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# An Extended Thermodynamics Study for Second-Grade Adiabatic Fluids

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**Abstract:** A 10-field theory for second-grade viscoelastic fluids is developed in the framework of Rational Extended Thermodynamics. The field variables are the density, the velocity, the temperature and the stress tensor. The particular case of an adiabatic fluid is considered. The field equations are determined by use of physical universal principles such as the Galilean and the Entropy Principles. As already proved, Rational Extended Thermodynamics is able to eliminate some inconsistencies with experiments that arise in Classical Thermodynamics. Moreover, the paper shows that, if the quadratic terms are taken into account, the classical constitutive relations for a second-grade fluid can be obtained as a limit case of the field equations of the present theory.

**Keywords:** extended thermodynamic; second-grade fluids; non-newtonian fluids

**MSC:** 80A17; 80A10; 76A10

## 1. Introduction

Rational Extended Thermodynamics (ET) [1–3] can describe processes where rapid time changes or a strong deviation from equilibrium occur. In fact, it has been shown that the field equations of ET can describe a range of various non-equilibrium phenomena such as light shattering, sound waves, heat waves and structure of shock waves [1–3]. ET has been applied, with many interesting results, to monoatomic gases [1] and mixtures [2–4], showing in particular the possibility of describing the thermal-diffusion effect. Recently, ET has been generalized to dense and rarefied polyatomic gases both in the classical [2,3,5] and in the relativistic framework [6–8], with respect to metal electrons [9], quantum systems [10], graphene [11], biological models [12–14], blood flow [15,16], heat transfer in different symmetries [17] and gas bubbles [18,19], providing in all cases relevant results. In the last 40 years, there were different attempts to describe viscoelastic fluids using the methods of Extended Thermodynamics.

Viscoelastic fluids are well known in the literature. The basic descriptions for these kind of fluids can be found in the classical papers and books [20–22], while the basis of molecular theory of non-Newtonian fluids goes back to Bird [23,24]. In the literature, there are also some controversial discussions about the coefficients and the stability criteria [25–27]. More recently, viscoelastic fluids have been described in different volumes in continuum mechanics like [28,29] and there have been a lot of different applications to different kind of flows with both analytical [30–34] and numerical solutions [35–37].

The first attempt to describe viscoelastic fluids within Rational Extended Thermodynamics was made by Müller and Wilmansky [38] in 1986, and was later described in detail in [28]; they constructed the first Extended Thermodynamic theory for viscoelastic materials moving in parallel with those for ideal gasses. Later, IS-Liu [39] in 1989 and Vignatti and Oliveira [40] in 2013 made another attempt to describe this kind of fluid using Rational Extended Thermodynamic theory. They used the more complex Eulerian structure to describe these materials. Also, Reitebuch in his dissertation [41], following the ideas in [42],

tried to move closer to the molecular theory of non-Newtonian fluids. All these attempts are discussed in detail in Chapter 16 of [1], showing the advances and the limitations of these theories. In particular, there was still a problem with respect to recovering the classical second-grade constitutive relations from the field equations of Extended Thermodynamics. Fazio in her master's thesis [43] studied second-grade adiabatic fluids from the Extended Thermodynamic point of view and tried to solve a problem that arises in the description of these fluids, as described in detail in [1] and later in the present paper. Therefore, here we want to present a first attempt to recover the classical constitutive relations as a limit case of Extended Thermodynamics.

In this paper, following [1], we derive a hyperbolic model for a non-Newtonian adiabatic second-grade fluid. We consider as field variables not only the classical ones (density, velocity and temperature) but also the stress tensor. The field equations are balance laws, closed by local and instantaneous constitutive relations. Their generality is reduced using physical universal laws like the Galilean and the Entropy Principles. The exploitation of the Galilean Principle furnishes the explicit velocity dependence of the constitutive relations, while the Entropy Principle implies some relations between the constitutive quantities. The Entropy Principle is exploited by use of the Lagrange Multipliers [39] or Main Fields [44].The calculations are carried out both in the linear and in the quadratic cases and it was shown that the two principles are able to determine all constitutive functions except for the thermal and caloric equations of state and two remaining arbitrary functions. The obtained model is hyperbolic at least in the neighborhood of the equilibrium state. Hyperbolicity guarantees finite speeds of propagation and hyperbolic models are better suited to describing transient regimes.

Once the field equations are obtained, the reduction to the classical constitutive relations using Maxwellian iteration is discussed. In the linear case, it was not possible to obtain the classical law as a limit case while, in the presence of quadratic constitutive relations, a possibility is shown.

The main aim of the present paper is to construct a hyperbolic set of field equations for the description of a second-grade fluid that is compatible with the Galilean and Entropy Principles in line with Rational Extended Thermodynamics. In addition, it was shown that, contrary to the previous Rational Extended Thermodynamics field equations, the herein obtained model is able to furnish the classical constitutive relations for second-grade fluids via second Maxwellian iterations if quadratic constitutive equations in the stress tensor are taken into account.

## 2. Classical Thermodynamics

In Classical Thermodynamics [45], the field variables are the density $\rho(\mathbf{x}, t)$, the velocity $\mathbf{v}(\mathbf{x}, t)$ and the temperature $T(\mathbf{x}, t)$. They describe the state of the fluid in position $\mathbf{x}$ and time $t$. The field equations for these variables are based on the conservation laws of mass, momentum and energy that are expressed by the following set of balance equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_j)}{\partial x_j} = 0,$$

$$\frac{\partial (\rho v_i)}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j v_i - \tau_{ij}) = \rho f_i, \tag{1}$$

$$\frac{\partial (\rho e)}{\partial t} + \frac{\partial}{\partial x_j}(\rho e v_j - \tau_{ij} v_i + q_j) = \rho v_i f_i,$$

where $\tau_{ij}$ represents the components of the stress tensor and $e$ the specific total energy, given by

$$\rho e = \rho \epsilon + \frac{1}{2}\rho v^2, \tag{2}$$

where $\epsilon$ is the specific internal energy, $q_i$ the components of heat flux and $f_i$ the components of the specific body force. System (1) is not closed for the occurrence of $\epsilon$, $\tau_{ij}$ and $q_i$ that must be expressed explicitly in terms of the field variables $\rho$, $v_i$ and $T$ through the so-called constitutive functions. In this paper, we consider an adiabatic second-grade viscoelastic fluid, so $q_i$ vanishes while the stress tensor $\tau$ can be written as the sum

$$\tau_{ij} = -p\delta_{ij} + \tau_{<ij>}, \tag{3}$$

where $p$ is the hydrostatic pressure, $\delta_{ij}$ are the components of the Kronecker symbol and $\tau_{<ij>}$ are the components of the traceless part of the stress tensor.

In second-grade fluids, the traceless part of the stress tensor depends on the temperature, the gradient of the velocity $L_{ij} = \partial v_i/\partial x_j$ and its derivative with respect to time $\dot{L}_{ij} = \mathrm{d}L_{ij}/\mathrm{d}t$:

$$\tau_{<ij>} = \hat{\tau}_{<ij>}(T, \mathbf{L}, \dot{\mathbf{L}}). \tag{4}$$

The functions $\hat{\tau}_{<ij>}$ are subject to the principle of the material frame indifference, which implies [46] that the dependence on $\mathbf{L}$ and $\dot{\mathbf{L}}$ must take place through the first two Rivlin–Ericksen tensors:

$$\begin{aligned}
\mathbf{A}^{(1)} &= \mathbf{L} + \mathbf{L}^T, \\
\mathbf{A}^{(2)} &= \dot{\mathbf{A}}^{(1)} + \mathbf{A}^{(1)}\mathbf{L} + \mathbf{L}^T\mathbf{A}^{(1)}
\end{aligned} \tag{5}$$

in the form

$$\mathbf{S} = \mu\mathbf{A}^{(1)} + \alpha_1\left[\mathbf{A}^{(2)} - tr\mathbf{A}^{(2)}\mathbf{I}\right] + \alpha_2\left[\left(\mathbf{A}^{(1)}\right)^2 - tr\left(\mathbf{A}^{(1)}\right)^2\mathbf{I}\right], \tag{6}$$

where we set for simplicity $S_{ij} = \tau_{<ij>}$. The coefficient $\mu$ represents the viscosity, while $\alpha_1$ and $\alpha_2$ are the normal stress coefficients that depend on the temperature $T$. As discussed in [1] (page 370), experiments show [47] that $\alpha_1 < 0$, while a stability thermodynamic criterion implies a positive coefficient $\alpha_1$ in contradiction with the experimental evaluations.

A possible way to solve this problem was presented by Müller and Wilmanski in [38], who introduced a differential equation for the stress tensor. In their attempt, they showed that Extended Thermodynamics is able to furnish the correct sign of the first normal coefficient. Then, Liu [39] introduced a new and more complex Extended Thermodynamic theory for viscoelastic fluids. This was a more precise theory for these fluids, but not a completely resolutive one [1].

In [1], it was shown that Extended Thermodynamics is able to recover the exact sign of $\alpha_1$, but it fails when it wants to re-obtain the constitutive relations (6) as a particular case. In this paper, a possible solution for this problem is presented.

## 3. Extended Thermodynamics

Extended Thermodynamics [1–3] considers as field variables not only the classical ones like the density $\rho$, the velocity $\mathbf{v}$ and the temperature $T$, but also some non-equilibrium variables. Here, since we deal with an adiabatic fluid, we add to the set of the classical variables only the stress tensor. So we have the following set of 10 field variables: $\rho$, $v_k$, $T$ and $\rho_{<ij>} = -\tau_{<ij>}$.

Following the guidelines of Rational Extended Thermodynamics, we assume that these 10 field variables must satisfy the following set of 10 field equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_k)}{\partial x_k} = 0,$$

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial F_{ik}}{\partial x_k} = \rho f_i, \tag{7}$$

$$\frac{\partial(\rho e_{ij})}{\partial t} + \frac{\partial F_{ijk}}{\partial x_k} = P_{<ij>} + \rho(v_i f_j + v_j f_i).$$

The first two equations represent the conservation law of mass and momentum. The trace of the third equation recovers the conservation law of energy, with $2e_{ll}$ being the total specific energy and $2F_{ill}$ its flux. The traceless part of $(7)_3$ is a new balance equation introduced with the aim of expressing the balance law of the stress tensor.

In order to recover from (7) the full set of field equations, it is necessary to express all quantities in them in terms of the 10 fields by constitutive relations. Here the unknown quantities are $F_{ik}$, $e_{ij}$, $F_{ijk}$ and $P_{<ij>}$.

Extended Thermodynamics assumes that the constitutive relations for these variables must be local and instantaneous, so at one point **x** and time $t$, they must depend only on the quantities at **x** and $t$ but not on their derivatives:

$$F_{ik} = F_{ik}(\rho, v_i, T, \rho_{<ij>}),$$

$$e_{ij} = e_{ij}(\rho, v_i, T, \rho_{<ij>}),$$

$$F_{ijk} = F_{ijk}(\rho, v_i, T, \rho_{<ij>}), \tag{8}$$

$$P_{<ij>} = P_{<ij>}(\rho, v_i, T, \rho_{<ij>}).$$

Once we know these functions, we can insert them into the balance Equation (7) in order to obtain a closed set of field equations, whose solution is called "Thermodynamic Process".

Clearly, relations (8) are not a priori known; for this reason, Rational Extended Thermodynamics reduces their generality by universal physical principles such as using the Galileian and Entropy Principles.

## 4. Galileian Principle

The Galileian invariance principle requires that the field equations must hold in every inertial frame, so they must be invariant under Galileian transformations. This requirement furnishes (see [1] page 35 and [48]) the dependence of all variables in (8) on the velocity field:

$$F_{ij} = p\delta_{ik} + \rho_{<ik>} + \rho v_i v_j,$$

$$e_{ij} = \rho \epsilon_{ij} + \rho v_i v_j,$$

$$F_{ijk} = \rho_{ijk} + \rho_{<jk>} v_i + \rho_{<ki>} v_j + \rho \epsilon_{ij} v_k + \rho v_i v_j v_k, \tag{9}$$

$$P_{<ij>} = p_{<ij>}.$$

The quantities $\rho_{ij}$, $\epsilon_{ij}$, $\rho_{ijk}$ and $p_{<ij>}$ do not depend on the velocity field, so they are called internal quantities and they are related to the common thermodynamic variables. In particular, in order to obtain from (7) and (9) the conservation laws of mass, momentum and energy, the quantities $\rho_{ij}$ must coincide with the components of stress tensor, $\rho \epsilon_{ll}$ must be twice the internal energy and $q_k = 2\rho_{kll}$ must be the heat flux components.

By substitution of relations (9) into the balance Equations (7), one has the balance equations for the internal quantities that read

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\partial v_k}{\partial x_k} = 0,$$

$$\rho\frac{\mathrm{d}v_i}{\mathrm{d}t} + \frac{\partial p}{\partial x_i} + \frac{\partial \rho_{<ik>}}{\partial x_k} = 0, \tag{10}$$

$$\frac{\mathrm{d}(\rho\epsilon_{ij})}{\mathrm{d}t} + \rho\epsilon_{ij}\frac{\partial v_k}{\partial x_k} + \frac{\partial \rho_{ijk}}{\partial x_k} + \rho_{jk}\frac{\partial v_i}{\partial x_k} + \rho_{ki}\frac{\partial v_j}{\partial x_k} = p_{<ij>},$$

where the derivative $\mathrm{d}/\mathrm{d}t$ represents the material derivative defined as $\mathrm{d}/\mathrm{d}t = \partial/\partial t + v_k\partial/\partial x_k$. The trace of Equation (10)$_3$ is the conservation law of energy:

$$\frac{\mathrm{d}(\rho\epsilon_{ll})}{\mathrm{d}t} + \rho\epsilon_{ll}\frac{\partial v_k}{\partial x_k} + 2\frac{\partial q_k}{\partial x_k} + 2\rho_{kl}\frac{\partial v_l}{\partial x_k} = 0. \tag{11}$$

Inspection of (10) reveals that in order to close the system of balance equations, one has to express the quantities $p$, $\epsilon_{ij}$, $\rho_{ijk}$ and $p_{<ij>}$ in terms of the fields. The pressure $p$ and the internal energy $\rho\epsilon_{ll}/2 = \rho\epsilon$ must be evaluated by means of the thermal and caloric equations of state, so it remains to evaluate the functions

$$\epsilon_{<ij>} = \epsilon_{<ij>}(\rho, T, \rho_{<ij>}),$$

$$\rho_{<ijk>} = \rho_{<ijk>}(\rho, T, \rho_{<ij>}), \tag{12}$$

$$p_{<ij>} = p_{<ij>}(\rho, T, \rho_{<ij>}).$$

As will be shown in the next section, we limit their generality by use of the Entropy Principle.

## 5. Entropy Principle

The Entropy Principle assumes the existence of state variable $h$, called entropy density, satisfying the balance equation

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x_k}(hv_k + \phi_k) = \Sigma \geq 0, \tag{13}$$

for all solutions of the field equations. The quantities $\phi_k$ and $\Sigma$ represent, respectively, the components of the entropy flux and the entropy production, that must also be determined in terms of the fields.

Following again the guidelines of Rational Extended Thermodynamics, we consider the field equations as constraints that the fields must satisfy. We take into consideration these constraints introducing the Lagrange Multipliers [49] or Main Fields [44], i.e.,

$$\Lambda, \quad \Lambda^i, \quad \Lambda_\epsilon, \quad \Lambda^{<ij>}. \tag{14}$$

So, the Entropy Inequality becomes

$$\frac{\mathrm{d}h}{\mathrm{d}t} + h\frac{\partial v_k}{\partial x_k} + \frac{\partial \phi_k}{\partial x_k} - \Lambda\left[\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\frac{\partial v_k}{\partial x_k}\right] +$$

$$- \Lambda^i\left[\frac{\mathrm{d}(\rho v_i)}{\mathrm{d}t} + \rho v_i\frac{\partial v_k}{\partial x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial \rho_{<ik>}}{\partial x_k} - \rho f_i\right] +$$

$$- \Lambda^\epsilon\left[\frac{\mathrm{d}(\rho\epsilon_{ll})}{\mathrm{d}t} + \rho\epsilon_{ll}\frac{\partial v_k}{\partial x_k} + 2\rho_{<kl>}\frac{\partial v_l}{\partial x_k} + 2p\frac{\partial v_s}{\partial x_s}\right] + \qquad (15)$$

$$- \Lambda^{<ij>}\left[\frac{\mathrm{d}(\rho\epsilon_{<ij>})}{\mathrm{d}t} + \rho\epsilon_{<ij>}\frac{\partial v_k}{\partial x_k} + \frac{\partial \rho_{<ij>k}}{\partial x_k} + \right.$$

$$\left. + 2\rho_{<k<i>}\frac{\partial v_{j>}}{\partial x_k} + 2p\frac{\partial v_{<i}}{\partial x_{j>}} - P_{<ij>}\right] = \Sigma \geq 0$$

which must be valid for all variables $\rho$, $\rho_{<ij>}$ and $T$ and not only for the solutions of the field Equation (10).

In order to exploit the Entropy Inequality (15), in this paper we assume that the constitutive relations must be at least of second order in the non-equilibrium variables. In [1] (page 374), the authors assumed only linear terms in the non-equilibrium field $\rho_{<ij>}$. In order to show more explicitly the evaluation of the Entropy Principle, we show it before (next section) the study of the linear terms and after the second-order ones.

## 6. Linear Constitutive Relations

Since it is very complicated to recover the general form of the constitutive relations (12) from the Entropy Inequality (15), we firstly limit our attention to linear constitutive functions, assuming

$$\rho\epsilon_{<ij>} = \beta_1(\rho, T)\rho_{<ij>},$$

$$\rho_{<ijk>} = 0, \qquad (16)$$

$$P_{<ij>} = P_1(\rho, T)\rho_{<ij>}.$$

In order to be coherent with the orders of all terms in the Entropy Inequality, we must consider some quadratic terms in the entropy quantities and in the Lagrange Multipliers, i.e.,

$$h = h_0(\rho, T) + h_1(\rho, T)\rho_{<ij>}\rho_{<ij>},$$

$$\phi_i = 0,$$

$$\Lambda^i = 0,$$

$$\Lambda = \Lambda_0(\rho, T) + \Lambda_1(\rho, T)\rho_{<ij>}\rho_{<ij>}, \qquad (17)$$

$$\Lambda^\epsilon = \Lambda_0^\epsilon(\rho, T) + \Lambda_1^\epsilon(\rho, T)\rho_{<ij>}\rho_{<ij>},$$

$$\Lambda^{<ij>} = H_0(\rho, T)\rho_{<ij>} + H_1(\rho, T)\rho_{<k<i>}\rho_{<j>k>}.$$

By substituting the constitutive relations (16) and (17) and equating the coefficients of the derivatives of all fields, it is possible to recover ten equations for the determination of the coefficients in (16) and (17). In particular, we have the following four differential equations with $\gamma_0 = 2\rho\epsilon$

$$\frac{\partial h_0}{\partial \rho} = \Lambda_0 + \Lambda_0^\epsilon \frac{\partial \gamma_0}{\partial \rho},$$

$$\frac{\partial h_0}{\partial T} = \Lambda_0^\epsilon \frac{\partial \gamma_0}{\partial T},$$

$$\frac{\partial h_1}{\partial \rho} = \Lambda_1 + \Lambda_1^\epsilon \frac{\partial \gamma_0}{\partial \rho} + H_0 \frac{\partial \beta_1}{\partial \rho}, \tag{18}$$

$$\frac{\partial h_1}{\partial T} = \Lambda_1^\epsilon \frac{\partial \gamma_0}{\partial T} + H_0 \frac{\partial \beta_1}{\partial T},$$

the five algebric equations

$$2h_1 = H_0 \beta_1,$$

$$h_0 = \Lambda_0 \rho + \Lambda_0^\epsilon \gamma_0 + 2\Lambda_0^\epsilon p,$$

$$h_1 = \Lambda_1 \rho + \Lambda_1^\epsilon \gamma_0 + 2\Lambda_1^\epsilon p + H_0 \beta_1 - \frac{2}{3} H_1 p, \tag{19}$$

$$H_0 + pH_1 = 0,$$

$$\Lambda_0^\epsilon + pH_0 = 0$$

and the residual inequality

$$\Sigma = P_1 H_0 \rho_{<ij>} \rho_{<ij>} \geq 0. \tag{20}$$

From Equations $(18)_{1,2}$, one has

$$dh_0 = \left( \Lambda_0 + \Lambda_0^\epsilon \frac{\partial \gamma_0}{\partial \rho} \right) d\rho + \Lambda_0^\epsilon \frac{\partial \gamma_0}{\partial T} dT, \tag{21}$$

which, compared with the well-known Gibbs equation of Classical Thermodynamics [46], implies the equilibrium values of the Lagrange Multipliers $\Lambda_0^\epsilon$ and $\Lambda_0$:

$$\Lambda_0^\epsilon = \frac{1}{2T}, \quad \Lambda_0 = -\frac{g}{T}, \tag{22}$$

where $g$ represents the chemical potential. From $(19)_{4,5}$, it is possible to obtain

$$H_0 = -\frac{1}{2pT}, \quad H_1 = \frac{1}{2p^2 T}, \tag{23}$$

while from $(19)_{1,3}$ it follows that

$$\beta_1 + \frac{4}{3} = 4pT \left[ \rho \Lambda_1 + 2\Lambda_1^\epsilon (\rho \epsilon + p) \right] \tag{24}$$

which can be used in order to evaluate $\Lambda_1^\epsilon$.

Finally, from the remaining equations, the expressions

$$\Lambda_1 = \frac{1}{2}\beta_1\frac{\partial H_0}{\partial\rho} - \frac{1}{2}H_0\frac{\partial\beta_1}{\partial\rho} - \Lambda_1^\epsilon\frac{\partial\gamma_0}{\partial\rho},$$

$$\beta_1\frac{\partial H_0}{\partial T} - H_0\frac{\partial\beta_1}{\partial T} - 2\Lambda_1^\epsilon\frac{\partial\gamma_0}{\partial T} = 0 \tag{25}$$

$$h_1 = -\frac{1}{4pT}\beta_1$$

can be determined together with the inequality

$$P_1 \leq 0. \tag{26}$$

When the thermal and caloric equations of state are determined, relation $(25)_3$ can be used in order to evaluate $\beta_1$ via integration except for an arbitrary function. Then, ignoring the terms of order higher than one in the fluxes and using the constitutive relations determined in this section, it is possible to obtain from $(10)_3$ the balance equation for the stress tensor of the form:

$$\frac{\mathrm{d}(\beta_1\rho_{<ij>})}{\mathrm{d}t} + \beta_1\rho_{<ij>}\frac{\partial v_k}{\partial x_k} + 2\rho_{<k<i>}\frac{\partial v_{j>}}{\partial x_k} + 2p\frac{\partial v_{<i}}{\partial x_{j>}} = -|P_1|\rho_{<ij>} \tag{27}$$

System $(10)_{1,2}$, (11) and (27) represents the system of 10 partial differential equations for the determination of the 10 fields $\rho$, $v_k$, $T$, $\rho_{<ij>}$. The system is completely explicit in terms of the fields except for the thermal and caloric equations of state and the arbitrary function in $\beta_1$, which was impossible to determine using the physical principles.

Equation (27) represents a generalization of the Navier–Stokes law for non-Newtonian fluids since it relates the stress tensor in terms of the gradient of the velocity field.

For a better comparison with the classical case, (27) is written in the equivalent form

$$\begin{aligned}
&\frac{\mathrm{d}(\beta_1\rho_{<ij>})}{\mathrm{d}t} + \beta_1\rho_{<ij>}\frac{\partial v_k}{\partial x_k} + 2\beta_1\rho_{<k<i>}\frac{\partial v_k}{\partial x_{j>}} + 2\beta_1\rho_{<k<j>}\frac{\partial v_k}{\partial x_{i>}}\\
&- 2\left(\beta_1 - \frac{1}{2}\right)\rho_{<k<i>}\left(\frac{\partial v_{j>}}{\partial x_k}\right)^s - 2\left(\beta_1 - \frac{1}{2}\right)\rho_{<k<j>}\left(\frac{\partial v_{i>}}{\partial x_k}\right)^s\\
&- 2\left(\beta_1 + \frac{1}{2}\right)\rho_{<k<i>}\left(\frac{\partial v_{j>}}{\partial x_k}\right)^a - 2\left(\beta_1 + \frac{1}{2}\right)\rho_{<k<j>}\left(\frac{\partial v_{i>}}{\partial x_k}\right)^a\\
&+ 2p\frac{\partial v_{<i}}{\partial x_{j>}} = -|P_1|\rho_{<ij>}.
\end{aligned} \tag{28}$$

The terms with the symbol $s$ refer to symmetric parts, while those with $a$ refer to antisymmetric ones.

In the next section, we will try to recover from this last balance equation the assumption (6) via the so-called Maxwellian iterations.

## 7. Maxwellian Iterations

As already said, Rational Extended Thermodynamics introduces balance equations like (28) for the additional non-equilibrium field variables, instead of assuming constitutive relations like (6). A possible way to re-obtain from (28) the classical constitutive relations (6) is using a method known as Maxwellian iterations, since Maxwell used it to solve a similar problem in kinetic theory of gases [50].

In the first iteration, one substitutes the equilibrium values in the left-hand side of the balance equations, obtaining in this case under investigation

$$2p\frac{\partial v_{<i}}{\partial x_{j>}} = -|P_1|\rho_{<ij>}. \tag{29}$$

This is the classical Navier–Stokes law, which assumes the traceless part of the stress tensor proportional to the gradient of the velocity field. The coefficient of proportionality

$$\mu = \frac{p}{|P_1|} \tag{30}$$

is the viscosity. In Rational Extended Thermodynamics, relations like (30) are often used to evaluate the viscosity coefficient if the production terms are known. For example, in gases or mixtures the kinetic theory is able to evaluate the production terms for particular kinds of molecules and from (30) it is possible to recover explicit expressions for the viscosity. Sometimes, relations like (30) are used to evaluate the production terms if the viscosity is known.

In non-Newtonian fluids, (6) represents the generalization of the Navier–Stokes law (29) already in the classical case. Therefore, the second iteration must be performed. So, we substitute the first iteration (29) in the left-hand side of Equations (28) and neglect the terms with $\partial v_k / \partial x_k$ since the classical case refers to incompressible fluids. In this way, it is possible to obtain an explicit expression for the stress tensor in terms of the products of the gradients of the velocity fields that are comparable with the classical relation (6). The terms coincide except for the terms that contain the antisymmetric parts of the gradient of the velocity, which cannot be present in (6). In principle, coefficient $\beta_1$ is not already fully determined but it cannot be chosen in such a way to eliminate these antisymmetric parts, since $\beta_1$ must be positive. So the antisymmetric part cannot be removed. This problem was already shown in [1] and in this paper we propose to solve it by taking into account the quadratic terms in the constitutive relations (12) instead of limiting the analysis to the linear terms (16).

## 8. Quadratic Constitutive Relations

We assume quadratic terms in the constitutive relations, so instead of (16), we have

$$\rho\epsilon_{<ij>} = \beta_1(\rho, T)\rho_{<ij>} + \beta_2(\rho, T)\rho_{<k<i}\rho_{<j>k>},$$

$$\rho_{<ijk>} = 0, \tag{31}$$

$$P_{<ij>} = P_1(\rho, T)\rho_{<ij>} + P_2(\rho, T)\rho_{<k<i}\rho_{<j>k>}$$

and, in order to be coherent in the Entropy Inequality, we must consider the entropic variables until third-order terms in the non-equilibrium variable $\rho_{<ij>}$, that is

$$h = h_0(\rho, T) + h_1(\rho, T)\rho_{<ij>}\rho_{<ij>} + h_2(\rho, T)\rho_{<ij>}\rho_{<jk>}\rho_{<ki>} \tag{32}$$

while, for the assumption of vanishing heat flux, $\phi_i = 0$ and $\Lambda^i = 0$ still hold and the Lagrange Multipliers assume the form

$$\Lambda = \Lambda_0(\rho, T) + \Lambda_1(\rho, T)\rho_{<ij>}\rho_{<ij>} + \Lambda_2(\rho, T)\rho_{<ij>}\rho_{<jk>}\rho_{<ki>},$$

$$\Lambda^\epsilon = \Lambda_0^\epsilon(\rho, T) + \Lambda_1^\epsilon(\rho, T)\rho_{<ij>}\rho_{<ij>} + \Lambda_2^\epsilon(\rho, T)\rho_{<ij>}\rho_{<jk>}\rho_{<ki>},$$

$$\Lambda^{<ij>} = H_0(\rho, T)\rho_{<ij>} + H_1(\rho, T)\rho_{<k<i>}\rho_{<j>k>} + \tag{33}$$

$$+ H_2(\rho, T)\rho_{<lv>}\rho_{<l<i}\rho_{<j>v>} + H_3(\rho, T)\rho_{<ls>}\rho_{<ls>}\rho_{<ij>}.$$

All relations (31)–(33) must be inserted in the Entropy Inequality (15) in order to recover the coefficients in (31)–(33). Equating the coefficients of the derivatives of all fields, it is possible to recover 16 equations. The first ten coincide with (18)–(20); the remaining ones are the two differential relations for the coefficients in the entropic quantities

$$\frac{\partial h_2}{\partial \rho} = \Lambda_2 + \Lambda_2^\epsilon \frac{\partial \gamma_0}{\partial \rho} + H_0 \frac{\partial \beta_2}{\partial \rho} + H_1 \frac{\partial \beta_1}{\partial \rho},$$

$$\frac{\partial h_2}{\partial T} = \Lambda_2^\epsilon \frac{\partial \gamma_0}{\partial T} + H_0 \frac{\partial \beta_2}{\partial T} + H_1 \frac{\partial \beta_1}{\partial T},$$

(34)

together with the four algebraic relations

$$3h_2 = 2H_0\beta_2 + H_1\beta_1,$$

$$h_2 = \Lambda_2\rho + \Lambda_2^\epsilon\gamma_0 + 2\Lambda_2^\epsilon p + H_0\beta_2 + H_1\beta_1 - \frac{2}{3}pH_2,$$

$$2\Lambda_1^\epsilon - \frac{2}{3}H_1 + 2pH_3 = 0,$$

$$2pH_2 + 2H_1 = 0.$$

(35)

Therefore, it is possible to obtain again the functions obtained in the previous section together with the two following ones

$$H_2 = -\frac{1}{2p^3T}, \quad H_3 = -\frac{\Lambda_1^\epsilon}{p} + \frac{1}{6p^2T},$$

(36)

the relations

$$h_2 = -\frac{\beta_2}{3pT} + \frac{\beta_1}{6p^2T},$$

$$\beta_2 - 2\frac{\beta_1}{p} - \frac{4}{3}\frac{1}{p} = 6pT\left[\rho\Lambda_2 + 2(\rho e + p)\Lambda_2^\epsilon\right],$$

$$\frac{1}{3}\beta_1\frac{\partial H_1}{\partial T} - \frac{2}{3}H_1\frac{\partial \beta_1}{\partial T} - \frac{1}{3}H_0\frac{\partial \beta_2}{\partial T} + \frac{1}{3}\beta_2\frac{\partial H_0}{\partial T} - \Lambda_2^\epsilon\frac{\partial \gamma_0}{\partial T} = 0$$

(37)

and the expression for $\Lambda_2$ that for simplicity we do not write here. From (37)$_2$, it is possible to obtain $\Lambda_2^\epsilon$ and, by integration of (37)$_3$, one has $\beta_2$ in terms of an arbitrary integration function. In this way, all coefficients (31)–(33) are known except for the equations of state and the two arbitrary integration functions in $\beta_1$ and $\beta_2$.

Taking into account all these functions and neglecting all terms of order higher than 2 in the non-equilibrium variable $\rho_{<ij>}$, Equation (10)$_3$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\beta_1\rho_{<ij>} + \beta_2\rho_{<k<i>}\rho_{<j>k>}\right) + \left(\beta_1\rho_{<ij>} + \beta_2\rho_{<k<i>}\rho_{<j>k>}\right)\frac{\partial v_k}{\partial x_k}$$

$$+ 2p\frac{\partial v_{<i}}{\partial x_{j>}} + 2\rho_{<k<i>}\frac{\partial v_{j>}}{\partial x_k} = -|P_1|\rho_{<ij>} + P_2\rho_{<k<i>}\rho_{<j>k>}.$$

(38)

As it is easy to see, the first iteration coincides with (29). Then, in order to evaluate the second iteration and compare it easily with the Classical Equation (6), we write (38) in a different form, neglecting for simplicity the terms with $\partial v_k/\partial x_k$ that vanish in the classical case,

$$
\begin{aligned}
&\frac{\mathrm{d}(\beta_1 \rho_{<ij>})}{\mathrm{d}t} + 2\beta_1 \rho_{<k<i>} \frac{\partial v_k}{\partial x_{j>}} + 2\beta_1 \rho_{<k<j>} \frac{\partial v_k}{\partial x_{i>}} + \\
&- 2\left(\beta_1 - \frac{1}{2}\right) \rho_{<k<i>} \left(\frac{\partial v_{j>}}{\partial x_k}\right)^s - 2\left(\beta_1 - \frac{1}{2}\right) \rho_{<k<j>} \left(\frac{\partial v_{i>}}{\partial x_k}\right)^s + \\
&- 2\left(\beta_1 + \frac{1}{2}\right) \rho_{<k<i>} \left(\frac{\partial v_{j>}}{\partial x_k}\right)^a - 2\left(\beta_1 + \frac{1}{2}\right) \rho_{<k<j>} \left(\frac{\partial v_{i>}}{\partial x_k}\right)^a + \\
&+ \frac{1}{2} \frac{\mathrm{d}\beta_2}{\mathrm{d}t} \rho_{<k<i>} \left(\frac{\partial v_{j>}}{\partial x_k}\right)^s + \frac{1}{2} \frac{\mathrm{d}\beta_2}{\mathrm{d}t} \rho_{<j<k>} \left(\frac{\partial v_{i>}}{\partial x_k}\right)^s + \\
&+ \frac{1}{2} \frac{\mathrm{d}\beta_2}{\mathrm{d}t} \rho_{<k<i>} \left(\frac{\partial v_{j>}}{\partial x_k}\right)^a + \frac{1}{2} \frac{\mathrm{d}\beta_2}{\mathrm{d}t} \rho_{<j<k>} \left(\frac{\partial v_{i>}}{\partial x_k}\right)^a + \\
&+ 2p \frac{\partial v_{<i}}{\partial x_{j>}} = P_1 \rho_{<ij>} + P_2 \rho_{<k<i>} \rho_{<j>k>}.
\end{aligned}
\tag{39}
$$

Furthermore, in order to better show the results, in the fourth and fifth lines the stress tensor components $\rho_{<jk>}$ and $\rho_{<ik>}$ are already evaluated in terms of the velocity derivative using the first Maxwellian iteration. Moreover, the derivatives of the stress tensor components are neglected since they are terms of higher order when compared with second-grade fluid. As can be easily seen, this last equation contains, in addition to the terms in (28), the terms with the derivatives of $\beta_2$. These additional terms can eliminate the antisymmetric parts of the velocity derivatives: since Equation (6) is valid for an incompressible fluid, we must suppose

$$
\frac{\mathrm{d}\beta_2}{\mathrm{d}t} = \frac{\partial \beta_2}{\partial T} \frac{\mathrm{d}T}{\mathrm{d}t},
\tag{40}
$$

so we can use the function $\beta_2$ in order to satisfy the relation

$$
\frac{1}{2} \frac{\partial \beta_2}{\partial T} \frac{\mathrm{d}T}{\mathrm{d}t} = 2\beta_1 + 1
\tag{41}
$$

and this choice eliminates the antisymmetric parts of the velocity derivatives. It is clear that, in order to satisfy relation (41), the temperature must change during the time. This is allowed also if the heat flux vanishes, as in the case of the present paper.

Therefore, inserting the first iteration (29) into the left-hand side of Equation (39), and assuming (41), we obtain the second iteration that has the same form as (6).

## 9. Conclusions and Final Remarks

In the paper, a model for a second-grade fluid is introduced in the context of Rational Extended Thermodynamics. The field equations are based on the conservation law of mass, momentum and energy and a balance equation for the stress tensor. The constitutive relations are determined via universal physical principles, like the Galileian and Entropy Principles. Firstly, linear constitutive relations are determined like in [1]. In this case, it was shown that, applying Maxwellian iterations, it is not possible to obtain the classical equations appropriate to a second-grade fluid (6) as a limiting case. Constitutive relations that are quadratic in the stress tensor can instead recover the classical relation (6).

An interesting further study could be the investigation of solutions for different kinds of flow for second-order fluids described by the obtained field equations and the comparison with the analogue classical results like the ones obtained [30–34].

It could be interesting to investigate the more general case of a heat conduction fluid in order to study the effect of heat conduction in the field equations. This study is already

under investigation, although it is more complex. Furthermore, third-grade fluid could also be an interesting subject for investigation. Probably in this case, the effect of the third Maxwellian iteration must be carried out. Surely, the third-order constitutive relations must be taken into account and the formula must become more complex.

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