al.* 2017) If X is an a-star-Lindelöf space with $|X| < \mathfrak{d}$ then X is sel-a-star-Lindelöf.

Corollary 2.2. (see also Song 2013) The following properties are equivalent:

(i) $\Psi(\mathscr{A})$ is selSSM
(ii) $\Psi(\mathscr{A})$ is aSSM
(iii) $\Psi(\mathscr{A})$ is SSM
(iv) $|\mathscr{A}| < \mathfrak{d}$.

Proof. Of course (i)⇒(ii)⇒(iii) and by Theorem **2.1** (iii)⇔(iv). Now suppose $|\mathscr{A}| < \mathfrak{d}$. Since $\Psi(\mathscr{A})$ is always a-star-Lindelöf, by the previous proposition we can conclude that $\Psi(\mathscr{A})$ is selSSM. □

Cuzzupé posed the following question:

Question 2.3. (Cuzzupé 2017) Is the product of a selSSM space with a compact first countable space a selSSM?

Now, under the assumption $\omega_1 < \mathfrak{d}$, we answer in the negative to the previous question.

Corollary 2.3. $(\omega_1 < \mathfrak{d})$ There is a selSSM space X and a compact first countable space Y such that $X \times Y$ is not selSSM.

Proof. Let $X = \mathbb{N} \cup \mathscr{A}$ be a Ψ-space with $|\mathscr{A}| = \omega_1$ and Y a compact first countable non ccc space. Then, by Corollay **2.2**, X is selSSM. Following Bonanzinga and Matveev (2001, Corollary 2.4), since X is a space having uncountable extent and Y is a non ccc space, we have that the product $X \times Y$ is not star-Lindelöf, hence not selSSM. □

Bonanzinga *et al.* (2017, Example 8) also constructed an a-star-Lindelöf non sel-a-star-Lindelöf space of cardinality $\mathfrak{d}$ and then, by Corollary **2.1**, the following result was obtained:

Corollary 2.4. (see Bonanzinga *et al.* 2017) The following conditions are equivalent:

(i) $\omega_1 < \mathfrak{d}$
(ii) Every a-star-Lindelöf space of cardinality ω_1 is sel-a-star-Lindelöf.

Question 2.4. Is the characterization of Corollary **2.4** true if "sel-a-star-Lindelöf" is replaced by with "aSSM"?

We can consider the following Hurewicz-type definition.

Definition 2.2. (Caserta *et al.* 2011) A space X is *absolutely strongly star-Hurewicz* (briefly, *aSSH*) if for each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X and each dense subspace D of X, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets of D such that each x belongs to all but finitely elements of $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$.

Recall the following characterization of Bonanzinga and Matveev:

Theorem 2.2. (Bonanzinga and Matveev 2009) The following properties are equivalent:

(i) $\Psi(\mathscr{A})$ is SSH
(ii) $|\mathscr{A}| < \mathfrak{b}$.

By the previous result and following step by step the proof of the implication (iii)⇒(i) in Proposition **2.2**, we obtain:

Proposition 2.3. If X is an a-star-Lindelöf space with $|X| < \mathfrak{b}$, then X is selSSH.

Corollary 2.5. If X is an aSSH space with $|X| < \mathfrak{b}$, then X is selSSH.

Then we have the following characterization.

Corollary 2.6. Let X be a topological space. Suppose $|X| < \mathfrak{b}$. Then all the following properties are equivalent in X:

(i) selSSH
(i) selSSM
(ii) sel-a-star-Lindelöf
(iii) a-star-Lindelöf
(iv) aSSH.

3. Relative selective star-selection principles

We introduce the following definitions:

Definition 3.1. A subspace Y of a space X is *relatively selSSM* in X if for each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X and each sequence $(D_n : n \in \mathbb{N})$ of dense subspaces of X, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\{st(F_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ is an open cover of Y (i.e. if $selSS_{fin}(\mathscr{O}_X, \mathscr{O}_{YX})$ holds).

Now we introduce the following technical property:

Definition 3.2. A subspace Y of a space X is *relatively closed selSSM* in X if it is closed in X and relatively selSSM in X. A space Y is said to be a *relatively closed selSSM*, briefly, *rel-cl selSSM*, if there is a larger space X such that Y is relatively closed selSSM in X.

Of course, every selSSM space is rel-cl selSSM. The following example shows that a rel-cl selSSM space need not to be selSSM.

Example 3.1. A rel-cl selSSM space which is not selSSM.

Let $\mathscr{A}$ be a almost disjoint family of cardinality $\omega_1 < \mathfrak{d}$. Since, by Corollary **2.2**, $\Psi(\mathscr{A})$ is selSSM and $\mathscr{A}$ is closed in $\Psi(\mathscr{A})$, we have that $\mathscr{A}$ is rel-cl selSSM in $\Psi(\mathscr{A})$. Since the subspace $\mathscr{A}$ of $\Psi(\mathscr{A})$ is the discrete subspace of cardinality ω_1, we have that $\mathscr{A}$ can not be selSSM.

We have the following result:

Theorem 3.1. Let Y be a subspace of X. If for every $n \in \mathbb{N}$, Y^n is relatively selSSM in X^n, then $selSS_{fin}(\mathscr{O}_X, \Omega_{YX})$ holds.

Proof. Let $(\mathscr{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X, $(D_n : n \in \mathbb{N})$ be a sequence of dense subspaces of X and $\mathbb{N} = N_1 \cup N_2 \dots$ be a partition on $\mathbb{N}$ into infinite (pairwise disjoint) sets. For each $\kappa \in \mathbb{N}$ and every $m \in N_\kappa$ let $\mathscr{W}_m = (\mathscr{U}_m)^\kappa$ and $E_m = (D_m)^\kappa$.

Then, $(\mathscr{W}_m : m \in N_\kappa)$ is a sequence of open covers of X^κ and $(E_m : m \in N_\kappa)$ is a sequence of dense subspaces of X^κ. Applying the fact that Y^κ is relatively selSSM in X^κ to these sequences, we can find a sequence $(F_m : n \in N_\kappa)$ of finite subsets $F_m \subset E_m$, $m \in N_\kappa$, such that $\{st(F_m, \mathscr{W}_m) : n \in N_\kappa\}$ is an open cover of Y^κ. For each $m \in N_\kappa$, let $S_m \subset X$ be the union of projections of F_m on all coordinates. Since each projection is finite and it is contained in D_m, we have that S_m is a finite subset of D_m. Also, for every $m \in N_\kappa$, $(S_m)^\kappa \supset F_m$. Now we prove that $\{st(S_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ is an ω-cover of Y. Let $H = \{y_1, ..., y_p\}$ be a finite subset of Y. Then $(y_1, ..., y_p) \in Y^p$. Then, there exists $n \in N_p$ such that $(y_1, ..., y_p) \in st(F_n, \mathscr{W}_n) \subset st((S_n)^p, \mathscr{W}_n)$ and consequently $H \subset st(S_n, \mathscr{U}_n)$. □

Corollary 3.1. Let X be a space. If for every $n \in \mathbb{N}$, X^n is selSSM, then X satisfies $selSS^*_{fin}(\mathscr{O}, \Omega)$.

Definition 3.3. A subspace Y of a space X is *relatively selSSH* in X if $selSS_{fin}(\mathscr{O}_X, \Gamma_{YX})$ holds.

Now we introduce the following technical property:

Definition 3.4. A subspace Y of a space X is *relatively closed selSSH* in X if it is closed in X and relatively selSSH in X. A space Y is said to be a *relatively closed selSSH*, briefly, *rel-cl selSSH*, if there is a larger space X such that Y is relatively closed selSSH in X.

Of course, every selSSH space is rel-cl selSSH. The following example shows that a rel-cl selSSH space need not to be selSSH.

Example 3.2. A rel-cl selSSH space which is not selSSH.

Let $\mathscr{A}$ be a almost disjoint family of cardinality $< \mathfrak{b}$. Since, by Proposition **2.3**, $\Psi(\mathscr{A})$ is selSSH and $\mathscr{A}$ is closed in $\Psi(\mathscr{A})$, we have that $\mathscr{A}$ is rel-cl selSSH in $\Psi(\mathscr{A})$. Since the subspace $\mathscr{A}$ of $\Psi(\mathscr{A})$ is the discrete supace of cardinality $\mathfrak{b}$, we have that $\mathscr{A}$ can not be selSSH.

Now we consider the relative versions of neighbourhood selective SSM and SSH properties. First we note the following easy characterizations.

Proposition 3.1. A space X is selNSM (selNSH) iff for each sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X and each sequence $(D_n : n \in \mathbb{N})$ of dense subspaces of X, there exists a sequence $(F_n : n \in \mathbb{N})$ of finite subsets $F_n \subset D_n$, $n \in \mathbb{N}$, so that for every $x \in X$ there exists $n \in \mathbb{N}$ such that $\overline{st(x, \mathscr{U}_n)} \cap F_n \neq \emptyset$ (respectively, for every $x \in X$, $\overline{st(x, \mathscr{U}_n)} \cap F_n \neq \emptyset$ for all but finite $n \in \mathbb{N}$).

Proof. It is enough to note that, if $x \in X$, then
$\overline{st(x, \mathscr{U}_n)} \cap F_n \neq \emptyset \Leftrightarrow$ for every open $O_n \supset F_n, n \in \mathbb{N}, st(x, \mathscr{U}_n) \cap O_n \neq \emptyset \Leftrightarrow$ for every open $O_n \supset F_n, n \in \mathbb{N}, x \in st(O_n, \mathscr{U}_n)$. □

Definition 3.5. A subspace Y of a space X is
relatively selNSM if $selNSS_{fin}(\mathscr{O}_X, \mathscr{O}_{YX})$ holds;
relatively selNSH if $selNSS_{fin}(\mathscr{O}_X, \Gamma_{YX})$ holds.

We introduce the following technical property:

Definition 3.6. A subspace Y of a space X is
relatively closed selNSM in X if it is closed in X and relatively selNSM in X; a space Y is said to be a *relatively closed selNSM*, briefly, *rel-cl selNSM* if there is a larger space X such that Y is relatively closed selNSM in X;

relatively closed selNSH in X if it is closed in X and relatively selNSH in X; a space Y is said to be a *relatively closed selNSH*, briefly, *rel-cl selNSH* if there is a larger space X such that Y is relatively closed selNSH in X.

Of course, every selNSM space is rel-cl selNSM and every selNSH space is rel-cl selNSH. The following examples show that assuming $\omega_1 < \mathfrak{d}$ a rel-cl selNSM space need not to be selNSM and that assuming $\omega_1 < \mathfrak{b}$ a rel-cl selNSH space need not to be selNSH.

Example 3.3. $(\omega_1 < \mathfrak{d})$ A rel-cl selNSM space which is not selNSM.

Example 3.4. $(\omega_1 < \mathfrak{b})$ A rel-cl selNSH space which is not selNSH.

Bonanzinga *et al.* (2009) considered the following space: let S be a subset of $\mathbb{R}$ such that for every open $U \subset \mathbb{R}$, $|S \cap U| = \omega_1$ (in particular $|S| = \omega_1$). Consider $X_S = S \times (\omega + 1)$ topologized as follows: a basic neighbourhood of a point $\langle x, n\rangle$, where $x \in S$ and $n \in \omega$, takes the form $((U \cap S) \setminus A) \times \{n\}$ where U is a neighbourhood of x in the usual topology of $\mathbb{R}$ and A is an arbitrary countable subset of S not containing x; a point $\langle x, \omega\rangle$, where $x \in S$, has basic neighbourhoods of the form $\{\langle x, \omega\rangle\} \cup (((U \cap S) \setminus A) \times (n, \omega))$ where U is a neighbourhood of x in the usual topology of $\mathbb{R}$ and A is an arbitrary countable subset of S. Bonanzinga *et al.* (2009) proved, under the assumption $\omega_1 < \mathfrak{d}$, that for every sequence $(\mathscr{U}_n : n \in \mathbb{N})$ of open covers of X_S there exists finite subset $C_n \subset X$ such that for every neighbourhood O_n of C_n, $n \in \mathbb{N}$, we have that $\bigcup\{st(O_n, \mathscr{U}_n) : n \in \mathbb{N}\}$ contains $S \times \{\omega\}$. Since each C_n, $n \in \mathbb{N}$, is contained in $S \times \omega$ and each dense subspaces of X_S contains $S \times \omega$, we can conlude that $S \times \{\omega\}$ is relatively selNSM in X_S. Since $S \times \{\omega\}$ is closed in X_S, it is rel-cl selNSM in X_S; similarly, under the assumption $\omega_1 < \mathfrak{b}$, we can prove that $S \times \{\omega\}$ is rel-cl selNSH in X_S. Since the subspace $S \times \{\omega\}$ is a discrete space, it is neither selNSM nor selNSH.

Question 3.1. Do there exist ZFC examples of spaces as in Examples **3.3** and **3.4**?

We have the following result:

Theorem 3.2. Let Y be a subspace of X. If for every $n \in \mathbb{N}$, Y^n is relatively selNSM in X^n, then *selNSS*$_{fin}(\mathscr{O}_X, \Omega_{YX})$ holds.

Proof. Let $(\mathscr{U}_n : n \in \mathbb{N})$ be a sequence of open covers of X, $(D_n : n \in \mathbb{N})$ be a sequence of dense subspaces of X and $\mathbb{N} = N_1 \cup N_2 ...$ be a partition on $\mathbb{N}$ into infinite (pairwise disjoint) sets. For each $\kappa \in \mathbb{N}$ and every $m \in N_\kappa$ let $\mathscr{W}_m = (\mathscr{U}_m)^\kappa$ and $E_m = (D_m)^\kappa$.

Then, $(\mathscr{W}_m : m \in N_\kappa)$ is a sequence of open covers of X^κ and $(E_m : m \in N_\kappa)$ is a sequence of dense subspaces of X^κ. Applying the fact that Y^κ is relatively selNSM in X^κ to these sequences, we can find a sequence $(F_m : m \in N_\kappa)$ of finite subsets $F_m \subset D_m$, $m \in N_\kappa$, such that for every sequence $(O_m(F_m) : m \in N_\kappa)$ of neighbourhoods of F_m, $m \in N_\kappa$ in X^κ, we have that $\{st((O_m(F_m), \mathscr{W}_m) :\ m \in N_\kappa\}$ is an open cover of Y^κ. For each $m \in N_\kappa$, let $S_m \subset X$ be the union of projections of F_m on all coordinates. Since each projection is finite and it is contained in D_m, we have that S_m is a finite subset of D_m. Also, for every $m \in N_\kappa$, $(S_m)^\kappa \supset F_m$. Let $(O'_n(S_n) : n \in \mathbb{N})$ be a sequence of neighbourhoods of S_n, $n \in \mathbb{N}$, in X. Now we prove that $\{st(O'(S_n), \mathscr{U}_n) :\ n \in \mathbb{N}\}$ is a ω-cover of Y. Let $H = \{y_1, ..., y_p\}$ be a finite subset of Y. Then $(y_1, ..., y_p) \in Y^p$. Then, there exists $n \in N_p$ such that $((O'_n(S_n))^p : n \in \mathbb{N})$ is a sequence of neighbourhoods of F_n, $n \in \mathbb{N}$ in X^p and $(y_1, ..., y_p) \in st((O'(S_n))^p, \mathscr{W}_n)$. Then $H \subset st(O'(S_n), \mathscr{U}_n)$. $\square$

Corollary 3.2. Let X be a space. If for every $n \in \mathbb{N}$, X^n is selNSM, then X satisfies $selNSS_{fin}(\mathscr{O}, \Omega)$.

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