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QUASI-LINEAR ELLIPTIC SYSTEMS WITH INTRINSIC OPERATORS

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Abstract. Quasi-linear elliptic problems involving intrinsic operators are considered for the first time in the system context. A special sub-supersolution approach is developed leading to the existence of solutions and a priori estimates. An explicit example illustrates the abstract result.

1. **Introduction.** In this article, we study the existence and location of solutions to the following quasilinear elliptic system with homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta_{p_1} u_1 = f_1\left(x, B_1 u_1, B_2 u_2, \nabla\left(B_1 u_1\right), \nabla\left(B_2 u_2\right)\right) & \text{in } \Omega \\ -\Delta_{p_2} u_2 = f_2\left(x, B_1 u_1, B_2 u_2, \nabla\left(B_1 u_1\right), \nabla\left(B_2 u_2\right)\right) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Problem (P) is stated on a nonempty bounded open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. In the left-hand sides of the equations in (P), we have the negative p_i-Laplacian $-\Delta_{p_i} : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ for $i = 1, 2$, where $p_i \in (1, +\infty)$ and $p_i' = p_i/(p_i - 1)$. These operators are given by

$$\langle -\Delta_{p_i} u, v\rangle = \int_\Omega |\nabla u(x)|^{p_i-2}\, \nabla u(x) \cdot \nabla v(x) dx \quad \text{for each } u, v \in W_0^{1,p_i}(\Omega).$$

In the statement of (P) we also have the continuous operators called intrinsic $B_i : W_0^{1,p_i}(\Omega) \to W_0^{1,p_i}(\Omega)$ for $i = 1, 2$ that satisfy the conditions (H_1) and (H_2) in Section 2. Moreover, the right-hand sides in problem (P) are expressed via Carathéodory functions $f_i : \Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ meaning that $f_i(\cdot, s_1, s_2, \xi_1, \xi_2)$ is measurable for all $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$ and $f_i(x, \cdot, \cdot, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$, with $i = 1, 2$. Such nonlinearities depending on the solution and its gradient are often called convection terms.

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The most noteworthy aspect is that there are involved nonlinearities in the form of convection composed with intrinsic operators applied to both the solutions and their gradients. In the case of equations, this study was initiated in [8] and further developed in [10] and [12]. Here, for the first time, we focus on systems exhibiting simultaneously convection and intrinsic operators. We observe that when B_1 and B_2 are identities, we retrieve the framework in earlier works [1, 3, 6, 7]. If $p_1 = p_2$, $f_1 = f_2$ and $B_1 = B_2$, we get back to [8] and [10]. A major challenge would be to investigate problems with convection and intrinsic operators arising in the setting of weighted Sobolev spaces (see [4, 5]).

Due to the dependence on the gradient in the nonlinearities, the application of variational methods is not feasible. That's why for studying problem (P) we adopt a non-variational methodology based on a version of sub-supersolution approach in the case of systems. Specifically, we introduce an auxiliary parametric problem (see (P_λ) in Section 3) containing the truncation operators (T_1, T_2) and cut-off functions (π_1, π_2) that correspond to a sub-supersolution $((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2))$. Contrary to equations, in the case of systems we cannot introduce separately a subsolution and a supersolution being needed to manage as a single body called sub-supersolution. Through the theory of pseudomonotone operators we establish the existence of at least one solution (u_1, u_2) to the auxiliary problem (P_λ) provided $\lambda > 0$ is sufficiently large. The main tool to resolve the auxiliary problem is the abstract surjectivity result quoted as Theorem 3.1. By means of comparison arguments, we prove that the solution (u_1, u_2) to the auxiliary problem is located in the ordered rectangle (called also trapping region) $\underline{u}_1 \leq u_1 \leq \overline{u}_1$ and $\underline{u}_2 \leq u_2 \leq \overline{u}_2$ a.e. in Ω. This location property ensures that (u_1, u_2) is actually a solution to the original problem (P).

The most difficult point is to integrate the intrinsic operators (B_1, B_2) in the sub-supersolution framework for system (P). The decisive step was to achieve the invariance of the trapping region with respect to the intrinsic operators. Due to this device, the needed estimates still hold under the presence of the intrinsic operators. We end the paper with a detailed example illustrating the applicability of our main result.

The paper is arranged as follows. In Section 2 we present the preliminaries and hypotheses needed for our results. In Section 3 we formulate the auxiliary problem and prove the existence of solutions for it. In Section 4 we show how the auxiliary problem permits to solve the original problem. Finally, in Section 5 we provide an example.

2. Preliminaries and hypotheses.

In this section we set forth basic facts related to system (P).

The function space associated to problem (P) is the product space $X = W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega)$ endowed with the norm

$$\|u\| = \|u_1\|_{W_0^{1,p_1}(\Omega)} + \|u_2\|_{W_0^{1,p_2}(\Omega)}, \quad u = (u_1, u_2),$$

where $\|\cdot\|_{W_0^{1,p_i}(\Omega)}$ denotes the usual norm of the Banach space $W_0^{1,p_i}(\Omega)$. In the sequel we suppose that $N > p_i$ for $i = 1, 2$, thus the Sobolev critical exponents are $p_i^* = \frac{Np_i}{N-p_i}$ for $i = 1, 2$. The case where $N \leq p_i$ can be treated in the same way.

We recall from [2] the following basic result.

Proposition 2.1. *The negative p_i-Laplacian $-\Delta_{p_i} : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ for $i = 1, 2$ is maximal monotone, strictly monotone (so, pseudomonotone) and satisfies the (S_+)-property, that is, any sequence $\{u_n\} \subset W_0^{1,p_i}(\Omega)$ for which $u_n \rightharpoonup u$ in $W_0^{1,p_i}(\Omega)$ and $\limsup_{n\to+\infty} \langle -\Delta_{p_i} u_n, u_n - u\rangle \leq 0$ fulfills $u_n \to u$ in $W_0^{1,p_i}(\Omega)$.*

By a weak solution to problem (P) we mean any pair $(u_1, u_2) \in X$ such that

$$f_i\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) v_i \in L^1(\Omega), \quad i = 1, 2,$$

and

$$\int_\Omega |\nabla u_1(x)|^{p_1-2}\, \nabla u_1(x) \cdot \nabla v_1(x) dx = \int_\Omega f_1\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) v_1 dx,$$

$$\int_\Omega |\nabla u_2(x)|^{p_2-2}\, \nabla u_2(x) \cdot \nabla v_2(x) dx = \int_\Omega f_2\left(x, B_1u_1, B_2u_2, \nabla\left(B_1u_1\right), \nabla\left(B_2u_2\right)\right) v_2 dx$$

for each $(v_1, v_2) \in X$.

We say that the pair $((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2)) \in X \times X$ is a sub-supersolution to problem (P) if $\underline{u}_i \leq \overline{u}_i$ a.e. in Ω, $\underline{u}_i \leq 0 \leq \overline{u}_i$ on $\partial\Omega$ for $i = 1, 2$, and

$$\int_\Omega |\nabla \underline{u}_1(x)|^{p_1-2}\, \nabla \underline{u}_1(x) \cdot \nabla v_1(x) dx - \int_\Omega f_1\left(x, B_1\underline{u}_1, B_2w_2, \nabla\left(B_1\underline{u}_1\right), \nabla\left(B_2w_2\right)\right) v_1 dx \leq 0,$$

$$\int_\Omega |\nabla \underline{u}_2(x)|^{p_2-2}\, \nabla \underline{u}_2(x) \cdot \nabla v_2(x) dx - \int_\Omega f_2\left(x, B_1w_1, B_2\underline{u}_2, \nabla\left(B_1w_1\right), \nabla\left(B_2\underline{u}_2\right)\right) v_2 dx \leq 0,$$

$$\int_\Omega |\nabla \overline{u}_1(x)|^{p_1-2}\, \nabla \overline{u}_1(x) \cdot \nabla v_1(x) dx - \int_\Omega f_1\left(x, B_1\overline{u}_1, B_2w_2, \nabla\left(B_1\overline{u}_1\right), \nabla\left(B_2w_2\right)\right) v_1 dx \geq 0,$$

$$\int_\Omega |\nabla \overline{u}_2(x)|^{p_2-2}\, \nabla \overline{u}_2(x) \cdot \nabla v_2(x) dx - \int_\Omega f_2\left(x, B_1w_1, B_2\overline{u}_2, \nabla\left(B_1w_1\right), \nabla\left(B_2\overline{u}_2\right)\right) v_2 dx \geq 0$$

for all $(v_1, v_2) \in X$, with $v_i \geq 0$ a.e. in Ω and all $(w_1, w_2) \in X$, and $\underline{u}_i \leq w_i \leq \overline{u}_i$ for $i = 1, 2$, provided all the integrals involving f_1 and f_2 that appear above exist.

If $((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2))$ constitutes a sub-supersolution, then the ordered rectangle $[\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$ is also called trapping region, where

$$[\underline{u}_i, \overline{u}_i] = \left\{ u \in W_0^{1,p_i}(\Omega) : \underline{u}_i \leq u \leq \overline{u}_i \text{ a.e. in } \Omega \right\}.$$

In order to simplify the presentation, for any real number $r \in]1, +\infty[$, we denote $r' = \dfrac{r}{r-1}$ (the Hölder conjugate of r).

We formulate the assumptions on the Carathéodory functions $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and the intrinsic operators $B_i : W_0^{1,p_i}(\Omega) \to W_0^{1,p_i}(\Omega)$ with $i = 1, 2$ admitting that a sub-supersolution $((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2))$ is fixed.

(H_0) There exist functions $\sigma_i \in L^{r_i'}(\Omega)$ with $r_i \in]1, p_i^*[$ and constants $a_i > 0$ and $\beta_i \in [0, \frac{p_i}{(p_i^*)'}[$ for $i = 1, 2$ such that

$$|f_1(x, s_1, s_2, \xi_1, \xi_2)| \leq \sigma_1(x) + a_1 \left(|\xi_1|^{\beta_1} + |\xi_2|^{\frac{\beta_1 p_2}{p_1}} \right)$$

and

$$|f_2(x, s_1, s_2, \xi_1, \xi_2)| \leq \sigma_2(x) + a_2 \left(|\xi_1|^{\frac{\beta_2 p_1}{p_2}} + |\xi_2|^{\beta_2} \right)$$

for a.e. $x \in \Omega$, for all $(s_1, s_2) \in [\underline{u}_1(x), \overline{u}_1(x)] \times [\underline{u}_2(x), \overline{u}_2(x)]$, all $\xi_1, \xi_2 \in \mathbb{R}^N$.

(H_1) The maps $B_i : W_0^{1,p_i}(\Omega) \to W_0^{1,p_i}(\Omega)$ are continuous and fulfill

$$\underline{u}_i \leq B_i v \leq \overline{u}_i$$

for a.e. $x \in \Omega$ and for each $v \in W_0^{1,p_i}(\Omega)$ with $\underline{u}_i \leq v \leq \overline{u}_i$, $i = 1, 2$.

(H_2) There exist positive constants K_1 and K_2 such that

$$\|B_i v\|_{W_0^{1,p_i}(\Omega)} \leq K_1 \|v\|_{W_0^{1,p_i}(\Omega)} + K_2$$

for all $v \in W_0^{1,p_i}(\Omega)$ and $i = 1, 2$.

Let the Nemytskij type operator $N_{f,B} : [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2] \to W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$ be defined by

$$N_{f,B}(u_1, u_2) = (N_{f_1,B}(u_1, u_2), N_{f_2,B}(u_1, u_2)),$$

with

$$\langle N_{f_1,B}(u_1, u_2), v_1 \rangle = \int_\Omega f_1 (x, B_1 u_1, B_2 u_2, \nabla (B_1 u_1), \nabla (B_2 u_2)) v_1 dx$$

for all $v_1 \in W_0^{1,p_1}(\Omega)$, and

$$\langle N_{f_2,B}(u_1, u_2), v_2 \rangle = \int_\Omega f_2 (x, B_1 u_1, B_2 u_2, \nabla (B_1 u_1), \nabla (B_2 u_2)) v_2 dx$$

for all $v_2 \in W_0^{1,p_2}(\Omega)$. These operators are well defined due to hypothesis (H_0).

At this point we consider the truncation operators $T_i : W_0^{1,p_i}(\Omega) \to W_0^{1,p_i}(\Omega)$ associated to the ordered intervals $[\underline{u}_i, \overline{u}_i]$ which are defined by

$$T_i u(x) = \begin{cases} \overline{u}_i(x) & \text{if} \quad u(x) > \overline{u}_i(x), \\ u(x) & \text{if} \quad \underline{u}_i(x) \leq u(x) \leq \overline{u}_i(x), \\ \underline{u}_i(x) & \text{if} \quad u(x) < \underline{u}_i(x) \end{cases} \tag{1}$$

for all $u \in W_0^{1,p_i}(\Omega)$ and $i = 1, 2$. It readily follows from (1) that the operator $T_i : W_0^{1,p_i}(\Omega) \to W_0^{1,p_i}(\Omega)$ is continuous and bounded (in the sense that it maps bounded sets into bounded sets), $i = 1, 2$. We introduce the Nemytskij type operator $\mathcal{N}_{f_i,B} : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ as

$$\mathcal{N}_{f_i,B} = N_{f_i,B} \circ T_i \tag{2}$$

for $i = 1, 2$, that is,

$$\langle \mathcal{N}_{f_1,B}(u_1, u_2), v_1 \rangle = \int_\Omega f_1 (x, B_1(T_1 u_1), B_2(T_2 u_2), \nabla (B_1(T_1 u_1)), \nabla (B_2(T_2 u_2))) v_1 dx$$

and

$$\langle \mathcal{N}_{f_2,B}(u_1, u_2), v_2 \rangle = \int_\Omega f_2 (x, B_1(T_1 u_1), B_2(T_2 u_2), \nabla (B_1(T_1 u_1)), \nabla (B_2(T_2 u_2))) v_2 dx.$$

They are well defined on $W_0^{1,p_1}(\Omega)$ and $W_0^{1,p_2}(\Omega)$, respectively, thanks to hypotheses (H_1) and (H_2).

We also define the cut-off functions $\pi_i : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows

$$\pi_i(x, s) = \begin{cases} (s - \overline{u}_i(x))^{\frac{\beta_i}{p_i - \beta_i}} & \text{if} \quad s > \overline{u}_i(x), \\ 0 & \text{if} \quad \underline{u}_i(x) \leq s \leq \overline{u}_i(x), \\ -(\underline{u}_i(x) - s)^{\frac{\beta_i}{p_i - \beta_i}} & \text{if} \quad s < \underline{u}_i(x), \end{cases} \tag{3}$$

with the constants β_i in hypothesis (H_0) for $i = 1, 2$. The functions π_i in (3) satisfy the growth

$$|\pi_i(x, s)| \leq |s|^{\frac{\beta_i}{p_i - \beta_i}} + \varrho_i(x) \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R}, \tag{4}$$

with $\varrho_i \in L^{\frac{p_i - \beta_i}{\beta_i} p_i^*}(\Omega)$, for $i = 1, 2$, given by $\varrho_i(x) = \max \left\{ |\underline{u}_i(x)|^{\frac{\beta_i}{p_i - \beta_i}}, |\overline{u}_i(x)|^{\frac{\beta_i}{p_i - \beta_i}} \right\}$.

The Nemytskij operator $\Pi_i : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ corresponding to the function π_i act as

$$\langle \Pi_i(u), v\rangle = \int_\Omega \pi_i(x,u)v dx. \tag{5}$$

Since $\beta_i < \frac{p_i}{(p_i^*)'}$, by (4) and the Rellich-Kondrachov embedding theorem, we infer that the Nemytskij operator $\Pi_i : W_0^{1,p_i}(\Omega) \to W^{-1,p_i'}(\Omega)$ with $i = 1, 2$ is completely continuous.

Furthermore, from (3) it is seen as in [11] that the following estimates hold

$$\int_\Omega \pi_i(x,u(x))u(x)\,dx \geq r_1^{(i)} \|u\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)}^{\frac{p_i}{p_i-\beta_i}} - r_2^{(i)} \quad \text{for all } u \in W_0^{1,p_i}(\Omega), \tag{6}$$

with positive constants $r_1^{(i)}$ and $r_2^{(i)}$, for $i = 1, 2$.

3. **Auxiliary problem.** This section is dedicated to the study of an auxiliary problem that will allow us to establish the existence and location of solutions for the original problem (P).

Let λ be a positive real parameter and consider the following auxiliary Dirichlet problem

$$\begin{cases} -\Delta_{p_1} u_1 + \lambda \Pi_1(u_1) = \mathcal{N}_{f_1,B}(u_1,u_2) & \text{in } \Omega \\ -\Delta_{p_2} u_2 + \lambda \Pi_2(u_2) = \mathcal{N}_{f_2,B}(u_1,u_2) & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{P_λ}$$

In the statement of problem (P_λ) the notation in (2) and (5) is used.

A weak solution to problem (P_λ) is any pair $(u_1, u_2) \in X$ such that

$$f_i\left(x, B_1u_1, B_2u_2, \nabla\left(B_1(u_1)\right), \nabla\left(B_2u_2\right)\right) v_i \in L^1(\Omega)$$

for $i = 1, 2$ and

$$\int_\Omega |\nabla u_1(x)|^{p_1-2}\, \nabla u_1(x)\cdot \nabla v_1(x) dx + \lambda \int_\Omega \pi_1(x,u_1)v_1 dx$$
$$= \int_\Omega f_1\left(x, B_1(T_1u_1), B_2(T_2u_2), \nabla\left(B_1(T_1u_1)\right), \nabla\left(B_2(T_2u_2)\right)\right) v_1 dx,$$

$$\int_\Omega |\nabla u_2(x)|^{p_2-2}\, \nabla u_2(x)\cdot \nabla v_2(x) dx + \lambda \int_\Omega \pi_2(x,u_2)v_2 dx$$
$$= \int_\Omega f_2\left(x, B_1(T_1u_1), B_2(T_2u_2), \nabla\left(B_1(T_1u_1)\right), \nabla\left(B_2(T_2u_2)\right)\right) v_2 dx$$

for each $(v_1, v_2) \in X$.

The solvability of problem (P_λ) can be guaranteed provided that $\lambda > 0$ is sufficiently large as will be shown in Theorem 3.2. To achieve this we need to apply the following theorem.

Theorem 3.1. *(see* [2, Theorem 2.99]*) Let X be a real reflexive Banach space and let $A : X \to X^*$ be a bounded, coercive and pseudomonotone operator. Then for every $b \in X^*$ the equation $Ax = b$ has at least one solution $x \in X$.*

We now state the theorem regarding the solvability for the auxiliary problem (P_λ).

Theorem 3.2. *Assume that the two pairs* $(\underline{u}_1, \underline{u}_2)$ *and* $(\overline{u}_1, \overline{u}_2)$ *in* X *form a sub-supersolution of problem* (P) *such that hypotheses* $(H_0) - (H_2)$ *are fulfilled. Then there exists* $\lambda_0 > 0$ *such that whenever* $\lambda \geq \lambda_0$ *there is a solution of auxiliary problem* (P_λ).

Proof. Our aim is to apply Theorem 3.1. Towards this, for each $\lambda > 0$ we introduce the operator $A_\lambda : W_0^{1,p_1}(\Omega) \times W_0^{1,p_2}(\Omega) \to W^{-1,p_1'}(\Omega) \times W^{-1,p_2'}(\Omega)$ defined by

$$\langle A_\lambda(u_1, u_2), (v_1, v_2)\rangle = A_\lambda^{(1)}(u_1, u_2)(v_1) + A_\lambda^{(2)}(u_1, u_2)(v_2), \tag{7}$$

where

$$A_\lambda^{(i)}(u_1, u_2) = \ -\Delta_{p_i} u_i + \lambda \Pi_i u_i - \mathcal{N}_{f_i,B}(u_1, u_2),$$

with $i = 1, 2$.

We claim that the operator A_λ is bounded, i.e., it maps bounded sets into bounded sets. From [8, Theorem 2] we know that $-\Delta_{p_i}$ for $i = 1, 2$ is bounded (see also [2, Lemma 2.111]). Moreover, hypotheses $(H_0) - (H_2)$ imply the boundedness of operators $\mathcal{N}_{f_i,B_i}$ with $i = 1, 2$. Therefore, combining with (4), we conclude that the operator A_λ is bounded.

In order to show that the operator A_λ in (7) is pseudomonotone, we consider a sequence $(u_{1,n}, u_{2,n}) \subset X$ satisfying

$$(u_{1,n}, u_{2,n}) \rightharpoonup (u_1, u_2) \text{ in } X \tag{8}$$

and

$$\limsup_{n\to\infty} \langle A_\lambda\left(u_{1,n}, u_{2,n}\right), \left(u_{1,n} - u_1, u_{2,n} - u_2\right)\rangle \leq 0. \tag{9}$$

We aim to show that

$$\begin{aligned}&\liminf_{n\to\infty} \langle A_\lambda\left(u_{1,n}, u_{2,n}\right), \left(u_{1,n} - v_1, u_{2,n} - v_2\right)\rangle \\ &\geq \langle A_\lambda\left(u_1, u_2\right), \left(u_1 - v_1, u_2 - v_2\right)\rangle, \ \forall (v_1, v_2) \in X.\end{aligned}$$

Hölder's inequality, (8) and the Rellich-Kondrachov embedding theorem yield

$$\int_\Omega \sigma_i |u_{i,n} - u_i| \, dx \leq \|\sigma_i\|_{L^{r_i'}(\Omega)} \|u_{i,n} - u_i\|_{L^{r_i}(\Omega)} \to 0 \text{ as } n \to +\infty. \tag{10}$$

By the definition of the truncation operator T_i in (1) and using Hölder's inequality we derive

$$\int_\Omega |\nabla B_i(T_i u_{i,n})|^{\beta_i} |u_{i,n} - u_i| dx \leq \|\nabla B_i\left(T_i u_{i,n}\right)\|_{L^{p_i}(\Omega)}^{\beta_i} \|u_{i,n} - u_i\|_{L^{\frac{p_i}{p_i - \beta_i}}(\Omega)}.$$

Then (H_1), (H_2), (8), the Rellich-Kondrachov embedding theorem and the inequality $\frac{p_i}{p_i - \beta_i} < p_i^*$ imply

$$\lim_{n\to\infty} \int_\Omega |\nabla B_i(T_i u_{i,n})|^{\beta_i} |u_{i,n} - u_i| \, dx = 0. \tag{11}$$

Likewise, for $i \neq j$, we have that

$$\int_\Omega |\nabla B_i(T_i u_{i,n})|^{\frac{\beta_j p_i}{p_j}} |u_{j,n} - u_j| dx \leq \|\nabla B_i\left(T_i u_{i,n}\right)\|_{L^{p_i}(\Omega)}^{\frac{\beta_j p_i}{p_j}} \|u_{j,n} - u_j\|_{L^{\frac{p_j}{p_j - \beta_j}}(\Omega)}.$$

As above, (H_1), (H_2), (8), the Rellich-Kondrachov embedding theorem and the inequality $\frac{p_j}{p_j - \beta_j} < p_j^*$ provide

$$\lim_{n\to\infty} \int_\Omega |\nabla B_i(T_i u_{i,n})|^{\frac{\beta_j p_i}{p_j}} |u_{j,n} - u_j| dx = 0. \tag{12}$$

Altogether, from (10), (11) and (12) we infer that

$$\lim_{n\to\infty}\int_\Omega f_i\left(x, B_i\left(T_i u_{i,n}\right), B_j\left(T_j u_{j,n}\right), \nabla B_i\left(T_i u_{i,n}\right), \nabla B_j\left(T_j u_{j,n}\right)\right)\left(u_{i,n}-u_i\right) dx = 0. \tag{13}$$

On the other hand from (4) and Hölder's inequality we derive

$$\int_\Omega |\pi_i(x,u(x))||v(x)|\,dx \le r_3^{(i)}\|u\|^{\frac{\beta_i}{p_i-\beta_i}}_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)}\|v\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)} + r_4^{(i)}\|v\|_{L^{\frac{p_i}{p_i-\beta_i}}(\Omega)} \tag{14}$$

for all $u, v \in W_0^{1,p_i}(\Omega)$, with positive constants $r_3^{(i)}$ and $r_4^{(i)}$ for $i = 1, 2$. Then, on the basis of (14), the inequality $\frac{p_i}{p_i-\beta_i} < p_i^*$ and the Rellich-Kondrachov embedding theorem, it turns out

$$\lim_{n\to\infty}\int_\Omega \pi(x,u_{i,n})(u_{i,n}-u_i)\,dx = 0. \tag{15}$$

At this point, (13), (15) and (9) result in

$$\limsup_{n\to\infty}\left(\langle -\Delta_{p_1}u_{1,n}, u_{1,n}-u_1\rangle + \langle -\Delta_{p_2}u_{2,n}, u_{2,n}-u_2\rangle\right) \le 0. \tag{16}$$

We claim that (16) reduces to

$$\limsup_{n\to\infty}\langle -\Delta_{p_i}u_{i,n}, u_{i,n}-u_i\rangle \le 0 \quad \text{for } i = 1, 2. \tag{17}$$

Indeed, following [9], suppose by contradiction that up to subsequences it holds

$$\limsup_{n\to\infty}\langle -\Delta_{p_1}u_{1,n}, u_{1,n}-u_1\rangle > 0$$

and

$$\limsup_{n\to\infty}\langle -\Delta_{p_2}u_{2,n}, u_{2,n}-u_2\rangle < 0.$$

The second inequality, along with the (S_+)−property of $-\Delta_{p_i}$ on $W_0^{1,p_i}(\Omega)$ as described in Proposition 2.1, leads to $u_{2,n} \to u_2$ in $W_0^{1,p_2}(\Omega)$, from which a contradiction with (16) arises. Hence (17) is proven.

According to (17) and the (S_+)−property of the operator $-\Delta_{p_i}$ on $W_0^{1,p_i}(\Omega)$, we are able to deduce that

$$\lim_{n\to\infty}\langle A_\lambda\left(u_{1,n}, u_{2,n}\right), \left(u_{1,n}-v_1, u_{2,n}-v_2\right)\rangle = \langle A_\lambda\left(u_1,u_2\right), \left(u_1-v_1, u_2-v_2\right)\rangle$$

for every $(v_1, v_2) \in X$, which establishes that the operator A_λ is pseudomonotone.

Finally, we check that the operator $A_\lambda : X \to X^*$ is coercive, that is, for every sequence $\{(u_{1,n}, u_{2,n})\} \subseteq X$ such that $\|(u_{1,n}, u_{2,n})\| \to +\infty$ we have

$$\lim_{n\to+\infty}\frac{\langle A_\lambda\left(u_{1,n},u_{2,n}\right), \left(u_{1,n},u_{2,n}\right)\rangle}{\|\left(u_{1,n},u_{2,n}\right)\|} = +\infty. \tag{18}$$

To this end, by (1) it holds $\underline{u}_i \le T_i u \le \overline{u}_i$ a.e. in Ω for every $u \in W_0^{1,p_i}(\Omega)$, which is essential to refer to hypothesis (H_0). Through Hölder's and Young's inequalities, the Sobolev embedding theorem and assumptions $(H_0) - (H_2)$, for each $\varepsilon > 0$ we obtain that

$$\begin{aligned} &\left|\int_\Omega f_1\left(x, B_1\left(T_1u_{1,n}\right), B_2\left(T_2u_{2,n}\right), \nabla B_1\left(T_1u_{1,n}\right), \nabla B_2\left(T_2u_{2,n}\right)\right)u_{1,n}\,dx\right| \\ \le\ &\int_\Omega\left[\sigma_1(x) + a_1\left(|\nabla B_1\left(T_1u_{1,n}\right)|^{\beta_1} + |\nabla B_2\left(T_2u_{2,n}\right)|^{\frac{\beta_1 p_2}{p_1}}\right)\right]|u_{1,n}|dx \end{aligned}$$

$$
\begin{aligned}
&\leq \|\sigma_1\|_{L^{r_1'}(\Omega)}\|u_{1,n}\|_{L^{r_1}(\Omega)} + \varepsilon\|\nabla B_1(T_1u_{1,n})\|^{p_1}_{L^{p_1}(\Omega)} + c(\varepsilon)\|u_{1,n}\|^{\frac{p_1}{p_1-\beta_1}}_{L^{\frac{p_1}{p_1-\beta_1}}(\Omega)} \\
&\quad +\varepsilon\|\nabla B_2(T_2u_{2,n})\|^{p_2}_{L^{p_2}(\Omega)} + c(\varepsilon)\|u_{1,n}\|^{\frac{p_1}{p_1-\beta_1}}_{L^{\frac{p_1}{p_1-\beta_1}}(\Omega)} \\
&\leq C_1\|u_{1,n}\|_{W_0^{1,p_1}(\Omega)} + \varepsilon\bar{K}_1\|T_1u_{1,n}\|^{p_1}_{W_0^{1,p_1}(\Omega)} + c(\varepsilon)\|u_{1,n}\|^{\frac{p_1}{p_1-\beta_1}}_{L^{\frac{p_1}{p_1-\beta_1}}(\Omega)} + D_1(\varepsilon) \\
&\quad +\varepsilon\bar{K}_2\|T_2u_{2,n}\|^{p_2}_{W_0^{1,p_2}(\Omega)} + c(\varepsilon)\|u_{1,n}\|^{\frac{p_1}{p_1-\beta_1}}_{L^{\frac{p_1}{p_1-\beta_1}}(\Omega)} + D_2(\varepsilon) \\
&\leq C_1\|u_{1,n}\|_{W_0^{1,p_1}(\Omega)} + \varepsilon\bar{K}_1\|u_{1,n}\|^{p_1}_{W_0^{1,p_1}(\Omega)} + \varepsilon\bar{K}_2\|u_{2,n}\|^{p_2}_{W_0^{1,p_2}(\Omega)} \\
&\quad +c(\varepsilon)\|u_{1,n}\|^{\frac{p_1}{p_1-\beta_1}}_{L^{\frac{p_1}{p_1-\beta_1}}(\Omega)} + D(\varepsilon),
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&\left|\int_\Omega f_2\left(x, B_1\left(T_1u_{1,n}\right), B_2\left(T_2u_{2,n}\right), \nabla B_1\left(T_1u_{1,n}\right), \nabla B_2\left(T_2u_{2,n}\right)\right)u_{2,n}\,dx\right| \\
&\leq C_2\|u_{2,n}\|_{W_0^{1,p_2}(\Omega)} + \varepsilon\bar{K}_1\|u_{1,n}\|^{p_1}_{W_0^{1,p_1}(\Omega)} + \varepsilon\bar{K}_2\|u_{2,n}\|^{p_2}_{W_0^{1,p_2}(\Omega)} \\
&\quad +c'(\varepsilon)\|u_{2,n}\|^{\frac{p_2}{p_2-\beta_2}}_{L^{\frac{p_2}{p_2-\beta_2}}(\Omega)} + D'(\varepsilon),
\end{aligned}
$$

with positive constants C_1, C_2, $\bar{K}_1$, $\bar{K}_2$, $c(\varepsilon)$, $c'(\varepsilon)$, $D_1(\varepsilon)$, $D_2(\varepsilon)$, $D(\varepsilon)$ and $D'(\varepsilon)$.

Combining with (6) ensures

$$
\begin{aligned}
\langle A_\lambda^{(1)}\left(u_{1,n}, u_{2,n}\right), u_{1,n}\rangle \geq\ & (1-\varepsilon\bar{K}_1)\|u_{1,n}\|^{p_1}_{W_0^{1,p_1}(\Omega)} + (\lambda r_1^{(1)} - c(\varepsilon))\|u_{1,n}\|^{\frac{p_1}{p_1-\beta_1}}_{L^{\frac{p_1}{p_1-\beta_1}}(\Omega)} \\
& -C_1\|u_{1,n}\|_{W_0^{1,p_1}(\Omega)} - \varepsilon\bar{K}_2\|u_{2,n}\|^{p_2}_{W_0^{1,p_2}(\Omega)} - D(\varepsilon)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle A_\lambda^{(2)}\left(u_{1,n}, u_{2,n}\right), u_{2,n}\rangle \geq\ & (1-\varepsilon\bar{K}_2)\|u_{2,n}\|^{p_2}_{W_0^{1,p_2}(\Omega)} + (\lambda r_1^{(2)} - c'(\varepsilon))\|u_{2,n}\|^{\frac{p_2}{p_2-\beta_2}}_{L^{\frac{p_2}{p_2-\beta_2}}(\Omega)} \\
& -C_2\|u_{2,n}\|_{W_0^{1,p_2}(\Omega)} - \varepsilon\bar{K}_1\|u_{1,n}\|^{p_1}_{W_0^{1,p_1}(\Omega)} - D'(\varepsilon).
\end{aligned}
$$

Now, if we choose $\varepsilon \in \left]0, \dfrac{1}{2\bar{K}_i}\right[$ for $i = 1, 2$ and $\lambda > \max\left\{\frac{c(\varepsilon)}{r_1^{(1)}}, \frac{c'(\varepsilon)}{r_1^{(2)}}\right\}$, we obtain from (7) that (18) holds because $p_i > 1$. Therefore the operator $A_\lambda : X \to X^*$ is coercive.

Consequently, owing to Theorem 3.1, there exists $(u_1, u_2) \in X$ such that $A_\lambda(u_1, u_2) = 0$. According to (7), (u_1, u_2) is a (weak) solution to auxiliary problem (P_λ), which completes the proof. □

4. **Main result.** Our main result reads as follows.

Theorem 4.1. *Let $((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2))$ be a sub-supersolution of problem (P) such that hypotheses $(H_0)-(H_2)$ are fulfilled. Then system (P) possesses a weak solution $(u_1, u_2) \in X$ located in the corresponding trapping region, that is, $\underline{u}_1 \leq u_1 \leq \overline{u}_1$ and $\underline{u}_2 \leq u_2 \leq \overline{u}_2$ almost everywhere in Ω.*

Proof. Fix a $\lambda > 0$ sufficiently large such that by Theorem 3.2 there exists a weak solution $(u_1, u_2) \in X$ of auxiliary problem (P_λ). We claim that (u_1, u_2) is a solution of the original problem (P). The proof will be conducted through comparison arguments.

First, let us show that $u_1 \leq \overline{u}_1$ a.e. in Ω. Using the positive part $(u_1 - \overline{u}_1)^+ \in W_0^{1,p_1}(\Omega)$ as a test function in the definition of sub-supersolution of problem (P) and taking $w_2 = T_2 u_2$ (which is meaningful in view of the definition of the truncation operator T_2) we find that

$$\begin{aligned}&\int_\Omega |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1 \cdot \nabla (u_1 - \overline{u}_1)^+ dx \\ &\geq \int_\Omega f_1\left(x, B_1\overline{u}_1, B_2(T_2u_2), \nabla\left(B_1\overline{u}_1\right), \nabla\left(B_2(T_2u_2)\right)\right)(u_1 - \overline{u}_1)^+ dx.\end{aligned}$$

Using now $(u_1 - \overline{u}_1)^+ \in W_0^{1,p_1}(\Omega)$ as a test function in the definition of solution to auxiliary problem (P_λ), we get

$$\begin{aligned}&\int_\Omega |\nabla u_1|^{p_1-2} \nabla u_1 \cdot \nabla (u_1 - \overline{u}_1)^+ dx + \lambda \int_\Omega \pi_1(x, u_1)(u_1 - \overline{u}_1)^+ dx \\ &= \int_\Omega f_1\left(x, B_1(T_1u_1), B_2(T_2u_2), \nabla\left(B_1(T_1u_1)\right), \nabla\left(B_2(T_2u_2)\right)\right)(u_1 - \overline{u}_1)^+ dx.\end{aligned}$$

By subtraction and thanks to (1) we derive

$$\begin{aligned}&\int_\Omega \left(|\nabla u_1|^{p_1-2} \nabla u_1 - |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1\right) \nabla (u_1 - \overline{u}_1)^+ \, dx \\ &+\lambda \int_\Omega \pi_1(x, u_1)(u_1 - \overline{u}_1)^+ \, dx \\ \leq \; & \int_\Omega \Big(f_1\left(x, B_1(T_1u_1), B_2(T_2u_2), \nabla\left(B_1(T_1u_1)\right), \nabla\left(B_2(T_2u_2)\right)\right) \\ &- f_1\left(x, B_1\overline{u}_1, B_2(T_2u_2), \nabla\left(B_1\overline{u}_1\right), \nabla\left(B_2(T_2u_2)\right)\right) \Big)(u_1 - \overline{u}_1)^+ \, dx \\ = \; & \int_{\{u_1 > \overline{u}_1\}} \Big(f_1\left(x, B_1(T_1u_1), B_2(T_2u_2), \nabla\left(B_1(T_1u_1)\right), \nabla\left(B_2(T_2u_2)\right)\right) \\ &- f_1\left(x, B_1\overline{u}_1, B_2(T_2u_2), \nabla\left(B_1\overline{u}_1\right), \nabla\left(B_2(T_2u_2)\right)\right) \Big)(u_1 - \overline{u}_1) \, dx = 0. \quad (19)\end{aligned}$$

We also note that

$$\begin{aligned}&\int_\Omega \left(|\nabla u_1|^{p_1-2} \nabla u_1 - |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1\right) \nabla (u_1 - \overline{u}_1)^+ \, dx \\ &= \int_{\{u_1 > \overline{u}_1\}} \left(|\nabla u_1|^{p_1-2} \nabla u_1 - |\nabla \overline{u}_1|^{p_1-2} \nabla \overline{u}_1\right) \nabla (u_1 - \overline{u}_1) \geq 0.\end{aligned}$$

In view of (3) and (19), this amounts to saying that

$$\int_{\{u_1 > \overline{u}_1\}} (u_1 - \overline{u}_1)^{\frac{p_1}{p_1 - \beta_1}} \, dx = \int_\Omega \pi_1(x, u_1)(u_1 - \overline{u}_1)^+ \, dx \leq 0,$$

thereby meas $\{x \in \Omega : u_1(x) > \overline{u}_1(x)\} = 0$, or equivalently $u_1 \leq \overline{u}_1$ a.e in Ω.

Carrying out similar comparison arguments enables us to deduce that $\underline{u}_1 \leq u_1$ and $\underline{u}_2 \leq u_2 \leq \overline{u}_2$ a.e in Ω. Hence we have $(u_1, u_2) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$.

Exploiting the inclusion $(u_1, u_2) \in [\underline{u}_1, \overline{u}_1] \times [\underline{u}_2, \overline{u}_2]$, we infer from (1) and (3) that $T_i u_i = u_i$ and $\Pi_i(u_i) = 0$ for $i = 1, 2$. Therefore (u_1, u_2) is a solution of the original problem (P), which concludes the proof. □

5. **Example.** In thee sequel the following equations will play a major part:

$$\begin{cases} -\Delta_{p_i}\overline{u}_i = 1 & \text{in } \Omega \\ \overline{u}_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{20}$$

which has a unique solution $\overline{u}_i \in W_0^{1,p_i}(\Omega)$ for $i = 1, 2$, and

$$\begin{cases} -\Delta_{p_i}\varphi_{1,p_i} = \lambda_{1,p_i}\varphi_{1,p_i}^{p_i-1} & \text{in } \Omega \\ \varphi_{1,p_i} = 0 & \text{on } \partial\Omega, \end{cases} \tag{21}$$

where λ_{1,p_i} stands for the first eigenvalue of $-\Delta_{p_i}$ on $W_0^{1,p_i}(\Omega)$ and the eigenfunction φ_{1,p_i} is chosen to be positive and normalized as $\|\varphi_{1,p_i}\|_{L^{p_i}(\Omega)} = 1$, $i = 1, 2$.

The aim of the present section is to prove the existence of solutions as well as a priori estimates for the Dirichlet problem

$$\begin{cases} -\Delta_{p_1}u_1 = \dfrac{C_1(\min\{u_1^+, \overline{u}_1\})^{h_1}}{(1+\min\{u_2^+, \overline{u}_2\})(1+|\nabla(\min\{u_1^+, \overline{u}_1\}|)} & \text{in } \Omega \\ -\Delta_{p_2}u_2 = \dfrac{C_2(\min\{u_2^+, \overline{u}_2\})^{h_2}}{(1+\min\{u_1^+, \overline{u}_1\})(1+|\nabla(\min\{u_2^+, \overline{u}_2\}|)} & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{S}$$

with nonnegative constants C_i and h_i, $i = 1, 2$.

We state our result on system (S).

Theorem 5.1. *If $h_i < p_i - 1$ and $C_i\overline{u}_i^{h_i} \le 1$ for $i = 1, 2$, then problem (S) admits a weak solution $(u_1, u_2) \in X$ satisfying the a priori estimate $u_i \le \overline{u}_i$ almost everywhere in Ω and $i = 1, 2$.*

Proof. The idea is to apply Theorem 4.1. To this end we have to show that system (S) can be written in the form of (P), thus identifying the maps f_i and B_i, with $i = 1, 2$. In this respect we set

$$f_1(x, s_1, s_2, \xi_1, \xi_2) = \frac{C_1|s_1|^{h_1}}{(1+|s_2|)(1+|\xi_1|)}$$

and

$$f_2(x, s_1, s_2, \xi_1, \xi_2) = \frac{C_2|s_2|^{h_2}}{(1+|s_1|)(1+|\xi_2|)}.$$

We see that $f_i : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ ($i = 1, 2$) is a Carathéodory function satisfying

$$|f_i(x, s_1, s_2, \xi_1, \xi_2)| \le C_i(1 + |s_i|^{p_i-1}). \tag{22}$$

Moreover, we introduce

$$B_i u = \min\{u^+, \overline{u}_i\}, \quad i = 1, 2,$$

which produces a continuous map $B_i : W_0^{1,p_i}(\Omega) \to W_0^{1,p_i}(\Omega)$ for $i = 1, 2$. Since for any $v \in W_0^{1,p_i}(\Omega)$ it holds

$$\nabla(B_i v)(x) = \nabla(\min\{v^+, \overline{u}_i\})(x) = \begin{cases} 0 & \text{if } v(x) < 0 \\ \nabla v(x) & \text{if } 0 \le v(x) \le \overline{u}_i(x) \\ \nabla\overline{u}_i(x) & \text{if } \overline{u}_i(x) < v(x), \end{cases}$$

it follows that

$$\|B_i v\|_{W_0^{1,p_i}(\Omega)} \le \|v\|_{W_0^{1,p_i}(\Omega)} + \|\overline{u}_i\|_{W_0^{1,p_i}(\Omega)}$$

for $i = 1, 2$, so condition ($H2$) is satisfied.

With the preceding notation, system (S) becomes of type (P), namely

$$\begin{cases} -\Delta_{p_1} u_1 = \dfrac{C_1 (B_1 u_1)^{h_1}}{(1+B_2 u_2)(1+|\nabla(B_1 u_1)|)} & \text{in } \Omega \\ -\Delta_{p_2} u_2 = \dfrac{C_2 (B_2 u_2)^{h_2}}{(1+B_1 u_1)(1+|\nabla(B_2 u_2)|)} & \text{in } \Omega \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{23}$$

Fix an $\varepsilon > 0$ such that

$$\varepsilon \varphi_{1,p_i} \leq \overline{u}_i, \quad i = 1, 2, \tag{24}$$

which is possible by applying the strong maximum principle to equations (20) and (21). Moreover, since $h_i < p_i - 1$ for $i = 1, 2$, we can choose $\varepsilon > 0$ so small to have

$$\varepsilon^{p_1-1-h_1} \varphi_{1,p_1}^{p_1-1-h_1} \lambda_{1,p_1} (1+\overline{u}_2)(1+\varepsilon|\nabla\varphi_{1,p_1}|) \leq C_1 \tag{25}$$

and

$$\varepsilon^{p_2-1-h_2} \varphi_{1,p_2}^{p_2-1-h_2} \lambda_{1,p_2} (1+\overline{u}_1)(1+\varepsilon|\nabla\varphi_{1,p_2}|) \leq C_2. \tag{26}$$

Set $\underline{u}_i = \varepsilon\varphi_{1,p_i}$ for $i = 1, 2$. We are going to prove that $((\underline{u}_1, \underline{u}_2), (\overline{u}_1, \overline{u}_2))$ is a sub-supersolution to system (23) (which actually is just (S)).

By (20) and the hypothesis $C_i \overline{u}_i^{h_i} \leq 1$, $i = 1, 2$, we note that

$$-\Delta_{p_1} \overline{u}_1 = 1 \geq C_1 \overline{u}_1^{h_1} \geq C_1 \frac{\overline{u}_1^{h_1}}{(1+w_2)(1+|\nabla \overline{u}_1|)} = C_1 \frac{(B_1 \overline{u}_1)^{h_1}}{(1+B_2 w_2)(1+|\nabla (B_1 \overline{u}_1|)}$$

for $\underline{u}_2 = \varepsilon\varphi_{1,p_2} \leq w_2 \leq \overline{u}_2$, and

$$-\Delta_{p_2} \overline{u}_2 = 1 \geq C_2 \overline{u}_2^{h_2} \geq C_2 \frac{\overline{u}_2^{h_2}}{(1+w_1)(1+|\nabla \overline{u}_2|)} = C_2 \frac{(C_2 \overline{u}_2)^{h_2}}{(1+B_1 w_1)(1+|\nabla (B_2 \overline{u}_2|)}$$

for $\underline{u}_1 = \varepsilon\varphi_{1,p_1} \leq w_1 \leq \overline{u}_1$.

By (21), (25) and (26) we arrive at

$$\begin{aligned} -\Delta_{p_1} \underline{u}_1 = \lambda_{1,p_1} \varepsilon^{p_1-1} \varphi_{1,p_1}^{p_1-1} &\leq C_1 \frac{\varepsilon^{h_1} \varphi_{1,p_1}^{h_1}}{(1+\overline{u}_2)(1+\varepsilon|\nabla\varphi_{1,p_1}|)} \\ &\leq C_1 \frac{(B_1 \underline{u}_1)^{h_1}}{(1+B_2 w_2)(1+|\nabla(B_1 \underline{u}_1|)} \end{aligned}$$

for $\underline{u}_2 = \varepsilon\varphi_{1,p_2} \leq w_2 \leq \overline{u}_2$, and

$$\begin{aligned} -\Delta_{p_2} \underline{u}_2 = \lambda_{1,p_2} \varepsilon^{p_2-1} \varphi_{1,p_2}^{p_1-1} &\leq C_2 \frac{\varepsilon^{h_2} \varphi_{1,p_2}^{h_2}}{(1+\overline{u}_1)(1+\varepsilon|\nabla\varphi_{1,p_2}|)} \\ &\leq C_2 \frac{(B_2 \underline{u}_2)^{h_2}}{(1+B_1 w_1)(1+|\nabla(B_2 \underline{u}_2|)} \end{aligned}$$

for $\underline{u}_1 = \varepsilon\varphi_{1,p_1} \leq w_1 \leq \overline{u}_1$. Our previous discussion and (24) entail that $((\underline{u}_1, \underline{u}_2)$, $(\overline{u}_1, \overline{u}_2))$ forms a sub-supersolution to system (23) (or equivalently (S)).

We point out that (22)implies that condition (H_0) holds true because it only involves $(s_1, s_2) \in [\underline{u}_1(x), \overline{u}_1(x)] \times [\underline{u}_2(x), \overline{u}_2(x)]$. Let us also notice that the operator B_i is the identity on the ordered interval $[\underline{u}_i, \overline{u}_i]$ with $i = 1, 2$. Consequently, condition (H_1) is verified, too.

All the requirements to apply Theorem 4.1 to system (23) are fulfilled. The desired conclusion regarding system (S) is thus achieved. □

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