UNIVERSITY OF MESSINA
DEPARTMENT OF MATHEMATICAL AND COMPUTER SCIENCES, PHYSICAL SCIENCES, AND EARTH SCIENCES

PHD JOINT PROGRAM:
UNIVERSITY OF CATANIA - UNIVERSITY OF PALERMO
XXXVIII CYCLE

DOCTORAL THESIS

SSD: MATH-02/A

GENERALIZED SKEW HOMODERIVATIONS IN PRIME RINGS

PhD Coordinator:
Prof. OLIVERI FRANCESCO

Supervisor:
Prof. DE FILIPPIS VINCENZO

Author:
ANDALORO MILENA

Co-Supervisor:
Prof. SCUDO GIOVANNI

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in

Mathematics and Computational Sciences

ACADEMIC YEAR: 2024/2025

Declaration of Authorship

I, Andaloro Milena, declare that this thesis titled

"Generalized skew homoderivation in prime rings"

and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: Andaloro Milena

Date: September 13, 2025

"In the middle of difficulty lies opportunity."

Albert Einstein

UNIVERSITY OF MESSINA

Abstract

Department of Mathematical and Computer Sciences,
Physical Sciences and Earth Sciences

Doctor of Philosophy

GENERALIZED SKEW HOMODERIVATIONS IN PRIME RINGS

by ANDALORO MILENA

Originating from the study of noncommutative polynomials vanishing under all substitutions from a given algebra, the classical theory of PIs has evolved into a rich framework that connects combinatorial and structural aspects of algebraic systems, with matrix algebras playing a paradigmatic role. A natural extension of this theory is provided by generalized polynomial identities (GPIs), in which coefficients may lie within the algebra itself. Since Amitsur's foundational work and Martindale's generalization to prime rings, GPIs have been investigated in increasingly broad contexts, incorporating involutions, automorphisms, derivations, and anti-automorphisms, thereby yielding deep characterizations of prime and semiprime structures. Within this lineage, derivations and their many generalizations, such as generalized derivations, skew derivations and generalized skew derivations, have been a persistent source of commutativity criteria and structural constraints. Since the second half of the twentieth century, numerous authors have investigated generalized skew derivations that satisfy specific generalized polynomial identities on non-central Lie ideals of prime or semiprime rings. Particularly notable is the concept of *homoderivation*, introduced by El-Sofy, which unifies aspects of endomorphisms and derivations. Recent developments, including ϵ-homoderivations, have expanded this idea, but the full structural potential of such mappings remains largely unexplored. This dissertation advances the theory by introducing the notion of *extended homoderivations*, additive maps from a prime ring R into its right Martindale ring of quotients, associated with scalar parameters in the extended centroid. These maps, together with their generalized, skew, and extended skew variants, subsume and unify classical concepts such as derivations, generalized derivations, homomorphisms, and their skew analogues. The study systematically develops the theory of these mappings, extending and reformulating cornerstone results of Posner, Herstein, and others, and establishing new commutativity theorems in this broadened setting. Special attention is devoted to the interaction between these mappings and structural components such as Lie ideals, centralizing conditions, and torsion-free hypotheses. The dissertation is organized to progressively build this framework: beginning with generalized homoderivations, it extends to the fully skew and extended contexts, culminating in a classification of the most general class considered, the *extended generalized skew homoderivations*. Applications illustrate the breadth of the theory, including analyses of subrings generated by images of the form $F(x)x$ and structural constraints arising from generalized Jordan-type identities.
More specifically, in the course of our analysis, we have undertaken a broad classification of the maps introduced, thereby providing a comprehensive and multifaceted picture of this emerging framework. Furthermore, we have presented an application

that integrates various concepts and elucidates the precise structure of two generalized skew derivations, F and G, in the context where they act as a natural generalization of Jordan homoderivations.

In summary, this work situates extended homoderivations and their variants as a unifying theme in the theory of additive mappings on prime and semiprime rings, demonstrating how classical results on derivations naturally extend to a richer, more flexible setting. By bridging and generalizing multiple strands of existing theory, it provides new tools and perspectives for the structural analysis of noncommutative rings.

Acknowledgements

A doctoral degree is a journey of personal and intellectual growth, marked by moments of discovery and reflection, where curiosity meets challenge and commitment transforms into understanding. This thesis represents the culmination of a path during which I have always felt the support, guidance, and encouragement of the extraordinary people who have accompanied me along the way. It is with sincere gratitude that I thank all those who have shaped and enriched this journey.

First, I would like to express my deepest and most heartfelt gratitude to Professor V. De Filippis. His kindness, generous spirit, and unwavering professionalism have been a constant guide throughout this experience. Even when the problem at hand seemed unsolvable, he showed me that there is always an original solution, a new answer, or an alternative path to reach the goal, whether the expected one or the surprisingly unexpected one. Thanks to his example, I have come to understand that the beauty of research lies not only in seeking answers but also in sharing ideas, building community, and feeling part of a research group that aspires to create something meaningful and useful together. This sense of belonging and collaboration transcends the academic environment and enriches not only our shared work but also the bonds we have built and our passion for mathematics. It has truly been an honor and a privilege to grow under his guidance. I thank him for his example and for the trust he has placed in me, a precious gift that has guided me every step of the way. I am also deeply grateful to Professor G. Scudo for his constant support and warm friendship. His encouragement and insightful guidance have been invaluable in times of uncertainty, helping me face challenges with renewed confidence and clarity. The kindness and trust he has shown me have made a significant difference during this demanding journey, and I will always cherish the collaboration and friendship we have built together.

I am deeply grateful to Professor F. Oliveri for his availability, professionalism, and kind support over these years, as well as to Professor R. Utano, who first inspired my love for algebra during my early university studies.

A warm and heartfelt thank you goes to my friend A. Hermas, whom I had the privilege of meeting during these years and with whom I had the pleasure of collaborating. His invaluable advice, generous spirit, and unfailing good humor have enriched this journey in countless ways. I hold him in the deepest esteem, for he is already an excellent mathematician, and I wholeheartedly hope that his dream of becoming a professor will soon be realized. It has been an honor to share part of this path with him, and I am truly grateful for his presence and friendship.

A heartfelt thank you goes to my family, who have been my unshakable foundation and my greatest source of strength throughout these years. I would especially like to thank my parents for their enduring bond and for the extraordinary example they have set for me through their tireless efforts to build, brick by brick, new bridges to the future. I am grateful to my brother for the laughter that effortlessly sweeps away dark thoughts, and for his determination and playful spirit, which inspire me to embrace life with joy and courage. Their presence and support have been a source of endless comfort and motivation. A special thank you goes to my beloved companion. His unwavering faith in my abilities, his immense support, and the countless hours spent by my side studying and reflecting have been invaluable. His patience, understanding, and love have been my safe harbor during the most difficult days. Without his presence, this journey would have been far more difficult, and I am infinitely grateful for the gift of his companionship.

Finally, I must pay heartfelt tribute to my dear grandmother. She has been my rock,

my unwavering pillar of strength throughout these years. Her steadfast faith in me, gentle encouragement, and profound love have been a guiding light in times of uncertainty and exhaustion. The resilience and confidence I carry today are deeply rooted in the loving example and support she so generously offered me.
To all of you, I extend my sincerest thanks. I consider myself incredibly fortunate to have had the opportunity to embark on this journey, a privilege not afforded to everyone, and I will forever cherish the quiet, respectful, and boundless kindness I have received.

Contents

Introduction

Polynomials in several variables are a central topic in commutative algebra. Their counterparts in noncommutative algebra are *noncommutative polynomials*, namely polynomials in indeterminates that do not commute. A *polynomial identity* for an $\mathbb{F}$-algebra A is a polynomial $f(\xi_1, \ldots, \xi_n) \in \mathbb{F}\langle \xi_1, \ldots, \xi_n \rangle$ such that $f(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in A$. If a nonzero polynomial is an identity of A, then A is called a *PI-algebra*. For instance, saying that A is commutative is equivalent to saying that A satisfies the identity $[\xi_1, \xi_2] = \xi_1\xi_2 - \xi_2\xi_1$; hence every commutative algebra is a PI-algebra. In particular, matrix algebras $M_n(\mathbb{F})$ occupy a central position in the theory of polynomial identities. Broadly speaking, this theory develops along two closely related lines. On the one hand, there is the *combinatorial* approach, which focuses on determining and describing the entire set of identities satisfied by a given algebra, with special attention devoted to the case of $M_n(\mathbb{F})$. On the other hand, the *structural* approach seeks to characterize prime PI-algebras, showing that they can be embedded, in a remarkably natural fashion, into a matrix algebra over a suitable field extension of the base field. The classical theory of polynomial identities (PIs) was soon enriched by the notion of *generalized polynomial identities* (GPIs), where the coefficients of the polynomial are allowed to come not only from the base field, but also from the algebra itself. Formally, for an algebra A over a field F, a GPI is a noncommutative polynomial with fixed coefficients in A that vanishes under every substitution of elements from A. The systematic study of GPIs began with Amitsur's seminal 1965 paper [1], which established a characterization of primitive rings satisfying such identities. This required working within the framework of associative rings R, with center $Z(R)$, and distinguishing between *prime rings*, those for which $xRy = (0)$ implies $x = 0$ or $y = 0$, and the more general *semiprime rings*, where $xRx = (0)$ implies $x = 0$. Clearly, every prime ring is semiprime, but the converse does not hold, as exemplified by $\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ denotes the ring of integers. Building on Amitsur's results, Martindale extended the theory to prime rings [26]. In doing so, he introduced two fundamental tools: the *extended centroid* C, the natural field of scalars associated with a prime ring, and the *right Martindale ring of quotients* Q_r, which allows one to embed R into a larger, more tractable structure. These notions are essential in the study of polynomial identities and will play a key role in our work. The theory of GPIs rapidly evolved in several directions: Martindale, Rowen and others investigated GPIs in rings with involution, while Beidar and Mikhalev extended the analysis to semiprime rings by developing the method of *orthogonal completion*, a powerful alternative to the traditional reduction to the prime case.

A further turning point came in the late 1970s, when Kharchenko introduced the study of identities in which variables are acted upon by derivations and automorphisms, now referred to as *generalized identities* (GIs). Here, derivations are additive maps $d : R \to R$ satisfying the Leibniz rule

$$d(xy) = d(x)y + xd(y), \quad \text{for all } x, y \in R,$$

In particular, a derivation is called inner if there exists an element $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. In this context, commutators play a central role. We adopt the standard notation $[x, y] = xy - yx$ for the Lie product, and recall that a crucial structure within R is that of a Lie ideal, namely an additive subgroup U such that $[U, R] \subseteq U$, where

$$[U, R] = \{[u, r] = ur - ru : u \in U, r \in R\}.$$

In particular, the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ will be frequently used throughout this work.
Kharchenko's framework not only unified the existing perspectives and techniques up to that point, but also revealed key phenomena such as X-inner automorphisms and derivations arising in Utumi's ring of quotients. His contributions were subsequently refined in the 1980s by Lanski, who considered prime rings with involution, and in the 1990s by Chuang, who generalized the theory to arbitrary anti-automorphisms. Chuang's introduction of *Frobenius (anti)automorphisms* provided a powerful structural description of GI-rings, extending Kharchenko's approach even in the purely derivational or automorphic setting. Parallel to this, mappings interacting with the multiplicative structure of rings became a central theme. Among them, derivations have been extensively studied, along with their generalizations: *generalized derivations* $F : R \to R$, introduced by Brešar [7], for which there exists a derivation d such that

$$F(xy) = F(x)y + xd(y), \quad \text{for all } x, y \in R,$$

and *skew derivations*, associated with an automorphism α and satisfying

$$d(xy) = d(x)y + \alpha(x)d(y), \quad \text{for all } x, y \in R.$$

While every derivation is a generalized derivation, the converse is not true: a fundamental example is given by maps of the form $F(x) = ax + xb$ for some fixed $a, b \in R$, which are known as inner generalized derivations. Moreover, the interaction between derivations and automorphisms naturally led to *generalized skew derivations*, further broadening the scope of the theory, which are additive maps $F : R \longrightarrow R$ such that

$$F(xy) = F(x)y + \alpha(x)d(y), \quad \text{for all } x, y \in R,$$

where d is a skew derivation and α is an automorphism of R.
We should also recall that, in the study of GIs, the structural properties of the ring R will be of particular significance. For instance, in our work we shall frequently assume that R is 2-torsion free, meaning that $2x = 0$ implies $x = 0$. Finally, it is worth noting that a map $f : R \to R$ is said to be centralizing if $[f(x), x] \in Z(R)$ for every $x \in R$, and commuting if $[f(x), x] = 0$ for every $x \in R$. These notions will play a decisive role in many of the results presented in this thesis.

Let us now delve into the core of our work and present the starting point of our analysis. From the historical overview just outlined, it becomes evident that the study of endomorphisms and derivations on rings has been a central theme in algebra since the mid-twentieth century. Researchers have extensively investigated the properties of prime and semiprime rings through the lens of derivations, often arriving at significant commutativity results. This line of inquiry has naturally evolved toward the study of rings equipped with homomorphisms, where analogous structural properties have likewise been explored. At the turn of the millennium, El-Sofy [19] proposed a new unification of homomorphisms and derivations, introducing the concept of *homoderivations*: additive maps $h : R \to R$ such that

$$h(xy) = h(x)h(y) + h(x)y + xh(y), \quad \text{for all } x, y \in R.$$

An illustrative example is the map $h(x) = f(x) - x$, where f is an endomorphism of R. Notably, the only additive map that is both a derivation and a homoderivation on a prime ring is the zero map. Over the years, many authors have investigated homoderivations and their generalizations on rings and near-rings, establishing structural results parallel to those obtained for derivations. More recently, Gselmann and Kiss [21] introduced the notion of an *ϵ-homoderivation*, defined by the condition

$$h_\epsilon(xy) = \epsilon h_\epsilon(x)h_\epsilon(y) + h_\epsilon(x)y + xh_\epsilon(y), \quad \text{for all } x, y \in R,$$

where ϵ belongs to the center $Z(R)$ of the ring R. This definition recovers El-Sofy's homoderivation as a special case in unital rings. Moreover, when the center $Z(R)$ is trivial, ϵ-homoderivations reduce to derivations.

In the present dissertation, we propose a further extension of this line of research by introducing the concept of an *extended homoderivation*. Specifically, we call an additive map $d : R \to Q_r$ an extended homoderivation associated with $\lambda, \mu \in C$ if

$$d(xy) = \lambda d(x)d(y) + \mu d(x)y + \mu xd(y), \quad \text{for all } x, y \in R.$$

In close analogy with the classical case of derivations, we also develop the notions of *extended generalized homoderivations* and *extended skew homoderivations*. Our objective is to investigate the algebraic structure of prime rings admitting such maps, thereby extending fundamental results established in the mid-20th century. In particular, we draw inspiration from Posner's celebrated theorems on derivations in prime rings [29]. His first theorem asserts that if R is a 2-torsion-free prime ring and d_1, d_2 are derivations such that their composition d_1d_2 is also a derivation, then $d_1 = 0$ or $d_2 = 0$. His second theorem provides a commutativity criterion: if R is prime and d is a derivation such that $[d(x), x]$ lies in the center for all $x \in R$, then either $d = 0$ or R is commutative. Mathieu later showed that the second theorem follows from the first [27], underlining their deep connection. Subsequent research has generalized these results in many directions: Ashraf and Siddeeque [3] extended the first theorem to prime rings with involution; Sandhu [31] treated the case of skew derivations and derivations of Lie ideals; while the commutativity theorems of Herstein [22] and Daif [15] have been adapted to settings involving homoderivations and generalized derivations [4]. In what follows, we aim to show that analogous structural and commutativity results continue to hold for rings equipped with extended homoderivations, thereby enlarging the scope of a classical theory in a new and natural direction.

Ultimately, the aim of this dissertation is to investigate the properties of homoderivations within the framework of prime and semiprime rings, building on classical results on derivations, while exploring new algebraic structures that arise from this unification. Our study seeks to explore, in particular, the structural implications that arise when a prime ring admits an extended homoderivation. In particular, we investigate how the fundamental properties of classical derivations are reflected and transformed within this generalized framework. In pursuit of these objectives, the dissertation is structured to progressively introduce, generalize, and explore novel

classes of additive mappings acting on associative rings, with particular emphasis on their structural and potential applications.
To this end, a detailed overview of the dissertation's structure is provided below, chapter by chapter, with the purpose of highlighting the logical progression and conceptual developments that underpin our investigation.

In Chapter 1, we establish the theoretical foundation by revisiting the notion of *homoderivation*, originally introduced by El-Sofy, and extending it to the broader class of *generalized homoderivations*. Although this concept has already appeared in various academic works, it has not yet been subjected to a systematic and thorough analysis. We provide explicit examples and focus on the characterization of those mappings that naturally arise from homoderivations. This chapter also introduces several technical tools that will prove essential for the subsequent development of the theory. In doing so, we lay down a solid framework for the study of more advanced algebraic structures.

Chapter 2 builds upon this foundation by introducing the notions of *extended homoderivations* and *extended generalized homoderivations*. These newly defined mappings are studied in detail, both in connection with arbitrary additive maps and in the more specific case when they originate from extended homoderivations. Remarkably, they succeed in generalizing, all at once, the notions of homomorphism and derivation, together with those of homoderivation and ϵ-homoderivation, thus providing a unified framework that embraces them all.

In Chapter 3, the theory is further enriched through the incorporation of automorphisms, giving rise to the class of *generalized skew homoderivations*. After presenting their formal definitions and illustrative examples, we examine their structural properties through precise characterizations, whether they are associated with additive maps or with homoderivations. This comprehensive analysis allows us to determine all admissible forms of such mappings as the coefficients appearing in their definitions vary.

Chapter 4 is devoted to the most general class of mappings considered in this dissertation: the *extended generalized skew homoderivations*, defined by the identity

$$F(xy) = \mu F(x)F(y) + F(x)y + \alpha(x)d(y),$$

where $\mu \in C$, α is an automorphism of the ring, and d is either an additive map or, in specific cases, an extended homoderivation. This chapter mirrors the structure of Chapter 2, while emphasizing the interplay between extension and skewness. We provide a complete classification of these mappings in both the general additive case and in the more particular scenario where d is an extended homoderivation. These results broaden the scope of classical structural theorems, placing them in a highly generalized and non-commutative setting.

In Chapter 5, we illustrate the applicability of the developed framework by analyzing subrings generated by expressions of the form $F(x)x$, where F belongs to one of the classes of maps introduced in the previous chapters. Specifically, we estimate the "size" of $F(R)$ by investigating the structure of the corresponding subring. This analysis demonstrates how the properties of the introduced mappings shape the ambient ring, thereby providing a concrete instance of their influence beyond purely theoretical considerations.

Within each of these broader frameworks, we have also established significant extensions of classical ring-theoretic results, including new formulations of Posner's and Herstein's theorems.

Finally, Chapter 6 presents an additional application that highlights the expressive power of the developed theory. Here, we study the behavior of two generalized skew derivations when they act as a *generalization of Jordan homoderivations*, that is, when they satisfy the identity

$$F(x^2) = G(x)^2 + G(x)x + xG(x).$$

This final example underscores the structural significance of the newly introduced mappings and reinforces their potential as powerful tools for further investigations in the theory of non-commutative rings.

Chapter 1

Generalized homoderivations

The examination of functional mappings on rings has constituted a fundamental topic in algebra since the mid-20th century, with particular emphasis on homomorphisms and derivations. The study of prime and semiprime rings under the influence of derivations has frequently yielded results concerning commutativity, naturally extending to the analysis of rings equipped with homomorphisms, where analogous structural attributes have been identified. At the dawn of the 21st century, a new framework integrating homomorphisms and derivations was introduced: homoderivations.

1.1 Homoderivations

Definition 1. [19] Let R be an associative ring.
An additive map $d : R \longrightarrow R$ is called a homoderivation if

$$d(xy) = d(x)d(y) + d(x)y + xd(y) \qquad \text{for all } x, y \in R.$$

Remark 1. *It is clear that a homoderivation d is also a derivation if $d(x)d(y) = 0$ for all $x, y \in R$. In this case, $d(x)Rd(y) = 0$ for all $x, y \in R$. Hence if R is a prime ring, then the only additive mapping which is both a derivation and a homoderivation is the zero mapping.*

Remark 2. *Let $d : R \longrightarrow R$ be a homoderivation. Then, for all $x, y \in R$*

$$\begin{aligned} d^2(xy) &= d(d(xy)) = \\ &= d\Big(d(x)d(y) + d(x)y + xd(y)\Big) = \\ &= d\Big(d(x)d(y)\Big) + d\Big(d(x)y\Big) + d\Big(xd(y)\Big) = \\ &= d^2(x)d^2(y) + d^2(x)d(y) + d(x)d^2(y) + d^2(x)d(y) + d^2(x)y + d(x)d(y) + \\ &\quad + d(x)d^2(y) + d(x)d(y) + xd^2(y) = \\ &= d^2(x)d^2(y) + 2d^2(x)d(y) + 2d(x)d^2(y) + d^2(x)y + xd^2(y) + 2d(x)d(y) \end{aligned}$$

Remark 3. *Let R be a prime ring, d a nonzero homoderivation of R, and $a \in R$ a fixed element.*
If $ad(x) = 0$ for all $x \in R$, then $a = 0$.

Proof:
Since $ad(xy) = 0$ for all $x, y \in R$, then $0 = a\Big(d(x)d(y) + d(x)y + xd(y)\Big) = axd(y)$.
Since R is a prime ring, then $a = 0$.

Remark 4. *Let R be a prime ring, d a nonzero homoderivation of R, and $a \in R$ a fixed element.*
If $d(x)a = 0$ for all $x \in R$, then $a = 0$.

Proof:
Since $d(xy)a = 0$ for all $x, y \in R$, then $0 = \Big(d(x)d(y) + d(x)y + xd(y)\Big)a = d(x)ya$.
Since R is a prime ring, then $a = 0$.

Proposition 1. *Let R be an associative unital ring, with unit 1_R, and let $d : R \longrightarrow R$ be a homoderivation. Then*

$$d(1_R)\Big(d(x) + x\Big) = 0 = \Big(d(x) + x\Big)d(1_R) \qquad \text{for all } x \in R$$

Proof:
By the definition of homoderivation, we have

$$d(xy) = d(x)d(y) + d(x)y + xd(y) \qquad \text{for all } x, y \in R. \tag{1.1}$$

Replacing x with 1_R in (1.1), we obtain $d(y) = d(1_R)d(y) + d(1_R)y + d(y)$ for all $y \in R$, that is,

$$0 = d(1_R)(d(y) + y) \qquad \text{for all } y \in R.$$

Replacing y with 1_R in (1.1), we obtain $d(x) = d(x)d(1_R) + d(x) + xd(1_R)$ for all $x \in R$, that is,

$$0 = (d(x) + x)d(1_R) \qquad \text{for all } x \in R.$$

Remark 5. *In a prime ring, the annihilator of an isomorphism is zero.*
So, if f were an isomorphism, then we would have $d(1_R) = 0$. In the case of homomorphism, the annihilator is not zero in general.

Remark 6. *In particular, from Proposition 1 it follows that*

$$d(1_R)\Big(d(1_R) + 1_R\Big) = 0$$

that is

$$d(1_R)d(1_R) = \Big(-d(1_R)\Big)\Big(-d(1_R)\Big) = -d(1_R)$$

Therefore element $-d(1_R)$ is idempotent.
So, if R is a prime ring and $d(1_R)$ is central, then $d(1_R) = -1$

1.1.1 Examples

Example 1. *Let $f : R \longrightarrow R$ be a homomorphism. The map $d : R \longrightarrow R$ such that*

$$d(x) = f(x) - x$$

is a homoderivation. In fact for all $x, y \in R$

$$\begin{aligned} d(xy) &= f(xy) - xy = \\ &= f(x)f(y) - xy = \\ &= f(x)f(y) + f(x)y - f(x)y + xf(y) - xf(y) + xy - xy - xy = \\ &= (f(x) - x)f(y) + (f(x) - x)y - (f(x) - x)y + x(f(y) - y) = \\ &= (f(x) - x)(f(y) - y) + (f(x) - x)y + x(f(y) - y) = \\ &= d(x)d(y) + d(x)y + xd(y) \end{aligned}$$

Conversely, the additive map $f(x) = d(x) + x$, with d a homoderivation, is a homomorphism. Indeed:

$$\begin{aligned} f(xy) &= d(xy) + xy = \\ &= d(x)d(y) + d(x)y + xd(y) + xy = \\ &= d(x)(d(y) + y) + x(d(y) + y) = \\ &= (d(x) + x)(d(y) + y) = \\ &= f(x)f(y) \end{aligned}$$

In particular, the map $d(x) = -x$ is a homoderivation.
It is such that $d(1_R) = -1_R$ so, in general, it is not true that $d(1_R) = 0_R$.

Example 2. *Let $R = \mathbb{Z}(\sqrt{2})$, a ring of all the real numbers of the from $m + n\sqrt{2}$ such that $m, n \in \mathbb{Z}$, the set of all the integers, under the usual addition and multiplication of real numbers. Then, the map $d : R \longrightarrow R$ defined by*

$$d(m + n\sqrt{2}) = -2n\sqrt{2}$$

is a homoderivation of R.

Example 3. *Let d be a homoderivation. In general, the composition $d(d(x))$ and the sum of homoderivations are not homoderivations. Instead, the map $g(x) = d(d(x)) + 2d(x)$ is a homoderivation.*
In fact, since $f(x) = d(x) + x$ is a homomorphism, then $f(f(x)) - x = g(x)$ is a homoderivation.
More generally, if d_1 and d_2 are homoderivations, then the two maps

$$g_{12}(x) = d_1(d_2(x)) + d_1(x) + d_2(x)$$

and

$$g_{21}(x) = d_2(d_1(x)) + d_1(x) + d_2(x)$$

are homoderivations.

Example 4. *Let R be an associateve ring and let e be an idempotent element of R.*
Then $d(x) = e - x$ is a homoderivation.

Example 5. *Let d be a homoderivation. Let $F(x) = -d(x) - 2x$ and $G(x) = d(x) + x - 1$. Then*

$$d(xy) = F(x)F(y) + F(x)y + xF(y)$$

$$d(xy) = G(x + y) + G(x)G(y) - xy$$

1.1.2 Homomultiplier

Definition 2. *An additive mapping $d : R \longrightarrow R$ is said to be homoright multiplier (or, homoleft multiplier) if*

$$d(xy) = d(x)d(y) + xd(y)$$
$$(or\ d(xy) = d(x)d(y) + d(x)y)$$

for all $x, y \in R$.
If d is both homoright multiplier as well as homoleft multiplier, then d will be called homomultiplier.

In [6], A. Boua and E. Farhan found that the only homoleft multiplier or homoright multiplier on a semiprime ring is the zero mapping.

Example 6. *Let R be a ring such that $char(R) \neq 2$. Define*

$$S = \left\{ A = xe_{12} + ye_{13} + ze_{23} \in M_3(R), \quad with\ x, y, z \in R \right\}$$

and $d_1, d_2, d_3 : S \longrightarrow S$ such that

$$d_1(A) = -xe_{12} - ze_{23}, \quad d_2(A) = xe_{12} - ze_{23}, \quad d_3(A) = -xe_{12} + ze_{23}, \quad for\ all\ A \in S.$$

We can easily see that:

- *d_1 is a homo multiplier which is not a homomorphism, nor a multiplier;*
- *d_2 is a homoleft multiplier which is not a homoright multiplier, nor a homomorphism, nor a left multiplier;*
- *d_3 is a homoright multiplier, which is neither a homoleft multiplier, nor homomorphism or right multiplier.*

1.1.3 Posner's and Herstein's theorems

Lemma 1. *Let R be a prime ring of characteristic different from 2 and let d be a homoderivation of R. By Remark 2,*

$$d^2(xy) = d^2(x)d^2(y) + 2d^2(x)d(y) + 2d(x)d^2(y) + d^2(x)y + xd^2(y) + 2d(x)d(y)$$

So if $d^2(x) = 0$ for all $x \in R$, by Remarks 3 and 4 it follows $d = 0$.

Theorem 1. *Let R be a prime ring of characteristic different from 2 and let d_1 and d_2 be two homoderivations of R. If $d_1(d_2(x)) = 0$ for all $x \in R$, then $d_1(x) = 0$ or $d_2(x) = 0$.*

Proof:
For all $x, y \in R$ we have

$$\begin{aligned} 0 =& d_1(d_2(xy)) = \\ =& d_1\Big(d_2(x)d_2(y) + d_2(x)y + xd_2(y)\Big) = \\ =& d_2(x)d_1(y) + d_1(x)d_2(y) \end{aligned}$$

Replacing x with $d_2(x)$ we have

$$0 = d_2(d_2(x))d_1(y) \tag{1.2}$$

Replacing y with yz and using (1.2) we have

$$\begin{aligned} 0 =& d_2(d_2(x))\Big(d_1(y)d_1(z) + d_1(y)z + yd_1(z)\Big) = \\ =& d_2(d_2(x))yd_1(z) \end{aligned}$$

Since R is a prime ring then $d_2(d_2(x)) = 0$ or $d_1(x) = 0$. That is, using Lemma 1, $d_2(x) = 0$ or $d_1(x) = 0$.

Theorem 2. *Let R be a prime ring and let d be a homoderivation of R. If $[d(x), x] = 0$ for all $x \in R$ then $d(x) = 0$ or $d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.*

Proof:
For all $x, y \in R$

$$0 = [d(x+y), x+y] = [d(x) + d(y), x+y] = [d(x), y] + [d(y), x] \tag{1.3}$$

Replacing y with xy and using (1.3) we have

$$\begin{aligned} 0 &= [d(x), xy] + [d(xy), x] = \\ &= x[d(x), y] + [d(x)d(y) + d(x)y + xd(y), x] = \\ &= -x[d(y), x] + [d(x)d(y) + d(x)y + xd(y), x] = \\ &= -[xd(y), x] + [d(x)d(y) + d(x)y + xd(y), x] = \\ &= [d(x)(d(y) + y), x] = \\ &= d(x)[d(y), x] + d(x)[y, x] = \\ &= -d(x)[d(x), y] + d(x)[y, x] = \\ &= d(x)[y, d(x) + x] \end{aligned} \tag{1.4}$$

Replacing y with yt and using (1.4) we have

$$0 = d(x)[yt, d(x) + x] = d(x)y[t, d(x) + x]$$

Since R is prime it follows that for each $x \in R$:

$$d(x) = 0 \quad \text{or} \quad f(x) := d(x) + x \in Z(R) \tag{1.5}$$

Suppose there exists $x_1 \in R$ with $d(x_1) \neq 0$.
By (1.5) we have $f(x_1) \in Z(R)$.
Replacing y with yz in (1.3) and using (1.3) we have

$$\begin{aligned} 0 =& [d(x), yz] + [d(yz), x] = \\ =& y[d(x), z] + [d(x), y]z + [d(y)d(z), x] + [d(y)z, x] + [yd(z), x] = \\ =& y[d(x), z] + [d(x), y]z + d(y)[d(z), x] + [d(y), x]d(z) + d(y)[z, x] + [d(y), x]z + \\ & + y[d(z), x] + [y, x]d(z) = \\ =& d(y)[d(z), x] + [d(y), x]d(z) + d(y)[z, x] + [y, x]d(z) = \\ =& d(y)[d(z) + z, x] + [d(y) + y, x]d(z) \end{aligned}$$

Replacing $y = x_1$, since $f(x_1) \in Z(R)$, we have

$$\begin{aligned}0 =& d(x_1)[d(z)+z,x] + [d(x_1)+x_1,x]d(z) = \\ =& d(x_1)[d(z)+z,x] + [f(x_1),x]d(z) = \\ =& d(x_1)[d(z)+z,x]\end{aligned}$$

Replacing x with xt we have

$$0 = d(x_1)[d(z)+z,xt] = d(x_1)x[d(z)+z,t] + d(x_1)[d(z)+z,x]t = d(x_1)x[d(z)+z,t]$$

Since $d(x_1) \neq 0$ and R is a prime ring, then

$$f(z) \in Z(R) \quad \text{for all } z \in R \tag{1.6}$$

In particular, if $f(z) = 0$ for every $z \in R$, then $d(z) = -z$.

Theorem 3. *Let R be a noncommutative prime unital ring of characteristic not 2. Suppose there exists a homoderivation $d : R \longrightarrow R$ such that the mapping*

$$x \mapsto [d(x), x]$$

is commuting on R. Then $d = 0$ or $d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.

Proof:
We have

$$\Big[[d(x),x],x\Big] = 0 \quad \text{for all } x \in R \tag{1.7}$$

In particular, replacing x with $x + 1_R$ we have

$$\Big[[d(1_R),x],x\Big] = 0 \quad \text{for all } x \in R \tag{1.8}$$

and then $d(1_R) \in Z(R)$.
If $d(1_R) \neq 0$ then, since $d(1_R)\Big(d(x)+x\Big) = 0 = \Big(d(x)+x\Big)d(1_R)$ for all $x \in R$, we have $d(x) = -x$.
Assume $d(1_R) = 0$.
Replacing x with $x + y$ in (1.7) and using (1.7) we get

$$0 = \Big[[d(x),y]+[d(y),x]+[d(y),y],x\Big] + \Big[[d(x),x]+[d(x),y]+[d(y),x],y\Big] \tag{1.9}$$

Replacing x with $-x$ in (1.9) we get

$$0 = \Big[[d(x),y]+[d(y),x]-[d(y),y],x\Big] + \Big[[d(x),x]-[d(x),y]-[d(y),x],y\Big] \tag{1.10}$$

Then by (1.9) and (1.10) we have

$$0 = 2\Big[[d(y),y],x\Big] + 2\Big[[d(x),y]+[d(y),x],y\Big]$$

Since $char(R) \neq 2$ then

$$\Big[[d(y),y],x\Big] + \Big[[d(x),y] + [d(y),x],y\Big] = 0 \tag{1.11}$$

Replacing x with yx in (1.11) and using (1.7) we get

$$\begin{aligned} 0 =& y\Big[[d(y),y],x\Big] + d(y)\Big[[d(x),y],y\Big] + [d(y),y][2d(x)+3x,y] + d(y)\Big[[x,y],y\Big] + \\ &+ y\Big[[d(x),y] + [d(y),x],y\Big] \end{aligned}$$

Using (1.11) we get

$$\begin{aligned} 0 =& d(y)\Big[[d(x),y],y\Big] + [d(y),y][2d(x)+3x,y] + d(y)\Big[[x,y],y\Big] = \\ =& d(y)\Big[[d(x)+x,y],y\Big] + [d(y),y][2d(x)+3x,y] \end{aligned} \tag{1.12}$$

Replacing x with xz in (1.12) we get

$$\begin{aligned} 0 =& d(y)(d(x)+x)\Big[[d(z)+z,y],y\Big] + 2d(y)[d(x)+x,y][d(z)+z,y] + \\ &+ d(y)\Big[[d(x)+x,y],y\Big](d(z)+z) + 2[d(y),y](d(x)+x)[d(z)+z,y] + \\ &+ 2[d(y),y][d(x)+x,y](d(z)+z) + [d(y),y]x[x,y] + [d(y),y][x,y]z \end{aligned}$$

By (1.12) we have

$$\begin{aligned} 0 =& d(y)(d(x)+x)\Big[[d(z)+z,y],y\Big] + 2d(y)[d(x)+x,y][d(z)+z,y] + \\ &+ 2[d(y),y](d(x)+x)[d(z)+z,y] - [d(y),y][x,y]d(z) + [d(y),y]x[x,y] \end{aligned} \tag{1.13}$$

Replacing x with $x + 1_R$ in (1.13), since $d(1_R) = 0$, we have

$$\begin{aligned} 0 =& d(y)(d(x)+x)\Big[[d(z)+z,y],y\Big] + d(y)\Big[[d(z)+z,y],y\Big] + 2d(y)[d(x)+x,y][d(z)+z,y] + \\ &+ 2[d(y),y](d(x)+x)[d(z)+z,y] + 2[d(y),y][d(z)+z,y] - [d(y),y][x,y]d(z) + \\ &+ [d(y),y]x[x,y] + [d(y),y][x,y] \end{aligned}$$

By (1.13) we have

$$d(y)\Big[[d(z)+z,y],y\Big] + 2[d(y),y][d(z)+z,y] + [d(y),y][x,y] = 0 \tag{1.14}$$

Then, replacing x in (1.12) with z and then comparing with (1.14) it follows that

$$0 = [d(y),y][x,y] - [d(y),y][z,y] = 0$$

Replacing z with $-x$, since $char(R) \neq 2$, we have

$$0 = [d(y),y][x,y] = 0$$

Replacing x with xt we have

$$0 = [d(y), y]x[t, y] + [d(y), y][x, y]t = [d(y), y]x[t, y]$$

Since R is prime it follows that for each $y \in R$:

$$[d(y), y] = 0 \quad \text{or} \quad y \in Z(R) \tag{1.15}$$

If there exist y_0 such that

$$[d(y_0), y_0] \neq 0 \tag{1.16}$$

then by (1.15) we have $y_0 \in Z(R)$, and this is in contradiction with (1.16).
Hence

$$[d(y), y] = 0 \quad \text{for all } y \in R$$

that is, by Theorem 2, $d = 0$ or $d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.

Theorem 4. *Let R be a noncommutative prime ring of characteristic different from 2 and 3. Suppose there exists a homoderivation $d : R \longrightarrow R$, such that the mapping*

$$x \mapsto [d(x), x]$$

is centralizing on R. Then $d = 0$ or $d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.

Proof:
The assumption of the theorem can be written as follows

$$\Big[[d(x), x], x\Big] \in Z(R) \quad \text{for all } x \in R \tag{1.17}$$

Replacing x with $x + y$ in (1.17) we get

$$\Big[[d(x), x] + [d(x), y] + [d(y), x] + [d(y), y], x\Big] + \\ + \Big[[d(x), x] + [d(x), y] + [d(y), x] + [d(y), y], y\Big] \in Z(R)$$

and using (1.17) we have

$$\Big[[d(x), y] + [d(y), x] + [d(y), y], x\Big] + \Big[[d(x), x] + [d(x), y] + [d(y), x], y\Big] \in Z(R) \tag{1.18}$$

Replacing y with x^2 in (1.18) and using (1.17) we obtain

$$\begin{aligned} &2\Big(6x^2 + 3x + 2xd(x) + d(x) + 2d(x)x\Big)\Big[[d(x), x], x\Big] + \\ &+ 2(2x + 1)[d(x), x]^2 + 4[d(x), x]^2 x \in Z(R) \end{aligned} \tag{1.19}$$

Replacing x with $-x$ we have

$$\begin{aligned} &2\Big(-6x^2 + 3x - 2xd(x) + d(x) - 2d(x)x\Big)\Big[[d(x), x], x\Big] + \\ &+ 2(-2x + 1)[d(x), x]^2 - 4[d(x), x]^2 x \in Z(R) \end{aligned} \tag{1.20}$$

Adding (1.19) and (1.20) we have

$$4\Big(3x+d(x)\Big)\Big[[d(x),x],x\Big]+4[d(x),x]^2\in Z(R)$$

that is

$$\Big(3x+d(x)\Big)\Big[[d(x),x],x\Big]+[d(x),x]^2\in Z(R) \tag{1.21}$$

Therefore, for all $x\in R$,

$$\begin{aligned}0=&\Big[\Big(3x+d(x)\Big)\Big[[d(x),x],x\Big]+[d(x),x]^2,x\Big]=\\ =&\Big[\Big(3x+d(x)\Big)\Big[[d(x),x],x\Big],x\Big]+\Big[[d(x),x]^2,x\Big]=\\ =&\Big[[d(x),x],x\Big][d(x),x]+[d(x),x]\Big[[d(x),x],x\Big]+\Big[[d(x),x],x\Big][d(x),x]=\\ =&3\Big[[d(x),x],x\Big][d(x),x]\end{aligned}$$

Since $char(R)\neq 3$ then

$$\Big[[d(x),x],x\Big][d(x),x]=0 \quad \text{for all } x\in R \tag{1.22}$$

Suppose that there exists $x_0\in R$ such that

$$\Big[[d(x_0),x_0],x_0\Big]\neq 0 \tag{1.23}$$

Since for hypothesis $\Big[[d(x_0),x_0],x_0\Big]\in Z(R)$, then by (1.22) we have $[d(x_0),x_0]=0$, and this is in contradiction with (1.23).
Therefore $\Big[[d(x),x],x\Big]=0$ for all $x\in R$ and by Theorem 3 it follows that $d=0$ or $d(x)=f(x)-x$ for all $x\in R$ with $f:R\longrightarrow Z(R)$ an homomorphism.

In [22], Herstein proved that if R is a prime ring of $char(R)\neq 2$ admitting a non-zero derivation d such that $[d(x),d(y)]=0$ for all $x,y\in R$, then R is commutative. Similarly, for homoderivations, we have the following:

Lemma 2. *Let R be a non-commutative unital prime ring of characteristic different from 2 and $d:R\longrightarrow R$ a homoderivation of R. If $[d(x),d(y)]=0$ for all $x,y\in R$ then $d=0$.*

Proof:
By Proposition 1 we have

$$d(1_R)\Big(d(x)+x\Big)=0=\Big(d(x)+x\Big)d(1_R) \qquad \text{for all } x\in R \tag{1.24}$$

From the hypothesis we have

$$[d(x),d(y)]=0 \quad \text{for all } x,y\in R \tag{1.25}$$

Substituting y with yz we have

$$[d(x), d(yz)] = 0 \quad \text{for all } x, y, z \in R$$

that is

$$[d(x), d(y)d(z) + d(y)z + yd(z)] = 0 \quad \text{for all } x, y, z \in R$$

By (1.25) we have

$$d(y)[d(x), z] + [d(x), y]d(z) = 0 \quad \text{for all } x, y, z \in R \tag{1.26}$$

Replacing z with zt we get

$$d(y)[d(x), z]t + d(y)z[d(x), t] + [d(x), y]d(z)d(t) + [d(x), y]d(z)t + [d(x), y]zd(t) = 0 \quad \text{for all } x, y, z, t \in R \tag{1.27}$$

Multiplying (1.26) by t on the right we have

$$d(y)[d(x), z]t + [d(x), y]d(z)t = 0 \quad \text{for all } x, y, z \in R \tag{1.28}$$

Subtracting (1.28) from (1.27) we get

$$d(y)z[d(x), t] + [d(x), y]d(z)d(t) + [d(x), y]zd(t) = 0 \quad \text{for all } x, y, z, t \in R$$

Replacing y with $d(u)$ we have

$$d^2(u)z[d(x), t] + [d(x), d(u)]d(z)d(t) + [d(x), d(u)]zd(t) = 0 \quad \text{for all } x, u, z, t \in R$$

By (1.25) we have

$$d^2(u)z[d(x), t] = 0 \quad \text{for all } x, u, t \in R$$

Since R is a prime ring then either

(a) $d^2(u) = 0$ for all $u \in R$

or

(b) $[d(x), t] = 0$ for all $x, t \in R$

$\boxed{\text{case (a)}}$ Suppose that $d^2(u) = 0$ for all $u \in R$. Then by Lemma 1 we have $d = 0$.

$\boxed{\text{case (b)}}$ Suppose that $[d(x), t] = 0$ for all $x, t \in R$, that is $d(x) \in Z(R)$ for all $x \in R$.
We can assume $Z(R) \neq 0$, otherwise it would be $d = 0$.
In particular, for each $x \in R$ we have both $d(x) \in Z(R)$ and also $d(x^2) \in Z(R)$.
Hence

$$d(x^2) = d(x)^2 + xd(x) + d(x)x = d(x)^2 + 2xd(x) \in Z(R)$$

and then, since $char(R) \neq 2$, $xd(x) \in Z(R)$.
Since $d(x) \in Z(R)$ then for all $x \in R$ we have either $d(x) = 0$ or $x \in Z(R)$.
In particular, if $x \notin Z(R)$ then $d(x) = 0$.
Since R is non-commutative, exists $0 \neq x_0 \notin Z(R)$ such that $d(x_0) = 0$.

Moreover, since $Z(R) \neq 0$ exists $0 \neq x_1 \in Z(R)$. Therefore

$$d(x_0x_1) \in Z(R) \quad \Rightarrow d(x_0)d(x_1) + d(x_0)x_1 + x_0d(x_1) \in Z(R)$$

then $x_0d(x_1) \in Z(R)$.
Since $0 \neq x_0 \notin Z(R)$ and $d(x_1) \in Z(R)$, we have $d(x_1) = 0$.
Therefore $d(x)$ =0 in any case.

1.2 Generalized homoderivations

Definition 3. *Let R be an associative ring and let $d : R \longrightarrow R$ be a homoderivation. An additive map $G : R \longrightarrow R$ is called a generalized homoderivation associated with the homoderivation $d : R \longrightarrow R$ if*

$$G(xy) = G(x)G(y) + G(x)y + xd(y) \qquad \textit{for all } x, y \in R.$$

Theorem 5. *Let R be a semiprime ring and let $G : R \longrightarrow R$ be a generalized homoderivation associated with the additive map $d : R \longrightarrow R$. If $G \neq 0$ then $d \neq 0$.*

Proof:
Suppose that $d = 0$ and that therefore

$$G(xy) = G(x)G(y) + G(x)y$$

for all $x, y \in R$.
For all $x, y, z \in R$ we have:

$$\begin{aligned} G(x(yz)) &= G((xy)z) \\ G(x)G(yz) + G(x)yz &= G(xy)G(z) + G(xy)z \\ G(x)[G(y)G(z) + G(y)z] + G(x)yz &= [G(x)G(y) + G(x)y]G(z) + [G(x)G(y) + G(x)y]z \\ G(x)G(y)G(z) + G(x)G(y)z + G(x)yz &= G(x)G(y)G(z) + G(x)yG(z) + G(x)G(y)z + G(x)yz \end{aligned}$$

Then

$$0 = G(x)yG(z) \qquad \text{for all } x, y, z \in R$$

that is $G(R)RG(R) = 0$.
Since R is semiprime then $G = 0$, a contradiction.

Remark 7. *Let R be a prime ring, G a nonzero generalized homoderivation associated with the homoderivation d and $a \in R$ a fixed element. If $aG(x) = 0$ for all $x \in R$, then $a = 0$.*

Proof:
Since $aG(x) = 0$ for all $x \in R$ then

$$\begin{aligned} 0 =& aG(xy) = \\ =& aG(x)G(y) + aG(x)y + axd(y) = \\ =& axd(y) \end{aligned}$$

Since R is prime then $a = 0$ or $d = 0$. Since $G \neq 0$ then, by Theorem 5, $d \neq 0$.

1.2.1 Examples

Example 7. *Let R be a prime unital ring, $\mu \in Z(R)$ and let $G : R \longrightarrow R$ be such that $G(x) = \mu x$. If G is a generalized homoderivation associated with the homoderivation d, then*

$$\begin{aligned} \mu xy =& G(xy) = \\ =& G(x)G(y) + G(x)y + xd(y) = \\ =& \mu x \mu y + \mu xy + xd(y) \end{aligned}$$

that is $0 = \mu^2 xy + xd(y)$ and for $x = 1_R$ we have $d(y) = -\mu^2 y$. Since d is a homoderivation then

$$\begin{aligned} d(xy) =& d(x)d(y) + d(x)y + xd(y) \\ -\mu^2 xy =& \mu^2 x \mu^2 y - \mu^2 xy - x\mu^2 y \\ 0 =& \mu^2(\mu^2 - 1) \end{aligned}$$

therefore $\mu = 0, \pm 1$.

Example 8. *Let $d : R \longrightarrow R$ be a homoderivation and let $G : R \longrightarrow R$ be such that $G(xy) = G(x)y + xd(y)$. If G is a generalized homoderivation associated with d then*

$$G(x)y + xd(y) = G(x)G(y) + G(x)y + xd(y)$$

that is

$$0 = G(x)G(y)$$

Replacing y with yt we have

$$0 = G(x)G(y)t + G(x)yd(t) = G(x)yd(t)$$

Since R is prime then $G = 0$ or $d = 0$.
If $d = 0$ then, by Theorem 5 we have $G = 0$.

Example 9. *If $d : R \longrightarrow R$ is a homoderivation then it is a generalized homoderivation.*

1.2.2 Characterization of generalized homoderivations associated with a homoderivation

Theorem 6. *Let R be a prime ring and $F : R \longrightarrow R$ a generalized homoderivation associated with the homoderivation $d : R \longrightarrow R$. Then*

$$F(xy) = xF(y) \qquad \text{for all } x, y \in R$$

or

$$F(x) = d(x) \qquad \text{for all } x \in R$$

Therefore a generalized homoderivation $F \neq 0$ associated with the homoderivation $d \neq 0$ is either a generalized derivation or a homoderivation.

Proof:
For all $x, y, z \in R$ we have:

$$F((xy)z) =$$
$$F(xy)F(z) + F(xy)z + xyd(z) =$$
$$[F(x)F(y) + F(x)y + xd(y)]F(z) + [F(x)F(y) + F(x)y + xd(y)]z + xyd(z) =$$
$$F(x)F(y)F(z) + F(x)yF(z) + xd(y)F(z) + F(x)F(y)z + F(x)yz + xd(y)z + xyd(z)$$

Moreover

$$F(x(yz)) =$$
$$F(x)F(yz) + F(x)yz + xd(yz) =$$
$$F(x)[F(y)F(z) + F(y)z + yd(z)] + F(x)yz + x[d(y)d(z) + d(y)z + yd(z)] =$$
$$F(x)F(y)F(z) + F(x)F(y)z + F(x)yd(z) + F(x)yz + xd(y)d(z) + xd(y)z + xyd(z)$$

Subtracting member from member we have

$$0 = F(x)yF(z) + xd(y)F(z) - F(x)yd(z) - xd(y)d(z)$$

$$0 = F(x)y\Big(F(z) - d(z)\Big) + xd(y)\Big(F(z) - d(z)\Big)$$

$$0 = \Big(F(x)y + xd(y)\Big)\Big(F(z) - d(z)\Big) \qquad \text{for all } x, y, z \in R \tag{1.29}$$

Replacing x with xt in (1.29), with $t \in R$, we have

$$0 = \Big(F(xt)y + xtd(y)\Big)\Big(F(z) - d(z)\Big)$$

$$0 = \Big(F(x)F(t)y + F(x)ty + xd(t)y + xtd(y)\Big)\Big(F(z) - d(z)\Big) \tag{1.30}$$

Replacing x with t in (1.29) we have

$$0 = \Big(F(t)y + td(y)\Big)\Big(F(z) - d(z)\Big) \qquad \text{for all } t, y, z \in R$$

Multiplying the latter on the left by $x \in R$ we have

$$0 = \Big(xF(t)y + xtd(y)\Big)\Big(F(z) - d(z)\Big) \qquad \text{for all } t, x, y, z \in R$$

$$xtd(y)\Big(F(z) - d(z)\Big) = -xF(t)y\Big(F(z) - d(z)\Big) \tag{1.31}$$

Then from (1.30) and (1.31) follows

$$0 = \Big(F(x)F(t)y + F(x)ty + xd(t)y + -xF(t)y\Big)\Big(F(z) - d(z)\Big)$$

that is

$$0 = \Big(F(x)F(t) + F(x)t + xd(t) - xF(t)\Big)y\Big(F(z) - d(z)\Big) \qquad \text{for all } x, y, t, z \in R$$

Since R is prime then

- $F(x)F(t) + F(x)t + xd(t) - xF(t) = 0$ for all $x, t \in R$ or
- $F(z) - d(z) = 0$ for all $z \in R$

that is

- $F(xt) - xF(t) = 0$ for all $x, t \in R$ or
- $F(z) - d(z) = 0$ for all $z \in R$

Theorem 7. *Let R be a unital prime ring and let $F : R \longrightarrow R$ be a generalized homoderivation associated with the homoderivation $d : R \longrightarrow R$.*
Let $a \in R$ such that $F(1_R) = a$. Then

$$d(x) = -x, \quad F(x) = x \qquad \textit{for all } x \in R$$

or

$$d(x) = -x, \quad F(x) = -x \qquad \textit{for all } x \in R$$

or

$$d(x) = [x, a] - axa, \quad F(x) = xa \qquad \textit{for all } x \in R\textit{, where, by Theorem 5, } a \neq 0$$

or

$$F(x) = d(x) \qquad \textit{for all } x \in R$$

and we have

$$a\Big(d(x) + x\Big) = 0 = \Big(d(x) + x\Big)a \qquad \textit{for all } x \in R$$

where $-a$ is idempotent.

Proof:
By Theorem 6 we have

$$F(tx) = tF(x) \qquad \text{for all } x, t \in R$$

or

$$F(x) = d(x) \qquad \text{for all } x \in R$$

$\boxed{\text{I case}}$ Suppose that $F(tx) = tF(x)$ for all $x, t \in R$.
For $x = 1_R$ we have

$$F(t) = ta \qquad \text{for all } t \in R\text{, where } 0 \neq a = F(1_R)$$

In particular

$$F(xy) = xya \qquad \text{for all } x, y \in R\text{, where } 0 \neq a = F(1_R)$$

Then

$$xya = F(xy) = F(x)F(y) + F(x)y + xd(y) = (xa)(ya) + (xa)y + xd(y)$$

from which

$$0 = -xya + (xa)(ya) + (xa)y + xd(y)$$

that is

$$0 = x\Big(-ya + aya + ay + d(y)\Big)$$

Since the ring R is prime, then the homoderivation associated with F is

$$d(y) = [y,a] - aya \qquad \text{for all } y \in R, \text{ where } 0 \neq a = F(1_R)$$

Therefore

$$d(xy) = [xy,a] - axya = xya - axy - axya \tag{1.32}$$

Moreover

$$\begin{aligned}
&d(xy) = \\
&d(x)d(y) + d(x)y + xd(y) = \\
&\Big([x,a] - axa\Big)\Big([y,a] - aya\Big) + \Big([x,a] - axa\Big)y + x\Big([y,a] - aya\Big) = \\
&\Big(xa - ax - axa\Big)\Big(ya - ay - aya\Big) + \Big(xa - ax - axa\Big)y + x\Big(ya - ay - aya\Big) = \\
&xaya - xa^2y - xa^2ya - axya + axay + axaya - axaya + axa^2y + axa^2ya + xay - axy - \\
&- axay + xya - xay - xaya.
\end{aligned}$$

Comparing with (1.32) we have

$$0 = xa^2y + xa^2ya - axa^2y - axa^2ya$$

$$0 = (1-a)xa^2ya + (1-a)xa^2y$$

$$0 = (1-a)(xa^2ya + xa^2y)$$

$$0 = (1-a)xa^2y(a+1)$$

Since R a prime ring then

$$\begin{array}{rlll}
a = 1 & \text{and then} & d(x) = -x, & F(x) = x \quad \text{or} \\
a = -1 & \text{and then} & d(x) = -x, & F(x) = -x \quad \text{or} \\
a^2 = 0 & \text{and then} & d(x) = [x,a] - axa, & F(x) = xa
\end{array}$$

$\boxed{\text{II case}}$ Suppose that $F(x) = d(x)$ for all $x \in R$. In this case F is a homoderivation and for the Proposition 1 we have

$$d(1_R)\Big(d(x) + x\Big) = 0 = \Big(d(x) + x\Big)d(1_R) \qquad \text{for all } x \in R$$

1.2.3 Posner's theorems

Lemma 3. *Let R be a prime unital ring of characteristic different from 2 and let F be a generalized homoderivation of R associated with the homoderivation d. If $F^2(x) = 0$ for all $x \in R$ then one of the following possibilities occurs:*

- $F(x) = 0$;

- *$F(x) = xa$ with $a = F(1_R)$ such that $a^2 = 0$.*

Proof:
Suppose $F(x) \neq xa$ with $a^2 = 0$. By Theorem 7 we have $F(x) = d(x)$ or $d(x) = -x$ and $F(x) = \pm x$.

- If $F(x) = d(x)$ then by Lemma 1 we have $F = 0$.
- If $F(x) = \pm x$ then $0 = F^2(x) = \mp F(x)$.

Theorem 8. *Let R be a prime unital ring of characteristic different from 2 and let F and G be two generalized homoderivations of R associated, respectively, with the homoderivations d and g. If $F(G(x)) = 0$ for all $x \in R$, then one of the following possibilities occurs:*

- *$F(x) = 0$;*
- *$G(x) = 0$;*
- *$d(x) = [x, a] - axa, F(x) = xa$ and $g(x) = [x, b] - bxb, G(x) = xb$ with $a^2 = 0$, $b^2 = 0$ and $ba = 0$.*

Proof:
Let $a = F(1_R)$ and $b = G(1_R)$. By Theorem 7 we have the following cases:

- $F(x) = d(x), G(x) = g(x)$ and by Theorem 1 we have $F = 0$ or $G = 0$.
- $F(x) = d(x), g(x) = [x, b] - bxb$, and $G(x) = xb$ with $b^2 = 0$.
  Since $F(G(x)) = 0$ for all $x \in R$ then $d(xb) = 0$ for all $x \in R$ and, in particular, $d(b) = 0$. Therefore
  $$0 = d(xb) = d(x)d(b) + d(x)b + xd(b) = d(x)b$$
  and by Remark 4 we have $b = 0$ that is $G = 0$.
- $d(x) = -x$ and $F(x) = x$ or $F(x) = -x$.
  Since $F(G(x)) = 0$ for all $x \in R$, whatever the from of G, then $G = 0$.
- $g(x) = -x, G(x) = x$ or $G(x) = -x$.
  Since $F(G(x)) = 0$ for all $x \in R$, whatever the from of F, then $F = 0$.
- $d(x) = [x, a] - axa, F(x) = xa$ and $G(x) = g(x)$ with $a^2 = 0$.
  If $F(G(x)) = 0$ for all $x \in R$ then $g(x)a = 0$ and by Remark 4 we have $a = 0$ that is $F = 0$.
- $d(x) = [x, a] - axa, F(x) = xa$ and $g(x) = [x, b] - bxb, G(x) = xb$ with $a^2 = 0$ and with $b^2 = 0$.
  If $F(G(x)) = 0$ for all $x \in R$ then $ba = 0$.

Theorem 9. *Let R be a prime unital ring of characteristic different from 2 and let F be a generalized homoderivation of R. If $[F(x), x] = 0$ for all $x \in R$ then $F(x) = 0$ or $F(x) = x$ or $F(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.*

Proof:
Suppose $F(x) \neq \pm x$ and let $a = F(1_R)$. By Theorem 7 we have $F(x) = d(x)$ for all $x \in R$ or $F(x) = xa$ for all $x \in R$ with $a^2 = 0$.

- If $F(x) = d(x)$ then by Theorem 2 we have $F = 0$ or $F(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.

- If $F(x) = xa$ then $0 = [F(x), x] = [xa, x] = x[a, x]$ and replacing x with $1 + x$ we have $0 = (1 + x)[a, 1 + x] = [a, x]$, that is a is central.
  Since a is nilpotent then $a = 0$, that is $F = 0$.

Chapter 2

Extended generalized homoderivations

In recent developments, homoderivations have been generalized and applied to various classes of rings and near-rings, yielding deeper structural insights.
In particular, in 2020, Eszter Gselmann and Gergely Kiss introduced the concept of ϵ-homoderivation [21], defining a function h_ϵ on a ring such that, for all x, y in the ring,

$$h_\epsilon(xy) = \epsilon h_\epsilon(x)h_\epsilon(y) + h_\epsilon(x)y + xh_\epsilon(y),$$

where ϵ belongs to the center of the ring. This mapping was characterized even in the absence of the additivity assumption:

Theorem 10. *Let P be a subring of the ring Q and assume that ϵ is an arbitrary nonzero element of the center of Q. The additive mapping $h_\epsilon : P \longrightarrow Q$ fulfills functional equation*

$$h_\epsilon(xy) = \epsilon h_\epsilon(x)h_\epsilon(y) + h_\epsilon(x)y + xh_\epsilon(y)$$

for all $x, y \in P$ if and only if there exists a homomorphism $\phi : P \longrightarrow Q$ such that

$$\epsilon h_\epsilon(x) = \phi(x) - x$$

for all $x \in P$.

Notably, in rings with a trivial center, an ϵ-homoderivation coincides with a derivation, whereas in unital rings, El-Sofy's homoderivation emerges as a particular instance of this broader construct.

We now introduce a new class of additive maps that generalizes both the map introduced by Eszter Gselmann and Gergely Kiss and the homoderivations introduced by El-Sofy.

2.1 Extended homoderivations

Definition 4. *Let R be an associative ring. An additive map $d : R \longrightarrow Q_r$ is called an extended homoderivation associated with $\lambda, \mu \in C$ if*

$$d(xy) = \lambda d(x)d(y) + \mu d(x)y + \mu xd(y) \qquad \text{for all } x, y \in R.$$

Remark 8. *An extended homoderivation associated with $\epsilon \in Z(R)$ and with $\mu = 1_R$ is a ϵ-homoderivation.*

Proposition 2. *Let $d : R \longrightarrow Q_r$ be an extended homoderivation on the unital ring R associated to $\lambda, \mu \in C$.*
Then $d(x) = d(1_R)x$ for all $x \in R$ with $d(1_R) \in C$ or $\mu = 1_R$.

Proof:
For all $x, y, z \in R$ we have:

$$\begin{aligned} &d(x(yz)) = \\ &\lambda d(x)d(yz) + \mu d(x)yz + \mu x d(yz) = \\ &\lambda^2 d(x)d(y)d(z) + \lambda\mu d(x)d(y)z + \lambda\mu d(x)yd(z) + \mu d(x)yz + \lambda\mu x d(y)d(z) + \\ &+ \mu^2 x d(y)z + \mu^2 xyd(z) \end{aligned} \tag{2.1}$$

Moreover

$$\begin{aligned} &d((xy)z) = \\ &\lambda d(xy)d(z) + \mu d(xy)z + \mu xyd(z) = \\ &\lambda^2 d(x)d(y)d(z) + \lambda\mu d(x)yd(z) + \lambda\mu x d(y)d(z) + \lambda\mu d(x)d(y)z + \mu^2 d(x)yz + \\ &+ \mu^2 x d(y)z + \mu xyd(z) \end{aligned} \tag{2.2}$$

Therefore comparing (2.1) and (2.2) we have

$$\mu d(x)yz + \mu^2 xyd(z) - \mu^2 d(x)yz - \mu xyd(z) = 0 \qquad \text{for all } x, y, z \in R.$$

If $\mu \neq 0$ then

$$d(x)yz + \mu xyd(z) - \mu d(x)yz - xyd(z) = 0 \qquad \text{for all } x, y, z \in R.$$

that is

$$(1-\mu)d(x)yz - (1-\mu)xyd(z) = 0 \qquad \text{for all } x, y, z \in R.$$

Hence, for $y = z = 1_R$ we get

$$(1-\mu)\Big(d(x) - xd(1_R)\Big) = 0 \quad \text{for all } x \in R$$

and for $x = y = 1_R$ we get

$$(1-\mu)\Big(d(1)z - d(z)\Big) = 0 \quad \text{for all } z \in R.$$

If $\mu \neq 1_R$, we have $d(x) = d(1_R)x$ for all $x \in R$, with $d(1_R) \in C$.

Remark 9. *Let $d : R \longrightarrow Q_r$ be an extended homoderivation on the unital ring R associated to $\lambda \in C$ and $\mu = 1_R$.*
Then $\delta(x) = \lambda d(x)$ is a homoderivation on R.

Remark 10. *Let R be a prime ring, $d : R \longrightarrow Q_r$ a nonzero extended homoderivation of R associated to $\lambda \in C$ and $\mu = 1_R$, and $a \in R$ a fixed element.*
If $ad(x) = 0$ for all $x \in R$, then $a = 0$.

Proof:
Since $ad(xy) = 0$ for all $x, y \in R$, then

$$0 = a\Big(\lambda d(x)d(y) + d(x)y + xd(y)\Big) = axd(y).$$

Since R is a prime ring, then $a = 0$.

Remark 11. *Let R be a prime ring, $d : R \longrightarrow Q_r$ a nonzero extended homoderivation of R associated to $\lambda \in C$ and $\mu = 1_R$, and $a \in R$ a fixed element.*
If $d(x)a = 0$ for all $x \in R$, then $a = 0$.

Proof:
Since $d(xy)a = 0$ for all $x, y \in R$, then

$$0 = \Big(\lambda d(x)d(y) + d(x)y + xd(y)\Big)a = d(x)ya.$$

Since R is a prime ring, then $a = 0$.

Proposition 3. *Let R be a unital ring and let $d : R \longrightarrow Q_r$ be a nonzero extended homoderivation associated to $\lambda \in C$ and $\mu = 1_R$. Then*

$$d(1_R)\Big(\lambda d(x) + x\Big) = 0 = \Big(\lambda d(x) + x\Big)d(1_R) \qquad \textit{for all } x \in R$$

Proof:
By the definition of extended homoderivation of R, we have

$$d(xy) = \lambda d(x)d(y) + d(x)y + xd(y) \qquad \text{for all } x, y \in R. \tag{2.3}$$

Replacing x with 1_R in (2.3), we obtain

$$d(y) = \lambda d(1_R)d(y) + d(1_R)y + d(y) \qquad \text{for all } y \in R$$

that is,

$$0 = d(1_R)(\lambda d(y) + y) \qquad \text{for all } y \in R.$$

Replacing y with 1_R in (2.3), we obtain

$$d(x) = \lambda d(x)d(1_R) + d(x) + xd(1_R) \qquad \text{for all } x \in R$$

that is,

$$0 = (\lambda d(x) + x)d(1_R) \qquad \text{for all } x \in R.$$

Remark 12. *Let R be a prime unital ring and let $d : R \longrightarrow Q_r$ be a nonzero extended homoderivation associated to $\lambda \in C$ and $\mu = 1_R$.*
If $0 \neq d(1_R) \in C$, since

$$d(1_R)\Big(\lambda d(x) + x\Big) = 0 = \Big(\lambda d(x) + x\Big)d(1_R) \qquad \textit{for all } x \in R$$

then $d(x) = -\lambda^{-1}x$ for all $x \in R$.
Therefore, if $d(x) \neq -\lambda^{-1}x$, then $d(1_R) = 0$ or $0 \neq d(1_R) \notin C$.
In case, $0 \neq d(1_R) \notin C$ we have

$$d(\eta) \notin C \qquad \textit{for all } \eta \in Z(R).$$

El-Sofy [19] proved that if R is a prime ring of $char(R) \neq 2$, and I is a nonzero right ideal of R, and h is a nonzero homoderivation on R such that $[x, h(x)] \in Z(R)$ for all $x \in I$, then h is commuting on I.

Similarly we have the following:

Theorem 11. *Let R be a semiprime unital ring of characteristics not 2, I be a nonzero left ideal of R and h be a nonzero extended homoderivation on R, associated with $\lambda, \mu = 1_R \in C$, such that h is centralizing.*
Then h is commuting on I.

Proof:
Let $x \in I$. Then $[x^2, h(x^2)] \in Z(R)$ by hypothesis. Since $[x, h(x)] \in Z(R)$ then

$$\begin{aligned}
[x^2, h(x^2)] =& [x^2, \lambda h(x)h(x) + xh(x) + h(x)x] = \\
=& [x^2, \lambda h(x)h(x) + 2xh(x) - [x, h(x)]] = \\
=& [x^2, \lambda h(x)h(x) + 2xh(x)] - [x^2, [x, h(x)]] = \\
=& [x^2, \lambda h(x)h(x) + 2xh(x)] = \\
=& [x^2, \lambda h(x)h(x)] + [x^2, 2xh(x)] = \\
=& \lambda x[x, h(x)h(x)] + \lambda[x, h(x)h(x)]x + x[x, 2xh(x)] + [x, 2xh(x)]x = \\
=& \lambda xh(x)[x, h(x)] + \lambda x[x, h(x)]h(x) + \lambda h(x)[x, h(x)]x + \lambda[x, h(x)]h(x)x + \\
&+ 2x^2[x, h(x)] + 2x[x, h(x)]x = \\
=& 2(2x^2 + \lambda h(x)x + \lambda xh(x))[x, h(x)] \in Z(R)
\end{aligned}$$

If $[x, h(x)] \neq 0$ then $2(2x^2 + \lambda h(x)x + \lambda xh(x)) \in Z(R)$.
Then $2[2x^2 + \lambda h(x)x + \lambda xh(x), x] = 0$.
Since $char(R) \neq 2$ then $[2x^2 + \lambda h(x)x + \lambda xh(x), x] = 0$ and we have

$$\begin{aligned}
0 =& [2x^2 + \lambda h(x)x + \lambda xh(x), x] = \\
=& \lambda[h(x)x, x] + \lambda[xh(x), x] = \\
=& 2\lambda x[h(x), x]
\end{aligned}$$

If $\lambda \neq 0$ then $x[h(x), x] = 0$.
Then $[x, h(x)]^3 = 0$.
Since the center of a semiprime ring contains no nonzero nilpotent elements, then $[x, h(x)] = 0$ for all $x \in I$. So, h is commuting.

2.1.1 Extended homomultiplier

Definition 5. *An additive mapping $F : R \longrightarrow Q_r$ is said to be an extended homoright multiplier (or, respectively, extended homoleft multiplier) associated with $\lambda, 0 \neq \mu \in C$ if*

$$F(xy) = \lambda F(x)F(y) + \mu xF(y)$$

$$\text{(or, respectively,} \qquad F(xy) = \lambda F(x)F(y) + \mu F(x)y) \qquad \text{for all } x, y \in R.$$

If F is both an extended homoright multiplier as well as an extended homoleft multiplier, then F will be called extended homomultiplier.

Theorem 12. *If R is a prime ring and F is an extended homoright multiplier (or, respectively, extended homoleft multiplier) then one of the following possibilities occurs:*

- $F = 0$;
- $F(xy) = xF(y)$ *(or, respectively,* $F(xy) = F(x)y$*);*
- $F(x) = \lambda^{-1}(1 - \mu)x$.

Proof:
Suppose that F is an extended homoright multiplier (a similar proof is valid for an extended homoleft multiplier).
For all $x, y, z \in R$, we have

$$\begin{aligned} &F(xyz) = \\ &\lambda F(xy)F(z) + \mu xyF(z) = \\ &\lambda^2 F(x)F(y)F(z) + \lambda\mu xF(y)F(z) + \mu xyF(z) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} &F(xyz) = \\ &\lambda F(x)F(yz) + \mu xF(yz) = \\ &\lambda^2 F(x)F(y)F(z) + \lambda\mu F(x)yF(z) + \mu\lambda xF(y)F(z) + \mu^2 xyF(z) \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), since $\mu \neq 0$, we have $\Big(\lambda F(x) + (\mu - 1)x\Big)yF(z) = 0$.
Since R is a prime ring then $F = 0$ or $\lambda F(x) = (1 - \mu)x$. In the latter case we have:

- if $\lambda = 0$ then $\mu = 1$ and, using the definition, $F(xy) = xF(y)$;
- if $\lambda \neq 0$ and $\mu = 1$ then $F = 0$;
- if $\lambda \neq 0$ and $\mu \neq 1$ then $F(x) = \lambda^{-1}(1 - \mu)x$.

2.1.2 Examples

Example 10. *Let $\lambda \in C$ and let $d : R \longrightarrow Q_r$ be an additive map such that*

$$d(x) = [x, a] - \lambda axa \quad \textit{for all } x \in R.$$

d is an extended homoderivation associated with λ, in fact:

$$\begin{aligned} &\lambda d(x)d(y) + d(x)y + xd(y) = \\ &\lambda\Big(xa - ax - \lambda axa\Big)\Big(ya - ay - \lambda aya\Big) + \Big(xa - ax - \lambda axa\Big)y + x\Big(ya - ay - \lambda aya\Big) = \\ &\lambda xaya - \lambda axya + \lambda axay + \lambda^2 axaya - \lambda^2 axaya + xay - axy - \lambda axay + xya - xay - \lambda xaya = \\ &-\lambda axya - axy + xya = \\ &[xy, a] - \lambda axya = \\ &d(xy) \end{aligned}$$

Example 11. *Let $d : R \longrightarrow Q_r$ be an extended homoderivation on the prime ring R associated to $\lambda \in C$ and $\mu = 1_R$. If $d(x) = \eta x$ for all $x \in R$, with $\eta \in C$, then for all $x, y \in R$*

$$\begin{aligned} d(xy) &= \lambda d(x)d(y) + d(x)y + xd(y) \\ \eta xy &= \lambda\eta^2 xy + \eta xy + \eta xy \\ 0 &= \eta(\lambda\eta + 1)xy \end{aligned}$$

that is $\eta = 0$ and $d = 0$ or $\eta = -\lambda^{-1}$ and $d(x) = -\lambda^{-1}x$.

Example 12. *Let R be a prime ring and let $g : R \longrightarrow Q_r$ be a generalized derivation associated with the derivation d such that it is also an extended homoderivation associated with $\lambda \in C, \mu = 1_R$. Then*

$$g(x)y + xd(y) = \lambda g(x)g(y) + g(x)y + xg(y)$$

that is

$$xd(y) = \lambda g(x)g(y) + xg(y). \tag{2.6}$$

Replacing x with tx we have

$$txd(y) = \lambda g(tx)g(y) + txg(y) \tag{2.7}$$

Moreover, multiplying (2.6) by t on the left gives

$$txd(y) = \lambda tg(x)g(y) + txg(y). \tag{2.8}$$

From (2.7) and (2.8) it then follows that

$$\lambda\Big(g(tx) - tg(x)\Big)g(y) = 0$$

If $\lambda = 0$ then by (2.6) we have $d = g$.
If $\lambda \neq 0$, then using Remarks 10 and 11, $g = 0$ or $g(tx) = tg(x)$. If $g(tx) = tg(x)$ for all $t, x \in R$ then

$$xg(y) = g(xy) = \lambda g(x)g(y) + g(x)y + xg(y)$$

that is

$$0 = g(x)(\lambda g(y) + y)$$

and using Remarks 10 and 11 we have $g = 0$ or $g(y) = -\lambda^{-1}y$.

2.1.3 Posner's and Herstein's theorems

Lemma 4. *Let R be a prime unital ring and let $h : R \longrightarrow Q_r$ be an extended homoderivation of R associated with $\lambda \in C$ and $\mu = 1_R$.*
If $h^2(x) = 0$ for all $x \in R$ then $h(\eta) \in C$ for all $\eta \in Z(R)$

Proof:
For all $\eta \in Z(R)$ and $x \in R$ we have

$$0 = h([\eta, x]) = \lambda h(\eta)h(x) - \lambda h(x)h(\eta) + h(\eta)x - xh(\eta) \tag{2.9}$$

Since $h^2(x) = 0$ for all $x \in R$, applying h to (2.9) we get

$$0 = h\Big(\lambda h(\eta)h(x) - \lambda h(x)h(\eta) + h(\eta)x - xh(\eta)\Big) = h(\eta)h(x) - h(x)h(\eta) \tag{2.10}$$

Therefore, by (2.9) and (2.10) we get $[h(\eta), x] = 0$ for all $x \in R$.

Lemma 5. *Let R be a prime unital ring of characteristic different from 2 and let $h : R \longrightarrow Q_r$ be an extended homoderivation of R associated with $\lambda, \mu \in C$.*
If $h^2(x) = 0$ for all $x \in R$ then $h(x) = 0$ or the following conditions hold

$$\begin{cases} h(\lambda x) + \lambda h(x) + 2x = 0 & \text{for all } x \in R \\ h(\lambda) + 2 = 0 \end{cases}$$

Conversely, if

$$\begin{cases} h(\lambda x) + \lambda h(x) + 2x = 0 & \text{for all } x \in R \\ h\Big(\lambda h(x)\Big) + 2h(x) = 0 & \text{for all } x \in R \end{cases}$$

then $h^2 = 0$.

Proof:
Suppose $h^2(x) = 0$ for all $x \in R$.
By Proposition 2, $h(x) = h(1_R)x$ for all $x \in R$, with $h(1_R) \in C$, or $\mu = 1_R$.
If $h(x) = h(1_R)x$ for all $x \in R$ then $0 = h^2(x) = (h(1_R))^2 x$ for all $x \in R$. Hence $(h(1_R))^2 = 0$, that is $h = 0$.
If $\mu = 1_R$ and $\lambda = 0$ then h is a derivation and so $h = 0$.
If $\mu = 1_R$ and $\lambda \neq 0$, for all $x, y \in R$ we have

$$0 = h^2(xy) = \Big(2 + h(\lambda)\Big)h(x)h(y)$$

By Remarks 10 and 11 we get $h(\lambda) = -2$ and therefore

$$h(\lambda x) + \lambda h(x) + 2x = 0 \quad \text{for all } x \in R$$

In particular $h(1) = -\lambda h(1) - 2$ that is $h(1) = 0$.
Conversely, suppose that the following conditions hold

$$\begin{cases} h(\lambda x) + \lambda h(x) + 2x = 0 & \text{for all } x \in R \\ h\Big(\lambda h(x)\Big) + 2h(x) = 0 & \text{for all } x \in R \end{cases}$$

Then

$$h(x) = -\lambda^{-1}h(\lambda x) - 2\lambda^{-1}x \quad \text{for all } x \in R \tag{2.11}$$

and we have

$$\begin{aligned} h^2(x) = -\lambda^{-1}h\Big(\lambda h(x)\Big) - 2\lambda^{-1}h(x) = \\ = 2\lambda^{-1}h(x) - 2\lambda^{-1}h(x) = 0 \end{aligned}$$

Lemma 6. *Let R be a prime unital ring of characteristic different from 2, f and g two extended homoderivation of R associated, respectively, with $\lambda, 1_R \in C$ and $\mu, 1_R \in C$.*
If $f(g(x)) = 0$ and $g^2 = 0$ for all $x \in R$ then $g = 0$ or $f(x) = \lambda^{-1}\mu g(x)$.

Proof:
If $\mu = 0$ then g is a derivation. Since $g^2 = 0$ then $g = 0$.
Assume $\mu \neq 0$.
If $g^2(x) = 0$ for all $x \in R$ then, by Lemma 5, $g(\mu) + 2 = 0$ and

$$g(\mu x) + \mu g(x) + 2x = 0 \quad \text{for all } x \in R \tag{2.12}$$

Applying f to (2.12) we have

$$f(\mu)g(x) + 2f(x) = 0 \tag{2.13}$$

Replacing x with xy in (2.13) we have

$$f(\mu)\Big(\mu g(x)g(y)+g(x)y+xg(y)\Big)+2\lambda f(x)f(y)+2f(x)y+2xf(y)=0$$

using (2.13) we have

$$-2\mu f(x)g(y)-2f(x)y+f(\mu)xg(y)+2\lambda f(x)f(y)+2f(x)y+2xf(y)=0$$

that is

$$-2\mu f(x)g(y)+f(\mu)xg(y)+2\lambda f(x)f(y)+2xf(y)=0 \tag{2.14}$$

Replacing x with μ in (2.14) we have

$$-\mu f(\mu)g(y)+2\lambda f(\mu)f(y)+2\mu f(y)=0$$

using (2.13) we get

$$2\mu f(y)+2\lambda f(\mu)f(y)+2\mu f(y)=0$$

that is

$$4\mu f(y)+2\lambda f(\mu)f(y)=0$$

Since $char(R)\neq 2$ then

$$\Big(2\mu+\lambda f(\mu)\Big)f(y)=0$$

and, by Remarks 2.1 and 2.2 we get $2\mu+\lambda f(\mu)=0$.
Since $\mu\neq 0$ and $char(R)\neq 2$ then $\lambda\neq 0$.
Therefore $f(\mu)=-2\lambda^{-1}\mu\in C$ and by (2.13) we have

$$2f(x)=-f(\mu)g(x)=2\lambda^{-1}\mu g(x)$$

that is

$$f(x)=\lambda^{-1}\mu g(x)$$

Theorem 13. *Let R be a prime unital ring of characteristic different from 2, f and g two extended homoderivation of R associated, respectively, with $\lambda,\eta\in C$ and $\mu,\rho\in C$.*
If $f(g(x))=0$ for all $x\in R$ then $g=0$ or $f=0$ or $f(x)=\lambda^{-1}\mu g(x)$ with $g^2=0$ and

$$\begin{cases} g(\lambda x)+\lambda g(x)+2x=0 & \text{for all } x\in R\\ g(\lambda)+2=0 \end{cases}$$

Proof:
By Proposition 2, $f(x)=f(1_R)x$ for all $x\in R$ or $\eta=1_R$ and $g(x)=g(1_R)x$ for all $x\in R$ or $\rho=1_R$.

$\boxed{\text{I case}}$ Assume $\eta=1_R$ and $\rho=1_R$, that is

$$f(xy)=\lambda f(x)f(y)+f(x)y+xf(y)$$

$$g(xy)=\mu g(x)g(y)+g(x)y+xg(y)$$

Firstly we observe that for all $x\in R$,

$$f(\mu g(x))=f(\mu)g(x)$$

and
$$f(g(x)\mu) = g(x)f(\mu)$$
hence
$$f(\mu)g(x) = g(x)f(\mu) \tag{2.15}$$
If $f(g(x)) = 0$ then
$$f(\mu)g(x)g(y) + g(x)f(y) + f(x)g(y) = 0 \tag{2.16}$$
Replacing x with $g(x)$ and using (2.15) we have
$$g^2(x)\Big(f(\mu)g(y) + f(y)\Big) = 0 \tag{2.17}$$
If $f(\mu) = 0$ then $g^2(x)f(y) = 0$ and, by Remarks 2.1 and 2.2 we get $g^2(x) = 0$ or $f(x) = 0$.
If $g^2(x) = 0$, the conclusions follow from Lemma 6.
Assume $f(\mu) \neq 0$.
Replacing x with $g(x)y$ in (2.15) we have
$$f(\mu)\Big(\mu g^2(x)g(y) + g^2(x)y + g(x)g(y)\Big) = \Big(\mu g^2(x)g(y) + g^2(x)y + g(x)g(y)\Big)f(\mu)$$
By (2.15) we get
$$f(\mu)g^2(x)y = g^2(x)yf(\mu)$$
that is
$$g^2(x)\delta(R) = 0 \quad \text{with } \delta(x) = [f(\mu), x] \text{ inner derivation induced by } f(\mu)$$
Then $g^2(x) = 0$ or $\delta = 0$.
If $g^2(x) = 0$, the conclusions follow from Lemma 6.
Assume $\delta = 0$, that is $f(\mu) \in Z(R)$.
Replacing x with μ in (2.16) we have
$$f(\mu)g(\mu)g(y) + g(\mu)f(y) + f(\mu)g(y) = 0$$
that is
$$g(\mu)g(y) + g(\mu)f(y) + g(y) = 0$$
Replacing y with μ in (2.17) we have
$$g^2(x)\Big(f(\mu)g(\mu) + f(\mu)\Big) = 0$$
that is
$$g^2(x)\Big(g(\mu) + 1\Big) = 0 \tag{2.18}$$
If $g(\mu) = 0$ then $g^2 = 0$ and the conclusions follow from Lemma 6.
Assume $g(\mu) \neq 0$.
Replacing x and y with μ in (2.16) we have
$$f(\mu)g^2(\mu) + 2g(\mu)f(\mu) = 0$$

that is
$$g^2(\mu) + 2g(\mu) = 0$$
hence
$$\Big(g(\mu) + 1\Big)g(\mu) = -g(\mu) \tag{2.19}$$
Multiplying (2.18) on the right by $g(\mu)$ and using (2.19) we have
$$g^2(x)\Big(g(\mu) + 1\Big)g(\mu) = 0$$
$$g^2(x)g(\mu) = 0$$
Therefore by (2.18) we have $g^2 = 0$ and the conclusions follow from Lemma 6.

$\boxed{\text{II case}}$ Assume $\eta = 1_R$ and $g(x) = g(1_R)x$.
If $f(g(x)) = 0$ for all $x \in R$ then
$$0 = \lambda f(g(1_R))f(x) + f(g(1_R))x + g(1_R)f(x) \tag{2.20}$$
Replacing x with $g(x)$ we have $0 = f(g(1_R))g(x)$ and, by Remarks 2.1 and 2.2 we get $f(g(1_R)) = 0$.
Therefore by (2.20) we have $0 = g(1_R)f(x)$ and, by Remarks 2.1 and 2.2 we get $g(1_R) = 0$, that is $g = 0$.

$\boxed{\text{III case}}$ Assume $f(x) = f(1_R)x$ for all $x \in R$ and $\rho = 1_R$.
If $f(g(x)) = 0$ for all $x \in R$ then $0 = f(1_R)g(x)$.
If $g \neq 0$, by Remarks 2.1 and 2.2 we get $f(1_R) = 0$, that is $f = 0$.

$\boxed{\text{IV case}}$ Assume $f(x) = f(1_R)x$ for all $x \in R$ and $g(x) = g(1_R)x$.
If $f(g(x)) = 0$ for all $x \in R$ then $f(1_R)g(1_R) = 0$ that is $f = 0$ or $g = 0$.

Theorem 14. *Let R be a prime unital ring of characteristic different from 2, $f : R \longrightarrow Q_r$ an extended homoderivation of R associated with $\lambda, 1_R \in C$.
If $[f(x), x] = 0$ for all $x \in R$ then $f = 0$ or $f(x) = -\lambda^{-1}x + \omega(x)$ with $\omega : R \longrightarrow Z(R)$.
Moreover $\omega = 0$ or $\omega(1_R) = \lambda^{-1}$.*

Proof:
An additive map defined on a prime ring and satisfying the condition $[f(x), x] = 0$ for all $x \in R$ takes the following from
$$f(x) = \eta x + \omega(x) \quad \text{with } \eta \in Z(R) \text{ and } \omega : R \longrightarrow Z(R)$$
Therefore
$$\begin{aligned}
&\eta xy + \omega(xy) = \\
&\lambda f(x)f(y) + f(x)y + xf(y) = \\
&\lambda\Big(\eta x + \omega(x)\Big)\Big(\eta y + \omega(y)\Big) + \Big(\eta x + \omega(x)\Big)y + x\Big(\eta y + \omega(y)\Big) = \\
&\lambda\eta^2 xy + \lambda\eta\omega(y)x + \lambda\eta\omega(x)y + \lambda\omega(x)\omega(y) + \eta xy + \omega(x)y + \eta xy + x\omega(y)
\end{aligned}$$

that is

$$\Big(\lambda\eta+1\Big)\eta xy+\Big(\lambda\eta+1\Big)\omega(y)x+\Big(\lambda\eta+1\Big)\omega(x)y+\lambda\omega(x)\omega(y)-\omega(xy)=0 \tag{2.21}$$

commuting with $x\in R$ we have

$$\Big(\lambda\eta+1\Big)\eta x[y,x]+\Big(\lambda\eta+1\Big)\omega(x)[y,x]=0$$

that is

$$\Big(\lambda\eta+1\Big)f(x)[y,x]=0$$

Replacing y with yz we have

$$\Big(\lambda\eta+1\Big)f(x)y[z,x]=0$$

Suppose $\lambda\eta+1\neq 0$.
Since R is a prime ring then, for all $x\in R$,

$$x\in Z(R) \quad \text{or} \quad (\lambda\eta+1)f(x)=0. \tag{2.22}$$

If there exist x_0 and y_0 such that $x_0\notin Z(R)$ and $f(y_0)\neq 0$ then by (2.22) we have $F(x_0)=0$ and $y_0\in Z(R)$.
If $x_0+y_0\in Z(R)$ then $x_0\in Z(R)$, a contradiction.
If $f(x_0+y_0)=0$ then $f(y_0)=0$, a contradiction.
Therefore either $x\in Z(R)$ for all $x\in R$, or $(\lambda\eta+1)f(x)=0$ for all $x\in R$.
Hence R is commutative or $f=0$.
Suppose $\lambda\eta+1=0$.
Then $\lambda=-\eta^{-1}$ and

$$f(x)=-\lambda^{-1}x+\omega(x) \tag{2.23}$$

Moreover, by (2.21) we have

$$\lambda\omega(x)\omega(y)-\omega(xy)=0 \tag{2.24}$$

In particular,

$$f(1_R)=-\lambda^{-1}+\omega(1_R) \tag{2.25}$$

and

$$\omega(1_R)=\lambda\omega(1_R)\omega(1_R)$$

that is

$$\omega(1_R)\Big(\lambda\omega(1_R)-1\Big)=0 \tag{2.26}$$

Since the evaluations of ω are central then $\omega(1_R)=0$ or $\omega(1_R)=\lambda^{-1}$.
If $\omega(1_R)=0$ then, by (2.24), we have $\omega=0$.
Therefore, by (2.23) we have $f(x)=-\lambda^{-1}x$.
If $\omega(1_R)=\lambda^{-1}$ then by (2.25), $f(1_R)=0$.

Lemma 7. *Let R be a non-commutative unital prime ring of characteristic different from 2 and $d:R\longrightarrow Q_r$ an extended homoderivation of R associated with $\lambda,1_R\in C$.*

If $[d(x), d(y)] = 0$ *for all* $x, y \in R$ *then* $d(x) = 0$.

Proof:
By Proposition 3 we have

$$d(1_R)(\lambda d(x) + x) = 0 = (\lambda d(x) + x)d(1_R) \qquad \text{for all } x \in R \tag{2.27}$$

From the hypothesis we have

$$[d(x), d(y)] = 0 \quad \text{for all } x, y \in R \tag{2.28}$$

Substituting y with yz we have

$$[d(x), d(yz)] = 0 \quad \text{for all } x, y, z \in R$$

that is

$$[d(x), \lambda d(y)d(z) + d(y)z + yd(z)] = 0 \quad \text{for all } x, y, z \in R$$

By (2.28) we have

$$d(y)[d(x), z] + [d(x), y]d(z) = 0 \quad \text{for all } x, y, z \in R \tag{2.29}$$

Replacing z with zt we get

$$d(y)[d(x), z]t + d(y)z[d(x), t] + \lambda[d(x), y]d(z)d(t) + [d(x), y]d(z)t + [d(x), y]zd(t) = 0$$
$$\text{for all } x, y, z, t \in R \tag{2.30}$$

Multiplying (2.29) by t on the right we have

$$d(y)[d(x), z]t + [d(x), y]d(z)t = 0 \quad \text{for all } x, y, z \in R \tag{2.31}$$

Subtracting (2.31) from (2.30) we get

$$d(y)z[d(x), t] + \lambda[d(x), y]d(z)d(t) + [d(x), y]zd(t) = 0 \quad \text{for all } x, y, z, t \in R$$

Replacing y with $d(u)$ we have

$$d^2(u)z[d(x), t] + \lambda[d(x), d(u)]d(z)d(t) + [d(x), d(u)]zd(t) = 0 \quad \text{for all } x, u, z, t \in R$$

By (2.28) we have

$$d^2(u)z[d(x), t] = 0 \quad \text{for all } x, u, t \in R$$

Since R is a prime ring then

(a) $d^2(u) = 0$ for all $u \in R$ or

(b) $[d(x), t] = 0$ for all $x, t \in R$

$\boxed{\text{case (a)}}$ Suppose that $d^2(u) = 0$ for all $u \in R$.
Then by Lemma 5 we have $d(x) = 0$ or

$$\begin{cases} d(\lambda x) + \lambda d(x) + 2x = 0 & \text{for all } x \in R \\ d(\lambda) + 2 = 0 & \end{cases} \tag{2.32}$$

Assume that the conditions (2.32) hold.
Replacing x with λx in (2.28) and using (2.32) we get

$$0 = [d(\lambda x), d(y)] = [-\lambda d(x) - 2x, d(y)]$$

Using (2.28) we have

$$0 = [2x, d(y)]$$

Replacing y with λy and using (2.32) we get

$$0 = [2x, -\lambda d(y) - 2y]$$

Using (2.28) we have

$$0 = [2x, 2y]$$

Since $char(R) \neq 2$ then $[x, y] = 0$. A contradiction since R is not commutative.

$\boxed{\text{case (b)}}$ Suppose that $[d(x), t] = 0$ for all $x, t \in R$, that is

$$d(x) \in Z(R), \quad \text{for all } x \in R$$

We can assume $Z(R) \neq 0$, otherwise it would be $d = 0$.
In particular, for each $x \in R$ we have both $d(x) \in Z(R)$ and also $d(x^2) \in Z(R)$. Hence

$$d(x^2) = \lambda d(x)^2 + xd(x) + d(x)x = \lambda d(x)^2 + 2xd(x) \in Z(R)$$

and then, since $char(R) \neq 2$, $xd(x) \in Z(R)$.
Since $d(x) \in Z(R)$ then for all $x \in R$ we have either $d(x) = 0$ or $x \in Z(R)$.
In particular, if $x \notin Z(R)$ then $d(x) = 0$.
Since R is non-commutative, exists $0 \neq x_0 \notin Z(R)$ such that $d(x_0) = 0$.
Moreover, since $Z(R) \neq 0$ exists $0 \neq x_1 \in Z(R)$. Therefore

$$d(x_0x_1) \in Z(R) \quad \Rightarrow \lambda d(x_0)d(x_1) + d(x_0)x_1 + x_0d(x_1) \in Z(R)$$

then $x_0d(x_1) \in Z(R)$.
Since $0 \neq x_0 \notin Z(R)$ and $d(x_1) \in Z(R)$, we have $d(x_1) = 0$.
Therefore $d(x)$ =0 in any case.

2.2 Extended generalized homoderivations

Definition 6. *Let R be an associative ring, $\lambda, \mu \in C$ and $d : R \longrightarrow Q_r$ an additive map. An additive map $F : R \longrightarrow Q_r$ is called an extended generalized homoderivation associated with $\lambda, \mu \in C$ if*

$$F(xy) = \lambda F(x)F(y) + \mu F(x)y + \mu x d(y) \qquad \textit{for all } x, y \in R$$

Remark 13. *Let $F : R \longrightarrow Q_r$ be an extended generalized homoderivation associated with $\lambda, 0 \neq \mu \in C$ and with the additive map d. If $d = 0$ then F is an extended homomultiplier and, by Theorem 12, one of the following possibilities occurs:*

- *$F = 0$;*
- *$F(xy) = F(x)y$;*

- $F(x) = \lambda^{-1}(1-\mu)x$.

Remark 14. *Let $F : R \longrightarrow Q_r$ be an extended generalized homoderivation associated with $\lambda, 0 \neq \mu \in C$ and with the additive map d and let $a = F(1_R)$.*
Observe that if R is a unital ring then by replacing, in the definition of F, x with 1_R we obtain

$$F(y) = \lambda a F(y) + \mu a y + \mu d(y)$$

that is

$$d(y) = \mu^{-1}(1-\lambda a)F(y) - ay, \tag{2.33}$$

and replacing, in the definition of F, y with 1_R we obtain

$$F(x) = \lambda F(x)a + \mu F(x) + \mu x d(1_R) = F(x)(\lambda a + \mu) + \mu x d(1_R)$$

that is

$$F(x)\Big(1 - \lambda a - \mu\Big) = \mu x d(1_R). \tag{2.34}$$

In particular, for $x = y = 1_R$ we have

$$\lambda a^2 + (\mu - 1)a + \mu d(1_R) = 0.$$

2.2.1 Characterization of extended generalized homoderivations associated with an additive map

Theorem 15. *Let R be a prime unital ring, $d : R \longrightarrow Q_r$ an additive map and $F : R \longrightarrow Q_r$ an extended generalized homoderivation associated with $\lambda, 0 \neq \mu \in C$ and with the additive map d.*
Let $a = F(1_R)$. Then one of the following possibilities holds:

- *$d = 0$ and $F = 0$;*
- *$\lambda = 0$, $d = 0$ and $F(x) = ax$;*
- *$d = 0$ and $F(x) = \lambda^{-1}(1-\mu)x$;*
- *$\mu = 1$, $\lambda = 0$, F is a generalized derivation associated with the derivation d;*
- *$\mu = 1$, $\lambda \neq 0$, $F = d$ is an extended homoderivation;*
- *$\mu \neq 1$, $\lambda \neq 0$, d is an extended homoderivation associated with $\lambda \in C$, that is $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$ and $F(x) = \lambda^{-1}(1-\mu)x + d(x)$;*
- *$F(x) = xa$ and $d(x) = \Big(\mu^{-1} - \mu^{-1}\lambda a\Big)xa - ax$.*

Proof:
If $d = 0$ then F is an extended homomultiplier and, by Theorem 12, one of the following possibilities occurs:

- $F = 0$;
- $\lambda = 0$ and $F(x) = ax$;
- $F(x) = \lambda^{-1}(1-\mu)x$.

Assume $d \neq 0$.
For all $x, y, z \in R$ we have

$$\begin{aligned}
&F((xy)z) = \\
&\lambda F(xy)F(z) + \mu F(xy)z + \mu xyd(z) = \\
&\lambda[\lambda F(x)F(y) + \mu F(x)y + \mu xd(y)]F(z) + \mu[\lambda F(x)F(y) + \mu F(x)y + \mu xd(y)]z + \\
&+ \mu xyd(z) = \\
&\lambda^2 F(x)F(y)F(z) + \lambda\mu F(x)yF(z) + \lambda\mu xd(y)F(z) + \lambda\mu F(x)F(y)z + \mu^2 F(x)yz + \\
&+ \mu^2 xd(y)z + \mu xyd(z)
\end{aligned}$$

Moreover

$$\begin{aligned}
&F(x(yz)) = \\
&\lambda F(x)F(yz) + \mu F(x)yz + \mu xd(yz) = \\
&\lambda F(x)[\lambda F(y)F(z) + \mu F(y)z + \mu yd(z)] + \mu F(x)yz + \mu xd(yz) = \\
&\lambda^2 F(x)F(y)F(z) + \lambda\mu F(x)F(y)z + \lambda\mu F(x)yd(z) + \mu F(x)yz + \mu xd(yz)
\end{aligned}$$

Subtracting member from member we have

$$\mu F(x)y\Big(\lambda F(z) + \mu z - z - \lambda d(z)\Big) + \mu x\Big(\mu d(y)z + yd(z) - d(yz) + \lambda d(y)F(z)\Big) = 0 \tag{2.35}$$

Since $\mu \neq 0$ and indicating

$$\Delta_1 = y\Big(\lambda F(z) + \mu z - z - \lambda d(z)\Big)$$

and

$$\Delta_2 = \mu d(y)z + yd(z) - d(yz) + \lambda d(y)F(z),$$

we have

$$F(x)\Delta_1 + x\Delta_2 = 0 \tag{2.36}$$

that is

$$x\Delta_2 = -F(x)\Delta_1 \tag{2.37}$$

Replacing x with xt in (2.36) we have

$$\Big(\lambda F(x)F(t) + \mu F(x)t + \mu xd(t)\Big)\Delta_1 + x\Delta_2 = 0. \tag{2.38}$$

Using (2.37) we get

$$F(x)\Big(\lambda F(t) + \mu t\Big)\Delta_1 + x\Big(\mu d(t) - F(t)\Big)\Delta_1 = 0$$

that is

$$\Big(\lambda F(x)F(t) + \mu F(x)t + \mu xd(t) - xF(t)\Big)\Delta_1 = 0,$$

hence

$$\Big(F(xt) - xF(t)\Big)\Delta_1 = 0.$$

Indicating

$$\Delta_3 = F(xt) - xF(t)$$

we have $\Delta_3\Delta_1 = 0$ that is

$$\Delta_3 y\Big(\lambda F(z) + \mu z - z - \lambda d(z)\Big) = 0.$$

Since R is a prime ring then

$$\Delta_3 = 0 \quad \text{or} \lambda F(z) + (\mu - 1)z - \lambda d(z) = 0$$

- Assume $\Delta_3 = 0$ that is

$$F(xt) - xF(t) = 0$$

For $t = 1_R$ we have $F(x) = xa$ with $a = F(1_R)$.
By (2.33) of Remark 14 we get

$$d(x) = \mu^{-1}(1 - \lambda a)xa - ax, \quad \text{for all } x \in R. \tag{2.39}$$

- Assume

$$\lambda F(x) + (\mu - 1)x - \lambda d(x) = 0 \qquad \text{for all } x \in R. \tag{2.40}$$

First note that:

- if $\mu = 1$ and $\lambda = 0$ then, by definition, F is a generalized derivation and then d is a derivation;
- if $\mu = 1$ and $\lambda \neq 0$ then, by (2.40), $F = d$. Then, by definition, F is an extended homoderivation.

Therefore, in what follows we will consider $\mu - 1 \neq 0$.
Furthermore, if $\mu \neq 1$, then $\lambda \neq 0$ because otherwise, by (2.40), we would have $(\mu - 1)z = 0$.
By (2.40), we have

$$d(x) = F(x) + \lambda^{-1}(\mu - 1)x \tag{2.41}$$

Moreover by (2.33) of Remark 14 we have

$$d(x) = \mu^{-1}(1 - \lambda a)F(x) - ax \tag{2.42}$$

Therefore by (2.41) and (2.42) we get

$$F(x) + \lambda^{-1}(\mu - 1)x = \mu^{-1}(1 - \lambda a)F(x) - ax,$$

that is

$$\Big(\mu^{-1}(1 - \lambda a) - 1\Big)F(x) + \Big(-\lambda^{-1}(\mu - 1) - a\Big)x = 0.$$

Indicating

$$\Delta_4 = \mu^{-1}(1 - \lambda a) - 1$$

and

$$\Delta_5 = -\lambda^{-1}(\mu - 1) - a$$

we have

$$\Delta_4 F(x) + \Delta_5 x = 0. \tag{2.43}$$

Note that if $\Delta_4 = 0$ then by (2.43), $\Delta_5 = 0$.
Furthermore, if $\Delta_5 = 0$ then by (2.43), $\Delta_4 F(x) = 0$. In this case, replacing x with xy we have

$$0 = \Delta_4 F(xy) = \mu \Delta_4 x d(y)$$

and since $d \neq 0$ and $\mu \neq 0$, then $\Delta_4 = 0$.
Therefore

$$\Delta_4 = 0 \text{ if and only if } \Delta_5 = 0. \tag{2.44}$$

Replacing x with xy in (2.43) we have

$$\Delta_4 \Big(\lambda F(x) F(y) + \mu F(x) y + \mu x d(y) \Big) + \Delta_5 xy = 0. \tag{2.45}$$

By (2.43) we have $\Delta_4 F(x) = -\Delta_5 x$.
Therefore by (2.45) we have

$$-\lambda \Delta_5 x F(y) - \mu \Delta_5 xy + \mu \Delta_4 x d(y) + \Delta_5 xy = 0$$

that is

$$\Delta_5 x \Big(-\lambda F(y) - \mu y + y \Big) + \mu \Delta_4 x d(y) = 0. \tag{2.46}$$

By (2.44), either $\Delta_4 = \Delta_5 = 0$ or $\Delta_4 \neq 0$ and $\Delta_5 \neq 0$.

- Assume $\Delta_4 = \Delta_5 = 0$, that is

$$\Delta_4 = \mu^{-1}(1 - \lambda a) - 1 = 0$$

and

$$\Delta_5 = -\lambda^{-1}(\mu - 1) - a = 0.$$

Therefore $a = \lambda^{-1}(1 - \mu) \in Z(R)$ and

$$1 - \lambda a - \mu = 0.$$

Then by (2.34) of Remark 14 we have

$$\mu x d(1_R) = F(x) \Big(1 - \lambda a - \mu \Big) = 0,$$

that is $d(1_R) = 0$.
Moreover by (2.41) we have

$$F(x) = ax + d(x), \qquad \text{with } a = \lambda^{-1}(1 - \mu) \in Z(R). \tag{2.47}$$

- Assume $\Delta_4 \neq 0$ and $\Delta_5 \neq 0$.

Then by (2.46) there exists $0 \neq \eta \in C$ such that

$$-\lambda F(y) - \mu y + y = \eta d(y)$$

that is

$$F(y) = \lambda^{-1} \Big(y - \mu y - \eta d(y) \Big) = \lambda^{-1}(1 - \mu) y - \lambda^{-1} \eta d(y) \tag{2.48}$$

By (2.48) and (2.41) we get

$$d(x) + \lambda^{-1}(1 - \mu) x = \lambda^{-1}(1 - \mu) x - \lambda^{-1} \eta d(x)$$

that is

$$-\lambda^{-1}\eta = 1$$

Therefore

$$d(x) = F(x) + \theta x, \qquad \theta = \lambda^{-1}(\mu - 1) \neq 0 \tag{2.49}$$

Replacing x with xy in (2.49) and using (2.49) we have

$$\begin{aligned}
d(xy) = F(xy) + \theta xy = \\
&= \lambda F(x)F(y) + \mu F(x)y + \mu x d(y) + \theta xy = \\
&= \lambda\Big(d(x) - \theta x\Big)\Big(d(y) - \theta y\Big) + \mu\Big(d(x) - \theta x\Big)y + \mu x d(y) + \theta xy = \\
&= \lambda d(x)d(y) - \lambda\theta d(x)y - \lambda\theta x d(y) + \lambda\theta^2 xy + \mu d(x)y - \mu\theta xy + \mu x d(y) + \theta xy = \\
&= \lambda d(x)d(y) + \Big(-\lambda\theta + \mu\Big)d(x)y + \Big(-\lambda\theta + \mu\Big)x d(y) + \Big(\lambda\theta^2 - \mu\theta + \theta\Big)xy
\end{aligned}$$

Replacing θ with $\lambda^{-1}(\mu - 1)$ we get $-\lambda\theta + \mu = 1$ and $\lambda\theta^2 - \mu\theta + \theta = 0$.
Then $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$, that is d is an extended homoderivation.

Remark 15. *Let R be a prime ring, G an extended generalized homoderivation associated with $\lambda, 0 \neq \mu \in C$ and with the extended homoderivation $d : R \longrightarrow Q_r$ and $a \in R$ a fixed element. If $aG(x) = 0$ for all $x \in R$, then $a = 0$ or $d = 0$.*

Proof:
Since $aG(x) = 0$ for all $x \in R$ then

$$0 = aG(xy) = \lambda aG(x)G(y) + \mu aG(x)y + \mu a x d(y) = \mu a x d(y)$$

Since R is prime then $a = 0$ or $d = 0$.

2.2.2 Characterization of extended generalized homoderivations associated with an extended homoderivation

Theorem 16. *Let R be a prime unital ring and $F : R \longrightarrow Q_r$ an extended generalized homoderivation associated with $\lambda \in C$, $0 \neq \mu \in C$ and with the extended homoderivation $d : R \longrightarrow Q_r$.*
Let $a = d(1_R)$ and $b = F(1_R)$. Then one of the following possibilities holds:

- *$\mu = 1_R$ and $d = 0$, $F(x) = bx$;*
- *$\lambda \neq 0$, $\mu \neq 1_R$ and $d = 0$, $F(x) = -\lambda^{-1}\big(\mu - 1\big)x$;*
- *$\lambda = \eta = 0$, $\mu = 1_R$ and F is a generalized derivation associated to the derivation d;*
- *$\lambda = \eta \neq 0$, $\mu = 1_R$ and $F(x) = d(x)$;*
- *$\lambda = \eta \neq 0$, $\mu \neq 1_R$ and $F(x) = xb$;*
- *$\lambda = \eta \neq 0$, $\mu \neq 1_R$ and $F(x) = d(x) - \lambda^{-1}\big(\mu - 1\big)x$ where $\lambda\big(a - b\big) = \mu - 1$;*
- *$0 \neq b \in Z(R)$ and $d(x) = (\mu^{-1}b - \lambda\mu^{-1}b^2 - b)x$, $F(x) = bx$;*
- *$0 \neq b \notin Z(R)$, $\lambda = \eta$ and $d(x) = x\mu^{-1}b - bx - \lambda\mu^{-1}bxb$, $F(x) = xb$ with $(1 - \mu)b - \lambda b^2 = 0$;*

- $\lambda = 0$, $\mu = 1_R$ *and* $d = 0$, $F(x) = bx$;
- $\lambda = 0$, $\mu \neq 1_R$ *and* $d(x) = ax$, $F(x) = (1-\mu)^{-1}\mu ax$;
- $\lambda \neq 0$ *and* $d(x) = ax$, $F(x) = bx$ *with* $a \in C$ *and* $b \in C$.

Proof:
By Proposition 2 we have

$$d(xy) = \eta d(x)d(y) + d(x)y + xd(y) \qquad \text{for all } x, y \in R$$

or

$$d(x) = d(1_R)x \qquad \text{for all } x \in R$$

$\boxed{\text{I case}}$ Suppose that $d(xy) = \eta d(x)d(y) + d(x)y + xd(y)$ for all $x, y \in R$.
For all $x, y, z \in R$ we have

$$\begin{aligned} F((xy)z) &= \lambda F(xy)F(z) + \mu F(xy)z + \mu xyd(z) = \\ &= \lambda[\lambda F(x)F(y) + \mu F(x)y + \mu xd(y)]F(z) + \mu[\lambda F(x)F(y) + \mu F(x)y + \mu xd(y)]z + \\ &+ \mu xyd(z) = \\ &= \lambda^2 F(x)F(y)F(z) + \lambda\mu F(x)yF(z) + \lambda\mu xd(y)F(z) + \lambda\mu F(x)F(y)z + \\ &+ \mu^2 F(x)yz + \mu^2 xd(y)z + \mu xyd(z) \end{aligned}$$

Moreover

$$\begin{aligned} F(x(yz)) &= \lambda F(x)F(yz) + \mu F(x)yz + \mu xd(yz) = \\ &= \lambda F(x)[\lambda F(y)F(z) + \mu F(y)z + \mu yd(z)] + \mu F(x)yz + \\ &+ \mu x[\eta d(y)d(z) + d(y)z + yd(z)] = \\ &= \lambda^2 F(x)F(y)F(z) + \lambda\mu F(x)F(y)z + \lambda\mu F(x)yd(z) + \mu F(x)yz + \mu\eta xd(y)d(z) + \\ &+ \mu xd(y)z + \mu xyd(z) \end{aligned}$$

Subtracting member from member we have

$$F(x)y\Big(\lambda\mu F(z) + \mu^2 z - \lambda\mu d(z) - \mu z\Big) + xd(y)\Big(\lambda\mu F(z) + \mu^2 z - \mu\eta d(z) - \mu z\Big) = 0$$

Indicating

$$\Delta_1(z) = \lambda\mu F(z) + \mu^2 z - \lambda\mu d(z) - \mu z$$
$$\Delta_2(z) = \lambda\mu F(z) + \mu^2 z - \mu\eta d(z) - \mu z$$

we have

$$F(x)y\Delta_1(z) + xd(y)\Delta_2(z) = 0 \tag{2.50}$$

Replacing x with xt in (2.50) we have

$$F(xt)y\Delta_1(z) + xtd(y)\Delta_2(z) = 0 \tag{2.51}$$

Substituting x with t in (2.50) and then multiplying on the left by x we get

$$xtd(y)\Delta_2(z) = -xF(t)y\Delta_1(z) \tag{2.52}$$

By (2.52) and (2.51) it follows that

$$F(xt)y\Delta_1(z) - xF(t)y\Delta_1(z) = 0$$

that is

$$\Big(F(xt) - xF(t)\Big)y\Delta_1(z) = 0$$

Since R is a prime ring then $\Delta_1(z) = 0$ for all $z \in R$ or $F(xt) = xF(t)$ for all $x, t \in R$.

$\boxed{\text{I subcase}}$ Suppose that $\Delta_1(z) = 0$ for all $z \in R$, that is

$$\mu\Big(\lambda\big(F(z) - d(z)\big) + (\mu - 1)z\Big) = 0. \tag{2.53}$$

In particular, if $\lambda \neq 0$ and $\mu = 1_R$ then $F(x) = d(x)$.
Since $\mu \neq 0$ by (2.53) we have

$$\lambda\big(F(z) - d(z)\big) + (\mu - 1)z = 0 \quad \text{for all } z \in R \tag{2.54}$$

Moreover, by (2.50) we have

$$d(y)\Delta_2(z) = 0$$

and replacing y with yz we get

$$d(y)z\Delta_2(z) = 0.$$

Since R is a prime ring then $d = 0$ or $\Delta_2(z) = 0$ for all $z \in R$.

$\boxed{\text{case (a)}}$ Suppose that $d = 0$.
By (2.54) we have

$$\lambda F(z) + (\mu - 1)z = 0 \quad \text{for all } z \in R \tag{2.55}$$

If $\mu = 1_R$ and $\lambda = 0$ then, by the definition of extended generalized homoderivation, we have

$$F(xy) = F(x)y \quad \text{for all } x, y \in R$$

and replacing x with 1_R we get $F(y) = F(1_R)y$ for all $y \in R$.
If $\mu = 1_R$ and $\lambda \neq 0$ then by (2.55) we get $F = 0$.
If $\mu \neq 1_R$ and $\lambda \neq 0$ then by (2.55) we get $F(z) = -\lambda^{-1}(\mu - 1)z$ for all $z \in R$.

$\boxed{\text{case (b)}}$ Suppose that $\Delta_2(z) = 0$ for all $z \in R$ that is

$$\mu\Big(\lambda F(z) - \eta d(z) + (\mu - 1)z\Big) = 0$$

Since $\mu \neq 0$ we have

$$(\mu - 1)z = -\lambda F(z) + \eta d(z).$$

Using the previous relation in (2.54) we have

$$\lambda\big(F(z) - d(z)\big) - \lambda F(z) + \eta d(z) = 0$$

hence

$$\lambda = \eta$$

If $\lambda = \eta = 0$ then d is a derivation and by (2.54) it follows that $\mu = 1_R$. Therefore F is a generalized derivation associated to the derivation d.
If $\lambda \neq 0$ and $\mu = 1_R$ then by (2.54) it follows that $F = d$.
Assume $\lambda \neq 0$ and $\mu \neq 1_R$.
Replacing z with xy in (2.54) we get

$$\lambda\Big(F(xy) - d(xy)\Big) + \Big(\mu - 1\Big)xy = 0$$

$$\lambda\Big(F(xy) - \lambda d(x)d(y) - d(x)y - xd(y)\Big) + (\mu - 1)xy = 0$$

$$\lambda F(xy) - \lambda^2 d(x)d(y) - \lambda d(x)y - \lambda xd(y) + (\mu - 1)xy = 0 \tag{2.56}$$

Evaluating (2.54) in y and then multiplying to the left by x we have

$$\lambda xF(y) - \lambda xd(y) + (\mu - 1)xy = 0 \tag{2.57}$$

Comparing (2.56) and (2.57) it follows that

$$\lambda F(xy) - \lambda^2 d(x)d(y) - \lambda d(x)y - \lambda xF(y) = 0$$

Since $\lambda \neq 0$ then

$$F(xy) = \lambda d(x)d(y) + d(x)y + xF(y) \tag{2.58}$$

Replacing y with 1_R in (2.58) we get

$$F(x) = d(x)c + xb \qquad \text{where } c = \lambda d(1_R) + 1 \text{ and } b = F(1_R) \tag{2.59}$$

Then by (2.57) we get

$$\lambda x\Big(d(y)c + yb\Big) - \lambda xd(y) + (\mu - 1)xy = 0$$

Replacing x with 1_R we have

$$d(y)(\lambda c - \lambda) + y(\lambda b + \mu - 1) = 0 \tag{2.60}$$

Replacing y with xy we have

$$d(xy)(\lambda c - \lambda) + xy(\lambda b + \mu - 1) = 0$$

$$\lambda d(x)d(y)(\lambda c - \lambda) + d(x)y(\lambda c - \lambda) + xd(y)(\lambda c - \lambda) + xy(\lambda b + \mu - 1) = 0$$

Using (2.60) we have

$$-\lambda d(x)y(\lambda b + \mu - 1) + d(x)y(\lambda c - \lambda) = 0$$

$$\lambda d(x)y(-\lambda b - \mu + c) = 0$$

Since $\lambda \neq 0$ and since R is a prime ring then $d = 0$ or $c = \lambda b + \mu$, that is $\lambda\Big(d(1_R) - F(1_R)\Big) = \mu - 1$.

Therefore, by (2.59)

$$F(x) = xb \qquad \text{where } b = F(1_R) \tag{2.61}$$

or, by (2.54),

$$F(x) = d(x) - \lambda^{-1}(\mu - 1)x \qquad \text{where } \lambda\Big(d(1_R) - F(1_R)\Big) = \mu - 1$$

that is

$$F(x) = d(x) - \Big(d(1_R) - F(1_R)\Big)x$$

$\boxed{\text{II subcase}}$ Suppose that $F(xt) = xF(t)$ for all $x, t \in R$.
Replacing t with 1_R we have $F(x) = xb$ for all $x \in R$, where $b = F(1_R)$.
If $b = 0$ then $F = 0$.
Suppose $b \neq 0$. Then

$$\begin{aligned} xyb = F(xy) &= \\ &= \lambda F(x)F(y) + \mu F(x)y + \mu x d(y) = \\ &= \lambda xbyb + \mu xby + \mu x d(y) \end{aligned}$$

Hence

$$x\Big(yb - \lambda byb - \mu by - \mu d(y)\Big) = 0$$

Therefore

$$yb - \lambda byb - \mu by - \mu d(y) = 0$$

Since $\mu \neq 0$ then

$$d(y) = \mu^{-1}yb - \lambda\mu^{-1}byb - by \tag{2.62}$$

Therefore, for all $x, y \in R$ we have

$$d(xy) = \mu^{-1}xyb - \lambda\mu^{-1}bxyb - bxy \tag{2.63}$$

Hence, by definition

$$\begin{aligned} &d(xy) = \eta d(x)d(y) + d(x)y + xd(y) = \\ &\eta\Big(\mu^{-1}xb - \lambda\mu^{-1}bxb - bx\Big)\Big(\mu^{-1}yb - \lambda\mu^{-1}byb - by\Big) + \\ &+ \Big(\mu^{-1}xb - \lambda\mu^{-1}bxb - bx\Big)y + x\Big(\mu^{-1}yb - \lambda\mu^{-1}byb - by\Big) \end{aligned}$$

that is

$$\begin{aligned} &d(xy) = \\ &\eta\mu^{-2}xbyb - \eta\mu^{-1}xb^2y - \eta\lambda\mu^{-2}xb^2yb - \mu^{-1}\eta bxyb + \eta bxby + \eta\lambda\mu^{-1}bxbyb - \\ &- \eta\lambda\mu^{-2}bxbyb + \eta\lambda\mu^{-1}bxb^2y + \eta\lambda^2\mu^{-2}bxb^2yb + \mu^{-1}xby - bxy - \lambda\mu^{-1}bxby + \\ &+ \mu^{-1}xyb - xby - \lambda\mu^{-1}xbyb \end{aligned} \tag{2.64}$$

comparing (2.63) and (2.64) we have

$$x\Big(\eta\mu^{-2}byb - \eta\mu^{-1}b^2y - \eta\lambda\mu^{-2}b^2yb + \mu^{-1}by - by - \lambda\mu^{-1}byb\Big) +$$
$$+ bx\Big(-\mu^{-1}\eta yb + \eta by + \eta\lambda\mu^{-1}byb - \eta\lambda\mu^{-2}byb + \eta\lambda\mu^{-1}b^2y + \eta\lambda^2\mu^{-2}b^2yb -$$
$$- \lambda\mu^{-1}by + \lambda\mu^{-1}yb\Big) = 0$$

Indicating

$$\Gamma_1(y) = \eta\mu^{-2}byb - \eta\mu^{-1}b^2y - \eta\lambda\mu^{-2}b^2yb + \mu^{-1}by - by - \lambda\mu^{-1}byb$$
$$\Gamma_2(y) = -\mu^{-1}\eta yb + \eta by + \eta\lambda\mu^{-1}byb - \eta\lambda\mu^{-2}byb + \eta\lambda\mu^{-1}b^2y + \eta\lambda^2\mu^{-2}b^2yb -$$
$$- \lambda\mu^{-1}by + \lambda\mu^{-1}yb$$

we have

$$x\Gamma_1(y) + bx\Gamma_2(y) = 0 \tag{2.65}$$

If $0 \neq b \in C$ then, by (2.62), we have

$$F(x) = \beta x, \qquad d(x) = \gamma x \quad \text{where } \beta = b \in C \text{ and } \gamma = \mu^{-1}b - \lambda\mu^{-1}b^2 - b \in C$$

Suppose $0 \neq b \notin C$. Then by (2.65) it follows that $\Gamma_1(y) = 0$ and $\Gamma_2(y) = 0$ that is

$$0 = \eta\mu^{-2}byb - \eta\mu^{-1}b^2y - \eta\lambda\mu^{-2}b^2yb + \mu^{-1}by - by - \lambda\mu^{-1}byb \tag{2.66}$$

and

$$0 = -\mu^{-1}\eta yb + \eta by + \eta\lambda\mu^{-1}byb - \eta\lambda\mu^{-2}byb + \eta\lambda\mu^{-1}b^2y + \eta\lambda^2\mu^{-2}b^2yb - \lambda\mu^{-1}by + \lambda\mu^{-1}yb$$

In particular, by (2.66) it follows that

$$0 = \Big(\eta\mu^{-2}b - \eta\lambda\mu^{-2}b^2 - \lambda\mu^{-1}b\Big)yb + \Big(-\eta\mu^{-1}b^2 + \mu^{-1}b - b\Big)y$$

Since $0 \neq b \notin C$, it follows that

$$\eta\mu^{-2}b - \eta\lambda\mu^{-2}b^2 - \lambda\mu^{-1}b = 0 \tag{2.67}$$

and

$$-\eta\mu^{-1}b^2 + \mu^{-1}b - b = 0 \tag{2.68}$$

Since $\mu, \lambda \neq 0$, by (2.69) we have

$$\lambda^{-1}\eta\mu^{-1}b - \eta\mu^{-1}b^2 - b = 0 \tag{2.69}$$

Comparing (2.69) and (2.68) we have $\lambda^{-1}\eta\mu^{-1}b = \mu^{-1}b$ that is $\lambda = \eta$.
Then by (2.69) we have $\mu^{-1}b - \lambda\mu^{-1}b^2 - b = 0$, that is $(1-\mu)b - \lambda b^2 = 0$.

$\boxed{\text{II case}}$ Suppose that $d(x) = \eta_0 x$ for all $x \in R$, with $\eta_0 = d(1_R) \in C$.

Then for all $x, y \in R$ we have

$$F(xy) = \lambda F(x)F(y) + \mu F(x)y + \mu x\eta_0 y = \lambda F(x)F(y) + \mu\Big(F(x) + \eta_0 x\Big)y$$

$\boxed{\text{case (a)}}$ Suppose that $\lambda = 0$. Then

$$F(xy) = \mu\Big(F(x) + \eta_0 x\Big)y$$

For $x = 1_R$ we have

$$F(y) = \mu\Big(F(1_R) + \eta_0\Big)y \tag{2.70}$$

and for $y = 1_R$ we have

$$F(x) = \mu F(x) + \mu\eta_0 x \tag{2.71}$$

Therefore by (2.70) and (2.71) it follows $\mu F(1_R)x + \mu\eta_0 x = \mu F(x) + \mu\eta_0 x$ that is

$$F(x) = F(1_R)x \tag{2.72}$$

In particular, by (2.70) we have $F(1_R) = \mu F(1_R) + \mu\eta_0$ that is

$$(1-\mu)F(1_R) = \mu\eta_0 \tag{2.73}$$

If $\mu = 1$ it follows $\eta_0 = 0$, that is $d = 0$.
Therefore $d = 0$ and $F(x) = F(1_R)x$ or $d(x) = \eta_0 x$ and, by (2.73), $F(x) = (1-\mu)^{-1}\mu\eta_0 x$

$\boxed{\text{case (b)}}$ Suppose that $\lambda \neq 0$. Then

$$F(xy) = \lambda F(x)F(y) + \mu F(x)y + \mu x\eta_0 y$$

For $x = 1_R$ we have

$$F(x) = \lambda F(1_R)F(y) + \mu F(1_R)y + \mu\eta_0 y \tag{2.74}$$

and for $y = 1_R$ we have

$$F(x) = \lambda F(x)F(1_R) + \mu F(x) + \mu x\eta_0 \tag{2.75}$$

Therefore by (2.74) and (2.75) it follows

$$\lambda F(1_R)F(x) + \mu F(1_R)x = \lambda F(x)F(1_R) + \mu F(x) \tag{2.76}$$

Moreover, for all $x, y, t \in R$ we have

$$\begin{aligned}
&F(x(yt)) = \\
&\lambda F(x)F(yt) + \mu F(x)yt + \mu\eta_0 xyt = \\
&\lambda F(x)[\lambda F(y)F(t) + \mu F(y)t + \mu\eta_0 yt] + \mu F(x)yt + \mu\eta_0 xyt = \\
&\lambda^2 F(x)F(y)F(t) + \lambda\mu F(x)F(y)t + \lambda\mu\eta_0 F(x)yt + \mu F(x)yt + \mu\eta_0 xyt
\end{aligned} \tag{2.77}$$

$$
\begin{aligned}
&F((xy)t) = \\
&\lambda F(xy)F(t) + \mu F(xy)t + \mu\eta_0 xyt = \\
&\lambda[\lambda F(x)F(y) + \mu F(x)y + \mu\eta_0 xy]F(t) + \mu[\lambda F(x)F(y) + \mu F(x)y + \mu\eta_0 xy]t + \\
&+ \mu\eta_0 xyt = \\
&\lambda^2 F(x)F(y)F(t) + \lambda\mu F(x)yF(t) + \lambda\mu\eta_0 xyF(t) + \lambda\mu F(x)F(y)t + \mu^2 F(x)yt + \\
&+ \mu^2\eta_0 xyt + \mu\eta_0 xyt
\end{aligned}
\tag{2.78}
$$

Comparing (2.77) and (2.78) we have

$$(\lambda\eta_0 + 1)F(x)yt = \lambda F(x)yF(t) + \lambda\eta_0 xyF(t) + \mu F(x)yt + \mu\eta_0 xyt$$

$$F(x)y\Big(\lambda F(t) + \mu t\Big) + \eta_0 xy\Big(\lambda F(t) + \mu t\Big) - (\lambda\eta_0 + 1)F(x)yt = 0$$

$$\Big(F(x) + \eta_0 x\Big)y\Big(\lambda F(t) + \mu t\Big) - (\lambda\eta_0 + 1)F(x)yt = 0$$

For every x_0 such that $(\lambda\eta_0 + 1)F(x_0) = 0$ then, since R is a prime ring, we have $F(x_0) + \eta_0 x_0 = 0$ or $\lambda F(t) + \mu t = 0$ for all $t \in R$.
For every x_0 such that $(\lambda\eta_0 + 1)F(x_0) \neq 0$ then $\{\lambda F(t) + \mu t, t\}$ is linearly C-dependent for all $t \in R$.
So in every case $\{F(x), x\}$ is linearly C-dependent for all $x \in R$.
In particular, $\{F(1_R), 1_R\}$ is linearly C-dependent that is $F(1_R) \in C$.
Therefore by (2.76) we have $F(x) = F(1_R)x$ with $F(1_R) \in C$.
Hence $d(x) = d(1)x$ and $F(x) = F(1_R)x$ with $d(1_R) \in C$ and $F(1_R) \in C$

2.2.3 Examples

Example 13. *Let d be a homoderivation and $F(x) = d(x) - x$.*
F is an extended generalized homoderivation associated with $\mu = 2$, $\lambda = 1_R$, and the homoderivation d.

Example 14. *Let $d(x) = \Big(\mu^{-1} - a\Big)xa - ax$.*

Then $F(x) = xa$ is an extended generalized homoderivation associated with $\lambda = \mu$ and the additive map d.

Example 15. *Let $d : R \longrightarrow Q_r$ be an extended homoderivation and let $F : R \longrightarrow Q_r$ be such that $F(xy) = F(x)y + xd(y)$. If F is an extended generalized homoderivation associated with $\lambda, 0 \neq \mu \in C$ and with the extended homoderivation d, then*

$$F(x)y + xd(y) = \lambda F(x)F(y) + \mu F(x)y + \mu xd(y) \tag{2.79}$$

that is

$$F(x)\Big((1-\mu)y - \lambda F(y)\Big) + (1-\mu)xd(y) = 0. \tag{2.80}$$

Replacing x with tx we have

$$F(tx)\Big((1-\mu)y - \lambda F(y)\Big) + (1-\mu)txd(y) = 0. \tag{2.81}$$

Moreover, multiplying (2.80) by t on the left gives

$$tF(x)\Big((1-\mu)y-\lambda F(y)\Big)+(1-\mu)txd(y)=0. \tag{2.82}$$

From (2.81) and (2.82) it then follows that

$$\Big(F(tx)-tF(x)\Big)\Big((1-\mu)y-\lambda F(y)\Big)=0. \tag{2.83}$$

Replacing y with yz we have $\Big(F(tx)-tF(x)\Big)\Big((1-\mu)yz-\lambda F(y)z-\lambda yd(z)\Big)=0$ and using (2.83) we get

$$\Big(F(tx)-tF(x)\Big)\lambda yd(z)=0.$$

Then $F(tx)=tF(x)$ or $\lambda=0$ or $d=0$.
If $\lambda=0$ then by (2.79) we have $(1-\mu)\Big(F(x)y+xd(y)\Big)=0$ for all $x,y\in R$ that is $(1-\mu)F(xy)=0$ for all $x,y\in R$. Therefore $\mu=1_R$ or $F=0$.
If $F(tx)=tF(x)$ then $F(x)=xF(1_R)$.
If $d=0$ then $F(xy)=F(x)y$ that is $F(x)=F(1_R)x$.

2.2.4 Posner's theorems

Lemma 8. *Let R be a prime unital ring of characteristic different from 2, $d:R\longrightarrow Q_r$ an additive map and $F:R\longrightarrow Q_r$ an extended generalized homoderivation associated with $\lambda,0\neq\mu\in C$ and the additive map $d:R\longrightarrow Q_r$. If $F^2(x)=0$ for all $x\in R$ then one of the following possibilities holds:*

- *$d=0$ and $F=0$;*
- *there exists $a\in Q_r$ such that either $F(x)=ax$ or $F(x)=xa$ with $a^2=0$;*
- *$\mu=1_R$, $F(\lambda x)+\lambda F(x)+2x=0$ for all $x\in R$ and $F(\lambda)+2=0$;*
- *$F(x)=d(x)+bx$, $d(xy)=\lambda d(x)d(y)+d(x)y+xd(y)$ and $d^2(x)=\gamma d(x)$ with $b=F(1_R)\in C$ and $\gamma=\lambda b^2-2b\in C$.*

Proof:
By Theorem 15, we have the following cases:

- $d=0$ and $F(x)=ax$.
  If $F^2=0$ then $a^2=0$.
- $d=0$ and $F(x)=\lambda^{-1}(1-\mu)x$.
  If $F^2=0$ then

  $$0=\lambda^{-1}(1-\mu)F(x)=\lambda^{-1}(1-\mu)\lambda^{-1}(1-\mu)x$$

  that is

  $$\lambda^{-1}(1-\mu)=0$$

  and $F=0$.

- $\mu = 1$, $\lambda = 0$, F is a generalized derivation associated with the derivation d. By [23, Corollary 5], $F^2(x) = 0$ if and only if there exists $a \in Q_r$ such that either $F(x) = ax$ or $F(x) = xa$ with $a^2 = 0$.

- $\mu = 1$, $\lambda \neq 0$, $F = d$ is an extended homoderivation. By Lemma 5, if $d^2(x) = 0$ then $d(x) = 0$ or the following conditions hold

$$\begin{cases} d(\lambda x) + \lambda d(x) + 2x = 0 & \text{for all } x \in R \\ d(\lambda) + 2 = 0 \end{cases}$$

- $F(x) = d(x) - \theta x$ and $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$ with $\lambda \neq 0$, $u = d(1_R)$, $b = F(1_R)$ and

$$\theta = u - b = \lambda^{-1}(\mu - 1) \in C, \quad \text{with} \quad \mu \neq 1_R. \tag{2.84}$$

If $F^2(x) = 0$ for all $x \in R$ we have $0 = F^2(1_R) = F(b) = d(b) - \theta b$ that is

$$d(b) = \theta b \tag{2.85}$$

Moreover, for all $x \in R$,

$$\begin{aligned} 0 = F^2(x) =& d\Big(d(x) - \theta x\Big) - \theta\Big(d(x) - \theta x\Big) = \\ =& d^2(x) - d(\theta x) - \theta d(x) + \theta^2 x = \\ =& d^2(x) - \lambda d(\theta)d(x) - d(\theta)x - 2\theta d(x) + \theta^2 x = \\ =& d^2(x) - (\lambda d(\theta) + 2\theta)d(x) - (d(\theta) - \theta^2)x \end{aligned}$$

Therefore

$$d^2(x) = cd(x) + px \quad \text{with} \quad c = \lambda d(\theta) + 2\theta, \quad p = d(\theta) - \theta^2. \tag{2.86}$$

By Proposition 3 we have

$$\Big(\lambda d(x) + x\Big)u = 0 \quad \text{for all } x \in R. \tag{2.87}$$

Then

$$d(x)u = -\lambda^{-1}xu. \tag{2.88}$$

Moreover, replacing x with $d(x)$ in (2.87) and using (2.86) and (2.88) we have

$$\begin{aligned} 0 =& \Big(\lambda d^2(x) + d(x)\Big)u = \\ =& \lambda cd(x)u + \lambda pxu + d(x)u = \\ =& -cxu + \lambda pxu - \lambda^{-1}xu = \Big(-c + \lambda p - \lambda^{-1}\Big)xu \end{aligned}$$

Since R is a prime ring then $u = 0$ or $-c + \lambda p - \lambda^{-1} = 0$ that is $\theta = -b$ or $c = \lambda p - \lambda^{-1}$.

  - If $d(1_R) = u = 0$ then, by (2.84), $\theta = -b \in C$. Then $F(x) = d(x) + bx$ and, by (2.86),
  $$0 = d(u) = d^2(1_R) = cd(1_R) + p = p$$

Hence by (2.85) we have $d(b) = \theta b = -b^2$ and

$$c = \lambda d(\theta) + 2\theta = -\lambda d(b) - 2b = \lambda b^2 - 2b \in C$$

and by (2.86)

$$d^2(x) = cd(x) + px = cd(x) = \Big(\lambda b^2 - 2b\Big) d(x)$$

- If $c = \lambda p - \lambda^{-1}$ then, by (2.86), we have

$$\begin{aligned}\lambda d(\theta) + 2\theta = c =& \lambda p - \lambda^{-1} = \\ =& \lambda(d(\theta) - \theta^2) - \lambda^{-1} = \\ =& \lambda d(\theta) - \lambda\theta^2 - \lambda^{-1}\end{aligned}$$

that is, by (2.84),

$$\begin{aligned}2\theta =& -\lambda\theta^2 - \lambda^{-1} \\ 2\lambda^{-1}(\mu - 1) =& -\lambda\lambda^{-1}(\mu-1)\lambda^{-1}(\mu-1) - \lambda^{-1} \\ 2(\mu - 1) =& -(\mu-1)(\mu-1) - 1 \\ 2\mu - 2 =& -\mu^2 + 2\mu - 2 \\ \mu =& 0\end{aligned}$$

a contradiction.

- $F(x) = xa$ and $d(x) = \Big(\mu^{-1} - \mu^{-1}\lambda a\Big)xa - ax$ with $a = F(1_R)$.
If $F^2 = 0$ then $0 = F(x)a = xa^2$ that is $a^2 = 0$.

Theorem 17. *Let R be a prime unital ring of characteristic different from 2, $d : R \longrightarrow Q_r$ an additive map and $F : R \longrightarrow Q_r$ an extended generalized homoderivation associated with $\lambda, 0 \neq \mu \in C$ and the additive map $d : R \longrightarrow Q_r$.*
If $[F(x), x] = 0$ for all $x \in R$ then one of the following possibilities holds:

- *$F = \theta x$, $\theta \in C$ and either $d = 0$ or $d(x) = \gamma x$ with $\gamma \in C$;*
- *$F = 0$ and $d = 0$;*
- *$F(x) = d(x) = -\lambda^{-1}x + \omega(x)$ with $\omega : R \longrightarrow Z(R)$. Moreover $\omega = 0$ or $\omega(1_R) = \lambda^{-1}$;*
- *$F(x) = -(\lambda^{-1} + \theta)x + \omega(x)$ and $d(x) = -\lambda^{-1}x + \omega(x)$ with $\theta \in C$, $\omega : R \longrightarrow Z(R)$ and $\omega = 0$ or $\omega(1_R) = \lambda^{-1}$.*

Proof:
By Theorem 15, we have the following cases:

- $d = 0$ and $F(x) = \lambda^{-1}(1 - \mu)x$ is an extended homoleft multiplier and, clearly, $[F(x), x] = 0$.
- $d = 0$ and $F(x) = ax$.
If $[F(x), x] = 0$ then $[ax, x] = 0$ and replacing x with $1 + x$ we have $0 = [a, x]$, that is a is central.

- $\mu = 1$, $\lambda = 0$, F is a generalized derivation associated to the derivation d. By [30, Corollary 3.2], if $[F(x), x] = 0$ for all $x \in R$, since R is not commutative, then $F = 0$.

- $\mu = 1$, $\lambda \neq 0$, $F(x) = d(x)$ is an extended homoderivation. By Theorem 14, since $[d(x), x] = 0$, then $d = 0$ or $d(x) = -\lambda^{-1}x + \omega(x)$ with $\omega : R \longrightarrow Z(R)$.
  Moreover $\omega = 0$ or $\omega(1_R) = \lambda^{-1}$.

- $F(x) = d(x) - \theta x$ and $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$ with $u = d(1_R)$, $b = F(1_R)$ and
$$\theta = u - b = \lambda^{-1}(\mu - 1) \in C, \quad \text{with} \quad \mu \neq 1_R.$$
  We have $0 = [F(x), x] = [d(x) - \theta x, x] = [d(x), x]$ and by Theorem 14 it follows that $d = 0$ or $d(x) = -\lambda^{-1}x + \omega(x)$ with $\omega : R \longrightarrow Z(R)$ and $\omega = 0$ or $\omega(1_R) = \lambda^{-1}$.
  If $d = 0$ then $F(x) = -\theta x$, $\theta \in C$.
  If $d(x) = -\lambda^{-1}x + \omega(x)$ then
$$F(x) = -\lambda^{-1}x + \omega(x) - \theta x = -(\lambda^{-1} + \theta)x + \omega(x)$$

- $F(x) = xa$ with $a = F(1_R)$, and $d(x) = \left(\mu^{-1} - \mu^{-1}\lambda a\right)xa - ax$.
  Since $0 = [F(x), x] = [xa, x] = x[a, x]$, then replacing x with $x + 1$ we have
$$0 = (x+1)[a, x+1] = (x+1)[a, x] = x[a, x] + [a, x] = [a, x]$$
  that is $a \in C$. Then $d(x) = \gamma x$, with $\gamma = \mu^{-1}a - \mu^{-1}\lambda a^2 - a \in C$.

Chapter 3

Generalized skew homoderivation

The purpose of this chapter is to generalize the maps introduced in previous sections by moving in a new direction, one shaped by the concept of automorphisms.
We will characterize these novel mappings, which we define as generalized skew homoderivations, by establishing a connection with an additive map, and subsequently, with a homoderivation. This detailed analysis will not only reveal all possible structural forms of these maps but also provide a generalization of Posner's well-known theorems.

Theorem 18. *Let R be a semiprime ring, let α be an automorphism of R and let $d : R \longrightarrow R$ be an additive map such that*

$$d(xy) = d(x)d(y) + d(x)y + \alpha(x)d(y) \qquad \textit{for all } x, y \in R$$

Then d is the null map or is a homoderivation (i.e. α is the identity automorphism)

Proof:
If $\alpha = 0$ then d is a homoleftmultiplier and by [6] we have $d = 0$.
Assume $\alpha \neq 0$. For all $x, y, z \in R$ we have

$$\begin{aligned}
&d(x(yz)) = \\
&d(x)d(yz) + d(x)yz + \alpha(x)d(yz) = \\
&d(x)\Big(d(y)d(z) + d(y)z + \alpha(y)d(z)\Big) + d(x)yz + \alpha(x)\Big(d(y)d(z) + d(y)z+ \\
&+ \alpha(y)d(z)\Big) = \\
&d(x)d(y)d(z) + d(x)d(y)z + d(x)\alpha(y)d(z) + d(x)yz + \alpha(x)d(y)d(z) + \alpha(x)d(y)z+ \\
&+ \alpha(x)\alpha(y)d(z)
\end{aligned}$$

Moreover

$$\begin{aligned}
&d((xy)z) = \\
&d(xy)d(z) + d(xy)z + \alpha(xy)d(z) = \\
&\Big(d(x)d(y) + d(x)y + \alpha(x)d(y)\Big)d(z) + \Big(d(x)d(y) + d(x)y + \alpha(x)d(y)\Big)z+ \\
&+ \alpha(x)\alpha(y)d(z) = \\
&d(x)d(y)d(z) + d(x)yd(z) + \alpha(x)d(y)d(z) + d(x)d(y)z + d(x)yz + \alpha(x)d(y)z+ \\
&+ \alpha(x)\alpha(y)d(z)
\end{aligned}$$

Subtracting member from member we have

$$0 = d(x)\alpha(y)d(z) - d(x)yd(z) \qquad \text{for all } x, y, z \in R. \tag{3.1}$$

Replacing x with xt we have

$$\begin{aligned}
0 &= d(xt)\alpha(y)d(z) - d(xt)yd(z) = \\
&= \Big(d(x)d(t) + d(x)t + \alpha(x)d(t)\Big)\alpha(y)d(z) - \Big(d(x)d(t) + d(x)t + \alpha(x)d(t)\Big)yd(z) = \\
&= d(x)d(t)\alpha(y)d(z) + d(x)t\alpha(y)d(z) + \alpha(x)d(t)\alpha(y)d(z) - d(x)d(t)yd(z) - d(x)tyd(z) - \\
&\quad - \alpha(x)d(t)yd(z)
\end{aligned}$$

By (3.1) we have

$$0 = d(x)t\alpha(y)d(z) - d(x)tyd(z) \qquad \text{for all } x, y, z, t \in R$$

that is

$$0 = d(x)t\Big(\alpha(y) - y\Big)d(z) \qquad \text{for all } x, y, z, t \in R. \tag{3.2}$$

Since R is a semiprime ring then d is the null map or

$$\Big(\alpha(y) - y\Big)d(z) = 0 \qquad \text{for all } y, z \in R$$

that is

$$\alpha(y)d(z) = yd(z) \qquad \text{for all } y, z \in R. \tag{3.3}$$

and substituting (3.3) in the expression of the map d we get the conclusion.

3.1 Generalized skew homoderivation

Definition 7. *Let R be an associative ring and let α be an automorphism of R. An additive map $F : R \longrightarrow R$ is called a generalized skew homoderivation associated with α and the additive map d if*

$$F(xy) = F(x)F(y) + F(x)y + \alpha(x)d(y) \qquad \textit{for all } x, y \in R$$

Remark 16. *Observe that if R is a unital ring then by replacing, in the definition of a generalized skew homoderivation F, x with 1_R we obtain*

$$F(y) = F(1_R)F(y) + F(1_R)y + d(y)$$

that is

$$d(y) = (1 - a)F(y) - ay, \qquad \textit{with } a = F(1_R). \tag{3.4}$$

and in particular

$$F(y) - d(y) = a\Big(F(y) + y\Big), \qquad \textit{with } a = F(1_R). \tag{3.5}$$

Moreover, replacing, in the definition of a generalized skew homoderivation F, y with 1_R we obtain

$$F(x) = F(x)F(1_R) + F(x) + \alpha(x)d(1_R)$$

that is

$$0 = F(x)a + \alpha(x)d(1_R), \qquad \text{with } a = F(1_R). \tag{3.6}$$

In particular

$$0 = a^2 + d(1_R), \qquad \text{with } a = F(1_R). \tag{3.7}$$

hence by (3.6) and (3.7) we have

$$F(x)a = \alpha(x)a^2, \qquad \text{with } a = F(1_R).$$

Remark 17. *Let R be a prime unital ring, α an automorphism of R, $F : R \longrightarrow R$ a generalized skew homoderivation associated with α and the additive map $d : R \longrightarrow R$ and $c \in R$ a fixed element.*
If $cF(x) = 0$ for all $x \in R$, then $c = 0$ or $d = 0$.

Proof:
Since $cF(x) = 0$ for all $x, y \in R$, then

$$\begin{aligned} 0 = cF(xy) &= \\ &= cF(x)F(y) + cF(x)y + c\alpha(x)d(y) = \\ &= c\alpha(x)d(y) \end{aligned}$$

Since R is a prime ring, then $c = 0$ or $d = 0$.

3.1.1 Examples

Example 16. *Let $G : R \longrightarrow Q_r$ be a generalized skew homoderivation on the prime ring R associated with the additive map d and the automorphism α. If $G(x) = \eta x$ for all $x \in R$, with $\eta \in C$, then for all $x, y \in R$*

$$\begin{aligned} G(xy) =& G(x)G(y) + G(x)y + \alpha(x)d(y) \\ \eta xy =& \eta x \eta y + \eta xy + \alpha(x)d(y) \\ 0 =& \eta^2 xy + \alpha(x)d(y) \end{aligned}$$

then for $x = 1_R$ we have $d(y) = -\eta^2 y$ and then for $y = 1_R$ we have

$$0 = \eta^2 x + \alpha(x)d(1_R) = \eta^2 x - \eta^2 \alpha(x)$$

Therefore $\eta = 0$ or $\alpha = id_R$.
Hence $G = 0$ or $\alpha = id_R$, $G = \eta x$ and $d(x) = -\eta^2 x$.

Example 17. *Let R be a prime unital ring, α an automorphism of R and let $G : R \longrightarrow R$ a generalized skew derivation associated with a skew derivation d, that is*

$$G(xy) = G(x)y + \alpha(x)d(y)$$

for all $x, y \in R$.
If G is a generalized skew homoderivation associated with α and d, then $G = 0$.
Proof:
For all $x, y \in R$ we have

$$G(x)y + \alpha(x)d(y) = G(x)G(y) + G(x)y + \alpha(x)d(y)$$

that is $0 = G(x)G(y)$.
In this case, replacing y *with* yz *we get* $0 = G(x)\Big(G(y)z + \alpha(y)d(z)\Big)$ *that is* $G(x)\alpha(y)d(z)$.
Since R *is a prime ring then* $G = 0$ *or* $d = 0$.
If $d = 0$ *then* $G(x) = ax$ *with* $a = G(1_R)$. *In this case we have*

$$\begin{aligned} G(xy) =& G(x)G(y) + G(x)y + \alpha(x)d(y) = G(x)G(y) + G(x)y \\ axy =& axay + axy \end{aligned}$$

Then $axay = 0$ *that is* $a = 0$ *and* $G = 0$.

3.1.2 Characterization of generalized skew homoderivations associated with an additive map

Theorem 19. *Let* R *be a prime unital ring,* α *an automorphism of* R *and* $F : R \longrightarrow R$ *a generalized skew homoderivation associated with* α *and the additive map* $d : R \longrightarrow R$.
Then one of the following possibilities holds:

- $F(x) = d(x) = 0$ *for all* $x \in R$;
- $F = d$ *is a homoderivation;*
- $F(x) = \alpha(x)a$ *and* $d(x) = (1-a)\alpha(x)a - ax$ *for all* $x \in R$ *with* $a = F(1_R)$;
- $\alpha(x) = qxq^{-1}$, $F(x) = \eta q^{-1}d(x)$, *with* $\eta \in C$,

  $$F(xy) = F(x)F(y) + F(x)y + q_0xF(y), \qquad d(xy) = d(x)d(y) + d(x)y + q_0^2xd(y)$$

  and

  $$\Big(F(x) + q_0x\Big)(1-q_0) = 0, \qquad \Big(d(x) + q_0^2x\Big)(1-q_0) = 0$$

  with $q_0 = \eta^{-1}q$.

Proof:
If $d = 0$ then F is a homoleft multiplier and, since the only homoleft multiplier on a semiprime ring is the zero mapping ([6]), then $F = 0$.
Assume $d \neq 0$.
For all $x, y, z \in R$ we have:

$$\begin{aligned} &F((xy)z) = \\ &F(xy)F(z) + F(xy)z + \alpha(xy)d(z) = \\ &F(x)F(y)F(z) + F(x)yF(z) + \alpha(x)d(y)F(z) + F(x)F(y)z + F(x)yz + \alpha(x)d(y)z + \\ &+ \alpha(x)\alpha(y)d(z) \end{aligned}$$

Moreover

$$\begin{aligned} &F(x(yz)) = \\ &F(x)F(yz) + F(x)yz + \alpha(x)d(yz) = \\ &F(x)F(y)F(z) + F(x)F(y)z + F(x)\alpha(y)d(z) + F(x)yz + \alpha(x)d(yz) \end{aligned}$$

Subtracting member from member we have

$$F(x)yF(z) + \alpha(x)d(y)F(z) + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - F(x)\alpha(y)d(z) - \alpha(x)d(yz) = 0$$

that is
$$F(x)\Delta_1 + \alpha(x)\Delta_2 = 0 \tag{3.8}$$
where $\Delta_1 = yF(z) - \alpha(y)d(z)$ and $\Delta_2 = d(y)F(z) + d(y)z + \alpha(y)d(z) - d(yz)$.
Replacing x with xt we have

$$F(x)F(t)\Delta_1 + F(x)t\Delta_1 + \alpha(x)d(t)\Delta_1 + \alpha(x)\alpha(t)\Delta_2 = 0$$

and using (3.8) we get

$$F(x)F(t)\Delta_1 + F(x)t\Delta_1 + \alpha(x)d(t)\Delta_1 - \alpha(x)F(t)\Delta_1 = 0$$

that is
$$F(x)\Big(F(t) + t\Big)\Delta_1 + \alpha(x)\Big(d(t) - F(t)\Big)\Delta_1 = 0$$

and using (3.5) of Remark 16 it follows that

$$F(x)\Big(F(t) + t\Big)\Delta_1 - \alpha(x)a\Big(F(t) + t\Big)\Delta_1 = 0, \qquad \text{with } a = F(1_R)$$

that is
$$\Delta_3\Big(yF(z) - \alpha(y)d(z)\Big) = 0$$

where $\Delta_3 = \Big(F(x) - \alpha(x)a\Big)\Big(F(t) + t\Big)$ and $\Delta_1 = \Big(yF(z) - \alpha(y)d(z)\Big)$.
Fix x and t in Δ_3.
Therefore
$$\Delta_3 yF(z) - \Delta_3\alpha(y)d(z) = 0 \qquad \text{for all } y, z \in R. \tag{3.9}$$
If α is outer, by [12, Theorem 3] it follows that R satisfies

$$\Delta_3 yF(z) - \Delta_3 y'd(z) = 0$$

where y, z and y' are distinct indeterminates. Since R is prime then $\Delta_3 = 0$ or $F(z) = d(z) = 0$.
If α is inner then there exists an invertible element $q \in R$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$ and by (3.9) it follows that R satisfies

$$\Delta_3 yF(z) - \Delta_3 qyq^{-1}d(z) = 0.$$

Then one of the following possibilities holds:

- $\Delta_3 = 0$;
- $F(z) = d(z) = 0$ for all $z \in R$;
- $\Delta_3 q = \eta\Delta_3$ and $F(z) = \eta q^{-1}d(z)$ for all $z \in R$ with $\eta = \eta(x,t)$ and $\alpha(x) = qxq^{-1}$.

Then:

- If $\Delta_3 = 0$ then

$$\Big(F(x) - \alpha(x)a\Big)\Big(F(t) + t\Big) = 0, \qquad \text{for all } x, t \in R, \text{ with } a = F(1_R)$$

that is

$$F(x)\Big(F(t)+t\Big)-\alpha(x)a\Big(F(t)+t\Big)=0$$

and using (3.5) of Remark 16 it follows that

$$F(x)\Big(F(t)+t\Big)-\alpha(x)\Big(F(t)-d(t)\Big)=0$$

$$F(x)F(t)+F(x)t-\alpha(x)F(t)+-\alpha(x)d(t)=0$$

that is

$$F(xt)-\alpha(x)F(t)=0.$$

Therefore

$$F(x)=\alpha(x)a, \qquad \text{for all } x\in R, \text{ with } a=F(1_R)$$

and using (3.4) of Remark 16 we have

$$d(x)=(1-a)\alpha(x)a-ax, \qquad \text{with } a=F(1_R).$$

- Assume

$$\Delta_3 q=\eta\Delta_3 \tag{3.10}$$

 and

$$F(x)=\eta q^{-1}d(x), \tag{3.11}$$

 with $\eta=\eta(x,t)$ and $\alpha(x)=qxq^{-1}$.

Then

$$d(x)=\eta^{-1}qF(x)=q_0F(x), \qquad \text{with } q_0=\eta^{-1}q. \tag{3.12}$$

Moreover, for all $x,y\in R$, we have

$$\alpha(x)d(y)=qxq^{-1}\eta^{-1}qF(y)=\eta^{-1}qxF(y)=q_0xF(y).$$

Hence

$$\begin{aligned} F(xy)=&F(x)F(y)+F(x)y+\alpha(x)d(y)=\\ =&F(x)F(y)+F(x)y+q_0xF(y) \qquad \text{with } q_0=\eta^{-1}q. \end{aligned} \tag{3.13}$$

In particular

$$\alpha(x)d(1_R)=\eta^{-1}qxF(1_R)=\eta^{-1}qxa, \tag{3.14}$$

that is

$$qxq^{-1}d(1_R)=qx\eta^{-1}a,$$

and therefore

$$q^{-1}d(1_R)=\eta^{-1}a.$$

By (3.6) of Remark 16 we have

$$F(x)a=-\alpha(x)d(1_R),$$

and using (3.14) we get

$$F(x)a=-\eta^{-1}qxa,$$

that is

$$\eta q^{-1}d(x)a=-\eta^{-1}qxa.$$

Then
$$d(x)a = -(\eta^{-1}q)^2xa = -q_0^2xa$$
that is
$$\Big(d(x) + q_0^2x\Big)a = 0 \tag{3.15}$$
and
$$\Big(q_0F(x) + q_0^2x\Big)a = 0.$$
Since $q_0 \neq 0$ then
$$\Big(F(x) + q_0x\Big)a = 0. \tag{3.16}$$
In particular, for $x = 1_R$ we have
$$a^2 + q_0a = 0. \tag{3.17}$$
Replacing x with xy in (3.16) we get
$$\Big(F(xy) + q_0xy\Big)a = 0$$
that is, using (3.13),
$$\Big(F(x)F(y) + F(x)y + q_0xF(y) + q_0xy\Big)a = 0$$
and for $x = 1_R$ we have
$$\Big(aF(y) + ay + q_0F(y) + q_0y\Big)a = 0$$
Using (3.16) we have
$$\Big(-aq_0y + ay - q_0^2y + q_0y\Big)a = 0$$
that is
$$\Big(-aq_0 + a - q_0^2 + q_0\Big)ya = 0.$$
Since R is a prime ring then $a = 0$ or $-aq_0 + a - q_0^2 + q_0 = 0$.

- Assume $a = 0$.
  Then by (3.5) of Remark 16 we have $F = d$.
  Then, by Theorem 18, $F = 0$ or F is a homoderivation.

- Assume $-aq_0 + a - q_0^2 + q_0 = 0$

that is
$$a = aq_0 + q_0^2 - q_0 \tag{3.18}$$
By (3.16) we have
$$\Big(F(x) + q_0x\Big)a = 0$$

and using (3.18) we get

$$\Big(F(x)+q_0x\Big)(q_0^2-q_0)=0.$$

Since $q_0\neq 0$ then

$$\Big(F(x)+q_0x\Big)(q_0-1)=0 \tag{3.19}$$

that is

$$\Big(d(x)+q_0^2x\Big)(q_0-1)=0. \tag{3.20}$$

By (3.10) we have $\Delta_3 q=\eta\Delta_3$ that is

$$\Big(F(x)-\alpha(x)a\Big)\Big(F(t)+t\Big)(q-\eta)=0$$

that is

$$\Big(F(x)F(t)+F(x)t-\alpha(x)aF(t)-\alpha(x)at\Big)(q-\eta)=0. \tag{3.21}$$

By (3.13) we have

$$\Big(F(xt)-q_0xF(t)-\alpha(x)aF(t)-\alpha(x)at\Big)(q-\eta)=0 \tag{3.22}$$

Replacing t with 1_R we have

$$\Big(F(x)-q_0xa-\alpha(x)a^2-\alpha(x)a\Big)(q-\eta)=0 \tag{3.23}$$

By (3.17) we get

$$\Big(F(x)-q_0xa+\alpha(x)q_0a-\alpha(x)a\Big)(q-\eta)=0$$

that is

$$\Big(F(x)-\eta^{-1}qxa+qxq^{-1}\eta^{-1}qa-qxq^{-1}a\Big)(q-\eta)=0$$

that is

$$\Big(F(x)-qxq^{-1}a\Big)(q-\eta)=0$$

$$\Big(F(x)-\alpha(x)a\Big)(q-\eta)=0 \tag{3.24}$$

Replacing x with xy we have

$$\Big(F(x)F(y)+F(x)y+q_0xF(y)-\alpha(x)\alpha(y)a\Big)(q-\eta)=0 \tag{3.25}$$

Using (3.24) we get

$$\Big(F(x)\alpha(y)a+F(x)y+q_0x\alpha(y)a-\alpha(x)\alpha(y)a\Big)(q-\eta)=0 \tag{3.26}$$

that is

$$\Big(F(x)+q_0x-\alpha(x)\Big)\alpha(y)a(q-\eta)+F(x)y(q-\eta)=0 \tag{3.27}$$

that is

$$\Big(F(x)+q_0x-qxq^{-1}\Big)qyq^{-1}a(q-\eta)+F(x)y(q-\eta)=0$$

Then one of the following possibilities holds:

- $q=\eta$.
  In that case, by (3.11) and (3.13), $F=d$ is a homoderivation.

- $q\neq\eta$.
  In that case, there exists $\theta\in C$ such that

$$q^{-1}a(q-\eta)=\theta(q-\eta) \tag{3.28}$$

Then, by (3.24) we get

$$\begin{aligned}0=&F(x)(q-\eta)-\alpha(x)a(q-\eta)=\\ =&F(x)(q-\eta)-qxq^{-1}a(q-\eta)=\\ =&F(x)(q-\eta)-qx\theta(q-\eta)\end{aligned}$$

that is

$$F(x)(q-\eta)=\theta qx(q-\eta) \tag{3.29}$$

Multiplying by η^{-1} we get

$$F(x)(q\eta^{-1}-1)=\theta qx(q\eta^{-1}-1)$$

$$F(x)(q_0-1)=\theta qx(q_0-1) \tag{3.30}$$

By (3.19) we have $F(x)(q_0-1)=-q_0x(q_0-1)$ that is

$$F(x)(q_0-1)=-\eta^{-1}qx(q_0-1)$$

Then, by (3.19) and (3.30) we have

$$-\eta^{-1}qx(q_0-1)=\theta qx(q_0-1) \tag{3.31}$$

that is

$$\Big(\eta^{-1}+\theta\Big)qx(q_0-1)=0 \tag{3.32}$$

Since $q\neq\eta$ then $q_0\neq 1_R$. Then, since R is a prime ring, by (3.32) we have

$$\Big(\eta^{-1}+\theta\Big)q=0$$

that is

$$\eta^{-1}=-\theta$$

Then by (3.28) we have

$$q^{-1}a(q-\eta)=-\eta^{-1}(q-\eta)$$

that is

$$a(q-\eta) = -\eta^{-1}q(q-\eta)$$
$$a(q-\eta) = -q_0(q-\eta) \tag{3.33}$$

Moreover, by (3.29) we have

$$F(x)(q-\eta) = \theta q x(q-\eta) = -\eta^{-1}qx(q-\eta) = -q_0 x(q-\eta)$$

Starting now from the definition (3.13) of F and using the condition (3.11) we get

$$\begin{aligned} F(xy) =& F(x)F(y) + F(x)y + q_0 x F(y) \\ \eta q^{-1}d(xy) =& \eta q^{-1}d(x)\eta q^{-1}d(y) + \eta q^{-1}d(x)y + q_0 x \eta q^{-1}d(y) \\ q_0^{-1}d(xy) =& q_0^{-1}d(x)q_0^{-1}d(y) + q_0^{-1}d(x)y + q_0 x q_0^{-1}d(y) \end{aligned}$$

that is

$$\begin{aligned} d(xy) =& d(x)q_0^{-1}d(y) + d(x)y + q_0^2 x q_0^{-1}d(y) \\ =& \left(d(x)q_0^{-1} + q_0^2 x q_0^{-1}\right)d(y) + d(x)y \end{aligned} \tag{3.34}$$

Moreover by (3.20) we have $\left(d(x) + q_0^2 x\right)q_0 = d(x) + q_0^2 x$, that is

$$d(x) + q_0^2 x = d(x)q_0^{-1} + q_0^2 x q_0^{-1}$$

Then, by (3.34) we have

$$d(xy) = \left(d(x) + q_0^2 x\right)d(y) + d(x)y$$

that is

$$d(xy) = d(x)d(y) + d(x)y + q_0^2 x d(y)$$

3.1.3 Characterization of generalized skew homoderivations associated with a homoderivation

Theorem 20. *Let R be a prime unital ring and α an automorphism of R.*
Let $d : R \longrightarrow R$ be a homoderivation of R, that is

$$d(xy) = d(x)d(y) + d(x)y + xd(y), \quad \textit{for all } x, y \in R,$$

and let $F : R \longrightarrow R$ be a generalized skew homoderivation associated with α and the homoderivation d, that is

$$F(xy) = F(x)F(y) + F(x)y + \alpha(x)d(y), \quad \textit{for all } x, y \in R.$$

Then one of the following possibilities holds:

- *$F = d$ is a homoderivation;*
- *$d(x) = -x$ and $F(x) = \alpha(x)$;*
- *$\alpha(x) = qxq^{-1}$, $d(x) = -x$ and $F(x) = ax$ with $a = F(1_R)$ and $a^2 = 1_R$;*

- $\alpha(x) = qxq^{-1}$, $\eta \in C$, $q_0 = \eta^{-1}q$, $q_0^2 = 1$, $d(x) = q_0F(x)$ *is a homoderivation and* $F(x) = q_0d(x)$. *Moreover, we have*

$$\Big(d(x) + x\Big)(q_0 - 1) = 0$$

Proof:
By Theorem 19, one of the following possibilities holds:

- $\alpha = id_R$ and $F = d$ is a homoderivation.

- $F(x) = \alpha(x)a$ and

$$d(x) = (1-a)\alpha(x)a - ax = (1-a)F(x) - ax \quad \text{with } a = F(1_R) \tag{3.35}$$

In this case we have

$$d(xy) = (1-a)\alpha(xy)a - axy = (1-a)\alpha(x)F(y) - axy \tag{3.36}$$

Moreover, since d is a homoderivation, then $d(xy) = d(x)d(y) + d(x)y + xd(y)$ and then, using (3.35), we have

$$\begin{aligned}
&d(xy) = \\
&= d(x)d(y) + d(x)y + xd(y) = \\
&= \Big(F(x) - aF(x) - ax\Big)\Big(F(y) - aF(y) - ay\Big) + \Big(F(x) - aF(x) - ax\Big)y + \\
&+ x\Big(F(y) - aF(y) - ay\Big) = \\
&= \Big(F(x)F(y) - F(x)aF(y) - F(x)ay - aF(x)F(y) + aF(x)aF(y) + aF(x)ay - \\
&- axF(y) + axaF(y) + axay\Big) + \Big(F(x) - aF(x) - ax\Big)y + x\Big(F(y) - aF(y) - ay\Big)
\end{aligned} \tag{3.37}$$

By (3.36) and (3.37) we get

$$\begin{aligned}
&(1-a)\alpha(x)F(y) = \\
&= \Big(F(x)F(y) - F(x)aF(y) - F(x)ay - aF(x)F(y) + aF(x)aF(y) + aF(x)ay - \\
&- axF(y) + axaF(y) + axay\Big) + F(x)y - aF(x)y + xF(y) - xaF(y) - xay
\end{aligned}$$

that is

$$\begin{aligned}
&\Big(-\alpha(x) + a\alpha(x) + F(x) - F(x)a - aF(x) + aF(x)a - ax + axa + x - xa\Big)F(y) + \\
&+ \Big(axa + F(x) - aF(x) - xa + aF(x)a - F(x)a\Big)y = 0
\end{aligned}$$

that is

$$\Delta_1(x)y + \Delta_2(x)F(y) = \Delta_1(x)y + \Delta_2(x)\alpha(y)a = 0 \tag{3.38}$$

with

$$\Delta_1(x) = axa + F(x) - aF(x) - xa + aF(x)a - F(x)a;$$
$$\Delta_2(x) = -\alpha(x) + a\alpha(x) + F(x) - F(x)a - aF(x) + aF(x)a - ax + axa + x - xa.$$

$\boxed{\text{I case}}$ If α is outer, by [12, Theorem 3] it follows that R satisfies

$$\Delta_1(x)y + \Delta_2(x)za = 0$$

where x, y and z are distinct indeterminates.
Since R is prime and $a \neq 0$ then $\Delta_1 = \Delta_2 = 0$ that is

$$\Delta_1(x) = axa + \alpha(x)a - a\alpha(x)a - xa + a\alpha(x)a^2 - \alpha(x)a^2 = 0;$$
$$\Delta_2(x) = -\alpha(x) + a\alpha(x) + \alpha(x)a - \alpha(x)a^2 - a\alpha(x)a + a\alpha(x)a^2 - ax + axa + x - xa = 0.$$

Since α is outer then

$$\Gamma_1 = axa + za - aza - xa + aza^2 - za^2 = 0;$$
$$\Gamma_2 = -z + az + za - za^2 - aza + aza^2 - ax + axa + x - xa = 0.$$

Replacing $z = 0$ in Γ_1 we get

$$axa - xa = (a-1)xa = 0$$

and then, since R is a prime ring and $a \neq 0$ we have $a = 1_R$.
Hence by (3.35) we have

$$d(x) = -x, \qquad F(x) = \alpha(x)$$

$\boxed{\text{II case}}$ If $\alpha(x) = qxq^{-1}$ is inner then by (3.38) we have

$$\Delta_1(x)y + \Delta_2(x)qyq^{-1}a = 0 \tag{3.39}$$

Then $q^{-1}a = \theta_0 \in C$ and then $a = \theta_0 q$.
Therefore

$$F(x) = \alpha(x)a = qxq^{-1}a = ax$$

and

$$d(x) = (1-a)F(x) - ax = (1-a)ax - ax = -a^2x$$

Since d is a homoderivation then

$$\begin{aligned} d(xy) &= d(x)d(y) + d(x)y + xd(y) \\ -a^2xy &= a^2xa^2y - a^2xy - xa^2y \\ 0 &= a^2xa^2y - xa^2y \\ 0 &= \left(a^2 - 1\right)xa^2y \end{aligned}$$

Since R is a prime ring then $a^2 = 0$ or $a^2 - 1 = 0$.
If $a^2 = 0$ then $d = 0$ and

$$\begin{aligned} F(xy) =& F(x)F(y) + F(x)y \\ axy =& axay + axy \\ 0 =& axay \end{aligned}$$

Then $a = 0$ and then $F = 0$.
If $a^2 = 1$ then we have

$$F(x) = ax, \qquad d(x) = -a^2x = -x$$

- $\alpha(x) = qxq^{-1}$, $d(x) = q_0F(x)$ and $F(x) = q_0^{-1}d(x)$, with $q_0 = \eta^{-1}q$ and $\eta \in C$,

  $$F(xy) = F(x)F(y) + F(x)y + q_0xF(y), \qquad d(xy) = d(x)d(y) + d(x)y + q_0^2xd(y), \tag{3.40}$$

  and

  $$\Big(F(x) + q_0x\Big)(1 - q_0) = 0, \qquad \Big(d(x) + q_0^2x\Big)(q_0 - 1) = 0 \tag{3.41}$$

  with $q_0 = \eta^{-1}q$.

Since d is a homoderivation then, by (3.40), we have

$$d(x)d(y) + d(x)y + q_0^2xd(y) = d(x)d(y) + d(x)y + xd(y)$$

that is

$$0 = (1 - q_0^2)xd(y)$$

By Remark 10 we have

$$0 = (1 - q_0^2)x \tag{3.42}$$

that is $q_0^2 = 1$ and we have that $d(xy) = d(x)d(y) + d(x)y + xd(y)$.
Hence $q_0 = q_0^{-1}$ and then

$$F(x) = q_0^{-1}d(x) = q_0d(x).$$

Moreover, by (3.41) we get

$$\Big(d(x) + x\Big)(q_0 - 1) = 0$$

3.1.4 Posner's theorems

Lemma 9. *Let R be a prime unital ring of characteristic different from 2, α an automorphism of R and $F : R \longrightarrow R$ a generalized skew homoderivation associated with α and the additive map $d : R \longrightarrow R$.*
If $F^2(x) = 0$ for all $x \in R$ then one of the following possibilities holds:

- $F(x) = d(x) = 0$;
- $F(x) = \alpha(x)a$, $\alpha(a)a = 0$.

Proof:
By Theorem 19, we have the following cases:

- $F(x) = d(x) = 0$ for all $x \in R$.
- $F = d$ is a homoderivation.
  Then by Lemma 1 we have $F = d = 0$.
- $F(x) = \alpha(x)a$ and $d(x) = (1-a)\alpha(x)a - ax$ for all $x \in R$ with $a = F(1_R)$.
  Then $0 = F^2(x) = \alpha(\alpha(x)a)a = \alpha^2(x)\alpha(a)a$ for all $x \in R$ that is $\alpha(a)a = 0$.
- $\alpha(x) = qxq^{-1}$, $F(x) = \eta q^{-1}d(x)$, $d(x) = q_0F(x)$ with $q_0 = \eta^{-1}q$, $\eta \in C$,

  $$F(xy) = F(x)F(y) + F(x)y + q_0xF(y), \qquad d(xy) = d(x)d(y) + d(x)y + q_0^2xd(y),$$

  $$\Big(F(x) + q_0x\Big)(1-q_0) = 0. \tag{3.43}$$

  and

  $$\Big(d(x) + q_0^2x\Big)(1-q_0) = 0.$$

  If $F^2(x) = 0$ then by (3.43) we have

  $$0 = \Big(F^2(x) + q_0F(x)\Big)(1-q_0) = q_0F(x)(1-q_0)$$

  that is

  $$d(x)(1-q_0) = 0 \tag{3.44}$$

  Then

  $$F(x)(1-q_0) = \eta q^{-1}d(x)(1-q_0) = 0 \tag{3.45}$$

  Moreover, multiplying the relation $F(xy) = F(x)F(y) + F(x)y + \alpha(x)d(y)$ on the right by $(1-q_0)$ and using (3.44) and (3.45) we have

  $$\begin{aligned} F(xy)(1-q_0) =& F(x)F(y)(1-q_0) + F(x)y(1-q_0) + \alpha(x)d(y)(1-q_0) \\ 0 =& F(x)y(1-q_0) \end{aligned}$$

  Since R is a prime ring then $F = 0$ or $q_0 = 1$.
  If $q_0 = 1$ then $q = \eta \in C$ and we have $F(x) = d(x)$. Then, by Theorem 18, $F = d = 0$ or F is a homoderivation.
  If F is a homoderivation, by Lemma 1 we have $F = 0$.

Theorem 21. *Let R be a prime unital ring of characteristic different from 2, α an automorphism of R and $F : R \longrightarrow R$ a generalized skew homoderivation associated with α and the additive map $d : R \longrightarrow R$.*
If $[F(x), x] = 0$ for all $x \in R$ then one of the following possibilities holds:

- *$F(x) = d(x) = 0$;*
- *$F(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism;*
- *$F(x) = xa$ and $d(x) = -a^2x$, $a \in C$.*

Proof:
By Theorem 19, we have the following cases:

- $F(x) = d(x) = 0$ for all $x \in R$.

- $F = d$ is a homoderivation.
  Then by Theorem 2 we have $F = d = 0$ or $F(x) = d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism..

- $F(x) = \alpha(x)a$ and $d(x) = (1-a)\alpha(x)a - ax$ for all $x \in R$ with $a = F(1_R)$.
  Then $0 = [F(x), x] = [\alpha(x)a, x]$ for all $x \in R$. Replacing x with $x+1$ we have $[a, x] = 0$, that is $a \in C$. Then $[\alpha(x), x] = 0$ and by [28] we have $\alpha = id_R$. Then $F(x) = xa$ and $d(x) = -a^2 x$.

- $\alpha(x) = qxq^{-1}$, $F(x) = \eta q^{-1} d(x)$, $d(x) = q_0 F(x)$ with $q_0 = \eta^{-1} q$, $\eta \in C$, and

$$\Big(F(x) + q_0 x\Big)(1 - q_0) = 0. \tag{3.46}$$

If $[F(x), x] = 0$ for all $x \in R$ then replacing x with $x + \gamma$, $\gamma \in C$, we have $[F(\gamma), x] = 0$ that is

$$F(\gamma) \in C \text{ for all } \gamma \in C. \tag{3.47}$$

By (3.46) we have

$$F(x)(1 - q_0) = -q_0 x(1 - q_0). \tag{3.48}$$

Assume $q_0 \neq 1$.
Multiplying $[F(x), x] = 0$ by $1 - q_0$ on the right and using (3.48) we have

$$\begin{aligned} 0 =& F(x)x(1-q_0) - xF(x)(1-q_0) = \\ =& F(x)x(1-q_0) + xq_0x(1-q_0) = \\ =& \Big(F(x) + xq_0\Big)x(1-q_0) \end{aligned} \tag{3.49}$$

Replacing x with $x+1$ and using (3.49) we have

$$\begin{aligned} 0 =& \Big((F(x) + xq_0) + (F(1_R) + q_0)\Big)x(1-q_0) + \Big((F(x) + xq_0) + \\ &+ (F(1_R) + q_0)\Big)(1-q_0) = \\ =& \Big(F(1_R) + q_0\Big)x(1-q_0) \end{aligned} \tag{3.50}$$

Since R is a prime ring and $q_0 \neq 1$ then $q_0 = -F(1_R)$.
If $q_0 = -F(1_R)$, by (3.47) we have $q_0 \in C$ and by (3.46) we get

$$F(x) = -q_0 x = F(1_R)x, \quad F(1_R) \in C,$$

and

$$d(x) = -F(1_R)^2 x, \quad F(1_R) \in C.$$

Assume $q_0 = 1$.
Then $F(x) = d(x)$ and by Theorem 18 we have $F = d = 0$ or F is a homoderivation.
If F is a homoderivation, by Theorem 2 we have $F = 0$ or $F(x) = d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism.

Chapter 4

Extended generalized skew homoderivation

The following paragraphs are devoted to the broadest class of mappings considered in this dissertation, referred to as *extended generalized skew homoderivations*.
These mappings satisfy the functional equation

$$F(xy) = \mu F(x)F(y) + F(x)y + \alpha(x)d(y),$$

where μ belongs to the center C, α is an automorphism of the ring, and d denotes either an additive map or, in particular cases, an extended homoderivation. The structure of the discussion mirrors that of earlier sections, placing special emphasis on the interplay between extension and skewness. A complete classification of these mappings is provided, addressing both the general additive case and the more specific scenario in which d is an extended homoderivation. These results serve to extend the classical structural theorems within a highly generalized and non-commutative context.

4.1 Extended skew homoderivation

Definition 8. *Let R be an associative ring, $\lambda, 0 \neq \mu \in C$, α an automorphism of R and $d : R \longrightarrow Q_r$ an additive map. d is called extended skew homoderivation associated with λ, μ and α if*

$$d(xy) = \lambda d(x)d(y) + \mu d(x)y + \mu\alpha(x)d(y) \qquad \text{for all } x, y \in R.$$

Theorem 22. *Let R be a semiprime ring, $\lambda, 0 \neq \mu \in C$ and α an automorphism of R.
If $d : R \longrightarrow Q_r$ is an extended skew homoderivation associated with λ, μ and α, then one of the following possibilities holds:*

- $\mu = 1_R$;
- $\mu \neq 1_R$, $\lambda \neq 0$, $d(x) = \lambda^{-1}(1 - 2\mu)x$ *and* $\alpha(x) = x$ *for all* $x \in R$;
- $\mu \neq 1_R$, $\lambda = 0$, $2\mu = 1_R$ *and* $d(x) = \alpha(x)d(1_R)$ *with* $\alpha(x)d(1_R) = d(1_R)x$.

Proof:
For all $x, y, z \in R$ we have

$$
\begin{aligned}
&d((xy)z) = \\
&= \lambda d(xy)d(z) + \mu d(xy)z + \mu\alpha(xy)d(z) = \\
&= \lambda\Big(\lambda d(x)d(y) + \mu d(x)y + \mu\alpha(x)d(y)\Big)d(z) + \mu\Big(\lambda d(x)d(y) + \mu d(x)y + \mu\alpha(x)d(y)\Big)z + \\
&\quad + \mu\alpha(x)\alpha(y)d(z) = \\
&= \lambda^2 d(x)d(y)d(z) + \lambda\mu d(x)yd(z) + \lambda\mu\alpha(x)d(y)d(z) + \lambda\mu d(x)d(y)z + \mu^2 d(x)yz + \\
&\quad + \mu^2\alpha(x)d(y)z + \mu\alpha(x)\alpha(y)d(z)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&d(x(yz)) = \\
&= \lambda d(x)d(yz) + \mu d(x)yz + \mu\alpha(x)d(yz) = \\
&= \lambda d(x)\Big(\lambda d(y)d(z) + \mu d(y)z + \mu\alpha(y)d(z)\Big) + \mu d(x)yz + \mu\alpha(x)\Big(\lambda d(y)d(z) + \\
&\quad + \mu d(y)z + \mu\alpha(y)d(z)\Big) = \\
&= \lambda^2 d(x)d(y)d(z) + \lambda\mu d(x)d(y)z + \lambda\mu d(x)\alpha(y)d(z) + \mu d(x)yz + \mu\alpha(x)\lambda d(y)d(z) + \\
&\quad + \mu^2\alpha(x)d(y)z + \mu^2\alpha(x)\alpha(y)d(z)
\end{aligned}
$$

Comparing the previous conditions it follows that

$$\mu\Big[\lambda d(x)\Big(y - \alpha(y)\Big) - \Big(\mu - 1\Big)\alpha(xy)\Big]d(z) + \mu\Big(\mu - 1\Big)d(x)yz = 0$$

Since $\mu \neq 0$ then

$$\Big[\lambda d(x)\Big(y - \alpha(y)\Big) - \Big(\mu - 1\Big)\alpha(xy)\Big]d(z) + \Big(\mu - 1\Big)d(x)yz = 0$$

For $x = y = 1_R$ we have

$$-\Big(\mu - 1\Big)d(z) + \Big(\mu - 1\Big)d(1_R)z = 0$$

that is

$$\Big(\mu - 1\Big)\Big(-d(z) + d(1_R)z\Big) = 0$$

then $\mu = 1$ or $d(z) = d(1_R)z$ for all $z \in R$.
Assume that

$$d(x) = d(1_R)x \quad \text{for all } x \in R. \tag{4.1}$$

Replacing, in the definition of d, x with 1_R and using (4.1) we obtain

$$
\begin{aligned}
d(y) &= \lambda d(1_R)d(y) + \mu d(1_R)y + \mu d(y) \\
d(y) &= \lambda d(1_R)d(y) + \mu d(y) + \mu d(y)
\end{aligned}
$$

that is

$$\Big(1-2\mu-\lambda d(1_R)\Big)d(y)=0 \tag{4.2}$$

Note that if $ad(x)=0$ for all $x\in R$, with $a\in Q_r$ then $a=0$ or $d=0$.
Then by (4.2) we have $d=0$ or $1-2\mu-\lambda d(1_R)=0$ that is $\lambda d(1_R)=1-2\mu\in C$.
If $\lambda=0$ then $2\mu=1$.
Moreover, replacing in the definition of d, x with $2x$ and y with 1_R we obtain

$$d(2x)=2\mu d(x)+2\mu\alpha(x)d(1_R)$$

Since $2\mu=1$ then

$$d(2x)=d(x)+\alpha(x)d(1_R)$$

that is

$$d(x)=\alpha(x)d(1_R)$$

If $\lambda\neq 0$ then we have $d(x)=d(1_R)x=\lambda^{-1}(1-2\mu)x$ and

$$\begin{aligned}
d(xy)=&\lambda d(x)d(y)+\mu d(x)y+\mu\alpha(x)d(y)\\
\lambda^{-1}(1-2\mu)xy=&\lambda\lambda^{-1}(1-2\mu)x\lambda^{-1}(1-2\mu)y+\mu\lambda^{-1}(1-2\mu)xy+\mu\alpha(x)\lambda^{-1}(1-2\mu)y\\
x=&x(1-2\mu)+\mu x+\mu\alpha(x)\\
0=&-\mu x+\mu\alpha(x)\\
\alpha(x)=&x
\end{aligned}$$

Theorem 23. *Let R be a prime ring, $\lambda\in C$ and α an automorphism of R.
If $d:R\longrightarrow Q_r$ is an additive map such that*

$$d(xy)=\lambda d(x)d(y)+d(x)y+\alpha(x)d(y)\qquad \textit{for all } x,y\in R$$

then $d=0$ or d is a skew derivation or d is an extended homoderivation.

Proof:
For all $x,y,z\in R$ we have

$$\begin{aligned}
d((xy)z)=&\lambda d(xy)d(z)+d(xy)z+\alpha(xy)d(z)=\\
=&\lambda\Big(\lambda d(x)d(y)+d(x)y+\alpha(x)d(y)\Big)d(z)+\Big(\lambda d(x)d(y)+d(x)y+\alpha(x)d(y)\Big)z+\\
&+\alpha(x)\alpha(y)d(z)=\\
=&\lambda^2d(x)d(y)d(z)+\lambda d(x)yd(z)+\lambda\alpha(x)d(y)d(z)+\lambda d(x)d(y)z+d(x)yz+\\
&+\alpha(x)d(y)z+\alpha(x)\alpha(y)d(z)
\end{aligned}$$

Moreover

$$\begin{aligned}
d(x(yz))=&\lambda d(x)d(yz)+d(x)yz+\alpha(x)d(yz)=\\
=&\lambda d(x)\Big(\lambda d(y)d(z)+d(y)z+\alpha(y)d(z)\Big)+d(x)yz+\\
&+\alpha(x)\Big(\lambda d(y)d(z)+d(y)z+\alpha(y)d(z)\Big)=\\
=&\lambda^2d(x)d(y)d(z)+\lambda d(x)d(y)z+\lambda d(x)\alpha(y)d(z)+d(x)yz+\alpha(x)\lambda d(y)d(z)+\\
&+\alpha(x)d(y)z+\alpha(x)\alpha(y)d(z)
\end{aligned}$$

Comparing the previous conditions it follows that

$$\lambda d(x)\Big(y-\alpha(y)\Big)d(z)=0 \tag{4.3}$$

If $\lambda=0$ then d is a skew derivation.
Suppose $\lambda\neq 0$.
Replacing x with xt in (4.3) we have

$$\lambda\Big(\lambda d(x)d(t)+d(x)t+\alpha(x)d(t)\Big)\Big(y-\alpha(y)\Big)d(z)=0$$

that is, using (4.3),

$$d(x)t\Big(y-\alpha(y)\Big)d(z)=0$$

Since R is a prime ring then $d=0$ or $\Big(y-\alpha(y)\Big)d(z)=0$, that is $d=0$ or $\alpha(y)d(z)=yd(z)$ for all $y,z\in R$.
If $\alpha(y)d(z)=yd(z)$ for all $y,z\in R$ then

$$d(xy)=\lambda d(x)d(y)+d(x)y+\alpha(x)d(y)=\lambda d(x)d(y)+d(x)y+xd(y)$$

that is d is an extended homoderivation.

4.2 Extended generalized skew homoderivation

Definition 9. *Let R be an associative ring, $\mu\in C$, α an automorphism of R and $d:R\longrightarrow Q_r$ an additive map. An additive map $F:R\longrightarrow Q_r$ is called extended generalized skew homoderivation associated with d, μ and α if*

$$F(xy)=\mu F(x)F(y)+F(x)y+\alpha(x)d(y)\qquad \textit{for all } x,y\in R$$

Remark 18. *Let R be a prime unital ring, $\mu\in C$, α an automorphism of R, $d:R\longrightarrow Q_r$ an additive map and $F:R\longrightarrow R$ an extended generalized skew homoderivation associated with μ, α and the additive map d.*
If $\mu=0$ then F is a generalized derivation associated with the derivation d.
Assume $\mu\neq 0$.
Observe that if R is a unital ring then by replacing, in the definition of F, x with 1_R we obtain

$$F(y)=\mu F(1_R)F(y)+F(1_R)y+d(y)$$

that is

$$d(y)=(1-\mu a)F(y)-ay,\qquad \textit{with } a=F(1_R), \tag{4.4}$$

and in particular

$$F(y)-d(y)=a\Big(\mu F(y)+y\Big),\qquad \textit{with } a=F(1_R). \tag{4.5}$$

Moreover, replacing, in the definition of F, y with 1_R we obtain

$$F(x)=\mu F(x)F(1_R)+F(x)+\alpha(x)d(1_R)$$

that is

$$0 = \mu F(x)a + \alpha(x)d(1_R), \qquad \text{with } a = F(1_R). \tag{4.6}$$

In particular

$$0 = \mu a^2 + d(1_R), \qquad \text{with } a = F(1_R). \tag{4.7}$$

Since $\mu \neq 0$, by (4.6) and (4.7) we have

$$F(x)a = \alpha(x)a^2, \qquad \text{with } a = F(1_R).$$

Remark 19. *Let R be a prime unital ring, $\mu \in C$, α an automorphism of R, $F : R \longrightarrow Q_r$ an extended generalized skew homoderivation associated with α, μ and the additive map $d : R \longrightarrow Q_r$ and $c \in R$ a fixed element.*
If $cF(x) = 0$ for all $x \in R$, then $c = 0$ or $d = 0$.

Proof:
Since $cF(x) = 0$ for all $x, y \in R$, then

$$0 = cF(xy) = c\mu F(x)F(y) + cF(x)y + c\alpha(x)d(y) = c\alpha(x)d(y)$$

Since R is a prime ring, then $c = 0$ or $d = 0$.

4.2.1 Example

Example 18. *Let $q \in Q_r$ be such that $q^2 = \lambda^2 \in C$ and let $\alpha(x) = qxq^{-1}$ be an automorphism of R.*
Then $F(x) = -q^{-1}x$ is an extended generalized skew homoderivation associated with $\lambda \in C$ and $d(x) = -\lambda^{-1}x$.

Example 19. *Let α be an automorphism of R and $d(x) = -2\mu^{-1}\alpha(x) + \mu^{-1}x$.*
Then $F(x) = -\mu^{-1}\alpha(x)$ is an extended generalized skew homoderivation associated with $0 \neq \mu \in C$, α and the additive map d.

4.2.2 Characterization of extended generalized skew homoderivations associated with an additive map

Theorem 24. *Let R be a prime unital ring, $\mu \in C$, α an automorphism of R, $d : R \longrightarrow Q_r$ an additive map and $F : R \longrightarrow Q_r$ an extended generalized skew homoderivation associated with $0 \neq \mu$, α and the additive map d. Then one of the following possibilities holds:*

- *$\alpha = id_R$ and $F = d$ is an extended homoderivation;*
- *$F(x) = \alpha(x)a$ and $d(x) = (1 - \mu a)\alpha(x)a - ax$ with $a = F(1_R)$;*
- *$\alpha(x) = qxq^{-1}$, $F(x) = q_0^{-1}d(x)$, with $q_0 = \eta^{-1}q$ and $\eta \in C$,*

  $$F(xy) = \mu F(x)F(y) + F(x)y + q_0xF(y), \qquad d(xy) = \mu d(x)d(y) + d(x)y + q_0^2xd(y),$$

  and

  $$\Big(\mu F(x) + q_0x\Big)(1 - q_0) = 0, \qquad \Big(d(x) + \mu^{-1}q_0^2x\Big)(q_0 - 1) = 0$$

  with $q_0 = \eta^{-1}q$.

Proof:
For all $x, y, z \in R$ we have:

$$
\begin{aligned}
&F((xy)z) = \\
&\mu F(xy)F(z) + F(xy)z + \alpha(xy)d(z) = \\
&\mu^2 F(x)F(y)F(z) + \mu F(x)yF(z) + \mu\alpha(x)d(y)F(z) + \mu F(x)F(y)z + F(x)yz + \alpha(x)d(y)z + \\
&+ \alpha(x)\alpha(y)d(z)
\end{aligned}
$$

Moreover

$$
\begin{aligned}
&F(x(yz)) = \\
&\mu F(x)F(yz) + F(x)yz + \alpha(x)d(yz) = \\
&\mu^2 F(x)F(y)F(z) + \mu F(x)F(y)z + \mu F(x)\alpha(y)d(z) + F(x)yz + \alpha(x)d(yz)
\end{aligned}
$$

Subtracting member from member we have

$$
\begin{aligned}
&\mu F(x)yF(z) + \mu\alpha(x)d(y)F(z) + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \\
&- \mu F(x)\alpha(y)d(z) - \alpha(x)d(yz) = 0
\end{aligned}
$$

that is

$$F(x)\Delta_1 + \alpha(x)\Delta_2 = 0 \tag{4.8}$$

where $\Delta_1 = \mu yF(z) - \mu\alpha(y)d(z)$ and $\Delta_2 = \mu d(y)F(z) + d(y)z + \alpha(y)d(z) - d(yz)$.
Replacing x with xt we have

$$\mu F(x)F(t)\Delta_1 + F(x)t\Delta_1 + \alpha(x)d(t)\Delta_1 + \alpha(x)\alpha(t)\Delta_2 = 0$$

and using (4.8) we get

$$\mu F(x)F(t)\Delta_1 + F(x)t\Delta_1 + \alpha(x)d(t)\Delta_1 - \alpha(x)F(t)\Delta_1 = 0$$

that is

$$F(x)\Big(\mu F(t) + t\Big)\Delta_1 + \alpha(x)\Big(d(t) - F(t)\Big)\Delta_1 = 0$$

and using (4.5) of Remark 18 it follows that

$$F(x)\Big(\mu F(t) + t\Big)\Delta_1 - \alpha(x)a\Big(\mu F(t) + t\Big)\Delta_1 = 0, \qquad \text{with } a = F(1_R)$$

that is

$$\Delta_3\Big(\mu yF(z) - \mu\alpha(y)d(z)\Big) = 0$$

where $\Delta_3 = \Big(F(x) - \alpha(x)a\Big)\Big(\mu F(t) + t\Big)$ and $\Delta_1 = \Big(\mu yF(z) - \mu\alpha(y)d(z)\Big)$.
Fix x and t in Δ_3.
Therefore

$$\Delta_3 yF(z) - \Delta_3\alpha(y)d(z) = 0 \qquad \text{for all } y, z \in R. \tag{4.9}$$

If α is outer, by [12, Theorem 3] it follows that R satisfies

$$\Delta_3 yF(z) - \Delta_3 y'd(z) = 0$$

where y, z and y' are distinct indeterminates.
Since R is prime then $\Delta_3 = 0$ or $F(z) = d(z) = 0$.
If α is inner then there exists an invertible element $q \in R$ such that $\alpha(x) = qxq^{-1}$ for all $x \in R$ and by (4.9) it follows that R satisfies

$$\Delta_3 yF(z) - \Delta_3 qyq^{-1}d(z) = 0.$$

Then one of the following possibilities holds:

- $\Delta_3 = 0$;
- $F(z) = d(z) = 0$ for all $z \in R$;
- $\Delta_3 q = \eta\Delta_3$ and $F(z) = \eta q^{-1}d(z)$ for all $z \in R$ with $\eta = \eta(x,t)$ and $\alpha(x) = qxq^{-1}$.

Then:

- If $\Delta_3 = 0$ then

$$\Big(F(x) - \alpha(x)a\Big)\Big(\mu F(t) + t\Big) = 0, \qquad \text{for all } x, t \in R, \text{ with } a = F(1_R)$$

that is

$$F(x)\Big(\mu F(t) + t\Big) - \alpha(x)a\Big(\mu F(t) + t\Big) = 0$$

and using (4.5) of Remark 18 it follows that

$$F(x)\Big(\mu F(t) + t\Big) - \alpha(x)\Big(F(t) - d(t)\Big) = 0$$

$$\mu F(x)F(t) + F(x)t - \alpha(x)F(t) + \alpha(x)d(t) = 0$$

that is

$$F(xt) - \alpha(x)F(t) = 0.$$

Therefore

$$F(x) = \alpha(x)a, \qquad \text{for all } x \in R, \text{ with } a = F(1_R)$$

and using (4.4) of Remark 18 we have

$$d(x) = (1 - \mu a)\alpha(x)a - ax, \qquad \text{with } a = F(1_R).$$

- Assume

$$\Delta_3 q = \eta\Delta_3 \tag{4.10}$$

and

$$F(x) = \eta q^{-1}d(x), \tag{4.11}$$

with $\eta = \eta(x,t)$ and $\alpha(x) = qxq^{-1}$.

Then

$$d(x) = \eta^{-1}qF(x) = q_0F(x), \qquad \text{with } q_0 = \eta^{-1}q. \tag{4.12}$$

Then, for all $x, y \in R$, we have

$$\alpha(x)d(y) = qxq^{-1}\eta^{-1}qF(y) = \eta^{-1}qxF(y) = q_0xF(y)$$

and

$$\begin{aligned} F(xy) =&\mu F(x)F(y) + F(x)y + \alpha(x)d(y) = \\ =&\mu F(x)F(y) + F(x)y + q_0 xF(y) \qquad \text{with } q_0 = \eta^{-1}q. \end{aligned} \tag{4.13}$$

In particular

$$\alpha(x)d(1_R) = \eta^{-1}qxF(1_R) = \eta^{-1}qxa \tag{4.14}$$

that is

$$qxq^{-1}d(1_R) = qx\eta^{-1}a$$

and therefore

$$q^{-1}d(1_R) = \eta^{-1}a.$$

By (4.6) of Remark 18 we have

$$\mu F(x)a = -\alpha(x)d(1_R).$$

Using (4.14) we get

$$\mu F(x)a = -\eta^{-1}qxa$$

that is

$$\mu\eta q^{-1}d(x)a = -\eta^{-1}qxa.$$

Then

$$\mu d(x)a = -(\eta^{-1}q)^2 xa = -q_0^2 xa$$

that is

$$\left(\mu d(x) + q_0^2 x\right)a = 0$$

that is

$$\left(d(x) + \mu^{-1}q_0^2 x\right)a = 0 \tag{4.15}$$

and

$$\left(q_0 F(x) + \mu^{-1}q_0^2 x\right)a = 0$$

Since $q_0 \neq 0$ then

$$\left(F(x) + \mu^{-1}q_0 x\right)a = 0$$

that is

$$\left(\mu F(x) + q_0 x\right)a = 0 \tag{4.16}$$

In particular, for $x = 1_R$ we have

$$\mu a^2 + q_0 a = 0 \tag{4.17}$$

Replacing x with xy in (4.16) we get

$$\left(\mu F(xy) + q_0 xy\right)a = 0$$

that is, using (4.13),

$$\Big(\mu^2 F(x)F(y) + \mu F(x)y + \mu q_0 x F(y) + q_0 xy\Big)a = 0$$

and for $x = 1_R$ we have

$$\Big(\mu^2 aF(y) + \mu ay + \mu q_0 F(y) + q_0 y\Big)a = 0$$

Using (4.16) we have

$$\Big(-\mu a q_0 y + \mu ay - q_0^2 y + q_0 y\Big)a = 0$$

that is

$$\Big(-\mu a q_0 + \mu a - q_0^2 + q_0\Big)ya = 0$$

Since R is a prime ring then $a = 0$ or $-\mu a q_0 + \mu a - q_0^2 + q_0 = 0$.

- Assume $a = 0$.
  Then by (4.5) of Remark 18 we have $F = d$.
  Then by Theorem 23, $F = 0$ or F is an extended homoderivation.

- Assume $-\mu a q_0 + \mu a - q_0^2 + q_0 = 0$

that is

$$\mu a = \mu a q_0 + q_0^2 - q_0 \tag{4.18}$$

Then by (4.16) we have

$$\begin{aligned}\Big(\mu F(x) + q_0 x\Big)a &= 0\\ \Big(\mu F(x) + q_0 x\Big)\mu a &= 0\\ \Big(\mu F(x) + q_0 x\Big)(q_0^2 - q_0) &= 0\end{aligned}$$

Since $q_0 \neq 0$ then

$$\Big(\mu F(x) + q_0 x\Big)(q_0 - 1) = 0 \tag{4.19}$$

that is

$$\Big(F(x) + \mu^{-1} q_0 x\Big)(q_0 - 1) = 0$$

that is

$$\Big(d(x) + \mu^{-1} q_0^2 x\Big)(q_0 - 1) = 0 \tag{4.20}$$

By (4.10) we have $\Delta_3 q = \eta \Delta_3$, that is

$$\Big(F(x) - \alpha(x)a\Big)\Big(\mu F(t) + t\Big)(q - \eta) = 0$$

that is
$$\Big(\mu F(x)F(t)+F(x)t-\mu\alpha(x)aF(t)-\alpha(x)at\Big)(q-\eta)=0. \tag{4.21}$$
By (4.13) we have
$$\Big(F(xt)-q_0xF(t)-\mu\alpha(x)aF(t)-\alpha(x)at\Big)(q-\eta)=0 \tag{4.22}$$
Replacing t with 1_R we have
$$\Big(F(x)-q_0xa-\mu\alpha(x)a^2-\alpha(x)a\Big)(q-\eta)=0 \tag{4.23}$$
By (4.17) we get
$$\Big(F(x)-q_0xa+\alpha(x)q_0a-\alpha(x)a\Big)(q-\eta)=0$$
that is
$$\Big(F(x)-\eta^{-1}qxa+qxq^{-1}\eta^{-1}qa-qxq^{-1}a\Big)(q-\eta)=0$$
that is
$$\Big(F(x)-qxq^{-1}a\Big)(q-\eta)=0$$
$$\Big(F(x)-\alpha(x)a\Big)(q-\eta)=0 \tag{4.24}$$
Replacing x with xy we have
$$\Big(\mu F(x)F(y)+F(x)y+q_0xF(y)-\alpha(x)\alpha(y)a\Big)(q-\eta)=0 \tag{4.25}$$
Using (4.24) we get
$$\Big(\mu F(x)\alpha(y)a+F(x)y+q_0x\alpha(y)a-\alpha(x)\alpha(y)a\Big)(q-\eta)=0 \tag{4.26}$$
that is
$$\Big(\mu F(x)+q_0x-\alpha(x)\Big)\alpha(y)a(q-\eta)+F(x)y(q-\eta)=0 \tag{4.27}$$
that is
$$\Big(\mu F(x)+q_0x-qxq^{-1}\Big)qyq^{-1}a(q-\eta)+F(x)y(q-\eta)=0$$
Then one of the following possibilities holds:

- $q=\eta$.
  Then $F=d$ is an extended homoderivation.
- $q\neq\eta$.
  In that case, by (4.27), there exists $\theta\in C$ such that
$$q^{-1}a(q-\eta)=\theta(q-\eta) \tag{4.28}$$

Then, by (4.24) we get

$$\begin{aligned}0 =&F(x)(q-\eta) - \alpha(x)a(q-\eta) = \\ =&F(x)(q-\eta) - qxq^{-1}a(q-\eta) = \\ =&F(x)(q-\eta) - qx\theta(q-\eta)\end{aligned}$$

that is

$$F(x)(q-\eta) = qx(q-\eta)\theta \tag{4.29}$$

Multiplying by η^{-1} we get

$$F(x)(q\eta^{-1}-1) = \theta qx(q\eta^{-1}-1)$$

$$F(x)(q_0-1) = \theta qx(q_0-1) \tag{4.30}$$

By (4.19) we have $F(x)(q_0-1) = -\mu^{-1}q_0x(q_0-1)$ that is

$$F(x)(q_0-1) = -\mu^{-1}\eta^{-1}qx(q_0-1)$$

Then, by (4.19) and (4.30) we have

$$-\mu^{-1}\eta^{-1}qx(q_0-1) = \theta qx(q_0-1) \tag{4.31}$$

that is

$$\Big(\mu^{-1}\eta^{-1}+\theta\Big)qx(q_0-1) = 0 \tag{4.32}$$

Since $q \neq \eta$ then $q_0 \neq 1_R$. Then, since R is a prime ring, by (4.32) we have

$$\Big(\mu^{-1}\eta^{-1}+\theta\Big)q = 0$$

that is

$$\mu^{-1}\eta^{-1} = -\theta$$

Then by (4.28) we have

$$q^{-1}a(q-\eta) = -\mu^{-1}\eta^{-1}(q-\eta)$$

that is

$$a(q-\eta) = -\mu^{-1}\eta^{-1}q(q-\eta)$$

$$a(q-\eta) = -\mu^{-1}q_0(q-\eta)$$

Moreover, by (4.29) we have

$$\begin{aligned}&F(x)(q-\eta) = \\ &\theta qx(q-\eta) = \\ &-\mu^{-1}\eta^{-1}qx(q-\eta) = \\ &-\mu^{-1}q_0x(q-\eta)\end{aligned}$$

Starting now from the definition (4.13) of F and using the condition (4.11) we get

$$\begin{aligned} F(xy) =& \mu F(x)F(y) + F(x)y + q_0xF(y) \\ \eta q^{-1}d(xy) =& \mu\eta q^{-1}d(x)\eta q^{-1}d(y) + \eta q^{-1}d(x)y + q_0x\eta q^{-1}d(y) \\ q_0^{-1}d(xy) =& \mu q_0^{-1}d(x)q_0^{-1}d(y) + q_0^{-1}d(x)y + q_0xq_0^{-1}d(y) \end{aligned}$$

that is

$$\begin{aligned} d(xy) =& \mu d(x)q_0^{-1}d(y) + d(x)y + q_0^2xq_0^{-1}d(y) \\ =& \left(\mu d(x)q_0^{-1} + q_0^2xq_0^{-1}\right)d(y) + d(x)y \end{aligned} \tag{4.33}$$

Moreover by (4.20) we have $\left(d(x) + \mu^{-1}q_0^2x\right)q_0 = d(x) + \mu^{-1}q_0^2x$, that is

$$d(x) + \mu^{-1}q_0^2x = d(x)q_0^{-1} + q_0^2\mu^{-1}xq_0^{-1}$$

$$\mu d(x) + q_0^2x = \mu d(x)q_0^{-1} + q_0^2xq_0^{-1}$$

Then, by (4.33) we have

$$d(xy) = \left(\mu d(x) + q_0^2x\right)d(y) + d(x)y$$

that is

$$d(xy) = \mu d(x)d(y) + d(x)y + q_0^2xd(y)$$

4.2.3 Characterization of extended generalized skew homoderivations associated with an extended homoderivation

Theorem 25. *Let R be a prime unital ring, $0 \neq \lambda, 0 \neq \mu \in C$ and α an automorphism of R.*
Let $d : R \longrightarrow Q_r$ be an extended homoderivation associated with λ, that is

$$d(xy) = \lambda d(x)d(y) + d(x)y + xd(y), \quad \text{for all } x, y \in R,$$

and let $F : R \longrightarrow Q_r$ be an extended generalized skew homoderivation associated with μ, α and the extended homoderivation d, that is

$$F(xy) = \mu F(x)F(y) + F(x)y + \alpha(x)d(y), \quad \text{for all } x, y \in R.$$

Then one of the following possibilities holds:

- *$F = d$ is an extended homoderivation;*
- *$d(x) = -\mu^{-1}x$ is an extended homoderivation associated with μ;*
  $F(x) = \mu^{-1}\alpha(x)$;
- *$d(x) = -\lambda^{-1}x$ is an extended homoderivation associated with λ;*
  $F(x) = ax$ with $a = F(1_R)$ and $a^2 = \mu^{-1}\lambda^{-1} \in Z(R)$;
- *$\alpha(x) = qxq^{-1}$, $\eta \in C$, $q_0 = \eta^{-1}q$, $q_0^2 = 1$;*
  $d(x) = q_0F(x)$ is an extended homoderivation associated with μ;
  $F(x) = q_0d(x)$;

- $\alpha(x) = qxq^{-1}$, $\eta \in C$, $\lambda q^2 = \eta^2\mu$, $q_0 = \eta^{-1}q$;
  $d(x) = -\mu^{-1}q_0^2 x = -\lambda^{-1}x$ *is an extended homoderivation associated with* $\lambda = \mu q_0^{-2}$;
  $F(x) = -\mu^{-1}q_0 x = -q^{-1}\eta\lambda^{-1}x$.

Proof:
By Theorem 24, one of the following possibilities holds:

- $\alpha = id_R$ and $F = d$ is an extended homoderivation.
- $F(x) = \alpha(x)a$ and

$$d(x) = (1-\mu a)\alpha(x)a - ax = (1-\mu a)F(x) - ax \quad \text{with } a = F(1_R) \tag{4.34}$$

In this case we have

$$d(xy) = (1-\mu a)\alpha(xy)a - axy = (1-\mu a)\alpha(x)F(y) - axy \tag{4.35}$$

Moreover, since d is an extended homoderivation, then $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$ and then, using (4.34), we have

$$\begin{aligned} d(xy) &= \lambda d(x)d(y) + d(x)y + xd(y) = \\ &= \lambda\Big(F(x) - \mu aF(x) - ax\Big)\Big(F(y) - \mu aF(y) - ay\Big) + \Big(F(x) - \mu aF(x) - ax\Big)y + \\ &\quad + x\Big(F(y) - \mu aF(y) - ay\Big) = \\ &= \lambda\Big(F(x)F(y) - \mu F(x)aF(y) - F(x)ay - \mu aF(x)F(y) + \mu^2 aF(x)aF(y) + \mu aF(x)ay - \\ &\quad - axF(y) + \mu axaF(y) + axay\Big) + \Big(F(x) - \mu aF(x) - ax\Big)y + x\Big(F(y) - \mu aF(y) - ay\Big) \end{aligned} \tag{4.36}$$

By (4.35) and (4.36) we get

$$\begin{aligned} &(1-\mu a)\alpha(x)F(y) = \\ &= \lambda\Big(F(x)F(y) - \mu F(x)aF(y) - F(x)ay - \mu aF(x)F(y) + \mu^2 aF(x)aF(y) + \mu aF(x)ay - \\ &- axF(y) + \mu axaF(y) + axay\Big) + F(x)y - \mu aF(x)y + xF(y) - \mu xaF(y) - xay \end{aligned}$$

that is

$$\begin{aligned} &\Big(-\alpha(x) + \mu a\alpha(x) + \lambda F(x) - \lambda\mu F(x)a - \lambda\mu aF(x) + \lambda\mu^2 aF(x)a - \lambda ax + \lambda\mu axa + x - \\ &- \mu xa\Big)F(y) + \Big(\lambda axa + F(x) - \mu aF(x) - xa + \lambda\mu aF(x)a - \lambda F(x)a\Big)y = 0 \end{aligned}$$

that is

$$\Delta_1(x)y + \Delta_2(x)F(y) = \Delta_1(x)y + \Delta_2(x)\alpha(y)a = 0 \tag{4.37}$$

with

$$\begin{aligned}
\Delta_1(x) &= \lambda axa + F(x) - \mu aF(x) - xa + \lambda\mu aF(x)a - \lambda F(x)a;\\
\Delta_2(x) &= -\alpha(x) + \mu a\alpha(x) + \lambda F(x) - \lambda\mu F(x)a - \lambda\mu aF(x) + \lambda\mu^2 aF(x)a - \lambda ax+\\
&\quad + \lambda\mu axa + x - \mu xa.
\end{aligned}$$

$\boxed{\text{I case}}$ If α is outer, by [12, Theorem 3] it follows that R satisfies

$$\Delta_1(x)y + \Delta_2(x)za = 0$$

where x, y and z are distinct indeterminates.
Since R is prime and $a \neq 0$ then $\Delta_1 = \Delta_2 = 0$ that is

$$\lambda axa + \alpha(x)a - \mu a\alpha(x)a - xa + \lambda\mu a\alpha(x)a^2 - \lambda\alpha(x)a^2 = 0$$

and

$$-\alpha(x) + \mu a\alpha(x) + \lambda\alpha(x)a - \lambda\mu\alpha(x)a^2 - \lambda\mu a\alpha(x)a + \lambda\mu^2 a\alpha(x)a^2 - \lambda ax + \lambda\mu axa+$$
$$+ x - \mu xa = 0.$$

Since α is outer then

$$\begin{aligned}
\Gamma_1 &= \lambda axa + za - \mu aza - xa + \lambda\mu aza^2 - \lambda za^2 = 0;\\
\Gamma_2 &= -z + \mu az + \lambda za - \lambda\mu za^2 - \lambda\mu aza + \lambda\mu^2 aza^2 - \lambda ax + \lambda\mu axa + x - \mu xa = 0.
\end{aligned}$$

Replacing $z = 0$ in Γ_1 we get

$$\lambda axa - xa = (\lambda a - 1)xa = 0$$

and then, since R is a prime ring and $a \neq 0$ we have $a = \lambda^{-1}$.
Then, replacing $a = \lambda^{-1}$ in Γ_2 we have

$$-z + \mu\lambda^{-1}z + z - \lambda^{-1}\mu z - \lambda^{-1}\mu z + \lambda^{-2}\mu^2 z - x + \lambda^{-1}\mu x + x - \mu\lambda^{-1}x = 0$$

that is

$$-\lambda^{-1}\mu z + \lambda^{-2}\mu^2 z = 0$$

Since $\lambda \neq 0$ and $\mu \neq 0$ then we have

$$\lambda^{-1}\mu - 1 = 0$$

that is $\mu = \lambda$.
Hence by (4.34)

$$d(x) = -\lambda^{-1}x \text{ is an extended homoderivation associated with } \lambda$$

and

$$F(x) = \lambda^{-1}\alpha(x)$$

$\boxed{\text{II case}}$ If $\alpha(x) = qxq^{-1}$ is inner then by (4.37) we have

$$\Delta_1(x)y + \Delta_2(x)qyq^{-1}a = 0 \tag{4.38}$$

Then $q^{-1}a = \theta_0 \in C$ and then $a = \theta_0 q$.
Therefore

$$F(x) = \alpha(x)a = qxq^{-1}a = ax$$

and
$$d(x) = (1-\mu a)F(x) - ax = (1-\mu a)ax - ax = -\mu a^2 x$$
Since d is an extended homoderivation then
$$\begin{aligned} d(xy) &= \lambda d(x)d(y) + d(x)y + xd(y) \\ -\mu a^2 xy &= \lambda \mu a^2 x \mu a^2 y - \mu a^2 xy - x\mu a^2 y \\ 0 &= \lambda \mu a^2 x \mu a^2 y - x\mu a^2 y \\ 0 &= \mu\Big(\lambda\mu a^2 - 1\Big) x a^2 y \end{aligned}$$

Since R is a prime ring then $a^2 = 0$ or $\lambda\mu a^2 - 1 = 0$.
If $a^2 = 0$ then $d = 0$ and
$$\begin{aligned} F(xy) =& \mu F(x)F(y) + F(x)y \\ axy =& \mu axay + axy \\ 0 =& \mu axay \end{aligned}$$

Since $\mu \neq 0$ then $a = 0$ and then $F = 0$.
If $\lambda\mu a^2 = 1$ then $a^2 = \lambda^{-1}\mu^{-1} \in C$ and we have
$$F(x) = ax$$

and
$$d(x) = -\mu a^2 x = -\lambda^{-1} x \text{ is an extended homoderivation associated with } \lambda$$

- $\alpha(x) = qxq^{-1}$, $F(x) = q_0^{-1} d(x)$, with $q_0 = \eta^{-1} q$ and $\eta \in C$,
$$F(xy) = \mu F(x)F(y) + F(x)y + q_0 x F(y), \qquad d(xy) = \mu d(x)d(y) + d(x)y + q_0^2 x d(y), \tag{4.39}$$
and
$$\Big(\mu F(x) + q_0 x\Big)(1-q_0) = 0, \qquad \Big(d(x) + \mu^{-1} q_0^2 x\Big)(q_0 - 1) = 0 \tag{4.40}$$
with $q_0 = \eta^{-1} q$.

Since d is an extended homoderivation associated with λ then, by (4.39), we have
$$\mu d(x)d(y) + d(x)y + q_0^2 x d(y) = \lambda d(x)d(y) + d(x)y + xd(y)$$
that is
$$0 = (\lambda - \mu)d(x)d(y) + (1 - q_0^2)xd(y)$$
By Remark 10 we have
$$0 = (\lambda - \mu)d(x) + (1 - q_0^2)x \tag{4.41}$$
Note that:
$-$ if $q_0^2 = 1$ then by (4.41) and Remark 10 we get $\lambda = \mu$;
$-$ if $\lambda = \mu$ then by (4.41) we get $q_0^2 = 1$.
In both cases we have that $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$.
Assume now $\lambda - \mu = \theta_0 \neq 0$ and $q_0^2 \neq 1$.
Then by (4.41) we have
$$d(x) = -\theta_0^{-1}(1 - q_0^2)x \tag{4.42}$$

Moreover by (4.40) we have

$$\left(d(x) + \mu^{-1}q_0^2 x\right)(q_0 - 1) = 0$$

and using (4.42) we have

$$\left(-\theta_0^{-1}(1 - q_0^2)x + \mu^{-1}q_0^2 x\right)(q_0 - 1) = 0$$

that is

$$\left(-\theta_0^{-1}(1 - q_0^2) + \mu^{-1}q_0^2\right)x(q_0 - 1) = 0$$

Since $q_0^2 \neq 1$ and R is a prime ring then

$$-\theta_0^{-1}(1 - q_0^2) + \mu^{-1}q_0^2 = 0$$

that is

$$-\theta_0^{-1}(1 - q_0^2) = -\mu^{-1}q_0^2 \tag{4.43}$$

that is

$$\left(\theta_0^{-1} + \mu^{-1}\right)q_0^2 = \theta_0^{-1} \tag{4.44}$$

Since $\theta_0 \neq 0$ and $q_0 \neq 0$ then $\theta_1 = \theta_0^{-1} + \mu^{-1} \neq 0$.
Then

$$q_0^2 = \theta_0^{-1}\theta_1^{-1} \in C$$

By (4.42) and (4.43) we have

$$d(x) = -\theta_0^{-1}(1 - q_0^2)x = -\mu^{-1}q_0^2 x$$

that is an extended homoderivation associated with $\mu q_0^{-2} = \mu\eta^2 q^{-2}$, and then

$$F(x) = q_0^{-1}d(x) = -\mu^{-1}q_0 x$$

Moreover by (4.44) we have

$$(\theta_0^{-1} + \mu^{-1})q_0^2 = \theta_0^{-1}$$

$$(1 + \mu^{-1}\theta_0)q_0^2 = 1$$

$$(1 + \mu^{-1}(\lambda - \mu))q_0^2 = 1$$

$$\mu^{-1}\lambda q_0^2 = 1$$

$$q_0^2 = \lambda^{-1}\mu \in C$$

Hence by (4.41) we have

$$0 = (\lambda - \mu)d(x) + (1 - \lambda^{-1}\mu)x$$

that is, multiplying by λ,

$$0 = \lambda(\lambda - \mu)d(x) + (\lambda - \mu)x$$

Since $\lambda - \mu = \theta_0 \neq 0$ then
$$0 = \lambda d(x) + x$$
and then
$$d(x) = -\lambda^{-1}x \text{ is an extended homoderivation associated with } \lambda$$
Moreover
$$F(x) = q_0^{-1}d(x) = -q^{-1}\eta\lambda^{-1}x$$
Since $q_0^2 = \lambda^{-1}\mu$ then $\eta^{-1}q\eta^{-1}q = \lambda^{-1}\mu$ that is $\lambda q^2 = \eta^2\mu$.

4.2.4 Posner's theorems

Lemma 10. *Let R be a prime unital ring of characteristic different from 2, $\mu \in C$, α an automorphism of R and $F : R \longrightarrow Q_r$ an extended generalized skew homoderivation associated with μ, α and the additive map $d : R \longrightarrow Q_r$.*
If $F^2(x) = 0$ for all $x \in R$ then one of the following possibilities holds:

- $F(x) = d(x) = 0$;
- $F(x) = bx$ *or* $F(x) = xb$ *with* $b \in Q_r$, $b^2 = 0$;
- $F(x) = \alpha(x)a$, $\alpha(a)a = 0$.

Proof:
By Theorem 24, we have the following cases:

- $F(x) = d(x) = 0$ for all $x \in R$.
- F is a generalized derivation associated with the derivation d.
  By [23, Corollary 5], $F^2(x) = 0$ if and only if there exists $b \in Q_r$ such that either $F(x) = bx$ or $F(x) = xb$ with $b^2 = 0$.
- $F(x) = \alpha(x)a$ and $d(x) = (1 - \mu a)\alpha(x)a - ax$ for all $x \in R$ with $a = F(1_R)$.
  Then $0 = F^2(x) = \alpha(\alpha(x)a)a = \alpha^2(x)\alpha(a)a$ for all $x \in R$ that is $\alpha(a)a = 0$.
- $\alpha(x) = qxq^{-1}$, $F(x) = \eta q^{-1}d(x)$, $d(x) = q_0F(x)$ with $q_0 = \eta^{-1}q$, $\eta \in C$, and
$$\Big(\mu F(x) + q_0 x\Big)(1 - q_0) = 0. \tag{4.45}$$
  If $F^2(x) = 0$ then by (4.45) we have
$$0 = \Big(\mu F^2(x) + q_0F(x)\Big)(1 - q_0) = q_0F(x)(1 - q_0)$$
  that is
$$d(x)(1 - q_0) = 0 \tag{4.46}$$
  Then
$$F(x)(1 - q_0) = \eta q^{-1}d(x)(1 - q_0) = 0 \tag{4.47}$$
  Moreover, multiplying the relation $F(xy) = \mu F(x)F(y) + F(x)y + \alpha(x)d(y)$ on the right by $(1 - q_0)$ and using (4.46) and (4.47) we have
$$\begin{aligned} F(xy)(1 - q_0) =& \mu F(x)F(y)(1 - q_0) + F(x)y(1 - q_0) + \alpha(x)d(y)(1 - q_0) \\ 0 =& F(x)y(1 - q_0) \end{aligned}$$

Since R is a prime ring then $F = 0$ or $q_0 = 1$.
If $q_0 = 1$ then $q = \eta \in C$ and we have $F(x) = d(x)$.
Then, by Theorem 18, $F = d = 0$ or F is a homoderivation.
If F is a homoderivation, by Lemma 1 we have $F = 0$.

Theorem 26. *Let R be a prime unital ring of characteristic different from 2, $\mu \in C$, α an automorphism of R and $F : R \longrightarrow Q_r$ an extended generalized skew homoderivation associated with μ, α and the additive map $d : R \longrightarrow Q_r$.*
If $[F(x), x] = 0$ for all $x \in R$ then one of the following possibilities holds:

- $F(x) = d(x) = 0$;
- $F(x) = f(x) - x$ *for all* $x \in R$ *with* $f : R \longrightarrow Z(R)$ *an homomorphism.*
- $F(x) = xa$ *and* $d(x) = -\mu a^2 x$, $a \in C$.

Proof:
By Theorem 19, we have the following cases:

- $F(x) = d(x) = 0$ for all $x \in R$.
- F is a generalized derivation associated with the derivation d.
  By [30, Corollary 3.2], if $[F(x), x] = 0$ for all $x \in R$, since R is not commutative, then $F = 0$.
- $F(x) = \alpha(x)a$ and $d(x) = (1 - \mu a)\alpha(x)a - ax$ for all $x \in R$ with $a = F(1_R)$.
  Then $0 = [F(x), x] = [\alpha(x)a, x]$ for all $x \in R$. Replacing x with $x + 1$ we have $[a, x] = 0$, that is $a \in C$. Then $[\alpha(x), x] = 0$ and by [28] we have $\alpha = id_R$.
  Then $F(x) = xa$ and $d(x) = -\mu a^2 x$.
- $\alpha(x) = qxq^{-1}$, $F(x) = \eta q^{-1} d(x)$, $d(x) = q_0 F(x)$ with $q_0 = \eta^{-1} q$, $\eta \in C$, and
  $$\Big(\mu F(x) + q_0 x\Big)(1 - q_0) = 0. \tag{4.48}$$
  If $[F(x), x] = 0$ for all $x \in R$ then replacing x with $x + \gamma$, $\gamma \in C$, we have $[F(\gamma), x] = 0$ that is
  $$F(\gamma) \in C \text{ for all } \gamma \in C. \tag{4.49}$$
  By (4.48) we have
  $$F(x)(1 - q_0) = -\mu^{-1} q_0 x(1 - q_0). \tag{4.50}$$
  Assume $q_0 \neq 1$.
  Multiplying $[F(x), x] = 0$ by $1 - q_0$ on the right and using (4.50) we have
  $$\begin{aligned} 0 =& F(x)x(1 - q_0) - xF(x)(1 - q_0) = \\ =& F(x)x(1 - q_0) + \mu^{-1} x q_0 x(1 - q_0) = \\ =& \Big(F(x) + \mu^{-1} x q_0\Big) x(1 - q_0) \end{aligned} \tag{4.51}$$

Replacing x with $x+1$ and using (4.51) we have

$$
\begin{aligned}
0 = & \Big((F(x) + \mu^{-1} x q_0) + (F(1_R) + \mu^{-1} q_0) \Big) x(1-q_0) + \\
& + \Big((F(x) + \mu^{-1} x q_0) + (F(1_R) + \mu^{-1} q_0) \Big) (1-q_0) = \\
= & \Big(F(1_R) + \mu^{-1} q_0 \Big) x(1-q_0)
\end{aligned}
$$

Since R is a prime ring and $q_0 \neq 1$ then $q_0 = -\mu F(1_R)$.
If $q_0 = -\mu F(1_R)$, by (4.49) we have $q_0 \in C$ and by (4.48) we get

$$F(x) = -\mu^{-1} q_0 x = F(1_R)x, \quad F(1_R) \in C,$$

and

$$d(x) = -\mu F(1_R)^2 x, \quad F(1_R) \in C.$$

Assume $q_0 = 1$.
Then $F(x) = d(x)$ and by Theorem 18 we have $F = d = 0$ or F is a homoderivation.
If F is a homoderivation, by Theorem 2 we have $F = 0$ or $F(x) = d(x) = f(x) - x$ for all $x \in R$ with $f : R \longrightarrow Z(R)$ an homomorphism..

Chapter 5

An application: the sub-ring generated by $F(x)x$

The study of subrings generated by images of derivations and their generalizations has long provided valuable insights into the structure theory of prime rings.
A classical result in this direction was established by J. Bergen, I.N. Herstein, and J.W. Kerr in 1981 [5].
They proved that if R is a prime ring of characteristic different from 2, and $d : R \to R$ is a non-zero derivation, then the subring A generated by the set $\{d(r) \mid r \in R\}$ contains a non-zero ideal of R, unless the derivation d satisfies $d^3 = 0$.
This result was later extended in 1993 by M. Brešar and J. Vukman [8], who considered the set of commutators $\{[d(r), r] \mid r \in R\}$, again under the assumption that R is a prime ring of characteristic not equal to 2, and that d is a non-zero derivation. They showed that the subring generated by these elements contains both a non-zero left ideal and a non-zero right ideal of R.
In 1994, M.A. Chebotar [10] further strengthened this statement by proving that, under the same assumptions, the subring generated by the set $\{[d(r), r] \mid r \in R\}$ actually contains a non-zero two-sided ideal of R.
An important contribution in the context of Lie structure was given by Y. Wang in 2005 [32] who proved that if R is a prime ring of characteristic different from 2 and 3, L is a non-zero Lie ideal of R, and φ is a non-trivial Lie automorphism of R such that $\varphi^2 - 1$ is not central on L, then the subring A generated by the set $\{[\varphi(v), v] \mid v \in L\}$ contains a non-zero two-sided ideal of R.
More recently, in 2021, H. Argaç and V. De Filippis considered generalized skew derivations [2]. They proved that if R is a prime ring of characteristic different from 2 and $F : R \to R$ is a non-zero generalized skew derivation associated with a skew derivation $d : R \to R$, defining $S = \{F(x)x \mid x \in R\}$ and letting A be the subring of R generated by S, then:

- if $d \neq 0$, then A contains a non-zero left ideal of the from $Rd(R)R[F(R)R, F(R)R]$;
- if $d = 0$, then A contains a non-zero right ideal of the from $F(R)[A, A]R$.

This chapter aims to further advance this line of investigation by situating it within a broader and more general framework. Specifically, our objective is to replace standard derivations and skew derivations with their extended counterparts, thereby enriching the structural analysis of prime rings. In doing so, we contribute to a deeper and more nuanced understanding of how operator-theoretic identities influence and reflect the ideal structure of such rings.

5.1 The case of a non-zero generalized homoderivation

Lemma 11. *Let R be a non-commutative unital prime ring of characteristic different from 2 and $d : R \longrightarrow R$ a homoderivation of R. If $[d(x)x, d(y)y] = 0$ for all $x, y \in R$ then $d = 0$.*

Proof:
By Proposition 1 we have

$$d(1_R)(d(x) + x) = 0 = (d(x) + x)d(1_R) \qquad \text{for all } x \in R \tag{5.1}$$

In particular

$$d(1_R)x = -d(1_R)d(x) \qquad \text{for all } x \in R \tag{5.2}$$

By hypothesis we have

$$[d(x)x, d(y)y] = 0 \quad \text{for all } x, y \in R \tag{5.3}$$

Replacing y with $y + z$ in (5.3) we have

$$[d(x)x, d(y + z)(y + z)] = 0 \quad \text{for all } x, y, z \in R$$

then

$$[d(x)x, d(y)y] + [d(x)x, d(y)z] + [d(x)x, d(z)y] + [d(x)x, d(z)z] = 0 \quad \text{for all } x, y, z \in R$$

By (5.3) we have

$$[d(x)x, d(y)z] + [d(x)x, d(z)y] = 0 \quad \text{for all } x, y, z \in R$$

Replacing y with 1_R we get

$$[d(x)x, d(1_R)z + d(z)] = 0 \quad \text{for all } x, z \in R$$

By (5.2) we have

$$[d(x)x, -d(1_R)d(z) + d(z)] = 0 \quad \text{for all } x, z \in R$$

that is

$$[d(x)x, (1 - d(1_R))d(z)] = 0 \quad \text{for all } x, z \in R$$

Indicate $b = 1_R - d(1_R)$ and then consider the identity

$$[d(x)x, bd(z)] = 0 \quad \text{for all } x, z \in R \tag{5.4}$$

By substituting x with $x + t$ we get

$$[d(x + t)(x + t), bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

then

$$[d(x)x, bd(z)] + [d(x)t, bd(z)] + [d(t)x, bd(z)] + [d(t)t, bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

By (5.4) we get

$$[d(x)t + d(t)x, bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

Replacing x with 1_R we have

$$[d(1_R)t + d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

By (5.2) we have

$$[-d(1_R)d(t) + d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

that is

$$[(1_R - d(1_R))d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

that is

$$[bd(t), bd(z)] = 0 \quad \text{for all } t, z \in R \tag{5.5}$$

Replacing z with zr in (5.5) we get

$$[bd(t), bd(zr)] = 0 \quad \text{for all } t, z, r \in R$$

that is

$$[bd(t), bd(z)d(r) + bd(z)r + bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

By (5.5) we have

$$bd(z)[bd(t), d(r)] + bd(z)[bd(t), r] + [bd(t), bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

Replacing t with 1_R we get

$$bd(z)[bd(1_R), d(r)] + bd(z)[bd(1_R), r] + [bd(1_R), bzd(r)] = 0 \quad \text{for all } z, r \in R$$

that is

$$-bd(z)(d(r) + r)bd(1_R) + bd(1_R)bzd(r) - bzd(r)bd(1_R) = 0 \quad \text{for all } z, r \in R$$

Multiplying to the right by $(d(p) + p)$ with $p \in R$ we have

$$\begin{aligned} &- bd(z)(d(r) + r)bd(1_R)(d(p) + p) + bd(1_R)bzd(r)(d(p) + p) \\ &- bzd(r)bd(1_R)(d(p) + p) = 0 \quad \text{for all } z, r, p \in R \end{aligned}$$

By (5.1) we have $d(1)(d(p) + p) = 0$ therefore

$$bd(1_R)bzd(r)(d(p) + p) = 0 \quad \text{for all } z, r, p \in R$$

Since R a prime ring then

- $bd(1_R)b = 0$, that is $\Big(1_R - d(1_R)\Big)d(1_R)\Big(1_R - d(1_R)\Big) = 0$, or
- $d(r)(d(p) + p) = 0$ for all $r, p \in R$

$\boxed{\text{I subcase}}$ Suppose that $\Big(1_R - d(1_R)\Big)d(1_R)\Big(1_R - d(1_R)\Big) = 0$.
By (5.1) we have $d(1_R)(d(1_R) + 1_R) = 0$ that is

$$-[d(1_R)]^2 = d(1_R)$$

Therefore we have:

$$\left(1_R - d(1_R)\right)d(1_R)\left(1_R - d(1_R)\right) = 0$$

$$\left(d(1_R) - [d(1_R)]^2\right)\left(1_R - d(1_R)\right) = 0$$

$$\left(d(1_R) + d(1_R)\right)\left(1_R - d(1_R)\right) = 0$$

$$2d(1_R)\left(1_R - d(1_R)\right) = 0$$

$$2d(1_R) - 2[d(1_R)]^2 = 0$$

$$2d(1_R) + 2d(1_R) = 0$$

$$4d(1_R) = 0$$

Since $char(R) \neq 2$ then $d(1_R) = 0$, therefore $b = 1_R - d(1_R) = 1_R$.
Then (5.5) is reduced to

$$[d(x), d(y)] = 0 \quad \text{for all } x, y \in R$$

and the conclusion follows from the Lemma 2.
$\boxed{\text{II subcase}}$ Suppose that $d(z)(d(t) + t) = 0$ for all $z, t \in R$.
By Remarks 3 and 4 we have $d(t) + t = 0$ for all $t \in R$, that is $d(t) = -t$ for all $t \in R$.
In this case, by (5.3) we get

$$[-x^2, -y^2] = 0 \quad \text{for all } x, y \in R$$

a contradiction since R is not commutative.

Theorem 27. *Let R be a non-commutative unital prime ring of characteristic different from 2, $F : R \longrightarrow R$ a non-zero generalized homoderivation associated with the homoderivation $d : R \longrightarrow R$ and $a \in R$ such that $F(1_R) = a$.*
Consider $S = \{F(x)x, x \in R\}$ and indicate with $A =< F(x)x >$ the sub-ring generated by S in R.
A contains a left non-trivial ideal I.

Proof:
Suppose that A contains neither non-trivial right ideals nor non-trivial left ideals.
For all $x, y \in R$, the element $F(x + y)(x + y)$ is in $S \subseteq A$ therefore

$$F(x)x + F(y)y + F(x)y + F(y)x \in A$$

Since $F(x)x + F(y)y \in A$ then

$$F(x)y + F(y)x \in A \qquad \text{for all } x, y \in R \tag{5.6}$$

Consider $z \in S$. Replacing y with yz, we have

$$F(x)yz + F(yz)x \in A \qquad \text{for all } x, y \in R, z \in S$$

that is

$$F(x)yz + F(y)F(z)x + F(y)zx + yd(z)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.7}$$

Since $z \in S$ then multiplying (5.6) to the right by $z \in S$ we still get an element of A

$$F(x)yz + F(y)xz \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.8}$$

Since the difference of elements of A is still an element of A then subtracting (5.7) and (5.8) we get the following element of A

$$F(y)[z,x] + \Big(F(y)F(z) + yd(z)\Big)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.9}$$

Consider $r \in S$. Since $x \in R$ is arbitrary, we can replace it with xr in (5.9)

$$F(y)[z,x]r + F(y)x[z,r] + \Big(F(y)F(z) + yd(z)\Big)xr \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.10}$$

Multiplying (5.9) by the element $r \in S$ to the right we get another element of A that is

$$F(y)[z,x]r + \Big(F(y)F(z) + yd(z)\Big)xr \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.11}$$

Subtracting (5.11) from (5.10) we get

$$F(y)x[z,r] \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.12}$$

Let $t \in R$. Since y is arbitrary, we can replace it with ty in (5.12). We have

$$F(ty)x[z,r] \in A \quad \text{for all } x, y, t \in R, z, r \in S$$

that is, for all $x, y, t \in R, z, r \in S$

$$F(t)(F(y)x)[z,r] + F(t)(yx)[z,r] + td(y)x[z,r] \in A \tag{5.13}$$

By (5.12), we have $F(t)F(y)x[z,r] + F(t)yz[z,r] \in A$ therefore

$$td(y)x[z,r] \in A \quad \text{for all } x, y, t \in R, z, r \in S \tag{5.14}$$

Then $Rd(R)R[S,S] \subseteq A$.
Since A does not contain non-zero left ideals then $Rd(R)R[S,S] = 0$.
Since R a prime ring then $d(R) = 0$ or $[S,S] = 0$.
By Theorem 5 we have $d \neq 0$ then $[S,S] = 0$ that is

$$[F(x)x, F(y)y] = 0 \quad \text{for all } x, y \in R \tag{5.15}$$

By Theorem 7, the cases to be analyzed are:

- $d(x) = -x, \quad F(x) = x$ for all $x \in R$;
- $d(x) = -x, \quad F(x) = -x$ for all $x \in R$;
- $d(x) = [x,a] - axa, \quad F(x) = xa$ for all $x \in R$, where $a^2 = 0$;
- $F(x) = d(x)$ for all $x \in R$ where $d(1_R)(d(x) + x) = 0 = (d(x) + x)d(1_R)$ for all $x \in R$.

$\boxed{\text{I case}}$ Suppose that $d(x) = -x$ and $F(x) = x$ for all $x \in R$.
Then by (5.15) we have

$$[x^2, y^2] = 0 \quad \text{for all } x, y \in R$$

A contradiction since R is not commutative.
$\boxed{\text{II case}}$ Suppose that $d(x) = -x$ and $F(x) = -x$ for all $x \in R$.
Then by (5.15) we have

$$[-x^2, -y^2] = 0 \quad \text{for all } x, y \in R$$

A contradiction since R is not commutative.
$\boxed{\text{III case}}$ Suppose that $d(x) = [x, a] - axa$ and $F(x) = xa$ for all $x \in R$, where $a^2 = 0$.
Then by (5.15) we have

$$[xax, yay] = 0 \quad \text{for all } x, y \in R \tag{5.16}$$

Since the field $Z(R)$ does not contain any nilpotent elements, if $a \in Z(R)$ then $a = 0$, a contradiction.
Thus $a \notin Z(R)$ and (5.16) is a generalized non-trivial polynomial identity. By Martindale's theorem [26], R is a primitive ring having a non-zero socle H, with the extended centroid C as the associated division ring.
In light of Jacobson's theorem [24, page 75], R is isomorphic to a dense ring of linear transformations on some vector space V over the extended centroid C.
Since R is non-commutative, we can assume $dim_D V \geq 2$. If $a \notin Z(R)$ then there exists $v \in V$ such that the set $\{v, av\}$ is linearly C-independent. For the density of R, there exist $x, y \in R$ such that

$$xv = 0,\ yv = v,\ yav = av,\ xav = v$$

Then by (5.16) we have $0 = [xax, yay]v = v$, a contradiction.
$\boxed{\text{IV case}}$ Suppose that $F(x) = d(x)$ for all $x \in R$.
By Lemma 11 we have $d = 0$, a contradiction for the Theorem 5.

5.2 The case of a non-zero extended generalized homoderivation

Lemma 12. *Let R be a unital prime ring of characteristic different from 2 and $\omega \in Q_r$. If*

$$[\omega x^2, \omega y^2] = 0 \quad \textit{for all } x, y \in R \tag{5.17}$$

then R is a commutative ring.

Proof:
If $\omega \in C$ we are done. Assume $\omega \notin C$. For $x = 1_R$ we have

$$[\omega, \omega y^2] = 0 \quad \text{for all } y \in R \tag{5.18}$$

Replacing x with $x + 1_R$ in (5.17) and using (5.17) and (5.18) we have

$$0 = [\omega(x+1)^2, \omega y^2] = [\omega x^2 + 2\omega x + \omega, \omega y^2] = [2\omega x, \omega y^2]$$

that is

$$[\omega x, \omega y^2] = 0 \tag{5.19}$$

and in particular

$$[\omega x, \omega] = 0 \tag{5.20}$$

Replacing y with $y + 1_R$ in (5.19) and using (5.19) and (5.20) we have

$$0 = [\omega x, \omega y^2 + 2\omega y + \omega] = [\omega x, 2\omega y]$$

that is

$$[\omega x, \omega y] = 0 \tag{5.21}$$

Since $\omega \notin Z(R)$ then (5.21) is a generalized non-trivial polynomial identity. By Martindale's theorem [26], R is a primitive ring having a non-zero socle H, with the extended centroid C as the associated division ring. In light of Jacobson's theorem [24, page 75], R is isomorphic to a dense ring of linear transformations on some vector space V over the extended centroid C.
Since R is non-commutative, we can assume $dim_D V \geq 2$. Since $\omega \notin Z(R)$ then there exists $v \in V$ such that the set $\{v, \omega v\}$ is linearly C-independent. For the density of R, there exist $x, y \in R$ such that

$$xv = 0,\ yv = v,\ x\omega v = v$$

Then by (5.21) we have $0 = [\omega x, \omega y]v = \omega x \omega y v - \omega y \omega x v = \omega v$, a contradiction.

Lemma 13. *Let R be a non-commutative unital prime ring of characteristic different from 2, $0 \neq \lambda \in C$ and $d : R \longrightarrow Q_r$ an extended homoderivation of R associated with λ. If*

$$[d(x)x, d(y)y] = 0 \quad \textit{for all } x, y \in R$$

then $d = 0$.

Proof:
By Proposition 3 we have

$$d(1_R)(\lambda d(x) + x) = 0 = (\lambda d(x) + x)d(1_R) \qquad \text{for all } x \in R \tag{5.22}$$

In particular

$$d(1_R)x = -\lambda d(1_R)d(x) \qquad \text{for all } x \in R \tag{5.23}$$

By hypothesis we have

$$[d(x)x, d(y)y] = 0 \quad \text{for all } x, y \in R \tag{5.24}$$

Replacing y with $y + z$ in (5.24) we have

$$[d(x)x, d(y+z)(y+z)] = 0 \quad \text{for all } x, y, z \in R$$

then

$$[d(x)x, d(y)y] + [d(x)x, d(y)z] + [d(x)x, d(z)y] + [d(x)x, d(z)z] = 0 \quad \text{for all } x, y, z \in R$$

By (5.24) we have

$$[d(x)x, d(y)z] + [d(x)x, d(z)y] = 0 \quad \text{for all } x, y, z \in R$$

Replacing y with 1_R we get

$$[d(x)x, d(1_R)z + d(z)] = 0 \quad \text{for all } x, z \in R$$

By (5.23) we have

$$[d(x)x, -\lambda d(1_R)d(z) + d(z)] = 0 \quad \text{for all } x, z \in R$$

that is

$$[d(x)x, (1 - \lambda d(1_R))d(z)] = 0 \quad \text{for all } x, z \in R$$

Indicate $b = 1_R - \lambda d(1_R)$ and then consider the identity

$$[d(x)x, bd(z)] = 0 \quad \text{for all } x, z \in R \tag{5.25}$$

By substituting x with $x + t$ we get

$$[d(x+t)(x+t), bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

then

$$[d(x)x, bd(z)] + [d(x)t, bd(z)] + [d(t)x, bd(z)] + [d(t)t, bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

By (5.25) we get

$$[d(x)t + d(t)x, bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

Replacing x with 1_R we have

$$[d(1_R)t + d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

By (5.23) we have

$$[-\lambda d(1_R)d(t) + d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

that is

$$[(1_R - \lambda d(1_R))d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

that is

$$[bd(t), bd(z)] = 0 \quad \text{for all } t, z \in R \tag{5.26}$$

Replacing z with zr in (5.26) we get

$$[bd(t), bd(zr)] = 0 \quad \text{for all } t, z, r \in R$$

that is

$$[bd(t), \lambda bd(z)d(r) + bd(z)r + bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

By (5.26) we have

$$\lambda bd(z)[bd(t), d(r)] + bd(z)[bd(t), r] + [bd(t), bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

Replacing t with 1_R we get

$$\lambda bd(z)[bd(1_R), d(r)] + bd(z)[bd(1_R), r] + [bd(1_R), bzd(r)] = 0 \quad \text{for all } z, r \in R$$

By (5.22) we have $d(1_R)(\lambda d(r) + r) = 0$ then

$$-bd(z)(\lambda d(r) + r)bd(1_R) + bd(1_R)bzd(r) - bzd(r)bd(1_R) = 0 \quad \text{for all } z, r \in R$$

Multiplying to the right by $(\lambda d(p) + p)$ with $p \in R$ we have

$$\begin{aligned} &- bd(z)(\lambda d(r) + r)bd(1_R)(\lambda d(p) + p) + bd(1_R)bzd(r)(\lambda d(p) + p) - \\ &- bzd(r)bd(1_R)(\lambda d(p) + p) = 0 \quad \text{for all } z, r, p \in R \end{aligned}$$

By (5.22) we have $d(1_R)(\lambda d(p) + p) = 0$ therefore

$$bd(1_R)bzd(r)(\lambda d(p) + p) = 0 \quad \text{for all } z, r, p \in R$$

Since R a prime ring then

- $bd(1_R)b = 0$, that is $\Big(1_R - \lambda d(1_R)\Big)d(1_R)\Big(1_R - \lambda d(1_R)\Big) = 0$, or
- $d(r)(\lambda d(p) + p) = 0$ for all $r, p \in R$

$\boxed{\text{I subcase}}$ Suppose that $\Big(1_R - \lambda d(1_R)\Big)d(1_R)\Big(1_R - \lambda d(1_R)\Big) = 0.$
By (5.22) we have $d(1_R)(\lambda d(1_R) + 1_R) = 0$ that is

$$-\lambda[d(1_R)]^2 = d(1_R)$$

Therefore we have:

$$\begin{aligned} \Big(1_R - \lambda d(1_R)\Big)d(1_R)\Big(1_R - \lambda d(1_R)\Big) &= 0 \\ \Big(d(1_R) - \lambda[d(1_R)]^2\Big)\Big(1_R - \lambda d(1_R)\Big) &= 0 \\ \Big(d(1_R) + d(1_R)\Big)\Big(1_R - \lambda d(1_R)\Big) &= 0 \\ 2d(1_R)\Big(1_R - \lambda d(1_R)\Big) &= 0 \\ 2d(1_R) - 2\lambda[d(1_R)]^2 &= 0 \\ 2d(1_R) + 2d(1_R) &= 0 \\ 4d(1_R) &= 0 \end{aligned}$$

Since $char(R) \neq 2$ then $d(1_R) = 0$, therefore $b = 1_R - \lambda d(1_R) = 1_R$. Then (5.26) is reduced to

$$[d(x), d(y)] = 0 \quad \text{for all } x, y \in R$$

and the conclusion follows from the Lemma 7.

$\boxed{\text{II subcase}}$ Suppose that $d(z)(\lambda d(t) + t) = 0$ for all $z, t \in R$.
By Remarks 10 and 11 we have $\lambda d(t) + t = 0$ for all $t \in R$, that is $d(t) = -\lambda^{-1}t$ for all $t \in R$.
In this case, by (5.24) we get

$$[-x^2, -y^2] = 0 \quad \text{for all } x, y \in R$$

a contradiction since R is not commutative.

Theorem 28. *Let R be a non-commutative unital prime ring of characteristic different from 2, $d : R \longrightarrow Q_r$ an extended homoderivation associated with $1_R, \eta \in C$ and $F : R \longrightarrow Q_r$ a non-zero extended generalized homoderivation associated with $\lambda \in C, 0 \neq \mu \in C$ and with the extended homoderivation d.*
Consider $S = \{F(x)x, x \in R\}$ and indicate with $A =< F(x)x >$ the sub-ring generated by S in R.
A contains a left non-trivial ideal I.

Proof:
Suppose that A contains neither non-trivial right ideals nor non-trivial left ideals. For all $x, y \in R$, the element $F(x+y)(x+y)$ is in $S \subseteq A$ therefore

$$F(x)x + F(y)y + F(x)y + F(y)x \in A.$$

Since $F(x)x + F(y)y \in A$ then

$$F(x)y + F(y)x \in A \qquad \text{for all } x, y \in R. \tag{5.27}$$

Consider $z \in S$. Replacing y with yz, we have

$$F(x)yz + F(yz)x \in A \qquad \text{for all } x, y \in R, z \in S,$$

that is

$$F(x)yz + \lambda F(y)F(z)x + \mu F(y)zx + \mu yd(z)x \in A \qquad \text{for all } x, y \in R, z \in S. \tag{5.28}$$

Since $z \in S$ then multiplying (5.27) to the right by $z \in S$ we still get an element of A

$$F(x)yz + F(y)xz \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.29}$$

Since the difference of elements of A is still an element of A then subtracting (5.28) and (5.29) we get the following element of A

$$F(y)(\mu zx - xz) + \Big(\lambda F(y)F(z) + \mu yd(z)\Big)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.30}$$

Consider $r \in S$. Since $x \in R$ is arbitrary, we can replace it with xr in (5.30)

$$F(y)(\mu zxr - xrz) + \Big(\lambda F(y)F(z) + \mu yd(z)\Big)xr \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.31}$$

Multiplying (5.30) by the element $r \in S$ to the right we get another element of A that is

$$F(y)(\mu zxr - xzr) + \Big(\lambda F(y)F(z) + \mu yd(z)\Big)xr \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.32}$$

Subtracting (5.32) from (5.31) we get

$$F(y)x[z, r] \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.33}$$

Then $F(R)R[S, S] \subseteq A$.
Since A does not contain non-zero left ideals then $F(R)R[S, S] = 0$.
Since R is a prime ring then $F(R) = 0$ or $[S, S] = 0$.

If $[S, S] = 0$ then

$$[F(x)x, F(y)y] = 0 \quad \text{for all } x, y \in R. \tag{5.34}$$

By Theorem 15, the cases to be analyzed are:

- F is a generalized derivation associated to the derivation d;
- $F(x) = d(x)$ with $\lambda = \eta \neq 0$;
- $F(x) = d(x) - \theta x$ and $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$ with $a = d(1_R)$, $b = F(1_R)$ and

$$\theta = a - b = \lambda^{-1}(\mu - 1) \in C, \quad \text{with} \quad \mu \neq 1_R;$$

- $F(x) = \theta x$ or $F(x) = x\theta$.

$\boxed{\text{I case}}$ Suppose that F is a generalized derivation associated to the derivation d. Then by (5.34) we have

$$[F(x)x, F(y)y] = 0 \quad \text{for all } x, y \in R. \tag{5.35}$$

By [20], $x^2 \in C$ for all $x \in R$, a contradiction since R is not commutative.

$\boxed{\text{II case}}$ Suppose that $F(x) = d(x)$ with $\lambda = \eta \neq 0$.
By Lemma 13 we have $F = d = 0$, a contradiction.

$\boxed{\text{III case}}$ Suppose that $F(x) = d(x) - \theta x$ and $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$ with $a = d(1_R)$, $b = F(1_R)$ and

$$\theta = a - b = \lambda^{-1}(\mu - 1) \in C, \quad \text{with} \quad \mu \neq 1_R.$$

If $\theta = 0$ then by Lemma 13 we have $F = d = 0$, a contradiction.
Assume $\theta \neq 0$.
By (5.34) we have

$$[d(x)x - \theta x^2, d(y)y - \theta y^2] = 0 \quad \text{for all } x, y \in R. \tag{5.36}$$

Replacing y with 1 and x with $x + 1$ and using (5.36) we have

$$\begin{aligned} 0 =& [d(x)x + d(x) + d(1_R)x + d(1_R) - \theta x^2 - 2\theta x - \theta, d(1_R) - \theta] = \\ =& [d(x) + d(1_R)x - 2\theta x, d(1_R)] \end{aligned} \tag{5.37}$$

Replacing x with $x + 1$ and using (5.37) we have

$$0 = [d(x) + d(1_R) + d(1_R)x + d(1_R) - 2\theta x - 2\theta, d(1_R)] = -2\theta[x, d(1_R)]$$

Since $char(R) \neq 2$ and $\theta \neq 0$ then $d(1_R) \in C$.
If $d(1_R) \neq 0$ then by Proposition 3 we have $d(x) = -\lambda^{-1}x$ for all $x \in R$.
In this case, by (5.36) we get

$$[-\lambda^{-1}x^2 - \theta x^2, -\lambda^{-1}y^2 - \theta y^2] = 0 \quad \text{for all } x, y \in R.$$

Since $\mu \neq 1$ then $\theta \neq -\lambda^{-1}$ and we have

$$(-\lambda^{-1} - \theta)^2[x^2, y^2] = 0 \quad \text{for all } x, y \in R,$$

a contradiction since R is not commutative.
Assume $d(1_R) = 0$.
Replacing x with $x+1$ in (5.36) and using (5.36) we have

$$[d(x) - 2\theta x, d(y)y - \theta y^2] = 0 \tag{5.38}$$

Replacing y with $y+1$ and using (5.38) we have

$$[d(x) - 2\theta x, d(y) - 2\theta y] = 0 \tag{5.39}$$

Replacing x with xz and using (5.39) we have

$$[\lambda d(x)d(z) + xd(z), d(y) - 2\theta y] = 0$$

then, for $x = 1_R$ we have

$$[d(z), d(y) - 2\theta y] = 0 \tag{5.40}$$

Replacing y with zy and using (5.40) we have

$$[d(z), \lambda d(z)d(y) + zd(y)] = 0 \tag{5.41}$$

that is

$$0 = [d(z), d(z)y] = d(z)[d(z), y] \tag{5.42}$$

By Remarks 10 and 11, $d = 0$ or $[d(z), y] = 0$.
If $d = 0$ then by (5.36) we have $[-x^2, -y^2] = 0$, a contradiction since R is not commutative.
Suppose that $[d(x), t] = 0$ for all $x, t \in R$, that is

$$d(x) \in Z(R), \quad \text{for all } x \in R$$

We can assume $Z(R) \neq 0$, otherwise it would be $d = 0$.
In particular, for each $x \in R$ we have both $d(x) \in Z(R)$ and also $d(x^2) \in Z(R)$.
Hence

$$d(x^2) = \lambda d(x)^2 + xd(x) + d(x)x = \lambda d(x)^2 + 2xd(x) \in Z(R)$$

and then, since $char(R) \neq 2$, $xd(x) \in Z(R)$.
Since $d(x) \in Z(R)$ then for all $x \in R$ we have either $d(x) = 0$ or $x \in Z(R)$.
In particular, if $x \notin Z(R)$ then $d(x) = 0$.
Since R is non-commutative, exists $0 \neq x_0 \notin Z(R)$ such that $d(x_0) = 0$.
Moreover, since $Z(R) \neq 0$ exists $0 \neq x_1 \in Z(R)$. Therefore

$$d(x_0x_1) \in Z(R) \quad \Rightarrow \lambda d(x_0)d(x_1) + d(x_0)x_1 + x_0d(x_1) \in Z(R)$$

then $x_0d(x_1) \in Z(R)$. Since $0 \neq x_0 \notin Z(R)$ and $d(x_1) \in Z(R)$, we have $d(x_1) = 0$.
Therefore $d(x) = 0$ in any case, a contradiction.

$\boxed{\text{IV case}}$ Suppose that $F(x) = \theta x$.
Then by (5.34) we have $[\theta x^2, \theta y^2] = 0$ for all $x, y \in R$.
By Lemma 12 we get a contradiction.

$\boxed{\text{V case}}$ Suppose that $F(x) = x\theta$.
Then by (5.34) we have

$$[x\theta x, y\theta y] = 0 \quad \text{for all } x, y \in R. \tag{5.43}$$

If $\theta \in C$ then $[x^2, y^2] = 0$, a contradiction since R is not commutative.
Thus $\theta \notin Z(R)$ and (5.43) is a generalized non-trivial polynomial identity. By Martindale's theorem [26], R is a primitive ring having a non-zero socle H, with the extended centroid C as the associated division ring.
In light of Jacobson's theorem [24, page 75], R is isomorphic to a dense ring of linear transformations on some vector space V over the extended centroid C.
Since R is non-commutative, we can assume $dim_D V \geq 2$. Since $\theta \notin Z(R)$ then there exists $v \in V$ such that the set $\{v, \theta v\}$ is linearly C-independent. For the density of R, there exist $x, y \in R$ such that

$$xv = 0, \ yv = v, \ y\theta v = \theta v, \ x\theta v = v$$

Then by (5.43) we have $0 = [x\theta x, y\theta y]v = v$, a contradiction.

5.3 The case of a non-zero generalized skew homoderivation

Theorem 29. *Let R be a non-commutative unital prime ring of characteristic different from 2, α an automorphism of R, $d : R \longrightarrow R$ a homoderivation and $F : R \longrightarrow R$ a non-zero generalized skew homoderivation associated with α and d.*
Consider $S = \{F(x)x, x \in R\}$ and indicate with $A =< F(x)x >$ the sub-ring generated by S in R.
A contains a left non-trivial ideal I.

Proof:
First of all, note that if $d = 0$ then F is a homoleft multiplier and, since the only homoleft multiplier on a semiprime ring is the zero mapping ([6]), then $F = 0$, a contradiction.
Then assume $d \neq 0$.
Suppose that A contains neither non-trivial right ideals nor non-trivial left ideals.
For all $x, y \in R$, the element $F(x+y)(x+y)$ is in $S \subseteq A$ therefore

$$F(x)x + F(y)y + F(x)y + F(y)x \in A$$

Since $F(x)x + F(y)y \in A$ then

$$F(x)y + F(y)x \in A \qquad \text{for all } x, y \in R \tag{5.44}$$

Consider $z \in S$. Replacing y with yz, we have

$$F(x)yz + F(yz)x \in A \qquad \text{for all } x, y \in R, z \in S$$

that is

$$F(x)yz + F(y)F(z)x + F(y)zx + \alpha(y)d(z)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.45}$$

Since $z \in S$ then multiplying (5.44) to the right by $z \in S$ we still get an element of A

$$F(x)yz + F(y)xz \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.46}$$

Since the difference of elements of A is still an element of A then subtracting (5.45) and (5.46) we get the following element of A

$$F(y)[z, x] + \Big(F(y)F(z) + \alpha(y)d(z)\Big)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.47}$$

Consider $r \in S$. Since $x \in R$ is arbitrary, we can replace it with xr in (5.47)

$$F(y)[z,x]r + F(y)x[z,r] + \Big(F(y)F(z) + \alpha(y)d(z)\Big)xr \in A \quad \text{for all } x,y \in R, z,r \in S \tag{5.48}$$

Multiplying (5.47) by the element $r \in S$ to the right we get another element of A that is

$$F(y)[z,x]r + \Big(F(y)F(z) + \alpha(y)d(z)\Big)xr \in A \quad \text{for all } x,y \in R, z,r \in S \tag{5.49}$$

Subtracting (5.49) from (5.48) we get

$$F(y)x[z,r] \in A \quad \text{for all } x,y \in R, z,r \in S \tag{5.50}$$

Let $t \in R$. Since y is arbitrary, we can replace it with ty in (5.50). We have

$$F(ty)x[z,r] \in A \quad \text{for all } x,y,t \in R, z,r \in S$$

that is, for all $x,y,t \in R, z,r \in S$

$$F(t)(F(y)x)[z,r] + F(t)(yx)[z,r] + \alpha(t)d(y)x[z,r] \in A \tag{5.51}$$

By (5.50), we have $F(t)F(y)x[z,r] + F(t)yx[z,r] \in A$ therefore

$$\alpha(t)d(y)x[z,r] \in A \quad \text{for all } x,y,t \in R, z,r \in S$$

and, replacing t with $\alpha^{-1}(t)$ we have

$$td(y)x[z,r] \in A \quad \text{for all } x,y,t \in R, z,r \in S$$

Then $Rd(R)R[S,S] \subseteq A$.
Since A does not contain non-zero left ideals then $Rd(R)R[S,S] = 0$.
Since R a prime ring then $d(R) = 0$ or $[S,S] = 0$.
Since $d \neq 0$ then $[S,S] = 0$ that is

$$[F(x)x, F(y)y] = 0 \quad \text{for all } x,y \in R \tag{5.52}$$

By Theorem 20, the cases to be analyzed are:

- $F = d$ is a homoderivation;
- $d(x) = -x$ and $F(x) = \alpha(x)$;
- $\alpha(x) = qxq^{-1}$, $d(x) = -x$ and $F(x) = ax$ with $a = F(1_R)$ and $a^2 = 1_R$;
- $\alpha(x) = qxq^{-1}$, $\eta \in C$, $q_0 = \eta^{-1}q$, $q_0^2 = 1$, $d(x) = q_0F(x)$ is a homoderivation and $F(x) = q_0d(x)$. Moreover, we have

$$\Big(d(x) + x\Big)(q_0 - 1) = 0$$

$\boxed{\text{I case}}$ Suppose that $F = d$ is a homoderivation.
Then by Lemma 11 we have $F = d = 0$, a contradiction.

$\boxed{\text{II case}}$ Suppose that $F(x) = \alpha(x)$ and $d(x) = -x$ for all $x \in R$.
Then by (5.52) we have

$$[\alpha(x)x, \alpha(y)y] = 0 \quad \text{for all } x, y \in R$$

By [20], $x^2 \in C$ for all $x \in R$, a contradiction since R is not commutative.

$\boxed{\text{III case}}$ Suppose that $\alpha(x) = qxq^{-1}$, $d(x) = -x$ and $F(x) = ax$ with $a = F(1_R)$ and $a^2 = 1_R$.
Then by (5.52) we have $[ax^2, ay^2] = 0$ for all $x, y \in R$.
By Lemma 12 we get a contradiction.

$\boxed{\text{IV case}}$ Suppose that $\alpha(x) = qxq^{-1}$, $\eta \in C$, $q_0 = \eta^{-1}q$, $q_0^2 = 1$, $d(x) = q_0F(x)$ is a homoderivation, $F(x) = q_0d(x)$ and we have

$$\Big(d(x) + x\Big)(q_0 - 1) = 0 \tag{5.53}$$

that is

$$d(x)q_0 = d(x) - x(q_0 - 1) \tag{5.54}$$

and in particular

$$d(1_R)q_0 = d(1_R) - (q_0 - 1) \tag{5.55}$$

Moreover, by Proposition 1 we have

$$d(1_R)(d(x) + x) = 0 = (d(x) + x)d(1_R) \qquad \text{for all } x \in R \tag{5.56}$$

In particular

$$d(1_R)x = -d(1_R)d(x) \qquad \text{for all } x \in R \tag{5.57}$$

By (5.52) we have

$$[q_0d(x)x, q_0d(y)y] = 0 \quad \text{for all } x, y \in R \tag{5.58}$$

Replacing y with $y + z$ in we have

$$[q_0d(x)x, q_0d(y + z)(y + z)] = 0 \quad \text{for all } x, y, z \in R$$

then

$$[q_0d(x)x, q_0d(y)y] + [q_0d(x)x, q_0d(y)z] + [q_0d(x)x, q_0d(z)y] + [q_0d(x)x, q_0d(z)z] = 0$$
for all $x, y, z \in R$

By (5.52) we have

$$[q_0d(x)x, q_0d(y)z] + [q_0d(x)x, q_0d(z)y] = 0 \quad \text{for all } x, y, z \in R$$

Replacing y with 1_R we get

$$[q_0d(x)x, q_0d(1_R)z + q_0d(z)] = 0 \quad \text{for all } x, z \in R$$

By (5.57) we have

$$[q_0d(x)x, -q_0d(1_R)d(z) + q_0d(z)] = 0 \quad \text{for all } x, z \in R$$

that is

$$[q_0 d(x)x, q_0(1 - d(1_R))d(z)] = 0 \quad \text{for all } x, z \in R$$

Indicate $b = q_0(1_R - d(1_R))$ and then consider the identity

$$[q_0 d(x)x, bd(z)] = 0 \quad \text{for all } x, z \in R \tag{5.59}$$

Replacing x with $x + t$ we get

$$[q_0 d(x + t)(x + t), bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

then

$$[q_0 d(x)x, bd(z)] + [q_0 d(x)t, bd(z)] + [q_0 d(t)x, bd(z)] + [q_0 d(t)t, bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

By (5.59) we get

$$[q_0 d(x)t + q_0 d(t)x, bd(z)] = 0 \quad \text{for all } x, t, z \in R$$

Replacing x with 1_R we have

$$[q_0 d(1_R)t + q_0 d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

By (5.57) we have

$$[-q_0 d(1_R)d(t) + q_0 d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

that is

$$[q_0(1_R - d(1_R))d(t), bd(z)] = 0 \quad \text{for all } t, z \in R$$

that is

$$[bd(t), bd(z)] = 0 \quad \text{for all } t, z \in R \tag{5.60}$$

Replacing z with zr we get

$$[bd(t), bd(zr)] = 0 \quad \text{for all } t, z, r \in R$$

that is

$$[bd(t), bd(z)d(r) + bd(z)r + bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

By (5.60) we have

$$bd(z)[bd(t), d(r)] + bd(z)[bd(t), r] + [bd(t), bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

Replacing t with 1_R we get

$$bd(z)[bd(1_R), d(r)] + bd(z)[bd(1_R), r] + [bd(1_R), bzd(r)] = 0 \quad \text{for all } z, r \in R$$

that is

$$-bd(z)(d(r) + r)bd(1_R) + bd(1_R)bzd(r) - bzd(r)bd(1_R) = 0 \quad \text{for all } z, r \in R$$

Multiplying to the right by $(d(p) + p)$ with $p \in R$ we have

$$\begin{aligned} &- bd(z)(d(r) + r)bd(1_R)(d(p) + p) + bd(1_R)bzd(r)(d(p) + p) - \\ &- bzd(r)bd(1_R)(d(p) + p) = 0 \text{for all } z, r, p \in R \end{aligned}$$

By (5.56) we have $d(1)(d(p)+p)=0$ therefore

$$bd(1_R)bzd(r)(d(p)+p)=0 \quad \text{for all } z,r,p\in R$$

Since R is a prime ring then

- $bd(1_R)b=0$, that is $q_0\Big(1_R-d(1_R)\Big)d(1_R)q_0\Big(1_R-d(1_R)\Big)=0$, or
- $d(r)(d(p)+p)=0$ for all $r,p\in R$

$\boxed{\text{I subcase}}$ Suppose that $q_0\Big(1_R-d(1_R)\Big)d(1_R)q_0\Big(1_R-d(1_R)\Big)=0$.
Since q_0 is a non-zero invertible element then

$$\Big(1_R-d(1_R)\Big)d(1_R)q_0\Big(1_R-d(1_R)\Big)=0$$

By (5.56) we have $d(1_R)(d(1_R)+1_R)=0$ that is

$$-[d(1_R)]^2=d(1_R)$$

Therefore by $\Big(1_R-d(1_R)\Big)d(1_R)q_0\Big(1_R-d(1_R)\Big)=0$ we have:

$$\Big(d(1_R)-[d(1_R)]^2\Big)q_0\Big(1_R-d(1_R)\Big)=0$$

$$\Big(d(1_R)+d(1_R)\Big)q_0\Big(1_R-d(1_R)\Big)=0$$

$$2d(1_R)q_0\Big(1_R-d(1_R)\Big)=0 \tag{5.61}$$

By (5.55) we have $d(1_R)q_0=d(1_R)-(q_0-1)$. Then

$$2\Big(d(1_R)-(q_0-1)\Big)\Big(1_R-d(1_R)\Big)=0$$

$$2\Big(d(1_R)+1-q_0\Big)\Big(1_R-d(1_R)\Big)=0$$

Since $char(R)\neq 2$ then

$$\Big(d(1_R)+1\Big)\Big(1_R-d(1_R)\Big)=q_0\Big(1_R-d(1_R)\Big)$$

$$d(1_R)-[d(1_R)]^2+1_R-d(1_R)=q_0\Big(1_R-d(1_R)\Big)$$

$$d(1_R)+d(1_R)+1_R-d(1_R)=q_0\Big(1_R-d(1_R)\Big)$$

$$d(1_R)+1_R=q_0\Big(1_R-d(1_R)\Big)$$

Multiplying on the right by $d(1_R)$ we then have

$$\Big(d(1_R)+1_R\Big)d(1_R) = q_0\Big(1_R - d(1_R)\Big)d(1_R)$$

$$0 = q_0\Big(d(1_R) - [d(1_R)]^2\Big)$$

$$0 = q_0\Big(d(1_R) + d(1_R)\Big)$$

$$0 = 2q_0 d(1_R)$$

Since q_0 is an invertible element and $char(R) \neq 2$ then $d(1_R) = 0$.
Moreover by (5.53) we have

$$0 = \Big(d(1_R)+1\Big)(q_0 - 1) = q_0 - 1$$

that is $q_0 = 1$.
Therefore $b = q_0(1_R + d(1_R)) = q_0 = 1$ and (5.60) is reduced to

$$[d(x), d(y)] = 0 \quad \text{for all } x, y \in R$$

and by Lemma 2 we get $d = 0$, a contradiction.
$\boxed{\text{II subcase}}$ Suppose that $d(z)(d(t)+t) = 0$ for all $z, t \in R$.
By Remarks 3 and 4 we have $d(t) + t = 0$ for all $t \in R$, that is $d(t) = -t$ for all $t \in R$.
In this case, by (5.58) we get $[q_0 x^2, q_0 y^2] = 0$ for all $x, y \in R$.
By Lemma 12 we get a contradiction.

5.4 The case of a non-zero extended generalized skew homoderivation

Theorem 30. *Let R be a non-commutative unital prime ring of characteristic different from 2, $0 \neq \lambda, 0 \neq \mu \in C$, α an automorphism of R, $d : R \longrightarrow Q_r$ a non-zero extended homoderivation associated with λ and $F : R \longrightarrow Q_r$ a non-zero extended generalized skew homoderivation associated with μ, α and d.*
Consider $S = \{F(x)x, x \in R\}$ and indicate with $A =< F(x)x >$ the sub-ring generated by S in R.
A contains a left non-trivial ideal I.

Proof:
Suppose that A contains neither non-trivial right ideals nor non-trivial left ideals.
For all $x, y \in R$, the element $F(x+y)(x+y)$ is in $S \subseteq A$ therefore

$$F(x)x + F(y)y + F(x)y + F(y)x \in A$$

Since $F(x)x + F(y)y \in A$ then

$$F(x)y + F(y)x \in A \qquad \text{for all } x, y \in R \tag{5.62}$$

Consider $z \in S$. Replacing y with yz, we have

$$F(x)yz + F(yz)x \in A \qquad \text{for all } x, y \in R, z \in S$$

that is

$$F(x)yz + \mu F(y)F(z)x + F(y)zx + \alpha(y)d(z)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.63}$$

Since $z \in S$ then multiplying (5.62) to the right by $z \in S$ we still get an element of A

$$F(x)yz + F(y)xz \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.64}$$

Since the difference of elements of A is still an element of A then subtracting (5.63) and (5.64) we get the following element of A

$$F(y)[z,x] + \Big(\mu F(y)F(z) + \alpha(y)d(z)\Big)x \in A \qquad \text{for all } x, y \in R, z \in S \tag{5.65}$$

Consider $r \in S$. Since $x \in R$ is arbitrary, we can replace it with xr in (5.65)

$$F(y)[z,x]r + F(y)x[z,r] + \Big(\mu F(y)F(z) + \alpha(y)d(z)\Big)xr \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.66}$$

Multiplying (5.65) by the element $r \in S$ to the right we get another element of A that is

$$F(y)[z,x]r + \Big(\mu F(y)F(z) + \alpha(y)d(z)\Big)xr \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.67}$$

Subtracting (5.67) from (5.66) we get

$$F(y)x[z,r] \in A \quad \text{for all } x, y \in R, z, r \in S \tag{5.68}$$

Let $t \in R$. Since y is arbitrary, we can replace it with ty in (5.68). We have

$$F(ty)x[z,r] \in A \quad \text{for all } x, y, t \in R, z, r \in S$$

that is, for all $x, y, t \in R, z, r \in S$

$$F(t)(\mu F(y)x)[z,r] + F(t)(yx)[z,r] + \alpha(t)d(y)x[z,r] \in A \tag{5.69}$$

By (5.68), we have $F(t)\mu F(y)x[z,r] + F(t)yx[z,r] \in A$ therefore

$$\alpha(t)d(y)x[z,r] \in A \quad \text{for all } x, y, t \in R, z, r \in S$$

and, replacing t with $\alpha^{-1}(t)$ we have

$$td(y)x[z,r] \in A \quad \text{for all } x, y, t \in R, z, r \in S$$

Then $Rd(R)R[S,S] \subseteq A$.
Since A does not contain non-zero left ideals then $Rd(R)R[S,S] = 0$.
Since R a prime ring then $d(R) = 0$ or $[S,S] = 0$.
Since $d \neq 0$ then $[S,S] = 0$ that is

$$[F(x)x, F(y)y] = 0 \quad \text{for all } x, y \in R \tag{5.70}$$

By Theorem 25, the cases to be analyzed are:

- $F = d$ is an extended homoderivation;
- $d(x) = -\mu^{-1}x$ is an extended homoderivation associated with μ;
  $F(x) = \mu^{-1}\alpha(x)$;
- $d(x) = -\lambda^{-1}x$ is an extended homoderivation associated with λ;
  $F(x) = ax$ with $a = F(1_R)$ and $a^2 = \mu^{-1}\lambda^{-1} \in C$;
- $\alpha(x) = qxq^{-1}$, $\eta \in C$, $q_0 = \eta^{-1}q$, $q_0^2 = 1$;
  $d(x) = q_0F(x)$ is an extended homoderivation associated with μ;
  $F(x) = q_0d(x)$ and we have

$$\Big(\mu F(x) + q_0 x\Big)(1 - q_0) = 0, \qquad \Big(d(x) + \mu^{-1}x\Big)(q_0 - 1) = 0.$$

- $\alpha(x) = qxq^{-1}$, $\eta \in C$, $\lambda q^2 = \eta^2\mu$;
  $d(x) = -\lambda^{-1}x$ is an extended homoderivation associated with λ;
  $F(x) = -q^{-1}\eta\lambda^{-1}x$.

$\boxed{\text{I case}}$ Suppose that $F = d$ is an extended homoderivation.
Then by Lemma 13 we have $F = d = 0$, a contradiction.

$\boxed{\text{II case}}$ Suppose that $F(x) = \mu^{-1}\alpha(x)$ and $d(x) = -\mu^{-1}x$ for all $x \in R$.
Then by (5.70) we have

$$[\alpha(x)x, \alpha(y)y] = 0 \quad \text{for all } x, y \in R$$

By [20], $x^2 \in C$ for all $x \in R$, a contradiction since R is not commutative.

$\boxed{\text{III case}}$ Suppose that $d(x) = -\lambda^{-1}x$ and $F(x) = ax$ with $a = F(1_R)$ and $a^2 \in C$.
Then by (5.70) we have $[ax^2, ay^2] = 0$ for all $x, y \in R$.
By Lemma 12 we get a contradiction.

$\boxed{\text{IV case}}$ Suppose that $\alpha(x) = qxq^{-1}$, $\eta \in C$, $q_0 = \eta^{-1}q$, $q_0^2 = 1$, $d(x) = q_0F(x)$ is an extended homoderivation associated with μ, $F(x) = q_0d(x)$ and we have

$$\Big(\mu F(x) + q_0 x\Big)(1 - q_0) = 0, \qquad \Big(d(x) + \mu^{-1}x\Big)(q_0 - 1) = 0 \tag{5.71}$$

Then

$$d(x)q_0 = d(x) - \mu^{-1}x(q_0 - 1) \tag{5.72}$$

and in particular

$$d(1_R)q_0 = d(1_R) - \mu^{-1}(q_0 - 1) \tag{5.73}$$

Moreover, by Proposition 3 we have

$$d(1_R)(\mu d(x) + x) = 0 = (\mu d(x) + x)d(1_R) \qquad \text{for all } x \in R \tag{5.74}$$

In particular

$$d(1_R)x = -\mu d(1_R)d(x) \qquad \text{for all } x \in R \tag{5.75}$$

By (5.70) we have

$$[q_0d(x)x, q_0d(y)y] = 0 \quad \text{for all } x, y \in R \tag{5.76}$$

Replacing y with $y+z$ in we have

$$[q_0d(x)x, q_0d(y+z)(y+z)] = 0 \quad \text{for all } x,y,z \in R$$

then

$$[q_0d(x)x, q_0d(y)y] + [q_0d(x)x, q_0d(y)z] + [q_0d(x)x, q_0d(z)y] + [q_0d(x)x, q_0d(z)z] = 0$$
for all $x,y,z \in R$

By (5.70) we have

$$[q_0d(x)x, q_0d(y)z] + [q_0d(x)x, q_0d(z)y] = 0 \quad \text{for all } x,y,z \in R$$

Replacing y with 1_R we get

$$[q_0d(x)x, q_0d(1_R)z + q_0d(z)] = 0 \quad \text{for all } x,z \in R$$

By (5.75) we have

$$[q_0d(x)x, -\mu q_0d(1_R)d(z) + q_0d(z)] = 0 \quad \text{for all } x,z \in R$$

that is

$$[q_0d(x)x, q_0(1-\mu d(1_R))d(z)] = 0 \quad \text{for all } x,z \in R$$

Indicate $b = q_0(1_R - \mu d(1_R))$ and then consider the identity

$$[q_0d(x)x, bd(z)] = 0 \quad \text{for all } x,z \in R \tag{5.77}$$

By substituting x with $x+t$ we get

$$[q_0d(x+t)(x+t), bd(z)] = 0 \quad \text{for all } x,t,z \in R$$

then

$$[q_0d(x)x, bd(z)] + [q_0d(x)t, bd(z)] + [q_0d(t)x, bd(z)] + [q_0d(t)t, bd(z)] = 0 \quad \text{for all } x,t,z \in R$$

By (5.77) we get

$$[q_0d(x)t + q_0d(t)x, bd(z)] = 0 \quad \text{for all } x,t,z \in R$$

Replacing x with 1_R we have

$$[q_0d(1_R)t + q_0d(t), bd(z)] = 0 \quad \text{for all } t,z \in R$$

By (5.75) we have

$$[-\mu q_0d(1_R)d(t) + q_0d(t), bd(z)] = 0 \quad \text{for all } t,z \in R$$

that is

$$[q_0(1_R - \mu d(1_R))d(t), bd(z)] = 0 \quad \text{for all } t,z \in R$$

that is

$$[bd(t), bd(z)] = 0 \quad \text{for all } t,z \in R \tag{5.78}$$

Replacing z with zr we get

$$[bd(t), bd(zr)] = 0 \quad \text{for all } t,z,r \in R$$

that is

$$[bd(t), \mu bd(z)d(r) + bd(z)r + bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

By (5.78) we have

$$\mu bd(z)[bd(t), d(r)] + bd(z)[bd(t), r] + [bd(t), bzd(r)] = 0 \quad \text{for all } t, z, r \in R$$

Replacing t with 1_R we get

$$\mu bd(z)[bd(1_R), d(r)] + bd(z)[bd(1_R), r] + [bd(1_R), bzd(r)] = 0 \quad \text{for all } z, r \in R$$

that is

$$-bd(z)(\mu d(r) + r)bd(1_R) + bd(1_R)bzd(r) - bzd(r)bd(1_R) = 0 \quad \text{for all } z, r \in R$$

Multiplying to the right by $(\mu d(p) + p)$ with $p \in R$ we have

$$\begin{aligned} &- bd(z)(\mu d(r) + r)bd(1_R)(\mu d(p) + p) + bd(1_R)bzd(r)(\mu d(p) + p) - \\ &- bzd(r)bd(1_R)(\mu d(p) + p) = 0 \quad \text{for all } z, r, p \in R \end{aligned}$$

By (5.74) we have $d(1_R)(\mu d(p) + p) = 0$ therefore

$$bd(1_R)bzd(r)(\mu d(p) + p) = 0 \quad \text{for all } z, r, p \in R$$

Since R is a prime ring then

- $bd(1_R)b = 0$, that is $q_0\Big(1_R - \mu d(1_R)\Big)d(1_R)q_0\Big(1_R - \mu d(1_R)\Big) = 0$, or
- $d(r)(\mu d(p) + p) = 0$ for all $r, p \in R$

$\boxed{\text{I subcase}}$ Suppose that $q_0\Big(1_R - \mu d(1_R)\Big)d(1_R)q_0\Big(1_R - \mu d(1_R)\Big) = 0$.
Since q_0 is a non-zero invertible element then

$$\Big(1_R - \mu d(1_R)\Big)d(1_R)q_0\Big(1_R - \mu d(1_R)\Big) = 0$$

By (5.74) we have $d(1_R)(\mu d(1_R) + 1_R) = 0$ that is

$$-\mu[d(1_R)]^2 = d(1_R)$$

Therefore by $\Big(1_R - \mu d(1_R)\Big)d(1_R)q_0\Big(1_R - \mu d(1_R)\Big) = 0$ we have:

$$\Big(d(1_R) - \mu[d(1_R)]^2\Big)q_0\Big(1_R - \mu d(1_R)\Big) = 0$$

$$\Big(d(1_R) + d(1_R)\Big)q_0\Big(1_R - \mu d(1_R)\Big) = 0$$

$$2d(1_R)q_0\Big(1_R - \mu d(1_R)\Big) = 0 \tag{5.79}$$

By (5.73) we have $d(1_R)q_0 = d(1_R) - \mu^{-1}(q_0 - 1)$. Then

$$2\Big(d(1_R) - \mu^{-1}(q_0 - 1)\Big)\Big(1_R - \mu d(1_R)\Big) = 0$$

$$2\Big(\mu d(1_R) + 1 - q_0\Big)\Big(1_R - \mu d(1_R)\Big) = 0$$

Since $char(R) \neq 2$ then

$$\Big(\mu d(1_R) + 1_R\Big)\Big(1_R - \mu d(1_R)\Big) = q_0\Big(1_R - \mu d(1_R)\Big)$$

$$\mu d(1_R) - \mu^2[d(1_R)]^2 + 1_R - \mu d(1_R) = q_0\Big(1_R - \mu d(1_R)\Big)$$

$$\mu d(1_R) + \mu d(1_R) + 1_R - \mu d(1_R) = q_0\Big(1_R - \mu d(1_R)\Big)$$

$$\mu d(1_R) + 1_R = q_0\Big(1_R - \mu d(1_R)\Big)$$

Multiplying on the right by $d(1_R)$ we then have

$$\Big(\mu d(1_R) + 1_R\Big)d(1_R) = q_0\Big(1_R - \mu d(1_R)\Big)d(1_R)$$

$$0 = q_0\Big(d(1_R) - \mu[d(1_R)]^2\Big)$$

$$0 = q_0\Big(d(1_R) + d(1_R)\Big)$$

$$0 = 2q_0 d(1_R)$$

Since q_0 is an invertible element and $char(R) \neq 2$ then $d(1_R) = 0$.
Moreover by (5.71) we have

$$0 = \Big(d(1_R) + \mu^{-1}\Big)(q_0 - 1) = \mu^{-1}(q_0 - 1)$$

that is $q_0 = 1$.
Therefore $b = q_0(1_R + \mu d(1_R)) = q_0 = 1$ and (5.78) is reduced to

$$[d(x), d(y)] = 0 \quad \text{for all } x, y \in R$$

and by Lemma 7 we get $d = 0$, a contradiction.

$\boxed{\text{II subcase}}$ Suppose that $d(z)(\mu d(t) + t) = 0$ for all $z, t \in R$.
By Remarks 10 and 11 we have $\mu d(t) + t = 0$ for all $t \in R$, that is

$$d(t) = -\mu^{-1}t$$

for all $t \in R$.
In this case, by (5.76) we get

$$[q_0 x^2, q_0 y^2] = 0 \quad \text{for all } x, y \in R$$

By Lemma 12 we get a contradiction.

$\boxed{\text{V case}}$ Suppose that $\alpha(x) = qxq^{-1}$, $\eta \in C$, $\lambda q^2 = \eta^2 \mu$, $d(x) = -\lambda^{-1} x$ is an extended homoderivation associated with λ and $F(x) = -q^{-1}\eta\lambda^{-1}x$.
By (5.70) we have

$$[q^{-1}x^2, q^{-1}y^2] = 0 \quad \text{for all } x, y \in R \tag{5.80}$$

By Lemma 12 we get a contradiction.

Chapter 6

An application: on the distance of generalized skew derivations to the Jordan homoderivations

To conclude this dissertation, we propose an application of the theoretical framework developed thus far.
As a preliminary step, we introduce the following definition:

Definition 10. *An additive map $d : R \longrightarrow R$ is called a Jordan homoderivation, if*

$$d(x^2) = d(x)^2 + d(x)x + xd(x) \quad \text{holds for all } x \in R.$$

The principal result we aim to establish is the following:

Theorem 31. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Let L be a non-central Lie ideal of R, F and G be generalized skew derivations of R associated with the same automorphism α such that*

$$F(x^2) = G(x)^2 + G(x)x + xG(x)$$

for any $x \in L$.
Then either R satisfies s_4 or one of the following conditions holds

(a) *there exists $\lambda \in C$ such that*

$$F(x) = \lambda(\lambda + 2)x \quad \text{and} \quad G(x) = \lambda x$$

for any $x \in R$;

(b) *there exist $\lambda \in C$ and an automorphism α of R such that*

$$F(x) = \lambda^2\alpha(x) - x \quad \text{and} \quad G(x) = \lambda\alpha(x) - x$$

for any $x \in R$.

Let us begin by noting that the statement of the Theorem 31 may be interpreted as follows: *what is the precise structure of the maps F and G when they behave as a generalization of Jordan homoderivations?*
Indeed, when $F = G$, we recover a Jordan homoderivation.

Furthermore, we observe that in the special case $F = G$, both conclusions derived from the Theorem 31 imply that the map in question is, in fact, a homoderivation. This naturally leads us to inquire whether, analogously to the classical correspondence between Jordan derivations and derivations in characteristic not equal to 2, every Jordan homoderivation is necessarily a homoderivation in this setting as well.

We now proceed by presenting a series of intermediate results that will be fundamental to establish the final proof of Theorem 31. Firstly we recall some well known results related to a prime ring R that will be useful to us later.

Preliminary results

Fact 1. Let R be a prime ring, then the following statements hold:

- Any automorphism of R can be uniquely extended to Q_r [11, Fact 2].
- Every generalized skew derivation of R can be uniquely extended to Q_r as follows: by a (right) generalized skew derivation we mean an additive mapping $G : Q_r \longrightarrow Q_r$ such that

$$G(xy) = G(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in Q_r,$$

  where d is a skew derivation of R and α is an automorphism of R. Moreover, there exists $G(1) = a \in Q_r$ such that $G(x) = ax + d(x)$ for all $x \in R$ [9, Lemma 2].

Fact 2. If $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, for D an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates [14].

Fact 3. Let $\Phi(x_i, \alpha(x_i))$ be a generalized polynomial identity for R, α an outer automorphim of R.
If $\alpha(x_i)$-word degree is $\leq m$, for any indeterminate x_i and either $char(R) = 0$ or $char(R) \geq m+1$, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates
[12, Theorem 3].

Fact 4. Let R be a prime ring and I be a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r [13].
Furthermore, I, R and Q_r satisfy the same generalized polynomial identities with automorphisms
[12, Theorem 1].

Fact 5. Let R be a prime ring, $\alpha, \beta \in Aut(Q_r)$ and d, δ be skew derivations of R associated with the automorphism α. If there exist $0 \neq \eta \in C$ and $p \in Q_r$ such that

$$\delta(x) = \eta d(x) + px - \beta(x)p \qquad \text{for all } x \in R,$$

then either $\alpha = \beta$ or $px - \beta(x)p = 0$ and $\delta(x) = \eta d(x)$ for any $x \in R$.
[17, Lemma 3.2].

Fact 6. Let C be a infinite field and $m \geq 2$.
If $A_1, \ldots, A_k$ are not scalar matrices in $M_m(C)$ then there exists some invertible matrix $P \in M_m(C)$ such that any matrix $PA_1P^{-1}, ..., PA_kP^{-1}$ has all non-zero entries [16, Lemma1.5].

Fact 7. Let C be a field of characteristic different from 2, $R = M_t(C)$ the matrix ring over C and $t \geq 3$. Let $a, b \in R$, with $a = \sum_{r,s=1}^{t} a_{rs}e_{rs}$ and $b = \sum_{r,s=1}^{t} b_{rs}e_{rs}$, for $a_{rs}, b_{rs} \in C$.
Suppose that a, b satisfy the following condition:
If $i \neq j$ are fixed integers such that $a_{ij}b_{ij} = 0$, then for any inner automorphism φ of R it follows $a'_{ij}b'_{ij} = 0$ where $\varphi(a) = \sum_{r,s=1}^{t} a'_{rs}e_{rs}$, $\varphi(b) = \sum_{r,s=1}^{t} b'_{rs}e_{rs}$.
Then, if $a_{ij}b_{ij} = 0$ for all $i \neq j$, either $a \in C$ or $b \in C$
[18, Proposition 1].

The inner case

In this section, we initiate our analysis by considering maps expressed in their inner forms.
We begin by focusing on the particular case where $L = [R, R]$, and where F and G are assumed to be inner generalized skew derivations of the ring R.

Recall that an example of a generalized derivation is a map of the from

$$F(x) = ax + xb,$$

for some $a, b \in R$; such maps are referred to as inner generalized derivations.
Furthermore, an additive map F is called a generalized skew derivation if there exists a skew derivation d of R, associated with an automorphism $\alpha \in Aut(R)$ such that

$$F(xy) = F(x)y + \alpha(x)d(y),$$

for all $x, y \in R$.
In this context, d is called the associated skew derivation of F and α is referred to as its associated automorphism. A standard example in this setting is the inner generalized skew derivation, given by

$$F(x) = ax + \alpha(x)b$$

for some $a, b \in R$ and $\alpha \in Aut(R)$.

Lemma 14. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid and $a, b, c, d \in Q_r$ such that*

$$(c-a)[x_1,x_2]^2 + [x_1,x_2]^2(d-b) + c[x_1,x_2]^2 d + [x_1,x_2](dc+d+c)[x_1,x_2] + \\ + c[x_1,x_2]c[x_1,x_2] + [x_1,x_2]d[x_1,x_2]d$$

is a generalized polynomial identity for R. If R does not satisfy any non-trivial generalized identity then $a, b, c, d \in C$ and $a + b = (c+d)(c+d+2)$.

Proof:
By our hypothesis,

$$(c-a)[x_1,x_2]^2 + [x_1,x_2]^2(d-b) + c[x_1,x_2]^2 d + [x_1,x_2](dc+d+c)[x_1,x_2] + c[x_1,x_2]c[x_1,x_2] + \\ + [x_1,x_2]d[x_1,x_2]d = 0 \in T = Q_r *_C C\{x_1,x_2\} \tag{6.1}$$

where $T = Q_r *_C C\{x_1, x_2\}$ is the free product over C of the C-algebra Q_r and the free C-algebra $C\{x_1, x_2\}$.

If $\{d,1\}$ are are linearly C-independent, then

$$(c+1)[x_1,x_2]^2d+[x_1,x_2]d[x_1,x_2]d=0\in T=Q_r*_C C\{x_1,x_2\}$$

that is

$$(c+1)[x_1,x_2]^2+[x_1,x_2]d[x_1,x_2]=0\in T=Q_r*_C C\{x_1,x_2\} \tag{6.2}$$

If $\{c,1\}$ are are linearly C-independent, then

$$(c+1)[x_1,x_2]^2=0\in T=Q_r*_C C\{x_1,x_2\}$$

a contradiction.
Then we may assume $c\in C$ and by (6.2) we have

$$[x_1,x_2](d+c+1)[x_1,x_2]=0\in T=Q_r*_C C\{x_1,x_2\}$$

that is $d+c+1=0$. Then $d=-c-1\in C$, a contradiction.

Consider now $d\in C$. Then by (6.1) we have

$$\begin{aligned}&(c-a+2d+d^2+cd)[x_1,x_2]^2-[x_1,x_2]^2b+(d+c+1)[x_1,x_2]c[x_1,x_2]=\\&=0\in T=Q_r*_C C\{x_1,x_2\}\end{aligned} \tag{6.3}$$

If $\{b,1\}$ are are linearly C-independent, then $[x_1,x_2]^2b=0\in T=Q_r*_C C\{x_1,x_2\}$, a contradiction.
Then $b\in C$ and by (6.3) we get

$$\begin{aligned}&(c-a+2d+d^2+cd-b)[x_1,x_2]^2+(d+c+1)[x_1,x_2]c[x_1,x_2]=\\&=0\in T=Q_r*_C C\{x_1,x_2\}\end{aligned} \tag{6.4}$$

If $\{c,1\}$ are are linearly C-independent, then

$$c[x_1,x_2](c+d+1)[x_1,x_2]=0\in T=Q_r*_C C\{x_1,x_2\}$$

Hence $c+d+1=0$ that is $c=-d-1\in C$, a contradiction.
Therefore $c\in C$ and by (6.4) we have

$$\Big(-a-b+(c+d)(c+d+2)\Big)[x_1,x_2]^2=0\in T=Q_r*_C C\{x_1,x_2\}\quad \text{with } b,c,d\in C \tag{6.5}$$

Hence we have $a\in C$ and $a+b=(c+d)(c+d+2)$.

Lemma 15. *Let R be a prime ring of characteristic different from 2 and $b,d\in Q_r$ be such that*

$$[x_1,x_2]^2b-[x_1,x_2]d[x_1,x_2]d=0 \tag{6.6}$$

is a non-trivial generalized polynomial identity for R. Then $b,d\in C$, $b=d^2$.

Proof:
Since (6.6) is a non-trivial generalized polynomial identity for R, in view of [24, Theorem 2.5 and Theorem 3.5], we know that both Q_r and $Q_r\otimes_C\bar{C}$ are centrally closed, where $\bar{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r\otimes_C\bar{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem, Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In

light of Jacobson's theorem [25, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C.
If $dim_C V = t$ is finite, $t \geq 2$, then $Q_r \cong M_t(C)$.
Taking e_{ii} and e_{ij} instead of x_1 and x_2 respectively in (6.6) with $i, j \in \{1, \ldots, t\}$, $i \neq j$ we have $e_{ij}de_{ij}d = 0$ that is $d \in C$. Then by (6.6) we get $[x_1, x_2]^2(b - d^2) = 0$ that is $b = d^2 \in C$.
Assume $dim_C V = \infty$ and that there exist $h_1, h_2 \in H$ such that $[h_1, b] \neq 0$ or $[h_2, d] \neq 0$. By Litoff's theorem there exists an idempotent element $e \in H$ such that

$$h_1, h_2, h_1b, bh_1, h_2d, dh_2 \in eQ_re \cong M_t(C)$$

for some integer t. Since

$$[x_1, x_2]^2(ebe) - [x_1, x_2](ede)[x_1, x_2](ede) = 0$$

is a generalized identity for $eQ_re \cong M_t(C)$ then by the finite dimensionality case we have $ede, ebe \in eCe$ and this contradicts the choices of h_1 and h_2.

Lemma 16. *Let R be a prime ring of characteristic different from 2 and $a, c \in Q_r$ be such that*

$$(c - a)[x_1, x_2]^2 + (c + 1)[x_1, x_2]c[x_1, x_2] = 0 \tag{6.7}$$

is a non-trivial generalized polynomial identity for R. Then $a, c \in C$, $a = c^2 + 2c$.

Proof:
Since (6.7) is a non-trivial generalized polynomial identity for R, in view of [24, Theorem 2.5 and Theorem 3.5], we know that both Q_r and $Q_r \otimes_C \bar{C}$ are centrally closed, where $\bar{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \otimes_C \bar{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem, Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [25, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C.
If $dim_C V = t$ is finite, $t \geq 2$, then $Q_r \cong M_t(C)$.
Taking e_{ii} and e_{ij} instead of x_1 and x_2 respectively in (6.7) with $i, j \in \{1, \ldots, t\}$, $i \neq j$ we have $(c + 1)e_{ij}ce_{ij} = 0$ that is $c \in C$. Then by (6.7) we get $(c^2 + 2c - a)[x_1, x_2]^2 = 0$ that is $a = c^2 + 2c \in C$.
Assume $dim_C V = \infty$ and that there exist $h_1, h_2 \in H$ such that $[h_1, a] \neq 0$ or $[h_2, c] \neq 0$. By Litoff's theorem there exists an idempotent element $e \in H$ such that

$$h_1, h_2, h_1a, ah_1, h_2c, ch_2 \in eQ_re \cong M_t(C)$$

for some integer t. Since

$$e(c - a)e[x_1, x_2]^2 + e(c + 1)e[x_1, x_2]ece[x_1, x_2] = 0$$

is a generalized identity for $eQ_re \cong M_t(C)$ then by the finite dimensionality case we have $eae, ece \in eCe$ and this contradicts the choices of h_1 and h_2.

Lemma 17. *Let R be a prime ring of characteristic different from 2 and $u, \omega, t, c, d \in Q_r$ be such that*

$$\phi(x_1, x_2) =$$
$$u[x_1, x_2]^2 + [x_1, x_2]^2\omega + c[x_1, x_2]^2d + [x_1, x_2]t[x_1, x_2] + c[x_1, x_2]c[x_1, x_2] + [x_1, x_2]d[x_1, x_2]d \tag{6.8}$$

is a non-trivial generalized polynomial identity for R.
Then $\omega + u = -(c^2 + d^2 + cd + t) \in C$ with $c, d, t \in C$ and, if $R \nsubseteq M_2(C)$, then $\omega, u \in C$.

Proof:
Since (6.8) is a non-trivial generalized polynomial identity for R as well as for Q_r, in view of [24, Theorem 2.5 and Theorem 3.5], both Q_r and $Q_r \otimes_C \bar{C}$ are centrally closed, where $\bar{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \otimes_C \bar{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem, Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring.
In light of Jacobson's theorem [25, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C.
If $dim_C V = t$ is finite then $Q_r \cong M_t(C)$ with $t \geq 2$.
Taking e_{ii} and e_{ij} instead of x_1 and x_2 respectively in (6.8) with $i, j \in \{1, \dots, t\}, i \neq j$ we have

$$e_{ij}te_{ij} + ce_{ij}ce_{ij} + e_{ij}de_{ij}d = 0 \tag{6.9}$$

Multiplying (6.9) to the left by e_{ij}, we obtain $e_{ij}ce_{ij}ce_{ij} = 0$ thus $(c_{ji})^2 = 0$ and then $c_{ji} = 0$ for all $i \neq j$. With standard arguments it follows that c is central.
Similarly, multiplying to the right (6.9) by e_{ij}, it follows that d is central.
Then by (6.8) we get

$$[x_1, x_2]^2\omega + [x_1, x_2]t[x_1, x_2] + (c^2 + d^2 + cd + u)[x_1, x_2]^2 = 0$$

and taking e_{ii} and e_{ij} instead of x_1 and x_2 we get $t \in C$.
Then R satisfies

$$[x_1, x_2]^2\omega + (c^2 + d^2 + cd + u + t)[x_1, x_2]^2 = 0 \tag{6.10}$$

In case $[x_1, x_2]^2$ is central valued on R, it follows that

$$\omega + u = -(c^2 + d^2 + cd + t) \in C$$

Thus we assume that $[x_1, x_2]^2$ is not central valued on R, that is $t \geq 3$.
Let S be the additive subgroup of R generated by $\{[x_1, x_2]^2 : x_i \in R\}$.
Since $[x_1, x_2]^2$ is not central and $char(R) \neq 2$, then S must contain a non-central Lie ideal L of R. Moreover, since L is not central, then $[R, R] \subseteq L$.
Therefore, by (6.10) we have

$$[r_1, r_2]\omega + (c^2 + d^2 + cd + u + t)[r_1, r_2] = 0$$

Taking e_{ii} and e_{ij} instead of r_1 and r_2 we have

$$e_{ij}\omega + (c^2 + d^2 + cd + u + t)e_{ij} = 0 \tag{6.11}$$

Multiplying (6.11) to the left by e_{jj}, we obtain $(c^2 + d^2 + cd + u + t)_{ji} = 0$ and with standard arguments it follows that $c^2 + d^2 + cd + u + t$ is central.
Multiplying (6.11) to the right by e_{ii}, we obtain $\omega_{ji} = 0$ and with standard arguments it follows that ω is central.
Moreover $\omega + u = -(c^2 + d^2 + cd + t) \in C$.

Now suppose that V has infinite dimension on C and suppose that none of the expected conclusions are true and show that a contradiction follows. Hence, by contradiction we assume there exist $r_1 \in H$ such that either $[c, r_1] \neq 0$ or $[d, r_1] \neq 0$ or $[t, r_1] \neq 0$ or $[\omega + u, r_1] \neq 0$.
By Litoff's Theorem, there exists an idempotent element $e \in H$ such that

$$r_1, cr_1, r_1c, r_1d, dr_1, r_1t, tr_1, r_1(\omega + u), (\omega + u)r_1, \in eQ_re$$

with $eQ_re \cong M_k(C)$, for some integer k. Of course $\phi(x_1, x_2)$ is a generalized identity for eQ_re, as well as

$$eue[x_1, x_2]^2 + [x_1, x_2]^2 e\omega e + ece[x_1, x_2]^2 ede + [x_1, x_2]ete[x_1, x_2]+$$
$$+ ece[x_1, x_2]ece[x_1, x_2] + [x_1, x_2]ede[x_1, x_2]ede$$

Then, by the finite dimensionality case, we have that $ece, ede, ete, e(\omega + u)e \in eCe$ and this contradicts the choice of r_1.

Lemma 18. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid and*

$$F(x) = ax + xb,\ G(x) = cx + xd$$

two inner generalized derivations of R such that R satisfies

$$\phi(x_1, x_2) = G([x_1, x_2])^2 + G([x_1, x_2])[x_1, x_2] + [x_1, x_2]G([x_1, x_2]) - F([x_1, x_2]^2)$$

Then one of the following holds:

- *there exists $\lambda \in C$ such that $F(x) = \lambda(\lambda + 2)x$ and $G(x) = \lambda x$, for any $x \in R$;*
- *$R \subseteq M_2(C)$ and there exists $\lambda \in C$ such that*

$$G(x) = \lambda x \quad and \quad F(x) = \Big(\lambda(\lambda + 2) - b\Big)x + xb$$

Proof:
In light of our assumption, for all $x_1, x_2 \in R$ we have $\phi(x_1, x_2) = 0$ with

$$\phi(x_1, x_2) =$$
$$(c - a)[x_1, x_2]^2 + [x_1, x_2]^2(d - b) + c[x_1, x_2]^2 d + [x_1, x_2](dc + d + c)[x_1, x_2]+$$
$$+ c[x_1, x_2]c[x_1, x_2] + [x_1, x_2]d[x_1, x_2]d$$

By Lemma 14 we may assume that $\phi(x_1, x_2)$ is a non-trivial generalized polynomial identity for R as well as for Q_r and by Lemma 17 it follows that $c, d, a + b \in C$ with $a + b = (c + d)(c + d + 2)$ and, if $R \nsubseteq M_2(C)$, then $a, b \in C$.
The required conclusions follow by setting $c + d = \lambda$ with $\lambda \in C$.

Lemma 19. *Let R be a prime ring of characteristic different from 2, $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in Q_r$ and q an invertible element of $Q_r \setminus C$ such that, for all $X = [x_1, x_2] \in [R, R]$*

$$a_1X^2 - qX^2a_2 + qXa_3X + a_4Xa_5X + qXa_6Xa_7 + a_4XqXa_7 = 0 \tag{6.12}$$

is a non trivial generalized polynomial identity for $[R, R]$.
Then one of the following occurs:

- *$a_2, a_7 \in C$ and there exist $\lambda, \mu \in C$ such that $a_1 = \mu q$, $a_4 = \lambda q$ and $\mu q - qa_2 + q(\lambda a_5 + a_7 a_6 + \lambda a_7 q + a_3) = 0$;*
- *$a_2, a_7, a_5 + a_7 q, a_3 + a_7 a_6 \in C$ and $a_1 - qa_2 + q(a_3 + a_7 a_6) + a_4(a_5 + a_7 q) = 0$;*
- *there exist $\lambda, \mu, \theta, \omega \in C$ such that $a_4 = -\mu q$, $a_6 = \mu q + \lambda$, $a_3 - \mu a_5 = \theta$, $\lambda a_7 - a_2 = \omega$, $a_1 = -(\theta + \omega)q$.*

Proof:
In light of our assumption, for all $X = [x_1, x_2] \in [R, R]$, (6.12) is a non trivial GPI for R and for Q_r.
Moreover Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [25, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume $dim_C V < \infty$. Then $Q_r \cong M_t(C)$, $t \geq 2$.

Suppose that C is infinite.

Assume that $a_6 + a_4$ and a_7 are not scalar matrices. Then by Fact 6 there exists some invertible matrix $P \in M_t(C)$ such that $P(a_6 + a_4)P^{-1}, Pa_7P^{-1}$ have all non-zero entries.
Since (6.12) is a non trivial GPI for R then, in particular, for all $X \in [R, R]$ we have

$$\begin{aligned} &Pa_1P^{-1}X^2 - PqP^{-1}X^2Pa_2P^{-1} + PqP^{-1}XPa_3P^{-1}X + \\ &+ Pa_4P^{-1}XPa_5P^{-1}X + PqP^{-1}XPa_6P^{-1}XPa_7P^{-1} + Pa_4P^{-1}XPqP^{-1}XPa_7P^{-1} = 0 \end{aligned} \tag{6.13}$$

Taking $X = e_{ij}$ instead of X in (6.13) and multiplying to the left and to the right by e_{ij} we get

$$(PqP^{-1})_{ji}(P(a_6 + a_4)P^{-1})_{ji}(Pa_7P^{-1})_{ji} = 0$$

a contradiction. Then $a_7 \in C$ or $a_6 + a_4 \in C$.

$\boxed{\text{I case:}}$ Assume $a_7 \in C$. Then by (6.12) we have

$$a_1X^2 - qX^2a_2 + qX(a_3 + a_7a_6)X + a_4X(a_5 + a_7q)X = 0 \tag{6.14}$$

Assume that $q^{-1}a_4$ and $a_5 + a_7q$ are not scalar matrices. Then by Fact 6 there exists some invertible matrix $Q \in M_t(C)$ such that $Q(q^{-1}a_4)Q^{-1}, Q(a_5 + a_7q)Q^{-1}$ have all non-zero entries.
Since (6.14) is a non trivial GPI for R then, in particular, for all $X \in [R, R]$ we have

$$\begin{aligned} &Qa_1Q^{-1}X^2 - QqQ^{-1}X^2Qa_2Q^{-1} + QqQ^{-1}XQ(a_3 + a_7a_6)Q^{-1}X + \\ &+ Qa_4Q^{-1}XQ(a_5 + a_7q)Q^{-1}X = 0 \end{aligned} \tag{6.15}$$

Taking $X = e_{ij}$ instead of X in (6.15) and multiplying to the left by $e_{ij}Qq^{-1}Q^{-1}$ we get

$$(Qq^{-1}a_4Q^{-1})_{ij}(Q(a_5 + a_7q)Q^{-1})_{ij} = 0$$

a contradiction. Then $q^{-1}a_4 \in C$ or $a_5 + a_7q \in C$.

If $a_5 + a_7 q \in C$ then by (6.14) we have

$$a_1 X^2 - qX^2 a_2 + qX(a_3 + a_7 a_6)X + a_4(a_5 + a_7 q)X^2 = 0 \tag{6.16}$$

Assume $a_3 + a_7 a_6 \notin C$. Then, as before, by Fact 6 there exists some invertible matrix $T \in M_t(C)$ such that $T(a_3 + a_7 a_6)T^{-1}$ has all non-zero entries.
Since (6.16) is a non trivial GPI for R then, in particular, for all $X \in [R, R]$ we have

$$Ta_1 T^{-1} X^2 - TqT^{-1} X^2 Ta_2 T^{-1} + TqT^{-1} XT(a_3 + a_7 a_6)T^{-1} X + Ta_4(a_5 + a_7 q)T^{-1} X^2 = 0 \tag{6.17}$$

For $X = e_{ij}$ in (6.17) and multiplying to the left by e_{ij} we get

$$e_{ij} TqT^{-1} e_{ij} T(a_3 + a_7 a_6)T^{-1} e_{ij} = 0 \tag{6.18}$$

a contradiction.
Then one of the following cases occurs:

$\boxed{\text{I subcase:}}$ $q^{-1}a_4 \in C$.
Then multiplying (6.14) to the left by q^{-1} we get

$$q^{-1} a_1 X^2 - X^2 a_2 + X(a_3 + a_7 a_6)X + q^{-1} a_4 X(a_5 + a_7 q)X = 0$$

and, since $q^{-1}a_4 \in C$, we have

$$q^{-1} a_1 X^2 - X^2 a_2 + X\Big((a_3 + a_7 a_6) + q^{-1} a_4(a_5 + a_7 q)\Big)X = 0 \tag{6.19}$$

Taking $X = e_{ij}$ instead of X we get

$$e_{ij}\Big((a_3 + a_7 a_6) + q^{-1} a_4(a_5 + a_7 q)\Big)e_{ij} = 0$$

and with standard arguments it follows that $\Delta = (a_3 + a_7 a_6) + q^{-1}a_4(a_5 + a_7 q)$ is central.
Hence by (6.19) we have

$$q^{-1} a_1 X^2 - X^2 a_2 + \Delta X^2 = 0 \tag{6.20}$$

Let S be the additive subgroup of R generated by $\{[x_1, x_2]^2 : x_i \in R\}$.
Since $[x_1, x_2]^2$ is not central and $char(R) \neq 2$, then S must contain a non-central Lie ideal L of R. Moreover, since L is not central, then $[R, R] \subseteq L$. Therefore, by (6.20) we have

$$q^{-1} a_1 X - X a_2 + \Delta X = 0 \tag{6.21}$$

Taking $X = e_{ij}$ instead of X we have

$$q^{-1} a_1 e_{ij} - e_{ij} a_2 + \Delta e_{ij} = 0 \tag{6.22}$$

Multiplying (6.22) to the left by e_{ij} we get $e_{ij} q^{-1} a_1 e_{ij} = 0$ and with standard arguments it follows that $q^{-1}a_1$ is central.
Multiplying (6.22) to the right by e_{ij} we get $e_{ij} a_2 e_{ij} = 0$ and with standard arguments it follows that a_2 is central.

Therefore by (6.21) we have $q^{-1}a_1 - a_2 + \Delta = 0$ that is

$$q^{-1}a_1 - a_2 + a_3 + a_7a_6 + q^{-1}a_4(a_5 + a_7q) = 0$$

that is

$$a_1 + q(-a_2 + a_3 + a_7a_6) + a_4(a_5 + a_7q) = 0 \quad \text{with } q^{-1}a_4 \in C, a_2 \in C, q^{-1}a_1 \in C$$

Indicating $\lambda = q^{-1}a_4$, $\mu = q^{-1}a_1$ then $a_4 = \lambda q$, $a_1 = \mu q$ and by (6.12) we have

$$\mu q X^2 - qX^2a_2 + qXa_3X + \lambda qXa_5X + qXa_6Xa_7 + \lambda qXqXa_7 = 0 \quad \text{with } a_2, a_7 \in C \tag{6.23}$$

Taking $X = e_{ij}$ instead of X we have

$$qe_{ij}\Big(a_3 + \lambda a_5 + a_6a_7 + \lambda qa_7\Big)e_{ij} = 0 \quad \text{with } a_2, a_7 \in C$$

and then, with standard arguments, it follows that $\eta = a_3 + a_6a_7 + \lambda(qa_7 + a_5) \in C$. Therefore by (6.23) we have

$$\mu qX^2 - qa_2X^2 + q\eta X^2 = 0$$

and then

$$\mu q - qa_2 + q\eta = 0$$

with $a_1 = \mu q, a_4 = \lambda q, a_2 \in C, a_7 \in C$ and $\eta = a_3 + a_6a_7 + \lambda(qa_7 + a_5) \in C$.

$\boxed{\text{II subcase:}}$ $a_5 + a_7q, a_3 + a_7a_6 \in C$.
By (6.14) we have

$$a_1X^2 - qX^2a_2 + q(a_3 + a_7a_6)X^2 + a_4(a_5 + a_7q)X^2 = 0 \tag{6.24}$$

Let S be the additive subgroup of R generated by $\{[x_1, x_2]^2 : x_i \in R\}$.
Since $[x_1, x_2]^2$ is not central and $char(R) \neq 2$, then S must contain a non-central Lie ideal L of R. Moreover, since L is not central, then $[R, R] \subseteq L$. Therefore, by (6.24) we have

$$a_1X - qXa_2 + q(a_3 + a_7a_6)X + a_4(a_5 + a_7q)X = 0 \tag{6.25}$$

Taking $X = e_{ij}$ and multiplying to the right by e_{ij} we get $-qe_{ij}a_2e_{ij} = 0$ and then, with standard arguments, it follows that $a_2 \in C$.
Then by (6.25) we have

$$a_1 - qa_2 + q(a_3 + a_7a_6) + a_4(a_5 + a_7q) = 0$$

$\boxed{\text{II case:}}$ Assume $a_6 + a_4 \in C$.
If $a_7 \in C$ then for the previous case we are done. Therefore assume $a_7 \notin C$.
Then

$$a_4 = \lambda - a_6 \quad \text{with } \lambda \in C \tag{6.26}$$

and by (6.12) we have

$$a_1X^2 - qX^2a_2 + qXa_3X + \lambda Xa_5X - a_6Xa_5X + qXa_6Xa_7 + \lambda XqXa_7 - a_6XqXa_7 = 0 \tag{6.27}$$

Assume $\lambda q^{-1} - q^{-1}a_6 \notin C$.

Then there exists some invertible matrix $S \in M_t(C)$ such that $S(\lambda q^{-1} - q^{-1}a_6)S^{-1}$ has all non-zero entries.
Since (6.27) is a non trivial GPI for R then, in particular, for all $X \in [R, R]$ we have

$$Sa_1S^{-1}X^2 - SqS^{-1}X^2Sa_2S^{-1} + SqS^{-1}XSa_3S^{-1}X + XS\lambda a_5S^{-1}X - Sa_6S^{-1}XSa_5S^{-1}X + \\ + SqS^{-1}XSa_6S^{-1}XSa_7S^{-1} + XS\lambda qS^{-1}XSa_7S^{-1} - Sa_6S^{-1}XSqS^{-1}XSa_7S^{-1} = 0 \tag{6.28}$$

Taking $X = e_{ij}$ and multiplying to the right by e_{ij} we get

$$SqS^{-1}e_{ij}Sa_6S^{-1}e_{ij}Sa_7S^{-1}e_{ij} + e_{ij}S\lambda qS^{-1}e_{ij}Sa_7S^{-1}e_{ij} - Sa_6S^{-1}e_{ij}SqS^{-1}e_{ij}Sa_7S^{-1}e_{ij} = 0 \tag{6.29}$$

Multiplying to the left by $e_{ij}Sq^{-1}S^{-1}$ we get

$$(S(\lambda q^{-1} - q^{-1}a_6)S^{-1})_{ji}(SqS^{-1})_{ji}(Sa_7S^{-1})_{ji} = 0$$

a contradiction. Then

$$q^{-1}a_6 = \lambda q^{-1} + \mu \quad \text{with } \mu \in C \tag{6.30}$$

and by (6.26) we have $q^{-1}a_4 \in C$.

Assume now that $a_6 - \mu q \notin C$. Then there exists some invertible matrix $N \in M_t(C)$ such that $N(a_6 - \mu q)N^{-1}$ has all non-zero entries.
Since (6.27) is a non trivial GPI for R then, in particular, for all $X \in [R, R]$ we have

$$Na_1N^{-1}X^2 - NqN^{-1}X^2Na_2N^{-1} + NqN^{-1}XNa_3N^{-1}X + XN\lambda a_5N^{-1}X - \\ - Na_6N^{-1}XNa_5N^{-1}X + NqN^{-1}XNa_6N^{-1}XNa_7N^{-1} + XN\lambda qN^{-1}XNa_7N^{-1} - \\ - Na_6N^{-1}XNqN^{-1}XNa_7N^{-1} = 0 \tag{6.31}$$

Taking $X = e_{ij}$ in (6.31) and multiplying to the right by e_{ij} and to the left by $Nq^{-1}N^{-1}$ we get

$$e_{ij}Na_6N^{-1}e_{ij}Na_7N^{-1}e_{ij} + N\lambda q^{-1}N^{-1}e_{ij}NqN^{-1}e_{ij}Na_7N^{-1}e_{ij} - \\ - Nq^{-1}a_6N^{-1}e_{ij}NqN^{-1}e_{ij}Na_7N^{-1}e_{ij} = 0$$

Using (6.30) we have

$$e_{ij}Na_6N^{-1}e_{ij}Na_7N^{-1}e_{ij} - e_{ij}N\mu qN^{-1}e_{ij}Na_7N^{-1}e_{ij} = 0$$

that is

$$(Na_6N^{-1})_{ji}(Na_7N^{-1})_{ji} - (N\mu qN^{-1})_{ji}(Na_7N^{-1})_{ji} = 0$$

a contradiction. Then

$$a_6 = \mu q + \eta \quad \text{with } \eta \in C \tag{6.32}$$

Then

$$q^{-1}a_6 = q^{-1}(\mu q + \eta)$$

and by (6.30) it follows that

$$\lambda q^{-1} + \mu = q^{-1}(\mu q + \eta)$$

that is

$$\lambda q^{-1} + \mu = \mu + q^{-1}\eta$$
$$\lambda = \eta$$

Then by (6.32) we have

$$a_6 = \mu q + \lambda \tag{6.33}$$

and then by (6.26) we have

$$a_4 = \lambda - a_6 = \lambda - \mu q - \lambda = -\mu q \tag{6.34}$$

Hence by (6.27) we have

$$a_1X^2 - qX^2a_2 + qXa_3X + \lambda Xa_5X - (\mu q + \lambda)Xa_5X + qX(\mu q + \lambda)Xa_7 + \\ + \lambda XqXa_7 - (\mu q + \lambda)XqXa_7 = 0$$

that is

$$a_1X^2 + qX^2(-a_2 + \lambda a_7) + qX(a_3 - \mu a_5)X = 0 \tag{6.35}$$

Taking $X = e_{ij}$ we have

$$qe_{ij}(a_3 - \mu a_5)e_{ij} = 0$$

that is

$$a_3 - \mu a_5 = \theta \quad \text{with } \theta \in C \tag{6.36}$$

Then by (6.35) we have

$$a_1X^2 + qX^2(-a_2 + \lambda a_7) + \theta qX^2 = 0 \tag{6.37}$$

Let S be the additive subgroup of R generated by $\{[x_1, x_2]^2 : x_i \in R\}$.
Since $[x_1, x_2]^2$ is not central and $char(R) \neq 2$, then S must contain a non-central Lie ideal L of R. Moreover, since L is not central, then $[R, R] \subseteq L$. Therefore, by (6.37) we have

$$a_1X + qX(-a_2 + \lambda a_7) + \theta qX = 0 \tag{6.38}$$

Taking $X = e_{ij}$ and multiplying to the right by e_{ij} we get

$$qe_{ij}(-a_2 + \lambda a_7)e_{ij} = 0$$

that is

$$-a_2 + \lambda a_7 = \omega \in C \tag{6.39}$$

and then by (6.38) we have

$$a_1 + q(\omega + \theta) = 0 \tag{6.40}$$

Suppose now that C is finite.

Let E be an infinite field which is an extension of the field C and let $\bar{R} = M_t(E) \cong R \otimes_C E$.
Consider the generalized polynomial

$$\phi(x_1, x_2) = a_1[x_1, x_2]^2 - q[x_1, x_2]^2a_2 + q[x_1, x_2]a_3[x_1, x_2] + \\ + a_4[x_1, x_2]a_5[x_1, x_2] + q[x_1, x_2]a_6[x_1, x_2]a_7 + a_4[x_1, x_2]q[x_1, x_2]a_7$$

which is a generalized polynomial identity for R. Moreover, it is multi-homogeneous of multi-degree $(2,2)$ in the indeterminates x_1, x_2. Hence the complete linearization of $\phi(x_1, x_2)$ is a multilinear generalized polynomial $\phi'(x_1, x_2, y_1, y_2)$. Moreover, $\phi'(x_1, x_2, y_1, y_2) = 2^2\phi(x_1, x_2)$. Clearly, the multilinear polynomial $\phi'(x_1, x_2, y_1, y_2)$ is a generalized polynomial identity for R and $\bar{R}$ too. Since $char(R) \neq 2$, we obtain $\phi(x_1, x_2) = 0$ for all $x_1, x_2 \in \bar{R}$, and the conclusion follows by the previous case.

Now suppose that V has infinite dimension on C.

By [33] R satisfies

$$\phi(x) = a_1x^2 - qx^2a_2 + qxa_3x + a_4xa_5x + qxa_6xa_7 + a_4xqxa_7. \tag{6.41}$$

Suppose that none of the expected conclusions are true and show that a contradiction follows.
Therefore, by contradiction, suppose that there exists $h_i \in H$, $i = 1, \dots, 14$, such that

- $[a_7, h_1] \neq 0$;
- $a_4h_2 \neq 0$;
- $[a_3 + a_7a_6, h_3] \neq 0$;
- $[a_2, h_4] \neq 0$;
- $(a_1 + q(a_3 + a_7a_6 - a_2))h_5 \neq 0$;
- $[a_5 + a_7q, h_6] \neq 0$;
- $[a_4, h_7] \neq 0$;
- $(a_1 + q(a_3 + a_7a_6 - a_2) + a_4(a_5 + a_7q))h_8 \neq 0$;
- $\{a_2h_9, a_7h_9, h_9\}$ are C-linearly independent;
- $\{a_1h_{10}, qh_{10}\}$ are C-linearly independent;
- $\{a_4h_{11}, qh_{11}\}$ are C-linearly independent;
- $\{a_6h_{12}, qh_{12}, h_{12}\}$ are C-linearly independent;
- $\{a_7h_{13}, a_2h_{13}, h_{13}\}$ are C-linearly independent;
- $[a_5, h_{14}] \neq 0$.

By Litoff's Theorem, there exists an idempotent element $e \in H$ such that

$$h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, h_9, h_{10}, h_{11}, h_{12}, h_{13}, h_{14},$$

$$a_7h_1, h_1a_7, a_4h_2, (a_3 + a_7a_6)h_3, h_3(a_3 + a_7a_6), a_2h_4, h_4a_2, (a_1 + q(a_3 + a_7a_6 - a_2))h_5,$$

$$(a_5 + a_7q)h_6, h_6(a_5 + a_7q), a_4h_7, h_7a_4, (a_1 + q(a_3 + a_7a_6 - a_2) + a_4(a_5 + a_7q))h_8,$$

$$a_2h_9, a_7h_9, a_1h_{10}, qh_{10}, a_4h_{11}, qh_{11}, a_6h_{12}, qh_{12}, a_7h_{13}, a_2h_{13}, a_5h_{14}, h_{14}a_5 \in eQ_re,$$

with $eQ_re \cong M_k(C)$, for some integer k.
Of course $\phi(x)$ is a generalized identity for eQ_re, as well as

$$\phi(x) = ea_1ex^2 - eqex^2ea_2e + eqexea_3ex + ea_4exea_5ex + eqexea_6exea_7e + ea_4exeqexea_7e.$$

Then from the finite dimensional case a series of contradictions follow.

Theorem 32. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, q an invertible element of $Q_r \setminus C$ and*

$$F(x) = ax + qxq^{-1}b,\ G(x) = cx + qxq^{-1}d$$

two inner generalized skew derivations of R such that R satisfies

$$\phi(x_1, x_2) = G([x_1, x_2])^2 + G([x_1, x_2])[x_1, x_2] + [x_1, x_2]G([x_1, x_2]) - F([x_1, x_2]^2)$$

Then one of the following holds:

(a) there exists $\lambda \in C$ such that

$$F(x) = \lambda(\lambda + 2)x \quad \text{and} \quad G(x) = \lambda x$$

for any $x \in R$;

(b) there exist $\mu \in C$ such that

$$F(x) = \mu^2 qxq^{-1} - x \quad \text{and} \quad G(x) = \mu qxq^{-1} - x$$

for any $x \in R$.

Proof:
In light of our assumption, for all $x_1, x_2 \in R$

$$\phi(x_1, x_2) = (c - a)[x_1, x_2]^2 - q[x_1, x_2]^2 q^{-1}b + q[x_1, x_2]q^{-1}d(c + 1)[x_1, x_2] + \\ + (c + 1)[x_1, x_2]c[x_1, x_2] + q[x_1, x_2]q^{-1}dq[x_1, x_2]q^{-1}d + (c + 1)[x_1, x_2]q[x_1, x_2]q^{-1}d \tag{6.42}$$

is a non trivial GPI for R and for Q_r.
Then, by Lemma 19, one of the following conditions holds:

- $q^{-1}b, q^{-1}d \in C$ and there exist $\lambda, \mu \in C$ such that

$$c - a = \mu q, \quad \text{and} \quad c + 1 = \lambda q \tag{6.43}$$

and

$$\mu q - b + q(\lambda c + q^{-1}dq^{-1}dq + \lambda q^{-1}dq + q^{-1}dc + q^{-1}d) = 0$$

that is, since $q^{-1}d \in C$,

$$\mu q - b + \lambda qc + d^2 + \lambda dq + dc + d = 0 \tag{6.44}$$

Using (6.43) in (6.44) we have

$$c - a - b + c^2 + c + d^2 + dc + d + dc + d = 0$$

that is

$$a + b = c^2 + d^2 + 2dc + 2(d + c)$$

Therefore we have

$$\begin{aligned} F(x) &= ax + qxq^{-1}b = \\ &= (a + b)x = \\ &= \Big(c^2 + d^2 + 2dc + 2(c + d)\Big)x \end{aligned}$$

and

$$\begin{aligned}G(x) &= cx + qxq^{-1}d = \\ &= (c+d)x\end{aligned}$$

Since R satisfies (6.42) then, for all $X = [x_1, x_2] \in [R, R]$, we have

$$(c+d)X(c+d)X + (c+d)X^2 + X(c+d)X - \Big(c^2 + d^2 + 2dc + 2(c+d)\Big)X^2 = 0$$

that is

$$(c+d+1)X(c+d)X - (c^2 + d^2 + 2dc + c + d)X^2 = 0$$

Then by Lemma 16 we have $c + d \in C$, $c^2 + d^2 + 2dc + 2c + 2d \in C$ and

$$c^2 + d^2 + 2dc + 2c + 2d = (c+d)^2 + 2(c+d)$$

that is $dc = cd$.
Therefore, indicating $\gamma = c + d$, we have

$$\begin{aligned}F(x) &= ax + qxq^{-1}b = \\ &= (a+b)x = \\ &= \gamma(\gamma+2)x\end{aligned}$$

and

$$\begin{aligned}G(x) &= cx + qxq^{-1}d = \\ &= \gamma x\end{aligned}$$

- $q^{-1}b, q^{-1}d, c+d \in C$ and

$$c - a - b + q(q^{-1}d(c+1) + q^{-1}dq^{-1}dq) + (c+1)(c+d) = 0$$

that is

$$a + b = (c+d)(c+d+2) \in C$$

Therefore, indicating $\gamma = c + d$, we have

$$\begin{aligned}F(x) &= ax + qxq^{-1}b = \\ &= (a+b)x = \\ &= \gamma(\gamma+2)x\end{aligned}$$

and

$$\begin{aligned}G(x) &= cx + qxq^{-1}d = \\ &= \gamma x\end{aligned}$$

- there exist $\lambda, \mu, \theta, \omega \in C$ such that

(1) $c + 1 = -\mu q$,

(2) $q^{-1}dq = \mu q + \lambda$, that is

$$q^{-1}d = \mu + \lambda q^{-1}$$

(3) $q^{-1}d(c+1) - \mu c = \theta$, and using (1) and (2) we have

$$\begin{aligned} q^{-1}d(c+1) - \mu c =& \theta \\ (\mu + \lambda q^{-1})(-\mu q) - \mu(-\mu q - 1) =& \theta \\ -\mu\lambda + \mu =& \theta \end{aligned}$$

(4) $\lambda q^{-1}d - q^{-1}b = \omega$, and using (2) we have

$$\begin{aligned} q^{-1}b =& \lambda q^{-1}d - \omega \\ =& \mu\lambda + \lambda^2 q^{-1} - \omega \end{aligned}$$

(5) $c - a = -(\theta + \omega)q$, that is

$$c + \theta q = a - q\omega$$

Therefore we have

$$\begin{aligned} & G(x) = cx + qxq^{-1}d \\ \text{and using (2)} \quad & G(x) = cx + \mu qx + \lambda qxq^{-1} \\ & G(x) = (c + \mu q)x + \lambda qxq^{-1} \\ \text{and using (1)} \quad & G(x) = -x + \lambda qxq^{-1} \end{aligned}$$

and

$$\begin{aligned} & F(x) = ax + qxq^{-1}b \\ \text{and using (4)} \quad & F(x) = ax + \mu\lambda qx + \lambda^2 qxq^{-1} - \omega qx \\ & F(x) = (a + \mu\lambda q - q\omega)x + \lambda^2 qxq^{-1} \\ \text{and using (5)} \quad & F(x) = (\mu\lambda q + c + \theta q)x + \lambda^2 qxq^{-1} \\ \text{and using (3)} \quad & F(x) = (\mu q + c)x + \lambda^2 qxq^{-1} \\ \text{and using (1)} \quad & F(x) = -x + \lambda^2 qxq^{-1} \end{aligned}$$

Theorem 33. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid and*

$$F(x) = ax + \alpha(x)b, \; G(x) = cx + \alpha(x)d$$

two non-zero inner generalized skew derivations of R, associated with the same automorphism α such that R satisfies

$$F([x_1, x_2]^2) = G([x_1, x_2])^2 + G([x_1, x_2])[x_1, x_2] + [x_1, x_2]G([x_1, x_2])$$

Then either R satisfies s_4 or one of the following holds:

(a) there exists $\lambda \in C$ such that $F(x) = \lambda(\lambda + 2)x$, $G(x) = \lambda x$, for any $x \in R$;

(b) there exist $\lambda \in C$ and an automorphism α of R such that $F(x) = \lambda^2\alpha(x) - x$, $G(x) = \lambda\alpha(x) - x$, for any $x \in R$.

Proof:
If $\alpha = id_R$ then the conclusions follow from Lemma 18.

If α is inner then the conclusions follow from Theorem 32.
Suppose then that $\alpha \neq id_R$ is outer. Then, by hypothesis R satisfies

$$a[x_1,x_2]^2 + \alpha([x_1,x_2])^2 b = \\ (c[x_1,x_2] + \alpha([x_1,x_2])d)^2 + (c[x_1,x_2] + \alpha([x_1,x_2])d)[x_1,x_2] + [x_1,x_2](c[x_1,x_2] + \alpha([x_1,x_2])d) \tag{6.45}$$

Since α is outer, by Fact 3, R also satisfies

$$a[x_1,x_2]^2 + [y_1,y_2]^2 b = \\ (c[x_1,x_2] + [y_1,y_2]d)^2 + (c[x_1,x_2] + [y_1,y_2]d)[x_1,x_2] + [x_1,x_2](c[x_1,x_2] + [y_1,y_2]d) \tag{6.46}$$

In particular, starting from (6.46) R satisfies simultaneously the following identities

$$[y_1,y_2]^2 b - [y_1,y_2]d[y_1,y_2]d = 0 \tag{6.47}$$

$$(c-a)[x_1,x_2]^2 + (c+1)[x_1,x_2]c[x_1,x_2] = 0 \tag{6.48}$$

By Lemmas 15 and 16 we have

$$a,b,c,d \in C, \quad a = c^2 + 2c, \quad b = d^2. \tag{6.49}$$

Then by (6.45) and (6.49) we have

$$\begin{aligned} & c^2[x_1,x_2]^2 + 2c[x_1,x_2]^2 + \alpha([x_1,x_2])^2 d^2 = \\ & = (c[x_1,x_2] + \alpha([x_1,x_2])d)^2 + (c[x_1,x_2] + \alpha([x_1,x_2])d)[x_1,x_2] + [x_1,x_2](c[x_1,x_2] + \\ & \quad + \alpha([x_1,x_2])d) = \\ & = c^2[x_1,x_2]^2 + cd[x_1,x_2]\alpha([x_1,x_2]) + cd\alpha([x_1,x_2])[x_1,x_2] + d^2\alpha([x_1,x_2])^2 + \\ & \quad + c[x_1,x_2]^2 + d\alpha([x_1,x_2])[x_1,x_2] + c[x_1,x_2]^2 + d[x_1,x_2]\alpha([x_1,x_2]) \end{aligned}$$

that is

$$0 = (c+1)d\Big([x_1,x_2]\alpha([x_1,x_2]) + \alpha([x_1,x_2])[x_1,x_2]\Big)$$

and then $d = 0$ or $c = -1$.
By (6.49) we have $d = b = 0$ or $c = a = -1$.

The main result

In this section, we present the proof of the final theorem of this dissertation.
First, recall that if F and G are generalized skew derivations of R, then we can write $F(x) = ax + d(x)$ and $G(x) = cx + \delta(x)$ for all $x \in R$ and suitable $a, c \in Q_r$ and d, δ skew derivations associated with the same automorphism α [9, Lemma 2].
We also remind that if d is a skew derivation of R then

$$d([x_1,x_2]) = d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)$$

and then

$d([x_1,x_2]^2) =$
$d([x_1,x_2])[x_1,x_2] + \alpha([x_1,x_2])d([x_1,x_2]) =$
$d(x_1)x_2[x_1,x_2] + \alpha(x_1)d(x_2)[x_1,x_2] - d(x_2)x_1[x_1,x_2] - \alpha(x_2)d(x_1)[x_1,x_2] +$
$+ \alpha([x_1,x_2])d(x_1)x_2 + \alpha([x_1,x_2])\alpha(x_1)d(x_2) - \alpha([x_1,x_2])d(x_2)x_1 - \alpha([x_1,x_2])\alpha(x_2)d(x_1)$

Theorem 34. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring and C its extended centroid. Let L be a non-central Lie ideal of R, F and G be generalized skew derivations of R associated with the same automorphism α such that*

$$F(x^2) = G(x)^2 + G(x)x + xG(x)$$

for any $x \in L$.
Then either R satisfies s_4 or one of the following holds:

(a) there exists $\lambda \in C$ such that

$$F(x) = \lambda(\lambda+2)x \quad \text{and} \quad G(x) = \lambda x$$

for any $x \in R$;

(b) there exist $\lambda \in C$ and an automorphism α of R such that

$$F(x) = \lambda^2\alpha(x) - x \quad \text{and} \quad G(x) = \lambda\alpha(x) - x$$

for any $x \in R$.

Proof:
By our main hypothesis, the non central Lie ideal L of R satisfies

$$F(x^2) = G(x)^2 + G(x)x + xG(x) \tag{6.50}$$

Since $char(R) \neq 2$, we may assume there exists a non-central two-sided ideal I of R such that $[I,R] \nsubseteq Z(R)$ and $[I,R] \subseteq L$. Hence the set $[I,R]$ satisfies (6.50) and we know that the prime ring R and its ideal I satisfy the same identities with automorphisms and derivations, so we have that

$$F(x^2) = G(x)^2 + G(x)x + xG(x), \quad \text{for all } x \in [R,R]$$

that is

$a[x_1,x_2]^2 + d([x_1,x_2]^2) =$
$\Big(c[x_1,x_2] + \delta([x_1,x_2])\Big)^2 + \Big(c[x_1,x_2] + \delta([x_1,x_2])\Big)[x_1,x_2] + [x_1,x_2]\Big(c[x_1,x_2] + \delta([x_1,x_2])\Big)$

that is

$$\psi\Big(x_1,x_2,d(x_1),d(x_2),\delta(x_1),\delta(x_2),\alpha(x_1),\alpha(x_2)\Big) =$$
$$a[x_1,x_2]^2 + d(x_1)x_2[x_1,x_2] + \alpha(x_1)d(x_2)[x_1,x_2] - d(x_2)x_1[x_1,x_2] - \alpha(x_2)d(x_1)[x_1,x_2]+$$
$$+\alpha([x_1,x_2])d(x_1)x_2 + \alpha([x_1,x_2])\alpha(x_1)d(x_2) - \alpha([x_1,x_2])d(x_2)x_1 - \alpha([x_1,x_2])\alpha(x_2)d(x_1)-$$
$$-\Big(c[x_1,x_2] + \delta(x_1)x_2 + \alpha(x_1)\delta(x_2) - \delta(x_2)x_1 - \alpha(x_2)\delta(x_1)\Big)^2 -$$
$$-\Big(c[x_1,x_2] + \delta(x_1)x_2 + \alpha(x_1)\delta(x_2) - \delta(x_2)x_1 - \alpha(x_2)\delta(x_1)\Big)[x_1,x_2]-$$
$$-[x_1,x_2]\Big(c[x_1,x_2] + \delta(x_1)x_2 + \alpha(x_1)\delta(x_2) - \delta(x_2)x_1 - \alpha(x_2)\delta(x_1)\Big) = 0 \tag{6.51}$$

Assume that d,δ are not simultaneously inner skew derivations, otherwise the conclusions follow from Theorem 33.

I case Assume that d and δ are C-linearly independent modulo inner skew derivations.
In (6.51) the following words occur:

$$d(x_1),d(x_2),\delta(x_1),\delta(x_2),\alpha(x_1),\alpha(x_2)$$

If we denote
$$d(x_1)=z_1, d(x_2)=z_2, \delta(x_1)=t_1, \delta(x_2)=t_2$$
then we have that $\psi\Big(x_1,x_2,z_1,z_2,t_1,t_2,\alpha(x_1),\alpha(x_2)\Big)=0$ is a generalized identity for Q_r, where z_i,t_i,x_i are independent variables.
In particular for $x_1=0$ we have $0=(t_1x_2-\alpha(x_2)t_1)^2$.
If α is outer by Fact 3, R satisfies $0=(t_1x_2-z_2t_1)^2$ and for $z_2=0$ we get the contradiction $0=(t_1x_2)^2$.
If $\alpha\neq id_R$ is inner then R satisfies $0=(t_1x_2-qx_2q^{-1}t_1)^2$ and replacing t_1 with qt_1 we have that $0=q[t_1,x_2]q[t_1,x_2]$, forcing the commutativity of R.

II case Assume that $d\neq 0$ and $\delta\neq 0$ are C-linearly dependent modulo inner skew derivations.
In this case, there exist $\lambda,\mu\in C$, $u\in Q_r$ and $\beta\in Aut(Q_r)$ such that

$$\lambda d(x)+\mu\delta(x)=ux-\beta(x)u \quad \text{for all } x\in R$$

I subcase Assume $\lambda\neq 0$ and $\mu\neq 0$.
Then
$$\delta(x)=\eta d(x)+px-\beta(x)p \quad \text{for all } x\in R$$
where $p=\mu^{-1}u$ and $\eta=-\mu^{-1}\lambda$.
Moreover, by Fact 5, either $p=0$ or $\alpha=\beta$.
If $p=0$, that is $\delta(x)=\eta d(x)$, then by (6.51) we have

$$\psi\Big(x_1,x_2,d(x_1),d(x_2),\eta d(x_1),\eta d(x_2),\alpha(x_1),\alpha(x_2)\Big)=0$$

Since d is outer, by Fact 2 we have

$$\psi\Big(x_1, x_2, z_1, z_2, \eta z_1, \eta z_2, \alpha(x_1), \alpha(x_2)\Big) = 0$$

and for $x_1 = 0$ we get

$$0 = \Big(\eta z_1 x_2 - \alpha(x_2)\eta z_1\Big)^2$$

If $\alpha = \beta$ then $\delta(x) = \eta d(x) + px - \alpha(x)p$ and

$$G([x_1, x_2]) =$$
$$c[x_1, x_2] + \Big(\eta d(x_1) + px_1 - \alpha(x_1)p\Big)x_2 + \alpha(x_1)\Big(\eta d(x_2) + px_2 - \alpha(x_2)p\Big) -$$
$$-\Big(\eta d(x_2) + px_2 - \alpha(x_2)p\Big)x_1 - \alpha(x_2)\Big(\eta d(x_1) + px_1 - \alpha(x_1)p\Big) =$$
$$(c + p)[x_1, x_2] + \eta d(x_1)x_2 - \eta d(x_2)x_1 + \eta\alpha(x_1)d(x_2) - \eta\alpha(x_2)d(x_1) - \alpha([x_1, x_2])p$$

and then by (6.51) we have

$$\psi_1\Big(x_1, x_2, d(x_1), d(x_2), \alpha(x_1), \alpha(x_2)\Big) =$$
$$a[x_1, x_2]^2 + d(x_1)x_2[x_1, x_2] + \alpha(x_1)d(x_2)[x_1, x_2] - d(x_2)x_1[x_1, x_2] -$$
$$-\alpha(x_2)d(x_1)[x_1, x_2] + \alpha([x_1, x_2])d(x_1)x_2 + \alpha([x_1, x_2])\alpha(x_1)d(x_2) -$$
$$-\alpha([x_1, x_2])d(x_2)x_1 - \alpha([x_1, x_2])\alpha(x_2)d(x_1) - \Big\{(c + p)[x_1, x_2] +$$
$$+\eta d(x_1)x_2 - \eta d(x_2)x_1 + \eta\alpha(x_1)d(x_2) - \eta\alpha(x_2)d(x_1) - \alpha([x_1, x_2])p\Big\}^2 -$$
$$-\Big\{(c + p)[x_1, x_2] + \eta d(x_1)x_2 - \eta d(x_2)x_1 + \eta\alpha(x_1)d(x_2) - \eta\alpha(x_2)d(x_1) -$$
$$-\alpha([x_1, x_2])p\Big\}[x_1, x_2] - [x_1, x_2]\Big\{(c + p)[x_1, x_2] + \eta d(x_1)x_2 - \eta d(x_2)x_1 +$$
$$+\eta\alpha(x_1)d(x_2) - \eta\alpha(x_2)d(x_1) - -\alpha([x_1, x_2])p\Big\} = 0$$

Since d is outer then by Fact 2, R also satisfies

$$\psi_1\Big(x_1, x_2, z_1, z_2, \alpha(x_1), \alpha(x_2)\Big) = 0$$

and for $x_1 = 0$ we have

$$0 = \Big(\eta z_1 x_2 - \eta\alpha(x_2)z_1\Big)^2 \tag{6.52}$$

Therefore both in the case where $p = 0$ and in the case where $\alpha = \beta$, R satisfies the identity (6.52). In that case, if α is outer then by (6.52) and by Fact 3, R satisfies $(\eta z_1 x_2 - t_2\eta z_1)^2 = 0$ and for $t_2 = 0$ we get the contradiction $0 = (\eta z_1 x_2)^2$. If $\alpha \neq id_R$ is inner then by (6.52) R satisfies $(\eta z_1 x_2 - \eta q x_2 q^{-1} z_1)^2 = 0$ and replacing z_1 with qz_1 we have that $0 = \eta^2 q[z_1, x_2]q[z_1, x_2]$, forcing the commutativity of R.

II subcase Assume $\lambda = 0$ and $\mu \neq 0$.
Then $\delta(x) = px - \beta(x)p$ and, since a generalized skew derivation admits a unique associated automorphism, then $\alpha = \beta$ and so $\delta(x) = px - \alpha(x)p$ for all $x \in R$. Then

$$G([x_1, x_2]) = (c + p)[x_1, x_2] - \alpha([x_1, x_2])p$$

If d were inner then the conclusion follows by Theorem 33.
If d is outer then by Fact 2, R satisfies

$$a[x_1, x_2]^2 + z_1x_2[x_1, x_2] + \alpha(x_1)z_2[x_1, x_2] - z_2x_1[x_1, x_2] - \alpha(x_2)z_1[x_1, x_2]+ \\ + \alpha([x_1, x_2])z_1x_2 + \alpha([x_1, x_2])\alpha(x_1)z_2 - \alpha([x_1, x_2])z_2x_1 - \alpha([x_1, x_2])\alpha(x_2)z_1 = \\ = \Big\{(c + p)[x_1, x_2] - \alpha([x_1, x_2])p\Big\}^2 + \Big\{(c + p)[x_1, x_2] - \alpha([x_1, x_2])p\Big\}[x_1, x_2]+ \\ + [x_1, x_2]\Big\{(c + p)[x_1, x_2] - \alpha([x_1, x_2])p\Big\}$$

For $z_1 = z_2 = 0$ we get

$$- a[x_1, x_2]^2 + \Big\{(c + p)[x_1, x_2] - \alpha([x_1, x_2])p\Big\}^2 + \Big\{(c + p)[x_1, x_2] - \alpha([x_1, x_2])p\Big\}[x_1, x_2]+ \\ + [x_1, x_2]\Big\{(c + p)[x_1, x_2] - \alpha([x_1, x_2])p\Big\} = 0$$

If $\alpha(x) = qxq^{-1}$, by Theorem 32 we have $q^{-1}p, c \in C$, $a = c(c + 2) \in C$ and then

$$\delta(x) = px - \alpha(x)p = px - qxq^{-1}p = px - qq^{-1}px = 0$$

a contradiction. Then α is outer and by Fact 3, R satisfies

$$- a[x_1, x_2]^2 + \Big\{(c + p)[x_1, x_2] - [t_1, t_2]p\Big\}^2 + \Big\{(c + p)[x_1, x_2] - [t_1, t_2]p\Big\}[x_1, x_2]+ \\ + [x_1, x_2]\Big\{(c + p)[x_1, x_2] - [t_1, t_2]p\Big\} = 0$$

and for $x_1 = x_2 = 0$ we get the contradiction $\Big\{-[t_1, t_2]p\Big\}^2 = 0$.

III subcase Assume $\lambda \neq 0$ and $\mu = 0$.
Then $d(x) = ux - \beta(x)u$ and since a generalized skew derivation admits a unique associated automorphism, then $\alpha = \beta$ and so $d(x) = ux - \alpha(x)u$ for all $x \in R$ where $u = \lambda^{-1}p$.
If δ were inner then the conclusion follows by Theorem 33.

If δ is outer then by (6.51), and using Fact 2, we have

$$(a+u)[x_1,x_2]^2-\alpha([x_1,x_2]^2)u=$$
$$=\Big(c[x_1,x_2]+t_1x_2+\alpha(x_1)t_2-t_2x_1-\alpha(x_2)t_1\Big)^2+\Big(c[x_1,x_2]+t_1x_2+\alpha(x_1)t_2-$$
$$-t_2x_1-\alpha(x_2)t_1\Big)[x_1,x_2]+[x_1,x_2]\Big(c[x_1,x_2]+t_1x_2+\alpha(x_1)t_2-t_2x_1-\alpha(x_2)t_1\Big)$$

and for $x_1=0$ we have $0=(t_1x_2-\alpha(x_2)t_1)^2$.
If α is outer then by Fact 3, R satisfies $0=(t_1x_2-y_2t_1)^2$ and for $y_2=0$ we get the contradiction $0=(t_1x_2)^2$.
If $\alpha\neq id_R$ is inner then R satisfies $0=(t_1x_2-qx_2q^{-1}t_1)^2$ and replacing t_1 with qt_1 we have that $0=q[t_1,x_2]q[t_1,x_2]$, forcing the commutativity of R.

III case Assume $d=0$ and $\delta\neq 0$.
Since $F\neq 0$ we may assume $a\neq 0$. Moreover δ is not an inner skew derivation of R, otherwise both F and G should be inner generalized skew derivations and the conclusion follows by Theorem 32. By (6.51) R satisfies

$$a[x_1,x_2]^2-\Big(c[x_1,x_2]+\delta(x_1)x_2+\alpha(x_1)\delta(x_2)-\delta(x_2)x_1-\alpha(x_2)\delta(x_1)\Big)^2-$$
$$-\Big(c[x_1,x_2]+\delta(x_1)x_2+\alpha(x_1)\delta(x_2)-\delta(x_2)x_1-\alpha(x_2)\delta(x_1)\Big)[x_1,x_2]-$$
$$-[x_1,x_2]\Big(c[x_1,x_2]+\delta(x_1)x_2+\alpha(x_1)\delta(x_2)-\delta(x_2)x_1-\alpha(x_2)\delta(x_1)\Big)=0$$

Using Fact 2, we have

$$a[x_1,x_2]^2-\Big(c[x_1,x_2]+t_1x_2+\alpha(x_1)t_2-t_2x_1-\alpha(x_2)t_1\Big)^2-$$
$$-\Big(c[x_1,x_2]+t_1x_2+\alpha(x_1)t_2-t_2x_1-\alpha(x_2)t_1\Big)[x_1,x_2]-$$
$$-[x_1,x_2]\Big(c[x_1,x_2]+t_1x_2+\alpha(x_1)t_2-t_2x_1-\alpha(x_2)t_1\Big)=0$$

and for $x_1=0$ we get $(t_1x_2-\alpha(x_2)t_1)^2=0$.
If α is outer by Fact 3, R satisfies $0=(t_1x_2-z_2t_1)^2$ and for $z_2=0$ we get the contradiction $0=(t_1x_2)^2$.
If $\alpha\neq id_R$ is inner then R satisfies $0=(t_1x_2-qx_2q^{-1}t_1)^2$ and replacing t_1 with qt_1 we have that $0=q[t_1,x_2]q[t_1,x_2]$, forcing the commutativity of R.

IV case Assume $d\neq 0$ and $\delta=0$.
Since $G\neq 0$, we may assume in $c\neq 0$. Moreover, we may also suppose d is not an inner skew derivation of R. (6.51) R satisfies

$$(a-c)[x_1,x_2]^2+d(x_1)x_2[x_1,x_2]+\alpha(x_1)d(x_2)[x_1,x_2]-d(x_2)x_1[x_1,x_2]-$$
$$-\alpha(x_2)d(x_1)[x_1,x_2]+\alpha([x_1,x_2])d(x_1)x_2+\alpha([x_1,x_2])\alpha(x_1)d(x_2)-$$
$$-\alpha([x_1,x_2])d(x_2)x_1-\alpha([x_1,x_2])\alpha(x_2)d(x_1)-(c+1)[x_1,x_2]c[x_1,x_2]=0$$

Using Fact 2, we have

$$\begin{aligned}&(a-c)[x_1,x_2]^2+z_1x_2[x_1,x_2]+\alpha(x_1)z_2[x_1,x_2]-z_2x_1[x_1,x_2]-\alpha(x_2)z_1[x_1,x_2]+\\&+\alpha([x_1,x_2])z_1x_2+\alpha([x_1,x_2])\alpha(x_1)z_2-\alpha([x_1,x_2])z_2x_1-\alpha([x_1,x_2])\alpha(x_2)z_1-\\&-(c+1)[x_1,x_2]c[x_1,x_2]=0\end{aligned} \tag{6.53}$$

and for $z_1=z_2=0$ we get $(a-c)[x_1,x_2]^2-(c+1)[x_1,x_2]c[x_1,x_2]=0$.
By Lemma 16 we have $a,c\in C$ with $a=c^2+2c$. Then, by (6.53) we have

$$\begin{aligned}&z_1x_2[x_1,x_2]+\alpha(x_1)z_2[x_1,x_2]-z_2x_1[x_1,x_2]-\alpha(x_2)z_1[x_1,x_2]+\\&+\alpha([x_1,x_2])z_1x_2+\alpha([x_1,x_2])\alpha(x_1)z_2-\alpha([x_1,x_2])z_2x_1-\alpha([x_1,x_2])\alpha(x_2)z_1=0\end{aligned}$$

For $z_1=0$ we have

$$\Big(\alpha(x_1)z_2-z_2x_1\Big)[x_1,x_2]+\alpha([x_1,x_2])\Big(\alpha(x_1)z_2-z_2x_1\Big)=0 \tag{6.54}$$

If α is inner then, replacing z_2 with qx_2 we have $2q[x_1,x_2]^2=0$, a contradiction.
If α is outer then, by (6.54) and by Fact 3 we have

$$\Big(t_1z_2-z_2x_1\Big)[x_1,x_2]+[t_1,t_2]\Big(t_1z_2-z_2x_1\Big)=0$$

and for $t_2=0$, $t_1=x_1$, $z_2=x_2$ we get the contradiction $[x_1,x_2]^2=0$.

Chapter 7

Summary of all possible characterizations of the analyzed maps

To conclude, we present a concise synthesis of the admissible configurations of the newly introduced maps, along with the corresponding parameters, considered within the framework of prime unital rings.

HOMODERIVATION and EXTENDED HOMODERIVATION If $d(xy) = \lambda d(x)d(y) + \mu d(x)y + \mu x d(y)$ then one of the following conditions holds		
μ	λ	d
$\mu = 1$	$\lambda = 1$	d is a homoderivation $d(xy) = d(x)d(y) + d(x)y + xd(y)$
$\mu = 1$	$\lambda \in C$	$d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$
$\mu \neq 1$	$\lambda \in C$	$d(x) = d(1_R)x$
EXTENDED HOMORIGHT MULTIPLIER and EXTENDED HOMOLEFT MULTIPLIER If $F(xy) = \lambda F(x)F(y) + \mu x F(y)$ (or $F(xy) = \lambda F(x)F(y) + \mu F(x)y$) then one of the following conditions holds		
$\mu \neq 0$	λ	F
$\mu = 1$	$\lambda = 0$	$F(xy) = xF(y)$ (or $F(xy) = F(x)y$)
$\mu = 1$	$\lambda \neq 0$	$F = 0$
$\mu \neq 1$	$\lambda \neq 0$	$F(x) = \lambda^{-1}(1-\mu)x$

GENERALIZED HOMODERIVATIONS ASSOCIATED WITH A HOMODERIVATION d

If $F(xy) = F(x)F(y) + F(x)y + xd(y),\ a = F(1_R),$
and $d(xy) = d(x)d(y) + d(x)y + xd(y)$
then $a\Big(d(x) + x\Big) = 0 = \Big(d(x) + x\Big)a$ for all $x \in R$ and one of the following conditions holds

		d	F
		$d(x) = -x$	$F(x) = x$
		$d(x) = -x$	$F(x) = -x$
		$d(x) = [x, a] - axa$, $a \neq 0$	$F(x) = xa$
		$F(x) = d(x)$	$F(x) = d(x)$

EXTENDED GENERALIZED HOMODERIVATIONS ASSOCIATED WITH AN ADDITIVE MAP d

If $F(xy) = \lambda F(x)F(y) + \mu F(x)y + \mu xd(y)$
then one of the following conditions holds

λ	$\mu \neq 0$	d	F
$\lambda \neq 0$	$\mu = 1$	$d = 0$	$F = 0$
$\lambda = 0$	$\mu = 1$	$d = 0$	$F = ax$ with $a = F(1_R)$
$\lambda \neq 0$	$\mu \neq 1$	$d = 0$	$F = \lambda^{-1}(1-\mu)x$
$\lambda = 0$	$\mu = 1$	d is a derivation	F is a generalized derivation associated with d
$\lambda \neq 0$	$\mu = 1$	$d = F$ is a homoderivation	$d = F$ is a homoderivation
$\lambda \neq 0$	$\mu \neq 1$	d is an extended homoderivation $d(xy) = \lambda d(x)d(y) + d(x)y + xd(y)$	$F(x) = \lambda^{-1}(1-\mu)x + d(x)$
		$d(x) = \Big(\mu^{-1} - \mu^{-1}\lambda a\Big)xa - ax$	$F(x) = xa$ with $a = F(1_R)$

EXTENDED GENERALIZED HOMODERIVATIONS ASSOCIATED WITH AN EXTENDED HOMODERIVATION

If $F(xy) = \lambda F(x)F(y) + \mu F(x)y + \mu xd(y)$
and $d(xy) = \eta d(x)d(y) + d(x)y + xd(y)$
then one of the following conditions holds

λ	μ	η	$d(1_R) = a$, $F(1_R) = b$	F and d
$\lambda \neq 0$	$\mu = 0$		$\eta a^2 + a = 0,$ $(\lambda b)^2 = \lambda b$	$F(xy) = \lambda F(x)F(y)$ $d(xy) = \eta d(x)d(y) + d(x)y + xd(y)$
$\lambda = 0$	$\mu = 1_R$		$a = 0$	$F(x) = bx$ $d(x) = 0$
$\lambda \neq 0$	$\mu \neq 0$ $\mu \neq 1_R$		$a = 0$	$F(x) = -\lambda^{-1}(\mu - 1)x$ $d(x) = 0$
$\lambda = 0$	$\mu \neq 0$	$\eta = 0$	$a = 0$	F is a generalized derivation associated with d
$\lambda = \eta \neq 0$	$\mu = 1_R$	$\lambda = \eta \neq 0$	$\eta a^2 + a = 0$	$F(x) = d(x)$
$\lambda = \eta \neq 0$	$\mu \neq 0$ $\mu \neq 1_R$	$\lambda = \eta \neq 0$	$\eta a^2 + a = 0$	$F(x) = xb$
$\lambda = \eta \neq 0$	$\mu \neq 0$ $\mu \neq 1_R$	$\lambda = \eta \neq 0$	$\eta a^2 + a = 0$	$F(x) = d(x) - \big(a - b\big)x$
	$\mu \neq 0$		$\eta a^2 + a = 0$ $0 \neq b \in Z(R)$	$F(x) = bx$ $d(x) = \big(\mu^{-1}b - \lambda\mu^{-1}b^2 - b\big)x$
$\lambda = \eta$	$\mu \neq 0$	$\lambda = \eta$	$\eta a^2 + a = 0$ $0 \neq b \notin Z(R)$ $(1-\mu)b - \lambda b^2 = 0$	$F(x) = xb$ $d(x) = \mu^{-1}xb - bx - \lambda\mu^{-1}bxb$

If $F(xy) = \lambda F(x)F(y) + \mu F(x)y + \mu xd(y)$
and $d(x) = -\eta^{-1}x,\ a = -\eta^{-1}$
then one of the following conditions holds

λ	μ	η	$d(1_R) = a$, $F(1_R) = b$	F and d
$\lambda \neq 0$			$a \in Z(R)$ $b \in Z(R)$	$F(x) = bx$ $d(x) = ax$
$\lambda = 0$	$\mu = 1_R$		$a = 0$	$F(x) = bx$ $d(x) = 0$
$\lambda = 0$	$\mu \neq 1_R$		$a \neq 0$	$F(x) = (1-\mu)^{-1}\mu ax$ $d(x) = ax$

GENERALIZED SKEW HOMODERIVATIONS ASSOCIATED WITH AN ADDITIVE MAP d
If $F(xy)=F(x)F(y)+F(x)y+\alpha(x)d(y)$ with $a=F(1_R)$
then one of the following conditions holds

α	d	F
	$d=0$	$F=0$
$\alpha\neq 0$	$d(x)=(1-a)\alpha(x)a-ax$	$F(x)=\alpha(x)a$
$q\in Q_r, \eta\in C$ $q_0=\eta^{-1}q$ $\alpha(x)=qxq^{-1}$	$d(x)=q_0F(x)$ $d(xy)=d(x)d(y)+d(x)y+q_0^2xd(y)$ $\left(d(x)+q_0^2x\right)(1-q_0)=0$	$F(x)=q_0^{-1}d(x)$ $F(xy)=F(x)F(y)+F(x)y+q_0xF(y)$ $\left(F(x)+q_0x\right)(1-q_0)=0$

EXTENDED GENERALIZED SKEW HOMODERIVATIONS ASSOCIATED WITH AN ADDITIVE MAP d
If $F(xy)=\mu F(x)F(y)+F(x)y+\alpha(x)d(y)$ with $a=F(1_R)$
then one of the following conditions holds

α	d	F
	$d=0$	$F=0$
	d is a derivation	F is a generalized derivation associated with d
$\alpha\neq 0$	$d(x)=(1-\mu a)\alpha(x)a-ax$	$F(x)=\alpha(x)a$
$q\in Q_r, \eta\in C$ $q_0=\eta^{-1}q$ $\alpha(x)=qxq^{-1}$	$d(x)=q_0F(x)$ $d(xy)=d(x)d(y)+d(x)y+q_0^2xd(y)$ $\left(d(x)+\mu^{-1}q_0^2x\right)(q_0-1)=0$	$F(x)=q_0^{-1}d(x)$ $F(xy)=\mu F(x)F(y)+F(x)y+q_0xF(y)$ $\left(\mu F(x)+q_0x\right)(1-q_0)=0$

EXTENDED GENERALIZED SKEW HOMODERIVATIONS ASSOCIATED WITH AN EXTENDED HOMODERIVATION
If $F(xy)=\mu F(x)F(y)+F(x)y+\alpha(x)d(y)$ with $a=F(1_R)$
and $d(xy)=\lambda d(x)d(y)+d(x)y+xd(y)$
then one of the following conditions holds

	d	F
	$F=d$ is an extended homoderivation	$F=d$ is an extended homoderivation
	$d(x)=-\mu^{-1}x$	$F(x)=\mu^{-1}\alpha(x)$
$a^2=\mu^{-1}\lambda^{-1}\in Z(R)$	$d(x)=-\lambda^{-1}x$	$F(x)=ax$
$\alpha(x)=qxq^{-1}$ and $q_0=\eta^{-1}q$ $\eta\in C$ and $q_0^2=1$	$d(x)=q_0F(x)$	$F(x)=q_0d(x)$
$\alpha(x)=qxq^{-1}$ and $q_0=\eta^{-1}q$ $\eta\in C$ and $\lambda q^2=\eta^2\mu$	$d(x)=-\lambda^{-1}x$	$F(x)=-q^{-1}\eta\lambda^{-1}x$

Bibliography

[1] S. A. Amitsur. "Generalized Polynomial Identities and Pivotal Monomials". In: *Transactions of the American Mathematical Society* 114.1 (1965), pp. 210–226.

[2] N. Argaç and V. De Filippis. "Power-Central Values and Engel Conditions in Prime Rings with Generalized Skew Derivations". In: *Mediterranean Journal of Mathematics* 18 (2021), p. 82.

[3] M. Ashraf and M. A. Saddeeque. "Posner's first theorem for *-ideals in prime rings with involution". In: *Kyungpook Mathematical Journal* 56 (2016), pp. 343–347.

[4] H. E. Bell and N. Rehman. "Generalized derivations with commutattivity and anticommutativity conditions". In: *Mathematical Journal of Okayama University* 49 (2007), pp. 139–147.

[5] J. Bergen, I. N. Herstein, and J. W. Kerr. "Lie ideals and derivations of prime rings". In: *Journal of Algebra* 71.1 (1981), pp. 259–267.

[6] A. Boua and E. Farhan. "Notes on rings and near-rings with homo multiplier". In: *Discussiones Mathematicae - General Algebra and Applications* 44 (2024), pp. 199–215.

[7] M. Brešar. "On the distance of the composition of two derivations to the generalized derivations". In: *Glasgow Mathematical Journal* 33 (1991), pp. 89–93.

[8] M. Brešar and J. Vukman. "On certain subrings of prime rings with derivations". In: *Journal of the Australian Mathematical Society Series A Pure Mathematics and Statistics* 54.1 (1993), pp. 133–141.

[9] J. C. Chang. "On the identitity $h(x) = af(x) + g(x)b$". In: *Taiwanese Journal of Mathematics* 7 (2003), pp. 103–113.

[10] M. A. Chebotar. "On certain subrings and ideals of prime rings". In: *First International Tainan-Moscow Algebra Workshop: Proceedings*. Tainan, Taiwan, 1994.

[11] C. L. Chuang. "Differential identities with automorphisms and antiautomorphisms I". In: *Journal of Algebra* 149 (1992), pp. 371–404.

[12] C. L. Chuang. "Differential identities with automorphisms and antiautomorphisms II". In: *Journal of Algebra* 160 (1993), pp. 130–171.

[13] C. L. Chuang. "GPIs having coefficients in Utumi quotient rings". In: *Proceedings of the American Mathematical Society* 103 (1988), pp. 723–728.

[14] C. L. Chuang and T. K. Lee. "Identities with a single skew derivation". In: *Journal of Algebra* 288 (2005), pp. 59–77.

[15] M. N. Daif. "Commutativity results for semiprime rings with derivation". In: *International Journal of Mathematics and Mathematical Sciences* 21.3 (1998), pp. 471–474.

[16] V. De Filippis. "Product of generalized derivations on polynomials in prime rings". In: *Collectanea Mathematica* 61.3 (2010), pp. 303–322.

[17] V. De Filippis and O. M. Di Vincenzo. "Generalized Skew Derivations and Nilpotent Values on Lie Ideals". In: *Algebra Colloquium* 26.4 (2019), pp. 589–614.

[18] V. De Filippis and G. Scudo. "Strong commutativity and Engel condition preserving maps in prime and semiprime rings". In: *Linear and Multilinear Algebra* 61.7 (2013), pp. 917–938.

[19] M. M. El-Soufi. *Rings with some kinds of mappings*. M.Sc. thesis, Cairo University, Branch of Fayoum, Egypt. 2000.

[20] M. P. Eroglu. "Generalized skew derivations with centralizer conditions on multilinear polynomials". In: *Beiträge zur Algebra und Geometrie* 59 (2018), pp. 61–76.

[21] Eszter Gselmann and Gergely Kiss. *Remarks on the notion of homo-derivations*. arXiv preprint arXiv:2001.03439. 2020.

[22] I. N. Herstein. "A note on derivations". In: *Canadian Mathematical Bulletin* 21 (1978), pp. 369–370.

[23] B. Hvala. "Generalized derivations in rings". In: *Communications in Algebra* 26.4 (1998), pp. 1147–1166.

[24] N. Jacobson. *Structure of Rings*. Vol. 37. Colloquium Publications. Providence, RI: American Mathematical Society, 1964.

[25] V. K. Kharchenko. *Automorphisms and Derivations of Associative Rings*. Vol. 69. Mathematics and Its Applications (Soviet Series). Dordrecht: Kluwer Academic Publishers, 1991.

[26] W. S. Martindale. "Prime rings satisfying a generalized polynomial identity". In: *Journal of Algebra* 12 (1969), pp. 576–584.

[27] M. Mathieu. "Posner's second theorem deduced from the first". In: *Proceedings of the American Mathematical Society* 114 (1992), pp. 601–602.

[28] J. H. Mayne. "Centralizing automorphisms of prime rings". In: *Canadian Mathematical Bulletin* 19 (1976), pp. 113–115.

[29] E. C. Posner. "Derivations in prime rings". In: *Proceedings of the American Mathematical Society* 8 (1957), pp. 1093–1100.

[30] N. Rehman. "On commutativity of rings with generalized derivations". In: *Mathematical Journal of Okayama University* 44 (2002), pp. 43–49.

[31] G. S. Sandhu. "A Note on Posner's Theorems". In: *Annals of Pure and Applied Mathematics* 17.1 (2018), pp. 147–150.

[32] Y. Wang. "A Note on Lie Automorphisms, Subrings, and Lie Ideals of Prime Rings". In: *Communications in Algebra* 33.11 (2005), pp. 4057–4062.

[33] T. L. Wong. "Derivations with power central values on multilinear polynomials". In: *Algebra Colloquium* 3.4 (1996), pp. 369–378.