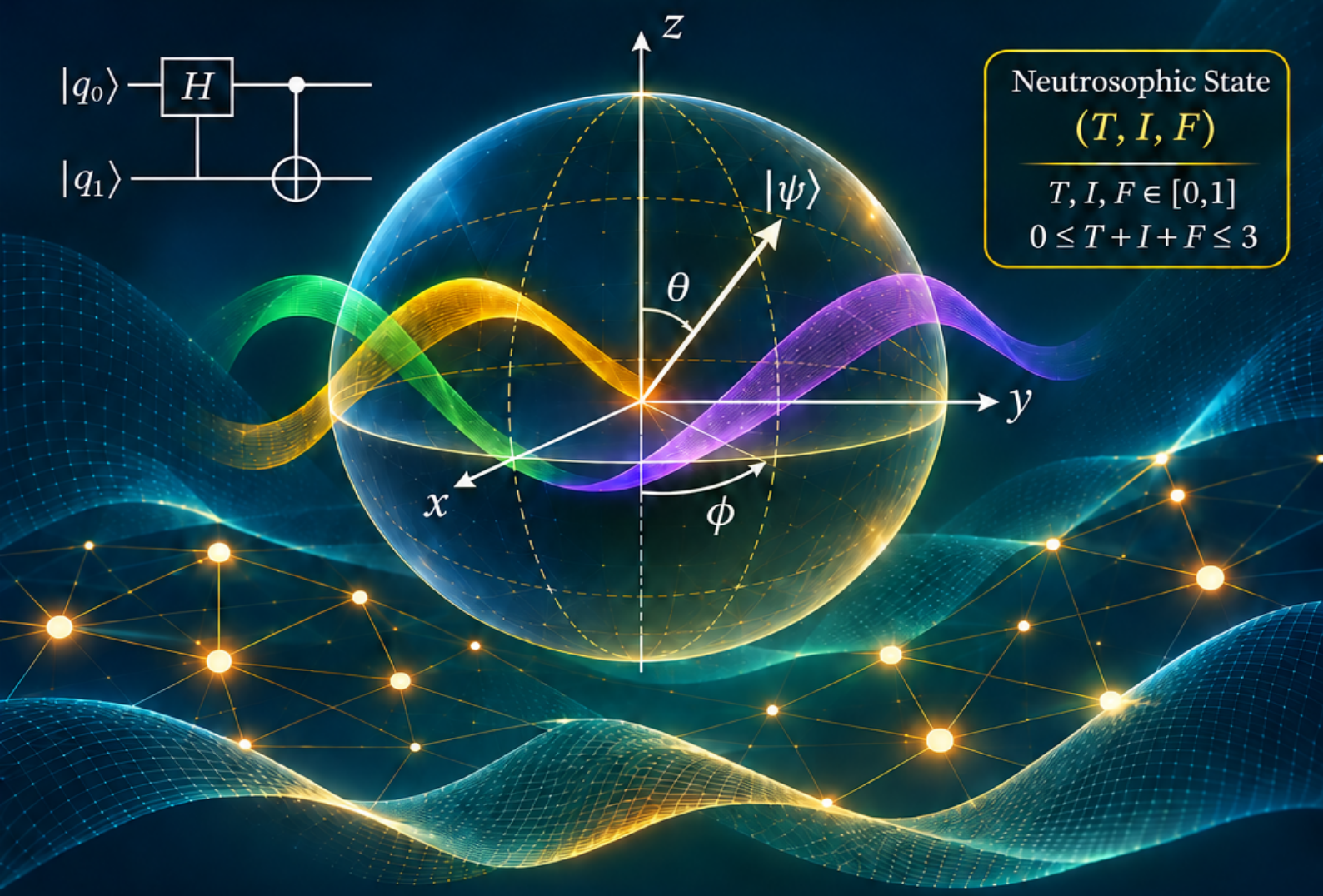
NEUTROSOPHIC QUANTUM COMPUTING

Mathematical Foundations, Quantum Information, and Computational Perspectives



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2026

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Preface

Quantum computing has emerged as one of the most important developments at the intersection of mathematics, physics, computer science, and information theory. By exploiting the principles of quantum mechanics, including superposition, entanglement, interference, and measurement, quantum computation provides a distinctive framework for processing information. Its mathematical foundations have inspired extensive research into quantum algorithms, quantum circuits, quantum communication, quantum information, and the physical realization of quantum computational systems.

At the same time, the study of quantum information raises important questions concerning incomplete knowledge, uncertainty in experimental data, imperfect state preparation, noisy operations, and the interpretation of measurement outcomes. Although the mathematical formalism of quantum mechanics provides precise rules for describing quantum states and their evolution, practical investigations often involve limited observations, incomplete information, and models whose parameters are not known with complete precision. These circumstances motivate the investigation of complementary mathematical frameworks capable of recording not only supported information but also indeterminacy and conflicting or unsupported information.

The present book, devoted to *neutrosophic quantum computing*, develops a mathematical perspective on these questions. Its central purpose is to investigate how ideas from neutrosophic logic and neutrosophic mathematics can be introduced into selected areas of quantum computation and quantum information theory. The book brings together fifteen connected studies that examine fundamental quantum-computational concepts and explore possible extensions through neutrosophic descriptions.

The guiding idea is to distinguish two levels of mathematical representation. The first is the conventional quantum-mechanical description, based on Hilbert spaces, state vectors, density operators, unitary transformations, quantum measurements, and the associated probability rules. The second is a neutrosophic annotation layer intended to represent the degree of support for a statement, the indeterminacy associated with it, and the degree of opposition or falsity assigned to it.

These two levels must be carefully distinguished. A neutrosophic triple is not automatically a quantum state, a probability distribution, or a physical observable. Rather, it can serve as an additional informational structure when its interpretation, assignment rules, and relation to the underlying quantum model are explicitly defined. The intention is not to replace quantum mechanics merely by changing its terminology, but to formulate precise definitions, investigate mathematical consequences, construct illustrative examples, identify limitations, and determine which neutrosophic extensions may offer meaningful benefits for the analysis of quantum information.

Scope and Organization of the Fifteen Studies

The fifteen studies on which this book is based approach neutrosophic quantum computing from complementary directions. Together, they are intended to establish a progression from

foundational concepts to more specialized structures and computational methods.

The opening studies introduce the mathematical language required for the subsequent development. They examine basic principles of quantum information and computation, including quantum states, qubits, superposition, measurement, and the mathematical structures used to describe quantum systems. The neutrosophic perspective is introduced as a framework for recording incomplete, indeterminate, or conflicting information about quantum states and computational processes. Particular attention is given to the need for consistency with the established mathematical principles of quantum theory.

The next group of studies investigates quantum information processing through quantum circuits and computational operations. Quantum gates, circuit composition, and transformations of quantum states provide settings in which to examine how neutrosophic annotations might be associated with operations, input information, intermediate computational results, and measurement records. These studies seek to distinguish uncertainty arising from physical noise from indeterminacy arising from incomplete knowledge or ambiguous interpretation.

The subsequent studies extend the discussion to quantum algorithms and related computational procedures. Their purpose is to investigate how neutrosophic information may be incorporated into the description, analysis, or evaluation of quantum computations without confusing informational annotations with the probabilities governing quantum measurements. Where appropriate, proposed constructions should be accompanied by explicit assumptions, mathematical examples, and comparisons with conventional formulations. Such comparisons are necessary to determine whether a neutrosophic extension supplies additional descriptive power or merely restates information already contained in a standard quantum model.

Another important theme is quantum entanglement and the mathematical characterization of multipartite quantum systems. Entanglement is a defining feature of quantum information theory and plays a central role in quantum communication, quantum computation, and measurement-based approaches to computation. A neutrosophic viewpoint offers a possible language for recording uncertainty about entanglement-related statements, experimental evidence, or incomplete state information. The challenge is to ensure that these annotations remain compatible with established entanglement criteria, separability conditions, stabilizer methods, and witness operators.

The final studies develop more specialized aspects of quantum computation, culminating in the investigation of *neutrosophic cluster-state quantum computing*. Cluster states are important resource states in measurement-based quantum computation, where computational operations are performed through appropriately chosen measurements on an entangled resource. Their graph-theoretic representation, stabilizer structure, and entanglement properties make them particularly suitable for examining the relationship between discrete mathematical structures and quantum information processing.

The neutrosophic approach considered in this setting provides a way to annotate graph descriptions, stabilizer-measurement records, and the interpretation of computational evidence when the available information is incomplete or indeterminate. Cluster-state computation thus brings together several themes developed throughout the book: quantum states, graph representations, entanglement, measurement, information processing, and the mathematical treatment of indeterminacy.

The Role of Neutrosophic Mathematics

Neutrosophic logic, introduced by Florentin Smarandache, provides a framework in which truth, indeterminacy, and falsity can be considered as distinct components of an informational description. In suitable applications, these components may be represented by a triple

$$(T, I, F),$$

where T denotes the degree of support for a proposition, I its associated indeterminacy, and F the degree of opposition or falsity. Depending on the neutrosophic model being used, these components may satisfy different restrictions. Their domains, normalization rules, and interpretation must therefore be specified for each application.

In this book, the triple is treated as an informational descriptor rather than as an automatic substitute for a quantum probability. For example, it may be used to record confidence in the claim that a quantum gate has been implemented correctly, uncertainty about a reconstructed quantum state, or incomplete evidence concerning a stabilizer relation. Each use requires an explicit procedure for assigning the three components and a clear account of how the resulting information is to be interpreted.

This methodological caution is central to the book. Quantum mechanics already contains rigorous methods for representing statistical mixtures, measurement probabilities, and uncertainty relations. Consequently, a neutrosophic extension must identify the type of information that is not adequately represented by the conventional model, explain how the additional structure is obtained, and demonstrate why it is useful. The framework must not conflate epistemic uncertainty, experimental noise, classical ignorance, and quantum indeterminacy, which are mathematically and physically distinct concepts.

The fifteen studies should therefore be understood as investigations into a developing mathematical programme rather than as a claim that every neutrosophic construction has already acquired an established physical interpretation. Some constructions may serve primarily as conceptual or organizational tools; others may lead to mathematical criteria, improved data descriptions, or computational procedures. Their value must be assessed individually and in relation to the relevant standard theory.

Mathematical Rigor and Research Methodology

A principal objective of this book is to maintain mathematical clarity throughout the proposed extensions. Definitions identify the objects under investigation; propositions and theorems establish consequences under stated assumptions; examples illustrate the scope of the constructions; and counterexamples, where appropriate, clarify the limitations of general claims. References to established results in quantum computation, quantum information theory, graph states, stabilizer methods, and neutrosophic mathematics provide the necessary scholarly context.

Equally important is the distinction between a mathematical proposal and a validated physical model. A formal extension may be internally consistent without necessarily describing a realizable quantum system or improving the performance of a quantum algorithm. Claims of computational or experimental advantage require additional evidence, such as complexity analysis, comparison with established methods, numerical experiments, or laboratory validation. The book therefore treats the development of definitions and the demonstration of their usefulness as separate but complementary tasks.

The intended result is a framework in which conventional quantum theory remains identifiable and its established results remain available, while neutrosophic structures are introduced only where their role can be stated precisely. This approach makes it possible to ask meaningful questions: Which forms of incomplete information benefit from neutrosophic annotation? Can annotations be propagated consistently through a quantum circuit? How should they be related to measurement records? Under what conditions do they contribute information beyond conventional statistical descriptions? Which mathematical or computational applications justify further investigation?

These questions also indicate directions for future research, including the development of operational assignment rules, the study of neutrosophic quantum channels, the treatment

of uncertainty in quantum-state reconstruction, and the investigation of graph-based and measurement-based quantum computation.

Intended Readership and Future Perspectives

This book is intended for researchers and advanced students in mathematics, theoretical physics, computer science, quantum information, and neutrosophic mathematics. Readers familiar with linear algebra and elementary quantum mechanics will find a mathematical route into the subject, while readers already working in quantum information or neutrosophic theory may use the studies to identify connections, open problems, and possible directions for collaboration.

The book does not assume that neutrosophic quantum computing constitutes a universally accepted or experimentally established replacement for conventional quantum computing. Rather, it presents the subject as a research direction whose mathematical formulations and practical significance must be examined critically. By making assumptions explicit and preserving the distinction between quantum-mechanical probabilities and neutrosophic annotations, the work seeks to support a careful and productive discussion of the subject.

The broader hope is that the interaction between quantum information theory and neutrosophic mathematics will encourage new questions about the representation of incomplete knowledge in advanced computational systems. Whether these investigations ultimately lead to new algorithms, improved methods of data interpretation, or primarily conceptual and mathematical insights, their progress will depend on precise definitions, rigorous analysis, reproducible examples, and meaningful comparisons with existing approaches.

Concluding Remarks

The fifteen studies collected in this volume reflect an ongoing interest in exploring mathematical structures that connect logic, information, computation, and mathematical physics. They are brought together here to provide a coherent starting point for the study of neutrosophic approaches to quantum computation, rather than a final or exhaustive account of the subject.

I hope that this book will stimulate further investigation, critical discussion, and collaboration among researchers interested in the mathematical foundations of quantum information and the possible applications of neutrosophic methods. Above all, I hope that it encourages the development of formulations that are not only conceptually appealing but also mathematically precise, operationally meaningful, and open to independent verification.

Contents

Authors and Affiliations ii

Preface iii
Scope and Organization of the Fifteen Studies . . . iii
The Role of Neutrosophic Mathematics . . . iv
Mathematical Rigor and Research Methodology . . . v
Intended Readership and Future Perspectives . . . vi
Concluding Remarks . . . vi

1 Introduction to Neutrosophic Quantum Computing 1
1.1 Introduction . . . 1
1.1.1 Scope and conventions . . . 2
1.2 Classical information as a starting point . . . 2
1.2.1 Bits and binary information . . . 2
1.2.2 Information content . . . 3
1.2.3 Shannon entropy . . . 3
1.2.4 A neutrosophic annotation of a classical bit . . . 3
1.2.5 From classical to quantum information . . . 4
1.3 Qubits and quantum states . . . 4
1.3.1 The computational basis . . . 4
1.3.2 Measurement and the Born rule . . . 4
1.3.3 Superposition and relative phase . . . 5
1.3.4 Neutrosophic interpretation of a qubit . . . 5
1.4 Neutrosophic representations of quantum outcomes . . . 5
1.4.1 Outcome annotations . . . 5
1.4.2 A worked example . . . 6
1.4.3 Consistency conditions . . . 6
1.5 Neutrosophic information and entropy . . . 6
1.5.1 Ordinary entropy of a qubit measurement . . . 6
1.5.2 A proposed neutrosophic uncertainty score . . . 7
1.5.3 Why two uncertainty measures may be useful . . . 7
1.6 Quantum gates and annotated information . . . 7
1.6.1 Unitary transformations . . . 7
1.6.2 An annotated gate operation . . . 8
1.6.3 Why ordinary gates cannot act directly on triples without further definitions 8
1.7 Density operators and mixed states . . . 9
1.7.1 Density operators . . . 9
1.7.2 Annotated density operators . . . 9
1.7.3 A proposition-dependent annotation . . . 9
1.8 Measurement in a hybrid framework . . . 10

1.8.1 Projective measurements . . . 10
1.8.2 Adding an annotation to a measurement record . . . 10
1.9 A simple two-stage example . . . 11
1.10 Possible applications and research questions . . . 11
1.10.1 Quantum-device diagnostics . . . 11
1.10.2 Hybrid quantum-classical learning . . . 11
1.10.3 Quantum information under incomplete evidence . . . 12
1.10.4 Research questions . . . 12
1.11 Limitations . . . 12
1.12 Conclusion . . . 12

2 Neutrosophic Qubits and Quantum States 14
2.1 Introduction . . . 15
2.2 Preliminaries on neutrosophic annotations . . . 15
2.2.1 Three-component descriptions . . . 15
2.2.2 Annotation operations . . . 16
2.3 The qubit and its mathematical representation . . . 17
2.3.1 The standard qubit . . . 17
2.3.2 The proposed annotated qubit . . . 17
2.3.3 A basic distinction . . . 18
2.4 Complex vector spaces and linear combinations . . . 18
2.4.1 Vector spaces . . . 18
2.4.2 Linear combinations of quantum states . . . 18
2.4.3 Annotated linear combinations . . . 19
2.5 Bases, dimension, and coordinates . . . 20
2.5.1 Linear independence . . . 20
2.5.2 Uniqueness of quantum coordinates . . . 20
2.5.3 Annotated coordinates . . . 20
2.5.4 A change of basis . . . 21
2.6 Inner products and bra–ket notation . . . 21
2.6.1 The complex inner product . . . 21
2.6.2 Bra–ket formalism . . . 21
2.6.3 An annotated inner-product record . . . 22
2.7 Orthonormality and Gram–Schmidt orthogonalization . . . 22
2.7.1 Orthonormal vectors . . . 22
2.7.2 The Gram–Schmidt process . . . 22
2.7.3 A worked example in $\mathbb{C}^2$ . . . 23
2.7.4 Annotations and orthogonalization . . . 24
2.8 Cauchy–Schwarz and triangle inequalities . . . 24
2.8.1 Cauchy–Schwarz inequality . . . 24
2.8.2 Triangle inequality . . . 25
2.8.3 A limitation of componentwise annotation rules . . . 25
2.9 Unitary transformations and annotation transport . . . 25
2.9.1 Unitary transformations . . . 25
2.9.2 The proposed annotation-update map . . . 26
2.9.3 A consistency condition for sequential gates . . . 26
2.10 Worked example: an annotated qubit . . . 27
2.11 Exercises . . . 27
2.12 Open problems . . . 28
2.13 Conclusion . . . 29

3 Neutrosophic Matrices and Operators 30
3.1 Introduction . . . 30
3.2 Preliminaries on neutrosophic annotations . . . 31
3.2.1 Neutrosophic triples . . . 31
3.2.2 Convex aggregation . . . 32
3.3 Matrices and linear operators . . . 32
3.3.1 Matrices as representations of operators . . . 32
3.3.2 Annotated operators . . . 32
3.3.3 Composition of annotated operators . . . 33
3.4 Outer products and the closure relation . . . 33
3.4.1 Outer products . . . 33
3.4.2 The closure relation . . . 34
3.4.3 Annotated projectors . . . 34
3.5 The Pauli operators . . . 35
3.5.1 Definition . . . 35
3.5.2 Pauli commutation relations . . . 35
3.5.3 An annotated Pauli operator . . . 35
3.6 Hermitian, unitary, and normal operators . . . 36
3.6.1 Hermitian operators . . . 36
3.6.2 Unitary operators . . . 36
3.6.3 Normal operators . . . 36
3.6.4 Annotations of operator properties . . . 36
3.7 Eigenvalues, eigenvectors, and spectral decomposition . . . 37
3.7.1 Eigenvalues and eigenvectors . . . 37
3.7.2 Spectral theorem . . . 37
3.7.3 An annotated spectral record . . . 37
3.8 The trace and its properties . . . 38
3.8.1 Definition . . . 38
3.8.2 Trace of an outer product . . . 38
3.8.3 Trace and annotations . . . 38
3.9 Expectation values . . . 38
3.9.1 Pure-state expectation values . . . 38
3.9.2 Density-operator formulation . . . 39
3.9.3 An annotated expectation record . . . 39
3.10 Projection operators and positive operators . . . 39
3.10.1 Projection operators . . . 39
3.10.2 Positive operators . . . 40
3.10.3 Annotations of positive operators . . . 40
3.11 Commutator algebra . . . 40
3.11.1 Commutators . . . 40
3.11.2 Annotated commutator records . . . 41
3.12 The Heisenberg uncertainty principle . . . 41
3.12.1 Variances . . . 41
3.12.2 Pauli example . . . 42
3.12.3 Quantum uncertainty versus neutrosophic indeterminacy . . . 42
3.12.4 A proposed annotation of uncertainty data . . . 42
3.13 Polar decomposition and singular values . . . 42
3.13.1 Polar decomposition . . . 42
3.13.2 Singular values . . . 43
3.13.3 Annotations of polar data . . . 43
3.14 Standard quantum-mechanical postulates . . . 43

3.14.1 State postulate . . . 43
3.14.2 Observable postulate . . . 43
3.14.3 Measurement postulate . . . 43
3.14.4 Evolution postulate . . . 44
3.14.5 The role of neutrosophic annotations . . . 44
3.15 A worked example: an annotated spin measurement . . . 44
3.16 Limitations and open problems . . . 45
3.17 Conclusion . . . 45

4 Neutrosophic Tensor Products and Composite Systems 47
4.1 Introduction . . . 47
4.2 Mathematical preliminaries . . . 48
4.2.1 Hilbert spaces and tensor products . . . 48
4.2.2 Neutrosophic triples . . . 48
4.3 Annotated quantum states . . . 50
4.3.1 Definition of an annotated state . . . 50
4.3.2 Composite annotated states . . . 50
4.3.3 An explicit two-qubit example . . . 51
4.4 Inner products and annotated inner-product records . . . 51
4.5 Tensor products of column vectors . . . 52
4.5.1 Kronecker product . . . 52
4.5.2 Annotated column vectors . . . 52
4.6 Operators acting on composite systems . . . 53
4.6.1 Local operators . . . 53
4.6.2 Action on annotated states . . . 54
4.7 Tensor products of matrices . . . 54
4.7.1 The Kronecker product of matrices . . . 54
4.7.2 Algebraic identities . . . 54
4.7.3 Annotated matrix products . . . 55
4.8 A two-qubit calculation . . . 55
4.9 Product states, entanglement, and annotations . . . 56
4.9.1 Product states and general composite states . . . 56
4.9.2 A limitation of subsystem annotations . . . 57
4.10 Mixed states and partial traces . . . 57
4.10.1 Density operators . . . 57
4.10.2 Annotated density operators . . . 57
4.10.3 Partial trace . . . 58
4.11 Consistency questions and possible extensions . . . 58
4.11.1 Choice of aggregation rule . . . 58
4.11.2 Compatibility with quantum evolution . . . 58
4.11.3 Compatibility with measurement . . . 59
4.11.4 Multisubsystem systems . . . 59
4.12 Further research problems . . . 59
4.13 Conclusion . . . 60

5 Neutrosophic Density Operators 61
5.1 Introduction . . . 61
5.2 Preliminaries on density operators . . . 62
5.2.1 The space of density operators . . . 62
5.2.2 Pure states . . . 63
5.2.3 Mixed states . . . 64

5.3 Bounded neutrosophic annotations . . . 64
5.3.1 Definition and interpretation . . . 64
5.3.2 Neutrosophically annotated density operators . . . 65
5.4 Pure states with neutrosophic annotations . . . 66
5.4.1 Annotated pure-state operators . . . 66
5.4.2 Example: a superposition state . . . 66
5.5 Mixed states and annotated ensembles . . . 66
5.5.1 An ensemble of annotated preparations . . . 66
5.5.2 Example: a two-state mixture . . . 67
5.6 Key properties and characterizations . . . 68
5.6.1 Trace, positivity, and normalization . . . 68
5.6.2 Purity . . . 68
5.6.3 A limitation of annotation-based purity . . . 68
5.7 Characterizing mixed states . . . 69
5.7.1 Spectral characterization . . . 69
5.7.2 Different ensembles can give the same density operator . . . 69
5.8 Partial trace and reduced density operators . . . 69
5.8.1 The standard partial trace . . . 69
5.8.2 An annotation rule for reduction . . . 70
5.8.3 An alternative rule for local annotations . . . 70
5.9 The density operator and the Bloch vector . . . 71
5.9.1 Bloch representation of a qubit . . . 71
5.9.2 Purity in Bloch coordinates . . . 71
5.9.3 Annotated Bloch representation . . . 72
5.9.4 Expectation values . . . 72
5.10 Worked examples . . . 73
5.10.1 Pure state . . . 73
5.10.2 Mixed state . . . 73
5.10.3 A reduced state from an entangled pure state . . . 73
5.11 Limitations and consistency requirements . . . 74
5.11.1 Annotations are not probabilities . . . 74
5.11.2 Annotations do not repair invalid density operators . . . 74
5.11.3 Nonuniqueness of annotation rules . . . 74
5.11.4 Operational interpretation . . . 75
5.12 Further research problems . . . 75
5.13 Conclusion . . . 75

6 Neutrosophic Quantum Measurement 77
6.1 Introduction . . . 77
6.2 Quantum states and measurement preliminaries . . . 78
6.2.1 Density operators . . . 78
6.2.2 The Born rule . . . 78
6.2.3 Outcome probabilities and informational records . . . 79
6.3 Bounded neutrosophic annotations . . . 79
6.3.1 Definition . . . 79
6.3.2 An aggregation operation . . . 79
6.4 Neutrosophically annotated measurements . . . 80
6.4.1 Definition . . . 80
6.4.2 A probability-weighted annotation summary . . . 81
6.5 Distinguishing quantum states . . . 81
6.5.1 The state-discrimination problem . . . 81

6.5.2 The Helstrom bound . . . 81
6.5.3 Example: two nonorthogonal pure states . . . 82
6.6 Projective measurements . . . 82
6.6.1 Measurement of an observable . . . 82
6.6.2 Post-measurement state . . . 83
6.6.3 Example: computational-basis measurement . . . 83
6.7 Measurements on composite systems . . . 84
6.7.1 Local measurements . . . 84
6.7.2 Annotated joint outcomes . . . 84
6.7.3 Example: measuring two qubits . . . 84
6.8 Generalized measurements and POVMs . . . 85
6.8.1 Definition of a POVM . . . 85
6.8.2 Neutrosophically annotated POVMs . . . 85
6.8.3 An unsharp qubit measurement . . . 86
6.9 Measurement operators and state update . . . 86
6.9.1 Measurement instruments . . . 86
6.9.2 An annotated measurement instrument . . . 87
6.10 Distinguishing quantum probabilities from annotations . . . 87
6.10.1 Quantum probability . . . 87
6.10.2 Indeterminacy in an information record . . . 87
6.10.3 Falsity and alternative outcomes . . . 88
6.10.4 Truth and successful classification . . . 88
6.11 Limitations and consistency conditions . . . 88
6.11.1 Dependence on the annotation convention . . . 88
6.11.2 No automatic change to the Born rule . . . 88
6.11.3 Consistency under repeated measurements . . . 88
6.11.4 Operational and empirical questions . . . 89
6.12 Further research problems . . . 89
6.13 Conclusion . . . 89

7 Neutrosophic Entanglement 90
7.1 Introduction . . . 90
7.2 Mathematical preliminaries . . . 91
7.2.1 Quantum states and composite systems . . . 91
7.2.2 Product states and separability . . . 92
7.3 Bipartite qubits and the Bell basis . . . 92
7.3.1 Two-qubit states . . . 92
7.3.2 The Bell states . . . 93
7.4 Schmidt decomposition . . . 93
7.4.1 An entropy-based entanglement measure . . . 94
7.5 Bell's theorem and CHSH correlations . . . 95
7.5.1 Local hidden-variable inequality . . . 95
7.5.2 The CHSH operator . . . 95
7.6 Concurrence and two-qubit entanglement . . . 95
7.6.1 Pure-state concurrence . . . 95
7.6.2 Mixed-state concurrence . . . 96
7.7 A neutrosophic annotation framework . . . 96
7.7.1 Annotation triples . . . 96
7.7.2 Basic consistency conditions . . . 97
7.7.3 A simple aggregation operation . . . 97
7.7.4 Averaging annotations . . . 98

7.8 Annotated entanglement examples . . . 99
7.8.1 A product state . . . 99
7.8.2 A Bell state . . . 99
7.8.3 A partially entangled state . . . 99
7.8.4 Comparison of the examples . . . 100
7.9 Entanglement fidelity . . . 100
7.9.1 A possible annotation of fidelity . . . 101
7.10 Purification and the role of an auxiliary system . . . 101
7.11 Propositions for annotated entanglement assessments . . . 102
7.12 Potential applications . . . 103
7.12.1 Incomplete state tomography . . . 103
7.12.2 Comparison of entanglement diagnostics . . . 103
7.12.3 Quantum communication . . . 103
7.13 Limitations and open problems . . . 103
7.14 Conclusion . . . 104

8 Neutrosophic Quantum Gates and Circuits 105
8.1 Introduction . . . 105
8.2 Preliminaries: qubits and unitary operations . . . 106
8.2.1 The state of a qubit . . . 106
8.2.2 Unitary gates . . . 107
8.2.3 Global phase . . . 107
8.3 Elementary single-qubit gates . . . 108
8.3.1 Pauli gates . . . 108
8.3.2 Hadamard gate . . . 108
8.3.3 Phase gates . . . 108
8.3.4 Rotation gates . . . 109
8.4 Multiqubit gates . . . 109
8.4.1 Tensor products . . . 109
8.4.2 Controlled-NOT gate . . . 109
8.4.3 Controlled phase . . . 110
8.4.4 Toffoli gate . . . 110
8.5 Quantum circuits as matrix products . . . 110
8.5.1 Circuit composition . . . 110
8.5.2 A circuit creating a Bell state . . . 110
8.5.3 Circuit equivalence . . . 111
8.6 Neutrosophic annotation space . . . 111
8.6.1 Definition . . . 111
8.6.2 Annotated circuits . . . 111
8.6.3 A possible aggregation rule . . . 112
8.6.4 Circuit-level annotation . . . 112
8.7 Examples of annotated gates and circuits . . . 112
8.7.1 An annotated Hadamard gate . . . 112
8.7.2 An annotated CNOT gate . . . 113
8.7.3 A two-gate circuit . . . 113
8.7.4 A three-gate circuit . . . 114
8.8 Measurement and output statistics . . . 114
8.8.1 The Born rule . . . 114
8.8.2 Measurement after Bell-state preparation . . . 114
8.9 Ideal gates and noisy implementations . . . 115
8.9.1 Quantum channels . . . 115

8.9.2 Example: a depolarizing channel . . . 115
8.9.3 Gate fidelity . . . 115
8.10 Consistency properties and propositions . . . 116
8.10.1 Closure of circuit annotations . . . 116
8.10.2 Dependence on the chosen annotation rule . . . 116
8.10.3 Global-phase invariance . . . 116
8.11 Potential applications . . . 117
8.11.1 Calibration records . . . 117
8.11.2 Circuit design and comparison . . . 117
8.11.3 Educational applications . . . 117
8.12 Limitations and open research problems . . . 117
8.13 Conclusion . . . 118

9 Neutrosophic Quantum Algorithms 119
9.1 Introduction . . . 119
9.2 Mathematical preliminaries for quantum algorithms . . . 120
9.2.1 Computational basis and quantum registers . . . 120
9.2.2 Quantum circuits and unitary transformations . . . 121
9.2.3 Oracle model . . . 121
9.3 Neutrosophic annotation framework . . . 121
9.3.1 Annotation triples . . . 121
9.3.2 Annotated algorithmic components . . . 122
9.3.3 A convenient annotation space . . . 122
9.3.4 Annotation and quantum probability . . . 122
9.4 Composition of neutrosophic annotations . . . 123
9.4.1 A separate reliability calculation . . . 124
9.5 Hadamard transformations and quantum interference . . . 124
9.5.1 The Hadamard gate . . . 124
9.5.2 Neutrosophic annotation of interference claims . . . 124
9.5.3 Quantum parallelism and its limitation . . . 125
9.6 The Deutsch–Jozsa algorithm . . . 125
9.6.1 Problem statement . . . 125
9.6.2 Derivation . . . 125
9.6.3 Neutrosophic treatment of the promise . . . 126
9.7 The quantum Fourier transform . . . 126
9.7.1 Definition . . . 126
9.7.2 A basic example . . . 127
9.7.3 Annotated Fourier transform . . . 127
9.8 Quantum phase estimation . . . 128
9.8.1 Basic formulation . . . 128
9.8.2 Uncertainty in the eigenphase model . . . 128
9.9 Shor's factoring algorithm . . . 128
9.9.1 Order finding . . . 128
9.9.2 Sources of uncertainty . . . 129
9.9.3 Verification as a separate stage . . . 129
9.10 Grover's quantum search algorithm . . . 129
9.10.1 Search problem . . . 129
9.10.2 Example . . . 130
9.10.3 Neutrosophic annotation of the oracle . . . 130
9.10.4 A limitation of annotation-based reliability claims . . . 131
9.11 An illustrative annotated algorithm . . . 131

9.12 From qualitative annotations to calibrated models . . . 131
9.12.1 Expert assessment . . . 132
9.12.2 Empirical calibration . . . 132
9.12.3 Formal verification . . . 132
9.13 Comparison with standard quantum analysis . . . 132
9.14 Limitations and open problems . . . 133
9.14.1 Non-uniqueness of the composition rule . . . 133
9.14.2 No automatic probabilistic interpretation . . . 133
9.14.3 Quantum channels and correlated noise . . . 133
9.14.4 Complexity analysis . . . 133
9.14.5 Research questions . . . 133
9.15 Conclusion . . . 134

10 Neutrosophic Teleportation and Superdense Coding 135
10.1 Introduction . . . 135
10.2 Preliminaries . . . 136
10.2.1 Qubits and quantum states . . . 136
10.2.2 Bell states . . . 137
10.2.3 Neutrosophic information triples . . . 137
10.2.4 Annotation propagation . . . 138
10.3 Quantum teleportation . . . 138
10.3.1 Standard teleportation protocol . . . 138
10.3.2 Neutrosophic annotation of teleportation . . . 139
10.3.3 A reliability model for classical messages . . . 140
10.3.4 Fidelity under a classical correction error . . . 140
10.4 Superdense coding . . . 140
10.4.1 Standard protocol . . . 140
10.4.2 Neutrosophic encoding records . . . 141
10.4.3 A noisy-message example . . . 141
10.5 Comparison of teleportation and superdense coding . . . 142
10.6 Entanglement swapping . . . 142
10.6.1 Standard entanglement swapping . . . 142
10.6.2 Neutrosophic description of swapping . . . 143
10.7 The Peres partial-transposition condition . . . 143
10.7.1 Partial transpose . . . 143
10.7.2 Bell-state example . . . 143
10.7.3 Neutrosophic annotations and positivity . . . 144
10.8 A channel-based formulation . . . 144
10.9 Information-theoretic interpretation . . . 145
10.10 Worked comparative example . . . 146
10.11 Limitations and research directions . . . 146
10.11.1 Operational meaning . . . 146
10.11.2 Consistency with quantum mechanics . . . 146
10.11.3 Composition of systems . . . 146
10.11.4 Experimental validation . . . 146
10.11.5 Potential research questions . . . 147
10.12 Conclusion . . . 147

11 Neutrosophic Quantum Cryptography **148**
11.1 Introduction . . . 148
11.2 Classical and quantum cryptographic preliminaries . . . 149
11.2.1 Classical encryption and RSA . . . 149
11.2.2 Quantum states and measurements . . . 149
11.3 Neutrosophic annotations . . . 150
11.3.1 Passive and operational annotations . . . 151
11.4 The BB84 protocol . . . 151
11.4.1 Standard protocol . . . 151
11.4.2 Why basis mismatch matters . . . 151
11.4.3 Neutrosophic annotation of BB84 records . . . 152
11.4.4 QBER as an operational statistic . . . 152
11.4.5 An ideal intercept-resend example . . . 152
11.5 A neutrosophic treatment of inconclusive records . . . 153
11.6 The B92 protocol . . . 153
11.6.1 Nonorthogonal signal states . . . 153
11.6.2 A simple unambiguous-discrimination model . . . 154
11.6.3 Neutrosophic record for B92 . . . 154
11.7 The E91 protocol and entanglement . . . 154
11.7.1 Entanglement-based quantum key distribution . . . 154
11.7.2 Neutrosophic annotations of correlations . . . 155
11.7.3 A limitation of the annotation approach . . . 155
11.8 A controlled-NOT attack model . . . 155
11.8.1 An ancilla interaction . . . 155
11.8.2 A neutrosophic attack record . . . 156
11.9 From neutrosophic records to statistical inference . . . 157
11.9.1 A simple confidence interval . . . 157
11.10 Security interpretation . . . 157
11.10.1 What a QKD security claim requires . . . 157
11.10.2 Passive annotations . . . 158
11.10.3 Operational annotations . . . 158
11.10.4 No automatic security enhancement . . . 158
11.11 Worked example . . . 158
11.12 Research directions . . . 159
11.12.1 Axioms for annotation propagation . . . 159
11.12.2 Finite-key security . . . 159
11.12.3 Device imperfections . . . 159
11.12.4 Comparison with conventional models . . . 160
11.13 Conclusion . . . 160

12 Neutrosophic Quantum Noise and Error Correction **161**
12.1 Introduction . . . 161
12.2 Quantum states and quantum operations . . . 162
12.2.1 Qubits and density operators . . . 162
12.2.2 Quantum channels . . . 163
12.3 Neutrosophic annotations for quantum records . . . 163
12.3.1 Basic definition . . . 163
12.3.2 Passive annotations . . . 164
12.4 Single-qubit errors . . . 164
12.4.1 Bit-flip errors . . . 164
12.4.2 Phase-flip errors . . . 164

12.4.3 Combined Pauli noise . . . 164
12.4.4 Neutrosophic annotation of an error event . . . 165
12.5 Quantum operations and the Kraus representation . . . 165
12.5.1 Annotations of Kraus events . . . 166
12.6 The depolarizing channel . . . 166
12.6.1 Definition . . . 166
12.6.2 Action on a pure state . . . 166
12.6.3 Neutrosophic description . . . 167
12.7 Amplitude damping . . . 167
12.7.1 Physical interpretation . . . 167
12.7.2 Example . . . 167
12.7.3 Neutrosophic annotation of relaxation . . . 168
12.8 Phase damping . . . 168
12.8.1 A phase-damping channel . . . 168
12.8.2 Comparison with amplitude damping . . . 168
12.9 The three-qubit bit-flip code . . . 168
12.9.1 Encoding a logical qubit . . . 168
12.9.2 Syndrome measurements . . . 169
12.9.3 Correction of one bit flip . . . 169
12.9.4 Limitations of the code . . . 169
12.10 Neutrosophic syndrome records . . . 169
12.10.1 Annotation of syndrome outcomes . . . 169
12.10.2 Passive syndrome annotations . . . 170
12.10.3 Operational syndrome annotations . . . 170
12.11 General quantum error-correction conditions . . . 170
12.11.1 What annotations can and cannot do . . . 171
12.12 A probabilistic model of indeterminate error records . . . 171
12.12.1 Three-category observation model . . . 171
12.12.2 Separating physical errors from record errors . . . 172
12.13 An annotated quantum-channel model . . . 172
12.13.1 When annotations influence recovery . . . 172
12.14 Logical error rate and fidelity . . . 173
12.15 Worked example . . . 173
12.16 Limitations and research directions . . . 173
12.16.1 Operational definitions . . . 173
12.16.2 Consistency with quantum mechanics . . . 174
12.16.3 Composition and correlated errors . . . 174
12.16.4 Comparison with established methods . . . 174
12.16.5 Possible research questions . . . 174
12.17 Conclusion . . . 174

13 Neutrosophic Quantum Information Theory 176
13.1 Introduction . . . 176
13.2 Quantum states and neutrosophic annotations . . . 177
13.2.1 Density operators . . . 177
13.2.2 Neutrosophic triples . . . 177
13.2.3 A basic annotation space . . . 178
13.3 The no-cloning theorem . . . 178
13.3.1 Statement of the theorem . . . 178
13.3.2 Neutrosophic interpretation . . . 179
13.4 Trace distance and state distinguishability . . . 179

13.4.1 Definition and properties . . . 179
13.4.2 Pure-state formula . . . 180
13.4.3 Annotation-aware reporting . . . 180
13.5 Fidelity . . . 180
13.5.1 Definition . . . 180
13.5.2 Relation to trace distance . . . 181
13.5.3 Fidelity with a neutrosophic annotation . . . 181
13.6 Entanglement of formation and concurrence . . . 181
13.6.1 Bipartite pure states . . . 181
13.6.2 Entanglement of formation . . . 182
13.6.3 Concurrence for two qubits . . . 182
13.6.4 Example: a maximally entangled state . . . 182
13.6.5 Neutrosophic annotations for entanglement reports . . . 183
13.7 Information content and quantum entropy . . . 183
13.7.1 Classical Shannon entropy . . . 183
13.7.2 Von Neumann entropy . . . 183
13.7.3 The Holevo quantity . . . 184
13.7.4 A neutrosophic information annotation . . . 184
13.7.5 A separate annotation-uncertainty functional . . . 184
13.8 An annotation-aware framework . . . 185
13.8.1 Record structure . . . 185
13.8.2 A reporting protocol . . . 185
13.8.3 A simple reporting example . . . 185
13.8.4 Monotonicity and physical consistency . . . 186
13.9 Limitations and open problems . . . 186
13.10Conclusion . . . 187

14 Neutrosophic Adiabatic Quantum Computation 188
14.1 Introduction . . . 188
14.2 Preliminaries on quantum Hamiltonians . . . 189
14.2.1 Hamiltonians and quantum states . . . 189
14.2.2 Time-dependent evolution . . . 189
14.2.3 The interpolation Hamiltonian . . . 190
14.3 Adiabatic evolution and the spectral gap . . . 190
14.3.1 The adiabatic principle . . . 190
14.3.2 Why the gap matters . . . 191
14.3.3 Schedule selection . . . 191
14.4 Neutrosophic annotations . . . 191
14.4.1 Definition of an annotation . . . 191
14.4.2 Annotation of the interpolation path . . . 192
14.4.3 A metric on annotation triples . . . 192
14.5 An annotated adiabatic computational model . . . 193
14.5.1 The four components of the model . . . 193
14.5.2 Separating physical and informational uncertainty . . . 193
14.5.3 A possible annotation-aware report . . . 193
14.6 Encoding an optimization problem . . . 194
14.6.1 Binary objective functions . . . 194
14.6.2 A two-variable example . . . 194
14.6.3 A neutrosophic record for the objective . . . 195
14.6.4 A necessary distinction . . . 195
14.7 Ground-state success and output interpretation . . . 196

14.7.1 Success probability . . . 196
14.7.2 Annotated output record . . . 196
14.8 Model perturbations and annotation updates . . . 196
14.8.1 Perturbed Hamiltonians . . . 196
14.8.2 A possible update rule . . . 197
14.9 Comparison with standard adiabatic computation . . . 197
14.10 Limitations and research problems . . . 198
14.11 Conclusion . . . 198

15 Neutrosophic Cluster-State Quantum Computing 200
15.1 Introduction . . . 200
15.2 Preliminaries . . . 201
15.2.1 Qubits and Pauli operators . . . 201
15.2.2 Controlled-phase operations . . . 202
15.3 Graphs and adjacency matrices . . . 202
15.3.1 Graph definition . . . 202
15.3.2 Neighborhoods . . . 202
15.3.3 A path graph . . . 203
15.4 Graph states and cluster states . . . 203
15.4.1 Definition of a graph state . . . 203
15.4.2 Computational-basis expression . . . 203
15.4.3 Preparation and physical implementation . . . 204
15.5 Stabilizer states . . . 204
15.5.1 Stabilizer operators . . . 204
15.5.2 The stabilizer group . . . 205
15.5.3 Stabilizer expectation values . . . 205
15.6 Entanglement and entanglement witnesses . . . 206
15.6.1 Entanglement . . . 206
15.6.2 Entanglement witnesses . . . 206
15.6.3 A graph-state witness . . . 206
15.6.4 Annotation of an entanglement claim . . . 207
15.7 Cluster-state processing . . . 207
15.7.1 Measurement-based quantum computation . . . 207
15.7.2 Single-qubit measurement bases . . . 207
15.7.3 Classical feed-forward . . . 208
15.7.4 An annotation-aware processing record . . . 208
15.8 Neutrosophic annotation framework . . . 208
15.8.1 The annotation space . . . 208
15.8.2 A metric on annotations . . . 209
15.8.3 Annotation of stabilizer data . . . 209
15.8.4 Annotation of graph specification . . . 210
15.9 Worked example: a three-qubit path cluster . . . 210
15.9.1 Ideal state and stabilizers . . . 210
15.9.2 A simple witness value . . . 210
15.9.3 A stronger stabilizer-based witness . . . 211
15.9.4 An annotation record . . . 211
15.10 Physical uncertainty and informational indeterminacy . . . 211
15.10.1 Quantum measurement uncertainty . . . 212
15.10.2 Statistical uncertainty . . . 212
15.10.3 Model uncertainty . . . 212
15.11 Limitations and research directions . . . 212

15.12Conclusion 213

Bibliography **214**

Chapter 1

Introduction to Neutrosophic Quantum Computing

Research context. Smarandache's original proposal for a neutrosophic quantum computer [72] and the subsequent mathematical development by Çevik, Topal and Smarandache [73] motivate the conceptual framework. Indeterminacy is discussed further in [75]; the physical realizability question is treated separately in the preprint [81].

This paper develops an introductory mathematical framework for studying the interaction between quantum information and neutrosophic representations of uncertainty. Classical information is first reviewed through bits, probability distributions, and Shannon entropy. The standard qubit formalism is then introduced, including superposition, measurement, unitary transformations, and density operators. On this foundation, a neutrosophic description is proposed in which a computational proposition is associated with truth, indeterminacy, and falsity degrees.

A central distinction is maintained between ordinary quantum probabilities and additional neutrosophic annotations. The latter may describe incomplete knowledge, conflicting evidence, or uncertainty about a model, but they cannot replace the Born rule without an explicitly specified alternative measurement theory. Two formal constructions are presented: a neutrosophic annotation of classical outcome distributions and an annotated quantum-state model. The paper also discusses entropy, elementary gates, measurement, limitations, and open mathematical problems. The framework is intended as a conceptual and mathematical starting point, not as an experimentally established quantum computing architecture.

1.1 Introduction

Quantum computing studies the processing of information by physical systems whose states obey the principles of quantum mechanics. Its mathematical foundations include complex vector spaces, unitary operators, tensor products, and probabilistic measurement. Standard introductions develop these ideas from classical information and probability toward qubits, gates, and quantum circuits [1, 2].

Neutrosophic logic, introduced by Smarandache, provides a framework for representing a proposition through three components: truth, indeterminacy, and falsity [6, 6]. These components are intended to represent aspects of information that need not be captured by a single truth value. In particular, indeterminacy can be useful when evidence is incomplete or inconsistent.

The purpose of this paper is to investigate how such a three-component representation can be used alongside the mathematics of quantum information. The word "alongside" is

important. A quantum state is not simply a proposition with three truth values. A qubit is represented by a unit vector, or more generally by a density operator, and its measurement statistics are constrained by quantum mechanics [2, 3].

Consequently, this paper separates two layers:

1. *The physical quantum layer*, consisting of states, transformations, and measurement probabilities.

2. *The neutrosophic information layer*, consisting of annotations that describe truth support, indeterminacy, and falsity relative to a specified proposition or information source.

The main contribution is a careful introductory formalization of this separation, together with worked examples and conditions that any more ambitious neutrosophic quantum model would need to satisfy.

1.1.1 Scope and conventions

Throughout, the standard quantum formalism is treated as the baseline theory. Neutrosophic constructions are explicitly identified as proposed extensions or annotations. No claim is made that the constructions below constitute a new physical theory or that they yield a computational speedup.

We use the term *neutrosophic annotation* for a triple

$$(N_T, N_I, N_F),$$

where the components represent degrees of truth support, indeterminacy, and falsity support for a specified proposition. The precise interpretation of these degrees must be stated for each application.

1.2 Classical information as a starting point

1.2.1 Bits and binary information

The fundamental unit of classical digital information is the bit. A bit takes one of two values,

$$b \in \{0, 1\}.$$

A classical register of n bits has 2^n possible configurations:

$$(b_1, \ldots, b_n) \in \{0, 1\}^n.$$

For example, a register with three bits can represent any of the eight strings

$$000,\ 001,\ 010,\ 011,\ 100,\ 101,\ 110,\ 111.$$

A classical computer manipulates such configurations through logical operations and physical implementations of those operations. When the register is uncertain, its description may be given by a probability distribution over the possible strings.

1.2.2 Information content

Suppose an event x occurs with probability $p(x) > 0$. The Shannon information content of the event is

$$\mathcal{J}(x) = -\log_2 p(x).$$

An event with small probability carries more information when it occurs than a highly probable event. For example,

$$p(x) = \frac{1}{2} \quad \Longrightarrow \quad \mathcal{J}(x) = 1,$$

whereas

$$p(x) = \frac{1}{4} \quad \Longrightarrow \quad \mathcal{J}(x) = 2.$$

This quantity is defined from a probability distribution. It does not measure neutrosophic indeterminacy directly. If an additional indeterminacy score is introduced, its relationship to probability must be specified rather than assumed.

1.2.3 Shannon entropy

For a finite random variable X with probabilities $p_1, \ldots, p_m$, the Shannon entropy is

$$H(X) = -\sum_{j=1}^{m} p_j \log_2 p_j,$$

where $0 \log_2 0$ is interpreted as zero [4, 5].

For a fair binary variable,

$$H(X) = -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2} = 1.$$

For a deterministic binary variable with probabilities $(1, 0)$, the entropy is zero.

Entropy quantifies uncertainty in a probability distribution. Neutrosophic indeterminacy, by contrast, is a separately specified component of a representation. The two concepts may be related in a particular model, but they are not identical by definition.

1.2.4 A neutrosophic annotation of a classical bit

Definition 1.2.1. Let P be a proposition concerning a classical bit. A neutrosophic annotation of P is a triple

$$\mathsf{N}(P) = (T(P), I(P), F(P)),$$

where $T(P), I(P), F(P) \in [0, 1]$ represent, respectively, truth support, indeterminacy, and falsity support.

In a normalized special case, we may require

$$T(P) + I(P) + F(P) = 1.$$

This condition is a modeling choice, not a universal requirement of neutrosophic logic. More general neutrosophic frameworks may allow the sum to differ from one [6, 6].

Example 1.2.2. Suppose a sensor reports evidence about the proposition $P = \{\text{the bit is 1}\}$. An application might assign

$$\mathsf{N}(P) = (0.7, 0.2, 0.1).$$

The values indicate truth support 0.7, indeterminacy 0.2, and falsity support 0.1, under the chosen interpretation of the sensor's evidence.

These numbers are not automatically probabilities for the bit to be 1, unknown, or 0. A separate calibration rule would be needed to turn them into a probability distribution.

1.2.5 From classical to quantum information

A classical bit is described by a definite value or by a probability distribution over definite values. A quantum bit, or qubit, is represented by a vector in a two-dimensional complex Hilbert space. Quantum amplitudes can interfere, and measurement probabilities are determined by the Born rule [2].

The next section recalls the standard formalism before introducing any neutrosophic extension.

1.3 Qubits and quantum states

1.3.1 The computational basis

The state space of a single qubit is

$$\mathcal{H} = \mathbb{C}^2.$$

The computational basis is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A pure qubit state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C},$$

with normalization

$$|\alpha|^2 + |\beta|^2 = 1.$$

The complex numbers α and β are probability amplitudes. They are not probabilities themselves.

1.3.2 Measurement and the Born rule

When the qubit is measured in the computational basis, the probabilities of the two outcomes are

$$\mathsf{P}(0) = |\alpha|^2, \qquad \mathsf{P}(1) = |\beta|^2.$$

They satisfy

$$\mathsf{P}(0) + \mathsf{P}(1) = 1.$$

For example, consider

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle.$$

The measurement probabilities are

$$\mathsf{P}(0) = \frac{1}{2}, \qquad \mathsf{P}(1) = \frac{1}{2}.$$

The state is a coherent superposition. It should not be identified with a classical statement that the bit is partly zero and partly one.

1.3.3 Superposition and relative phase

A general pure qubit may be written, up to an overall global phase, as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle,$$

where $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$.

The parameter ϕ is a relative phase. Although it does not change the computational-basis probabilities

$$\mathsf{P}(0) = \cos^2\frac{\theta}{2}, \qquad \mathsf{P}(1) = \sin^2\frac{\theta}{2},$$

it can affect measurement statistics in another basis. This is one reason that a quantum state contains more information than a pair of classical outcome probabilities [2, 3].

1.3.4 Neutrosophic interpretation of a qubit

A neutrosophic annotation can be attached to a proposition about a qubit, such as

$$P = \{\text{a computational-basis measurement returns } 1\}.$$

One possible annotation is

$$\mathsf{N}(P) = (T_P, I_P, F_P).$$

The interpretation must distinguish the physical probability $\mathsf{P}(1)$ from the annotation I_P. For example, $\mathsf{P}(1) = 1/2$ does not imply $I_P = 1/2$. A fair quantum measurement can have a perfectly known probability distribution, while an information source may still be fully informed about that distribution.

Conversely, uncertainty about the preparation procedure may justify an indeterminacy annotation even when a model predicts a definite measurement outcome.

Remark 1.3.1. The neutrosophic triple does not replace the normalized complex amplitudes of a qubit. It records additional information about a proposition or model. Any proposal to make the triple itself a physical state requires its own state space, dynamics, and measurement postulates.

1.4 Neutrosophic representations of quantum outcomes

1.4.1 Outcome annotations

Consider a measurement with outcomes

$$\Omega = \{0, 1\}$$

and ordinary probabilities

$$p_0 = \mathsf{P}(0), \qquad p_1 = \mathsf{P}(1), \qquad p_0 + p_1 = 1.$$

For each outcome $j \in \{0, 1\}$, suppose an application also supplies an annotation

$$\mathbf{n}_j = (T_j, I_j, F_j) \in [0, 1]^3.$$

The probabilities p_j describe the standard quantum measurement statistics. The triples $\mathbf{n}_j$ describe an additional assessment of evidence, model reliability, or interpretive uncertainty.

The complete annotated record can be written

$$\mathcal{R} = \big((p_0, \mathbf{n}_0), (p_1, \mathbf{n}_1)\big).$$

It is a hybrid representation, not a new quantum state space by itself.

1.4.2 A worked example

Consider the state

$$|\psi\rangle = \sqrt{0.8}|0\rangle + \sqrt{0.2}|1\rangle.$$

A computational-basis measurement gives

$$p_0 = 0.8, \qquad p_1 = 0.2.$$

Suppose a separate diagnostic process assigns the following illustrative annotations:

$$\mathbf{n}_0 = (0.90, 0.08, 0.02), \qquad \mathbf{n}_1 = (0.60, 0.25, 0.15).$$

Both triples have components summing to one. The annotation for outcome 1 expresses greater indeterminacy in the diagnostic assessment than the annotation for outcome 0.

The physical measurement probabilities remain 0.8 and 0.2. The diagnostic values do not alter those probabilities unless an explicit, physically justified model connects them.

1.4.3 Consistency conditions

A useful annotated model should state which conditions it requires. A minimal list is:

(C1) **Probability normalization:**

$$p_j \geq 0, \qquad \sum_j p_j = 1.$$

(C2) **Annotation bounds:**

$$0 \leq T_j, I_j, F_j \leq 1.$$

(C3) **Interpretive clarity:** each component must have an explicitly defined meaning.

(C4) **Operational separation:** ordinary quantum probabilities must not be silently replaced by neutrosophic components.

(C5) **Transformation rules:** if annotations are updated after a quantum gate or measurement, the update rule must be specified.

The optional normalization

$$T_j + I_j + F_j = 1$$

may be imposed when the three components are intended to form a partition of an assessment. It should not be imposed without justification in applications that allow overlapping or independent evidence.

1.5 Neutrosophic information and entropy

1.5.1 Ordinary entropy of a qubit measurement

For a computational-basis measurement of a pure qubit, the outcome entropy is

$$H_Z(\psi) = -|\alpha|^2 \log_2 |\alpha|^2 - |\beta|^2 \log_2 |\beta|^2.$$

For the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$

we obtain

$$H_Z(\psi) = 1.$$

For $|\psi\rangle = |0\rangle$, we obtain

$$H_Z(\psi) = 0.$$

This entropy refers to the distribution of outcomes in the specified measurement basis. It is not a complete measure of quantum coherence or of all information in the state.

1.5.2 A proposed neutrosophic uncertainty score

For applications that use normalized annotations, one might define a simple score

$$U_{\mathrm{N}}(\mathbf{n}) = I + \lambda\big(1 - |T - F|\big),$$

where $\lambda \geq 0$ is a chosen parameter and $\mathbf{n} = (T, I, F)$.

This expression is only an illustrative proposal. It is not a standard neutrosophic entropy and is not asserted to satisfy the axioms of an information measure. It can exceed one when λ is sufficiently large.

If a bounded score is desired, an alternative illustrative choice is

$$\widetilde{U}_{\mathrm{N}}(\mathbf{n}) = \frac{I + \lambda(1 - |T - F|)}{1 + \lambda}, \qquad 0 \leq \lambda < \infty.$$

Since $I \in [0, 1]$ and $1 - |T - F| \in [0, 1]$, this score lies in $[0, 1]$.

The score measures a chosen combination of indeterminacy and balance between truth and falsity support. It should not be interpreted as physical quantum entropy.

1.5.3 Why two uncertainty measures may be useful

A hybrid information system may distinguish:

- uncertainty inherent in the distribution of measured outcomes;
- uncertainty in the evidence used to construct a model;
- uncertainty about whether the model assumptions are valid;
- conflicting evidence from multiple sensors or data sources.

Shannon entropy can quantify the first item when a probability distribution is specified. Neutrosophic annotations may be used to represent the latter items, provided their meanings and update rules are defined.

There is no universal formula converting Shannon entropy into neutrosophic indeterminacy. Any such conversion is application-dependent.

1.6 Quantum gates and annotated information

1.6.1 Unitary transformations

In the circuit model, a quantum gate is represented by a unitary operator U satisfying

$$U^\dagger U = U U^\dagger = \mathbb{I}.$$

The state transforms according to

$$|\psi'\rangle = U|\psi\rangle.$$

Unitarity preserves normalization and inner products [2].

The Pauli-X gate is

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and exchanges the computational basis states:

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle.$$

The Hadamard gate is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It satisfies

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

1.6.2 An annotated gate operation

Suppose a qubit is initially in $|0\rangle$, and a Hadamard gate is applied. The resulting state is

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

A computational-basis measurement has equal probabilities for both outcomes.

One may attach an annotation to a proposition such as

$$P = \{\text{the preparation device produced the intended state}\}.$$

For example,

$$\mathsf{N}(P) = (0.95, 0.04, 0.01)$$

might describe a calibration assessment.

This annotation concerns confidence in the preparation process, not the mathematical action of the Hadamard gate. If the annotation is to be propagated through the circuit, the model must say how gate calibration, state preparation errors, and measurement errors affect the triple.

1.6.3 Why ordinary gates cannot act directly on triples without further definitions

A unitary gate is a linear operator on a complex Hilbert space. A neutrosophic triple belongs to $[0, 1]^3$. These objects have different mathematical types.

Consequently, the expression

$$U(T, I, F)^{\mathsf{T}}$$

is not, in general, a meaningful quantum gate operation: the dimensions and the mathematical roles do not match.

A hybrid framework may instead define an annotated state

$$(\rho, \mathbf{n}),$$

where ρ is a quantum density operator and $\mathbf{n}$ is an annotation. A transformation rule could then have the form

$$(\rho, \mathbf{n}) \longmapsto (U\rho U^{\dagger}, \Phi_U(\mathbf{n}, \rho)),$$

where Φ_U is a separately specified annotation-update rule.

The quantum component follows standard unitary evolution. The map Φ_U requires independent mathematical justification and may depend on the physical meaning of the annotation.

1.7 Density operators and mixed states

1.7.1 Density operators

A general quantum state, including a mixed state, can be represented by a density operator ρ satisfying

$$\rho^\dagger = \rho, \qquad \rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1.$$

For a pure state,

$$\rho = |\psi\rangle\langle\psi|.$$

For example, the pure state

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

has density operator

$$\rho_+ = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

The diagonal entries describe computational-basis outcome probabilities, while the off-diagonal entries encode coherence in that basis.

By contrast, the equal classical mixture of $|0\rangle$ and $|1\rangle$ is represented by

$$\rho_{\mathrm{mix}} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both states yield equal probabilities in a computational-basis measurement, but they are different quantum states [2, 3].

1.7.2 Annotated density operators

A possible hybrid representation is

$$\mathfrak{S} = (\rho, \mathbf{n}), \qquad \mathbf{n} = (T, I, F).$$

Here ρ specifies the physical quantum state, and $\mathbf{n}$ describes an additional proposition, model, or information source.

This pair is not a density operator on a larger Hilbert space unless an explicit construction establishes such an interpretation. In particular, the triple should not be inserted into a density matrix merely because both representations concern uncertainty.

1.7.3 A proposition-dependent annotation

The same density operator may be annotated differently depending on the question being asked. For instance, an annotation about the reliability of a state-preparation device is different from an annotation about the outcome of a particular measurement.

Thus the notation should include the proposition when necessary:

$$\mathsf{N}(P \mid \rho, \mathcal{E}),$$

where $\mathcal{E}$ denotes the available evidence. This notation emphasizes that an annotation depends on the specified proposition and evidence, not solely on the quantum state.

1.8 Measurement in a hybrid framework

1.8.1 Projective measurements

For a projective measurement with projectors $\{\Pi_j\}$ satisfying

$$\Pi_j^\dagger = \Pi_j, \qquad \Pi_j^2 = \Pi_j, \qquad \sum_j \Pi_j = \mathbb{I},$$

the probability of outcome j for a state ρ is

$$p_j = \mathrm{Tr}(\Pi_j \rho).$$

The probabilities are nonnegative and sum to one.

More generally, a quantum measurement may be described by positive operators $\{E_j\}$ satisfying

$$E_j \geq 0, \qquad \sum_j E_j = \mathbb{I},$$

with outcome probabilities

$$p_j = \mathrm{Tr}(E_j \rho).$$

This is the positive-operator-valued measure formalism [2, 3].

1.8.2 Adding an annotation to a measurement record

A hybrid measurement record may be written as

$$\mathcal{M}(\rho) = \{(j, p_j, \mathbf{n}_j) : j \in \Omega\}.$$

Here p_j is the Born-rule probability and $\mathbf{n}_j$ is an annotation associated with outcome j.

The annotation could represent uncertainty in the interpretation of a detector signal, a disagreement between classifiers, or an estimate of calibration reliability. These are possible applications rather than consequences of quantum mechanics itself.

Proposition 1.8.1. *Suppose ρ is a density operator and $\{E_j\}_{j=1}^m$ is a valid quantum measurement. If*

$$p_j = \mathrm{Tr}(E_j \rho),$$

then

$$p_j \geq 0, \qquad \sum_{j=1}^m p_j = 1.$$

Appending arbitrary annotation triples $\mathbf{n}_j \in [0,1]^3$ to the records does not change these probabilities.

Proof. Positivity of E_j and ρ implies $\mathrm{Tr}(E_j \rho) \geq 0$. By linearity of the trace,

$$\sum_{j=1}^m p_j = \sum_{j=1}^m \mathrm{Tr}(E_j \rho) = \mathrm{Tr}\left(\left(\sum_{j=1}^m E_j\right) \rho\right).$$

Since $\sum_j E_j = \mathbb{I}$ and $\mathrm{Tr}(\rho) = 1$, the sum equals one. The annotation triples are additional data and do not appear in the formula for p_j. Hence they do not alter the Born-rule probabilities. □

Remark 1.8.2. The proposition concerns an annotated record only. It does not establish that the annotations can be physically measured, that they can be propagated consistently through arbitrary circuits, or that they yield a computational advantage.

1.9 A simple two-stage example

Consider the initial state $|0\rangle$ and the circuit consisting of a Hadamard gate followed by a computational-basis measurement.

The state after the gate is

$$|\psi_1\rangle = H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The measurement probabilities are

$$p_0 = p_1 = \frac{1}{2}.$$

Suppose an external calibration assessment supplies

$$\mathbf{n}_0 = (0.92, 0.06, 0.02), \qquad \mathbf{n}_1 = (0.85, 0.10, 0.05).$$

These values are illustrative and are not derived from the Hadamard matrix.

The annotated record is

$$\mathcal{R} = \left\{ \left(0, \frac{1}{2}, \mathbf{n}_0\right), \left(1, \frac{1}{2}, \mathbf{n}_1\right) \right\}.$$

Three observations follow.

1. The physical outcome probabilities are equal.
2. The two annotations need not be equal, because they may describe different evidence about the two detector outcomes.
3. The annotations cannot be used to infer a different pair of measurement probabilities unless an additional model is explicitly introduced and justified.

This example illustrates the principal methodological distinction of the paper: quantum randomness and neutrosophic indeterminacy can be represented together without being identified with one another.

1.10 Possible applications and research questions

1.10.1 Quantum-device diagnostics

A quantum device produces data through preparation, control, and measurement procedures. In practice, the interpretation of these data may involve calibration uncertainty, incomplete records, and conflicting estimates.

A neutrosophic annotation might record separate assessments of support, indeterminacy, and opposition for a diagnostic proposition. To be scientifically useful, the components would need a reproducible operational definition and a validation procedure.

1.10.2 Hybrid quantum-classical learning

Hybrid quantum-classical algorithms combine quantum circuits with classical data processing. A proposed neutrosophic component could annotate classical labels, model confidence, or disagreement between data sources.

Such an approach should be evaluated against suitable baselines. Relevant questions include whether the annotations improve calibration, robustness, interpretability, or performance on held-out data. The mere addition of three-valued annotations does not imply improved learning.

1.10.3 Quantum information under incomplete evidence

In some applications, the quantum state is not known exactly, or the available evidence does not determine a unique state. Standard quantum information theory already provides mathematical tools for mixed states, statistical inference, and uncertainty about state preparation [2, 3].

A neutrosophic description may supplement these tools by recording how different sources of evidence support or oppose particular modeling assumptions. A useful theory must explain what information this annotation contributes beyond the density operator and the available statistical model.

1.10.4 Research questions

The following problems appear suitable for further study.

RQ1. Under which axioms can an annotation-update map Φ_U be defined consistently for unitary circuits?

RQ2. Can an annotated quantum-state model be constructed so that equivalent descriptions of the same physical state yield equivalent predictions?

RQ3. Which properties should a neutrosophic information measure satisfy, and how should it relate to Shannon and von Neumann entropy?

RQ4. Can annotation rules be made invariant under changes of measurement representation where such invariance is physically required?

RQ5. In which experimentally testable settings do neutrosophic annotations improve the interpretation of noisy quantum data compared with conventional statistical methods?

These questions should be answered before claims about new quantum-computational capabilities are made.

1.11 Limitations

The framework presented here has several limitations.

First, the neutrosophic annotations are not derived from quantum mechanics. They are additional mathematical data whose interpretation depends on the application.

Second, no unique map from a quantum density operator to a neutrosophic triple has been established. Different propositions and evidence sources can legitimately require different annotations.

Third, the proposed uncertainty score is illustrative. It has not been established as an entropy satisfying the usual axioms of information theory.

Fourth, a hybrid record does not automatically define a new quantum circuit model. A complete model would need precise definitions of states, transformations, measurements, composition, and operational predictions.

Finally, no complexity-theoretic advantage or experimental improvement is proved in this paper. Such claims require formal analysis and, where appropriate, empirical evidence.

1.12 Conclusion

We have presented an introductory framework for considering neutrosophic representations alongside quantum information. The discussion began with classical bits, information content,

probability, and Shannon entropy, and then reviewed qubits, superposition, quantum gates, density operators, and measurement.

The principal construction is an annotated measurement record in which standard Born-rule probabilities are accompanied by truth, indeterminacy, and falsity assessments. This construction allows uncertainty about evidence or model reliability to be distinguished from the probability distribution generated by a quantum measurement.

A more complete theory would require mathematically justified annotation-update rules, a suitable information measure, and an operational account of how the annotations affect scientific inference. Establishing those properties remains a task for future work.

Chapter 2

Neutrosophic Qubits and Quantum States

Related literature. Neutrosophic quantum states and their gates have been studied in [73]; possible physical encodings are explored as a preprint in [81]. The distinction between a three-valued information descriptor and a normalized quantum state is essential.

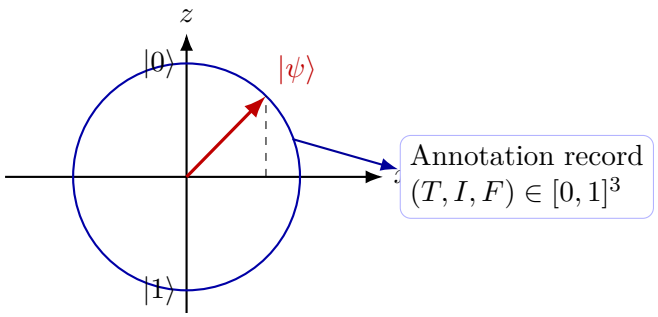


Figure 2.1: Schematic Bloch-sphere cross-section with an independent neutrosophic annotation record. The annotation is not itself a physical qubit.

We develop an introductory mathematical framework for studying neutrosophic annotations of qubits and quantum states. The standard theory of qubits is based on complex vector spaces, normalized vectors, inner products, and orthonormal bases. Neutrosophic logic, in contrast, describes a proposition through truth, indeterminacy, and falsity components. Since these mathematical structures have different roles, a precise formulation must distinguish quantum amplitudes from neutrosophic degrees.

We introduce annotated qubits, define operations on their annotations, and investigate the relationship between linear combinations, basis representations, inner products, and orthogonality. We prove several elementary results concerning normalization, uniqueness of coordinates, Gram–Schmidt orthogonalization, and the preservation of quantum-state geometry under unitary transformations. We also identify conditions under which neutrosophic annotations can be transported consistently through quantum-state transformations. A worked example illustrates the difference between an ordinary superposition and an annotated quantum state.

The proposed construction is an information-annotation framework built on the standard Hilbert-space formalism. It does not replace the Born rule, and it does not assert that neutrosophic degrees are physical probability amplitudes. The final section formulates problems for developing a more complete mathematical theory.

2.1 Introduction

Quantum computation uses quantum-mechanical systems to represent and process information. Its elementary unit of information is the qubit, whose state is represented by a normalized vector in a two-dimensional complex Hilbert space. The mathematical description of quantum states therefore depends on linear algebra, complex numbers, inner products, and orthonormal bases [1, 2].

Neutrosophic logic provides a three-component description of a proposition in terms of truth, indeterminacy, and falsity [6, 6]. Depending on the chosen neutrosophic framework, these components may be normalized or may be allowed to vary independently.

The purpose of this paper is to investigate how a neutrosophic description can accompany the standard mathematical representation of a qubit. We focus on the linear-algebraic foundations needed to understand quantum states rather than on quantum algorithms or physical implementations.

The main methodological distinction is the following: a quantum state determines the amplitudes used to calculate measurement probabilities, whereas a neutrosophic annotation describes an additional assessment of a specified proposition, model, or information source.

In particular, the triple

$$\mathbf{n} = (T, I, F)$$

is not identified with the complex vector

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The two objects have different mathematical structures. A hybrid representation must preserve this distinction.

The organization follows a pedagogical progression through the mathematical concepts underlying qubits and quantum states. The general subject matter is motivated by McMahon's introductory presentation of quantum states [1]; the neutrosophic constructions and propositions below are formulated independently.

The principal objectives are:

1. to recall the standard mathematical definition of a qubit and a quantum state;
2. to introduce a precise notion of a neutrosophically annotated qubit;
3. to distinguish quantum superposition from neutrosophic indeterminacy;
4. to formulate annotated linear combinations and coordinate representations;
5. to study inner products, orthogonality, and Gram–Schmidt orthogonalization in the hybrid setting;
6. to identify conditions required for consistent annotation updates.

2.2 Preliminaries on neutrosophic annotations

2.2.1 Three-component descriptions

Definition 2.2.1. A neutrosophic annotation is an ordered triple

$$\mathbf{n} = (T, I, F), \qquad T, I, F \in [0, 1],$$

where T, I, and F represent truth support, indeterminacy, and falsity support, respectively, for a specified proposition.

The interpretation of these quantities depends on the application. For example, they may describe evidence supporting a proposition, uncertainty in a classification, and evidence opposing that proposition.

Remark 2.2.2. The three components do not necessarily form a probability distribution. In the normalized special case one imposes

$$T + I + F = 1.$$

This is an additional modeling assumption, not a universal requirement of neutrosophic logic [6, 6].

Definition 2.2.3. The set of normalized neutrosophic annotations is

$$\mathcal{N}_1 = \left\{(T, I, F) \in [0, 1]^3 : T + I + F = 1\right\}.$$

The set of unrestricted bounded annotations is

$$\mathcal{N}_3 = [0, 1]^3.$$

The distinction between $\mathcal{N}_1$ and $\mathcal{N}_3$ will be important when defining operations on annotations.

2.2.2 Annotation operations

For two annotations

$$\mathbf{n} = (T, I, F), \qquad \mathbf{m} = (T', I', F'),$$

we may define the componentwise average by

$$\mathbf{n} \oplus_{\text{av}} \mathbf{m} = \left(\frac{T + T'}{2}, \frac{I + I'}{2}, \frac{F + F'}{2}\right).$$

This operation represents an arithmetic average of two annotations. It is not automatically a logical disjunction or a quantum operation.

Proposition 2.2.4. *If* $\mathbf{n}, \mathbf{m} \in \mathcal{N}_1$*, then*

$$\mathbf{n} \oplus_{\text{av}} \mathbf{m} \in \mathcal{N}_1.$$

Proof. All three coordinates of the average belong to $[0, 1]$. Moreover,

$$\begin{aligned}\frac{T + T'}{2} + \frac{I + I'}{2} + \frac{F + F'}{2} \\ = \frac{(T + I + F) + (T' + I' + F')}{2} = 1.\end{aligned}$$

Thus the average belongs to $\mathcal{N}_1$. □

For unrestricted annotations, componentwise averaging also remains in $\mathcal{N}_3$, because the cube $[0, 1]^3$ is convex.

2.3 The qubit and its mathematical representation

2.3.1 The standard qubit

Definition 2.3.1. A pure qubit state is a unit vector in $\mathbb{C}^2$. Using the computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

a qubit has the representation

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C},$$

where

$$|\alpha|^2 + |\beta|^2 = 1.$$

The coefficients α and β are called probability amplitudes. The Born rule assigns the computational-basis measurement probabilities

$$p(0) = |\alpha|^2, \qquad p(1) = |\beta|^2.$$

The normalization condition guarantees that

$$p(0) + p(1) = 1$$

[2, 3].

Example 2.3.2. The vector

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

is a valid qubit because

$$\left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = 1.$$

A computational-basis measurement gives each outcome with probability $1/2$.

2.3.2 The proposed annotated qubit

Definition 2.3.3. An annotated qubit is a pair

$$\mathfrak{q} = (|\psi\rangle, \mathbf{n}),$$

where $|\psi\rangle \in \mathbb{C}^2$ is a normalized quantum state and $\mathbf{n} \in \mathcal{N}_3$ is a neutrosophic annotation associated with a specified proposition about the state, its preparation, or its interpretation.

The annotation should always be understood relative to a proposition P. A more explicit notation is

$$\mathfrak{q} = (|\psi\rangle, \mathbf{n}_P).$$

For example, P might be the proposition that a preparation device has operated within its calibration tolerance. Its annotation describes evidence about that proposition rather than the amplitude of the qubit.

Remark 2.3.4. The pair $(|\psi\rangle, \mathbf{n}_P)$ is a hybrid mathematical object. It is not itself a vector in $\mathbb{C}^2$, nor is it automatically a physical state in a larger Hilbert space. A physical interpretation of the annotation would require an explicit measurement and dynamics model.

2.3.3 A basic distinction

Suppose

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

and the annotation of a preparation proposition is

$$\mathbf{n}_P = (0.95, 0.04, 0.01).$$

The equal amplitudes give equal computational-basis measurement probabilities. The annotation indicates a separate assessment of the proposition P. It does not imply that the state has a 95% amplitude for $|0\rangle$, nor does the indeterminacy value 0.04 change the Born-rule probabilities.

This distinction will be maintained in all subsequent definitions.

2.4 Complex vector spaces and linear combinations

2.4.1 Vector spaces

Definition 2.4.1. A complex vector space is a set V equipped with vector addition and scalar multiplication by complex numbers, satisfying the usual vector-space axioms.

The space $\mathbb{C}^2$ is a two-dimensional complex vector space. Its vectors have the form

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \alpha, \beta \in \mathbb{C}.$$

For vectors $u, v \in \mathbb{C}^2$ and scalars $a, b \in \mathbb{C}$, the linear combination

$$au + bv$$

is again a vector in $\mathbb{C}^2$.

The set of all linear combinations of a collection of vectors is called its span.

Definition 2.4.2. For vectors $v_1, \ldots, v_k \in V$, define

$$\operatorname{span}\{v_1, \ldots, v_k\} = \left\{ \sum_{j=1}^{k} a_j v_j : a_j \in \mathbb{C} \right\}.$$

2.4.2 Linear combinations of quantum states

If $|\psi\rangle$ and $|\phi\rangle$ are normalized quantum states, then

$$a|\psi\rangle + b|\phi\rangle$$

is a vector in the same Hilbert space. However, it is not necessarily normalized.

A valid pure quantum state is obtained when

$$\|a|\psi\rangle + b|\phi\rangle\| = 1.$$

This condition must be checked using the Hilbert-space inner product.

Proposition 2.4.3. *Let $|\psi\rangle$ and $|\phi\rangle$ be normalized vectors in a complex Hilbert space. Then*

$$\begin{aligned} \|a|\psi\rangle + b|\phi\rangle\|^2 = |a|^2 + |b|^2 \\ + 2\,\mathrm{Re}\left(\overline{a}b\langle\psi \mid \phi\rangle\right). \end{aligned}$$

Proof. Using the inner product,

$$\begin{aligned}\|a|\psi\rangle + b|\phi\rangle\|^2 &= \langle a\psi + b\phi \mid a\psi + b\phi\rangle \\ &= |a|^2\langle\psi \mid \psi\rangle + |b|^2\langle\phi \mid \phi\rangle \\ &\quad + \overline{a}b\langle\psi \mid \phi\rangle + \overline{b}a\langle\phi \mid \psi\rangle.\end{aligned}$$

Since both vectors are normalized and

$$\langle\phi \mid \psi\rangle = \overline{\langle\psi \mid \phi\rangle},$$

the stated identity follows. □

2.4.3 Annotated linear combinations

We now define a particular annotation rule for a restricted class of hybrid operations.

Definition 2.4.4. Let

$$\mathfrak{q}_1 = (|\psi_1\rangle, \mathbf{n}_1), \qquad \mathfrak{q}_2 = (|\psi_2\rangle, \mathbf{n}_2)$$

be annotated quantum states. Suppose that

$$a|\psi_1\rangle + b|\psi_2\rangle \neq 0.$$

Define the normalized quantum component

$$|\chi\rangle = \frac{a|\psi_1\rangle + b|\psi_2\rangle}{\|a|\psi_1\rangle + b|\psi_2\rangle\|}.$$

When $|a|^2 + |b|^2 = 1$, define the proposed annotation average

$$\mathbf{n}_\chi = |a|^2\mathbf{n}_1 + |b|^2\mathbf{n}_2,$$

where the operations on triples are componentwise. The resulting annotated vector is

$$\mathfrak{q}_\chi = (|\chi\rangle, \mathbf{n}_\chi).$$

This definition applies only when the coefficient condition holds. For general coefficients, one possible extension is to use normalized weights

$$w_1 = \frac{|a|^2}{|a|^2 + |b|^2}, \qquad w_2 = \frac{|b|^2}{|a|^2 + |b|^2},$$

provided a and b are not both zero.

The rule is a proposed averaging convention. It is not implied by the superposition principle of quantum mechanics. In particular, it does not account for interference terms in the annotation, because the annotation is not a quantum amplitude.

Proposition 2.4.5. *If $\mathbf{n}_1, \mathbf{n}_2 \in \mathcal{N}_1$ and $w_1, w_2 \geq 0$ with $w_1 + w_2 = 1$, then*

$$\mathbf{n}_\chi = w_1\mathbf{n}_1 + w_2\mathbf{n}_2$$

belongs to $\mathcal{N}_1$.

Proof. Every component of $\mathbf{n}_\chi$ is a convex combination of numbers in $[0, 1]$, and therefore belongs to $[0, 1]$. Furthermore,

$$\begin{aligned}T_\chi + I_\chi + F_\chi &= w_1(T_1 + I_1 + F_1) \\ &\quad + w_2(T_2 + I_2 + F_2) \\ &= w_1 + w_2 = 1.\end{aligned}$$

Thus $\mathbf{n}_\chi \in \mathcal{N}_1$. □

2.5 Bases, dimension, and coordinates

2.5.1 Linear independence

Definition 2.5.1. Vectors $v_1, \ldots, v_k$ are linearly independent if

$$\sum_{j=1}^{k} a_j v_j = 0$$

implies

$$a_1 = \cdots = a_k = 0.$$

Definition 2.5.2. A basis of a vector space V is a linearly independent set of vectors that spans V. The dimension of a finite-dimensional vector space is the number of vectors in any basis.

The computational basis

$$\mathcal{B} = \{|0\rangle, |1\rangle\}$$

is a basis of $\mathbb{C}^2$. Every vector

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

has a unique coordinate representation

$$[\psi]_{\mathcal{B}} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

2.5.2 Uniqueness of quantum coordinates

Theorem 2.5.3. *If*

$$\alpha|0\rangle + \beta|1\rangle = \alpha'|0\rangle + \beta'|1\rangle,$$

then

$$\alpha = \alpha', \qquad \beta = \beta'.$$

Proof. Subtracting the two expressions gives

$$(\alpha - \alpha')|0\rangle + (\beta - \beta')|1\rangle = 0.$$

The computational basis is linearly independent. Consequently, both coefficients must vanish:

$$\alpha - \alpha' = 0, \qquad \beta - \beta' = 0.$$

□

2.5.3 Annotated coordinates

An annotated coordinate representation may be recorded as

$$\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, (T, I, F) \right).$$

The uniqueness theorem applies to the quantum coordinate vector. It does not establish uniqueness of the annotation as a description of physical reality. Multiple annotations may be assigned to the same quantum vector if they concern different propositions or evidence sources.

Remark 2.5.4. To obtain a unique hybrid representation, one must specify both the quantum state and the proposition to which the annotation refers. The proposition is part of the modeling context, even if it is suppressed in the notation.

2.5.4 A change of basis

Let $\mathcal{B} = \{v_1, v_2\}$ and $\mathcal{C} = \{u_1, u_2\}$ be two bases of $\mathbb{C}^2$. A state has coordinate vectors

$$[\psi]_{\mathcal{B}} \quad \text{and} \quad [\psi]_{\mathcal{C}}.$$

A nonsingular change-of-basis matrix S relates them:

$$[\psi]_{\mathcal{C}} = S[\psi]_{\mathcal{B}},$$

with the precise form of S determined by the coordinate convention.

The annotation does not automatically transform by the same matrix, since its components are real-valued information degrees rather than complex vector coordinates.

A hybrid change-of-basis operation may therefore be written

$$([\psi]_{\mathcal{B}}, \mathbf{n}) \longmapsto (S[\psi]_{\mathcal{B}}, \Phi_S(\mathbf{n}, \psi)),$$

where Φ_S is an independently specified annotation rule.

2.6 Inner products and bra–ket notation

2.6.1 The complex inner product

For vectors

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

the standard inner product is

$$\langle u, v \rangle = \overline{u_1} v_1 + \overline{u_2} v_2.$$

The norm of v is

$$\|v\| = \sqrt{\langle v, v \rangle}.$$

The inner product satisfies conjugate symmetry, linearity in one argument under the chosen convention, and positive definiteness. In quantum mechanics the corresponding notation is

$$\langle \psi \mid \phi \rangle = \langle \psi, \phi \rangle.$$

2.6.2 Bra–ket formalism

A ket $|\psi\rangle$ denotes a vector in a Hilbert space. Its corresponding bra $\langle\psi|$ is the conjugate transpose of that vector.

For

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

we have

$$\langle\psi| = \begin{pmatrix} \overline{\alpha} & \overline{\beta} \end{pmatrix}.$$

The inner product is

$$\langle \psi \mid \phi \rangle = \langle \psi || \phi \rangle,$$

and the outer product is

$$|\psi\rangle\langle\phi|.$$

These are fundamental operations in quantum mechanics [2, 3].

2.6.3 An annotated inner-product record

For annotated states

$$\mathfrak{q}_1 = (|\psi_1\rangle, \mathbf{n}_1), \qquad \mathfrak{q}_2 = (|\psi_2\rangle, \mathbf{n}_2),$$

the quantum inner product remains

$$\langle \psi_1 \mid \psi_2 \rangle.$$

For supplementary information, one may record the pair

$$\mathfrak{I}(\mathfrak{q}_1, \mathfrak{q}_2) = \left(\langle \psi_1 \mid \psi_2 \rangle, \mathbf{n}_1, \mathbf{n}_2\right).$$

This notation keeps the complex inner product separate from the annotation data.

Remark 2.6.1. There is no canonical complex inner product on $\mathcal{N}_3$ merely because $\mathcal{N}_3$ is a subset of $\mathbb{R}^3$. One could introduce an auxiliary metric or similarity function on triples, but it would be a new definition and would not automatically have a quantum mechanical interpretation.

2.7 Orthonormality and Gram–Schmidt orthogonalization

2.7.1 Orthonormal vectors

Definition 2.7.1. Vectors $v_1, \ldots, v_k$ in a complex inner-product space are orthonormal if

$$\langle v_i, v_j \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

The computational basis satisfies

$$\langle 0 \mid 0 \rangle = 1, \qquad \langle 1 \mid 1 \rangle = 1, \qquad \langle 0 \mid 1 \rangle = \langle 1 \mid 0 \rangle = 0.$$

Orthonormality makes the coordinate representation of a state especially convenient. If $\{e_1, \ldots, e_n\}$ is an orthonormal basis, then

$$|\psi\rangle = \sum_{j=1}^{n} c_j |e_j\rangle$$

has coefficients

$$c_j = \langle e_j \mid \psi \rangle.$$

2.7.2 The Gram–Schmidt process

Suppose $v_1, \ldots, v_k$ are linearly independent vectors in an inner-product space. Define

$$u_1 = v_1$$

and, recursively,

$$u_j = v_j - \sum_{i=1}^{j-1} \frac{\langle u_i, v_j \rangle}{\langle u_i, u_i \rangle} u_i.$$

Then define

$$e_j = \frac{u_j}{\|u_j\|}.$$

The vectors $e_1, \ldots, e_k$ form an orthonormal set spanning the same subspace as the original vectors [7].

Theorem 2.7.2. *Let $v_1, \dots, v_k$ be linearly independent vectors in a finite-dimensional inner-product space. The Gram–Schmidt process produces an orthonormal set $e_1, \dots, e_k$ satisfying*

$$\operatorname{span}\{e_1, \dots, e_j\} = \operatorname{span}\{v_1, \dots, v_j\}$$

for each $j = 1, \dots, k$.

Proof. We proceed inductively. The first vector is

$$e_1 = \frac{v_1}{\|v_1\|},$$

which is well-defined because $v_1 \neq 0$.

Assume that $u_1, \dots, u_{j-1}$ are nonzero and mutually orthogonal. By the definition of u_j,

$$u_j = v_j - \sum_{i=1}^{j-1} \frac{\langle u_i, v_j \rangle}{\langle u_i, u_i \rangle} u_i.$$

For any $r < j$, orthogonality gives

$$\begin{aligned} \langle u_r, u_j \rangle &= \langle u_r, v_j \rangle \\ &\quad - \sum_{i=1}^{j-1} \frac{\langle u_i, v_j \rangle}{\langle u_i, u_i \rangle} \langle u_r, u_i \rangle \\ &= \langle u_r, v_j \rangle - \frac{\langle u_r, v_j \rangle}{\langle u_r, u_r \rangle} \langle u_r, u_r \rangle \\ &= 0. \end{aligned}$$

Moreover, $u_j \neq 0$, since otherwise v_j would belong to the span of $v_1, \dots, v_{j-1}$, contrary to linear independence. Thus $e_j = u_j / \|u_j\|$ is well-defined and is orthogonal to the preceding vectors. Normalization gives $\|e_j\| = 1$.

At each step, u_j differs from v_j by a linear combination of the preceding vectors. Hence the span of the first j constructed vectors equals the span of the first j original vectors. This completes the induction. □

2.7.3 A worked example in $\mathbb{C}^2$

Consider

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The first normalized vector is

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Next,

$$u_2 = v_2 - \langle e_1, v_2 \rangle e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since $\|u_2\| = 1$,

$$e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus the process recovers the computational basis.

2.7.4 Annotations and orthogonalization

Suppose each input vector has an annotation:

$$(v_j, \mathbf{n}_j).$$

The Gram–Schmidt process acts on the vectors v_j. The resulting vectors are linear combinations of the original vectors, so an annotation-update rule is needed if annotations are to accompany the new vectors.

For instance, one may define

$$\mathbf{m}_j = \sum_{i=1}^{j} w_{ji}\mathbf{n}_i, \qquad w_{ji} \geq 0, \qquad \sum_{i=1}^{j} w_{ji} = 1.$$

This is a convex annotation rule. The weights may be chosen according to an application, but they are not determined by Gram–Schmidt orthogonalization itself.

In particular, the complex coefficients used in Gram–Schmidt cannot be used directly as probabilities for averaging annotations. They may be negative or complex, whereas the convex weights must be real and nonnegative.

Proposition 2.7.3. *If* $\mathbf{n}_1, \ldots, \mathbf{n}_j \in \mathcal{N}_1$ *and*

$$w_{ji} \geq 0, \qquad \sum_{i=1}^{j} w_{ji} = 1,$$

then

$$\mathbf{m}_j = \sum_{i=1}^{j} w_{ji}\mathbf{n}_i$$

belongs to $\mathcal{N}_1$.

Proof. Each component of $\mathbf{m}_j$ is a convex combination of numbers in $[0, 1]$, and therefore belongs to $[0, 1]$. Also,

$$\begin{aligned} T(\mathbf{m}_j) + I(\mathbf{m}_j) + F(\mathbf{m}_j) &= \sum_{i=1}^{j} w_{ji}(T(\mathbf{n}_i) + I(\mathbf{n}_i) + F(\mathbf{n}_i)) \\ &= \sum_{i=1}^{j} w_{ji} = 1. \end{aligned}$$

Hence $\mathbf{m}_j \in \mathcal{N}_1$. □

2.8 Cauchy–Schwarz and triangle inequalities

2.8.1 Cauchy–Schwarz inequality

For vectors u, v in an inner-product space,

$$|\langle u, v\rangle| \leq \|u\| \, \|v\| \, .$$

This inequality is central to the mathematical consistency of quantum-state calculations [7, 2].

Proposition 2.8.1. *If* $|\psi\rangle$ *and* $|\phi\rangle$ *are normalized quantum states, then*

$$|\langle \psi \mid \phi\rangle| \leq 1.$$

Equality holds if and only if the two vectors are linearly dependent.

Proof. The Cauchy–Schwarz inequality gives

$$|\langle \psi \mid \phi\rangle| \leq \|\psi\| \, \|\phi\| = 1.$$

Its equality condition states that the two vectors are linearly dependent. □

2.8.2 Triangle inequality

The norm of a sum satisfies

$$\|u+v\| \le \|u\| + \|v\|\,.$$

More generally,

$$\left\| \sum_{j=1}^{k} a_j v_j \right\| \le \sum_{j=1}^{k} |a_j|\,\|v_j\|\,.$$

These inequalities are relevant when determining whether a linear combination must be normalized before it is used as a quantum state.

2.8.3 A limitation of componentwise annotation rules

Suppose

$$\mathbf{n}_1 = (0.9, 0.1, 0), \qquad \mathbf{n}_2 = (0.2, 0.3, 0.5).$$

Both are normalized annotations. A convex average with weights $1/2, 1/2$ gives

$$\mathbf{m} = (0.55, 0.20, 0.25).$$

However, the inner product $\langle \psi_1 \mid \psi_2 \rangle$ may be zero, nonzero, or equal to one, depending on the quantum states. The average $\mathbf{m}$ does not determine the inner product and cannot replace it.

Therefore, the geometry of quantum states and the geometry of annotations must be studied separately.

2.9 Unitary transformations and annotation transport

2.9.1 Unitary transformations

A linear operator $U : \mathcal{H} \to \mathcal{H}$ is unitary if

$$U^\dagger U = U U^\dagger = \mathrm{id}\,.$$

For a quantum state $|\psi\rangle$, the transformed state is

$$|\psi'\rangle = U|\psi\rangle.$$

A unitary transformation preserves norms and inner products:

$$\|U\psi\| = \|\psi\|\,, \qquad \langle U\psi \mid U\phi \rangle = \langle \psi \mid \phi \rangle.$$

Proposition 2.9.1. *Let U be unitary and let $|\psi\rangle$ be normalized. Then $U|\psi\rangle$ is normalized.*

Proof. We have

$$\begin{aligned} \|U\psi\|^2 &= \langle \psi | U^\dagger U | \psi \rangle \\ &= \langle \psi | \,\mathrm{id}\, | \psi \rangle \\ &= \|\psi\|^2 = 1. \end{aligned}$$

□

2.9.2 The proposed annotation-update map

For an annotated state

$$\mathfrak{q} = (|\psi\rangle, \mathbf{n}),$$

a hybrid transformation may be defined by

$$\mathfrak{U}(\mathfrak{q}) = (U|\psi\rangle, \Phi_U(\mathbf{n}, \psi)) .$$

Here Φ_U is an annotation-update map. It is not part of the standard definition of a unitary operator.

A simple special case is

$$\Phi_U(\mathbf{n}, \psi) = \mathbf{n},$$

which leaves the annotation unchanged. This may be appropriate if the annotation refers to a proposition unaffected by the transformation.

If the annotation refers to a proposition that changes under U, keeping the same triple may be inappropriate. The model should then define how the proposition itself transforms.

Definition 2.9.2. An annotation-update rule is norm-independent if its value depends on the annotation and the specified transformation but not on the overall normalization chosen for a nonzero representative of the same quantum ray.

The term above is a proposed modeling condition. A more complete theory should also specify how annotations behave under equivalent descriptions of the same physical state.

2.9.3 A consistency condition for sequential gates

Suppose U and V are two unitary transformations. A natural consistency condition is

$$\Phi_{VU}(\mathbf{n}, \psi) = \Phi_V \left(\Phi_U(\mathbf{n}, \psi), U\psi\right) .$$

This condition requires the annotation obtained from the combined transformation VU to agree with the annotation obtained by applying U and then V.

It is a proposed composition law. It does not follow automatically from the linearity of quantum mechanics.

Proposition 2.9.3. *Suppose a family of annotation-update maps satisfies*

$$\Phi_{VU}(\mathbf{n}, \psi) = \Phi_V \left(\Phi_U(\mathbf{n}, \psi), U\psi\right)$$

for every admissible pair of unitary transformations. Then annotations are independent of how a fixed unitary circuit is grouped into successive stages.

Proof. Consider a circuit composed of three transformations U, V, W. Applying the consistency condition twice gives

$$\begin{aligned} \Phi_{W(VU)}(\mathbf{n}, \psi) &= \Phi_W \left(\Phi_{VU}(\mathbf{n}, \psi), VU\psi\right) \\ &= \Phi_W \left(\Phi_V(\Phi_U(\mathbf{n}, \psi), U\psi), VU\psi\right) . \end{aligned}$$

Applying the condition instead to WV first yields the same nested expression, because operator composition is associative. Thus the annotation update does not depend on the grouping of the successive gates. The same argument extends by induction to any finite circuit. □

This proposition supplies a basic consistency requirement for a proposed annotation calculus. It does not prove that a suitable nontrivial family of maps exists for every desired interpretation.

2.10 Worked example: an annotated qubit

Consider the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle.$$

Its normalization is

$$\|\psi\|^2 = \frac{1}{2} + \frac{1}{2} = 1.$$

In the computational basis,

$$[\psi]_{\mathcal{B}} = \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}.$$

The Born-rule probabilities are

$$p(0) = \frac{1}{2}, \qquad p(1) = \frac{1}{2}.$$

Now let P be the proposition that the preparation procedure meets a specified calibration requirement. Suppose the available diagnostic information leads to the annotation

$$\mathbf{n}_P = (0.88, 0.09, 0.03).$$

The hybrid representation is

$$\mathfrak{q} = \left(\frac{|0\rangle + i|1\rangle}{\sqrt{2}}, (0.88, 0.09, 0.03) \right).$$

This record contains two different kinds of information:

1. The complex vector determines quantum measurement statistics.
2. The triple records a separate assessment of the calibration proposition.

The relative phase i does not appear in the annotation unless the chosen application explicitly makes the annotation depend on that phase.

For instance, the phase can affect measurements in a different basis. It is therefore possible for a model to update the annotation after a basis change, but such an update must be stated as an additional rule rather than inferred from the three components alone.

2.11 Exercises

The following exercises are intended to reinforce the mathematical ideas developed in the paper.

Exercise 2.11.1. Verify that

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{1}{2}|1\rangle$$

is normalized and calculate its computational-basis measurement probabilities.

Exercise 2.11.2. Let

$$\mathbf{n}_1 = (0.8, 0.1, 0.1), \qquad \mathbf{n}_2 = (0.4, 0.4, 0.2).$$

Calculate their componentwise average and verify whether it belongs to $\mathcal{N}_1$.

Exercise 2.11.3. Prove that the computational basis is linearly independent and spans $\mathbb{C}^2$.

Exercise 2.11.4. Apply the Gram–Schmidt process to

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Determine the resulting orthonormal vectors.

Exercise 2.11.5. Explain why the expression

$$\alpha T + \beta I$$

does not automatically define a neutrosophic degree when $\alpha, \beta \in \mathbb{C}$ and $T, I \in [0, 1]$. Specify conditions under which a real-valued combination of annotation components could be used instead.

Exercise 2.11.6. Suppose two unitary transformations U and V are applied successively. State and verify the composition condition that a proposed annotation-update map must satisfy.

2.12 Open problems

The framework suggests several problems that require further mathematical development.

Problem 1. Canonical annotation. Can a useful class of quantum-state propositions be identified for which the annotation is determined by a reproducible rule from experimental evidence?

Problem 2. Basis covariance. Find conditions on Φ_U ensuring that the annotation transforms consistently under changes of orthonormal basis.

Problem 3. Annotation geometry. Can a metric or divergence on $\mathcal{N}_3$ be constructed with clear operational meaning for quantum-information applications?

Problem 4. Composite systems. Develop a mathematically consistent annotation rule for tensor products of quantum systems. Determine whether the annotation of a composite state should depend on the annotations of its subsystems, on correlations, or on both.

Problem 5. Measurement compatibility. Determine which annotation-update rules remain consistent with ordinary quantum measurement statistics.

Problem 6. Information measures. Construct and analyze a neutrosophic information measure with explicitly stated axioms. Compare it with Shannon entropy and von Neumann entropy without identifying these different quantities by definition.

Problem 7. Operational testing. Identify an experimental or computational setting in which neutrosophic annotations provide a measurable benefit over standard uncertainty and calibration methods.

These questions are necessary steps toward a theory that goes beyond an annotation convention.

2.13 Conclusion

We have developed an introductory mathematical framework for neutrosophically annotated qubits and quantum states. The treatment begins with complex vector spaces and normalized quantum states, then develops linear combinations, bases, coordinates, inner products, orthonormality, and Gram–Schmidt orthogonalization.

The proposed hybrid object

$$\mathfrak{q} = (|\psi\rangle, T, I, F)$$

keeps the standard quantum vector separate from its three-component information annotation. This separation prevents neutrosophic indeterminacy from being confused with quantum superposition, probability amplitude, or measurement probability.

We proved elementary results for convex combinations of annotations, uniqueness of quantum coordinates, orthonormalization, and preservation of state norms under unitary transformations. We also formulated a composition condition for annotation-update maps.

The constructions in this paper provide a starting point rather than a complete theory of neutrosophic quantum computing. A full theory requires a justified interpretation of annotations, well-defined transformation rules, compatibility with quantum measurement, and demonstrable applications. Establishing these properties is an important direction for future research.

Chapter 3

Neutrosophic Matrices and Operators

Related literature. Matrix formulations of neutrosophic gates are developed in [73], while operator-theoretic formulations of neutrosophic quantum channels and density operators appear in the preprints [79, 82].

Matrices and linear operators provide the mathematical language for quantum observables, transformations, measurement probabilities, and quantum-state evolution. This paper develops an introductory framework for neutrosophic annotations of matrices and operators while retaining the standard complex Hilbert-space formalism. The principal objects of quantum mechanics, including Pauli matrices, outer products, Hermitian and unitary operators, spectral decompositions, traces, expectation values, projections, positive operators, and commutators, are reviewed and illustrated.

A neutrosophic annotation associates truth support, indeterminacy, and falsity support with a specified proposition concerning an operator, its calibration, its mathematical model, or its interpretation. We define annotated operators as pairs consisting of an ordinary linear operator and an annotation. Several elementary results concerning convex annotation aggregation, unitary invariance, expectation values, and uncertainty are established. The paper also discusses the Heisenberg uncertainty relation, polar decomposition, singular values, and the standard postulates of quantum mechanics.

The proposed framework is deliberately conservative: neutrosophic triples are not identified with complex matrices, quantum amplitudes, or physical observables. The standard Born rule and operator-theoretic results remain unchanged. The new constructions are information annotations whose operational meaning must be specified in each application. This distinction provides a mathematically controlled starting point for future research on neutrosophic quantum information.

3.1 Introduction

Quantum information theory describes quantum states and their transformations using complex Hilbert spaces, linear operators, and probabilistic measurement rules. Matrices are particularly useful when the state space is finite-dimensional. They allow quantum observables, gates, projectors, and measurement operations to be computed explicitly [1, 2].

A fundamental role is played by Hermitian operators, which represent observables in the standard formulation of quantum mechanics. Unitary operators describe closed quantum evolution, while positive operators occur in quantum measurement theory. Spectral decomposition connects an observable with its possible measurement outcomes and corresponding eigenspaces [8, 3].

Neutrosophic logic introduces a three-component description of a proposition through truth, indeterminacy, and falsity [6, 6]. Such a description may be useful for recording incomplete evidence, conflicting information, or uncertainty in the interpretation of a mathematical or experimental model.

The purpose of this paper is to develop a precise framework in which neutrosophic annotations accompany quantum operators. We do not replace a complex matrix with a triple of real numbers. Instead, we distinguish the operator itself from information about a proposition associated with it.

For example, a matrix may represent a calibrated measurement apparatus, while an annotation describes the evidence supporting the proposition that its calibration remains valid. The annotation does not change the matrix unless a separate physical model explicitly connects calibration uncertainty to the operator.

The main objectives are:

1. to review the standard algebra of quantum matrices and operators;
2. to define neutrosophically annotated operators;
3. to examine Pauli matrices, outer products, and operator representations;
4. to study Hermitian, unitary, normal, and positive operators;
5. to formulate annotated spectral and expectation records;
6. to distinguish ordinary quantum uncertainty from uncertainty in a neutrosophic annotation;
7. to identify mathematical conditions required for consistent annotation updates.

The standard quantum formalism is based on established operator theory [2, 3]. The annotation constructions introduced here are proposed extensions and should not be confused with established physical postulates.

3.2 Preliminaries on neutrosophic annotations

3.2.1 Neutrosophic triples

Definition 3.2.1. A neutrosophic annotation of a proposition P is a triple

$$\mathbf{n}(P) = (T(P), I(P), F(P)),$$

where

$$T(P), I(P), F(P) \in [0, 1].$$

The components represent truth support, indeterminacy, and falsity support for P, respectively.

The interpretation of the three components must be specified for the application. They may represent expert assessments, evidence from calibration experiments, or confidence in a modeling assumption.

Definition 3.2.2. The set of bounded neutrosophic triples is

$$\mathcal{N}_3 = [0, 1]^3.$$

The normalized subclass is

$$\mathcal{N}_1 = \left\{(T, I, F) \in [0, 1]^3 : T + I + F = 1\right\}.$$

The normalization condition is optional. General neutrosophic frameworks do not require the three components to sum to one [6, 6].

3.2.2 Convex aggregation

For annotations

$$\mathbf{n}_j = (T_j, I_j, F_j), \qquad j = 1, \dots, m,$$

and weights $w_j \geq 0$ satisfying

$$\sum_{j=1}^{m} w_j = 1,$$

define

$$\overline{\mathbf{n}} = \sum_{j=1}^{m} w_j \mathbf{n}_j = \left(\sum_j w_j T_j,\ \sum_j w_j I_j,\ \sum_j w_j F_j \right).$$

Proposition 3.2.3. *If $\mathbf{n}_j \in \mathcal{N}_1$ for every j, then*

$$\overline{\mathbf{n}} \in \mathcal{N}_1.$$

Proof. Each coordinate of $\overline{\mathbf{n}}$ is a convex combination of numbers in $[0, 1]$. Therefore all coordinates belong to $[0, 1]$. Furthermore,

$$\begin{aligned} \overline{T} + \overline{I} + \overline{F} &= \sum_j w_j (T_j + I_j + F_j) \\ &= \sum_j w_j = 1. \end{aligned}$$

Thus $\overline{\mathbf{n}} \in \mathcal{N}_1$. □

This aggregation rule is useful for combining annotations when the weights have an explicitly justified meaning. It is not a universal neutrosophic logical operation.

3.3 Matrices and linear operators

3.3.1 Matrices as representations of operators

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. A linear operator $A : \mathcal{H} \to \mathcal{H}$ satisfies

$$A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha A|\psi\rangle + \beta A|\phi\rangle$$

for all $\alpha, \beta \in \mathbb{C}$ and $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.

After choosing an orthonormal basis, the operator is represented by a complex matrix. Matrix multiplication represents the composition of operators.

If A and B are represented in the same basis, then

$$(AB)|\psi\rangle = A(B|\psi\rangle).$$

The order matters in general:

$$AB \neq BA.$$

3.3.2 Annotated operators

Definition 3.3.1. An annotated operator is a pair

$$\mathfrak{A} = (A, \mathbf{n}_P),$$

where $A : \mathcal{H} \to \mathcal{H}$ is a linear operator and $\mathbf{n}_P \in \mathcal{N}_3$ is an annotation of a specified proposition P concerning A, its implementation, its calibration, or its interpretation.

The proposition P is part of the meaning of the annotation. For instance, it could assert that a measurement apparatus implements A within a specified error tolerance.

Remark 3.3.2. The triple $\mathbf{n}_P$ is not an alternative matrix representation of A. The complex operator and the real-valued annotation have different mathematical roles.

3.3.3 Composition of annotated operators

Let

$$\mathfrak{A} = (A, \mathbf{n}_P), \qquad \mathfrak{B} = (B, \mathbf{n}_Q).$$

Their operator composition is AB. To define an annotated composition, one must also specify a rule for combining the annotations.

For a fixed application, suppose

$$\Phi_{A,B} : \mathcal{N}_3 \times \mathcal{N}_3 \to \mathcal{N}_3$$

is a prescribed update rule. Define

$$\mathfrak{A} \circ_\Phi \mathfrak{B} = (AB, \Phi_{A,B}(\mathbf{n}_P, \mathbf{n}_Q)).$$

The operator order follows the usual convention: B acts first, followed by A.

Proposition 3.3.3. *If the update rule is chosen so that*

$$\begin{aligned}&\Phi_{AB,C}(\Phi_{A,B}(\mathbf{n}_P, \mathbf{n}_Q), \mathbf{n}_R)\\ &\quad = \Phi_{A,BC}(\mathbf{n}_P, \Phi_{B,C}(\mathbf{n}_Q, \mathbf{n}_R)),\end{aligned}$$

then annotated composition is associative.

Proof. The operator components satisfy

$$(AB)C = A(BC)$$

by associativity of operator composition. The stated condition ensures that the annotation components obtained from the two groupings are equal. Hence the annotated compositions agree. □

This condition is necessary for a consistent annotation calculus. It does not specify a unique choice of $\Phi_{A,B}$.

3.4 Outer products and the closure relation

3.4.1 Outer products

For vectors $|u\rangle$ and $|v\rangle$, the outer product

$$|u\rangle\langle v|$$

is an operator. Its action on a vector $|\psi\rangle$ is

$$|u\rangle\langle v||\psi\rangle = \langle v \mid \psi\rangle|u\rangle.$$

The outer product is different from the inner product $\langle v \mid u\rangle$, which is a scalar.

For a normalized vector $|\psi\rangle$, the operator

$$P_\psi = |\psi\rangle\langle\psi|$$

is the rank-one projector onto the subspace spanned by $|\psi\rangle$.

Proposition 3.4.1. *If $|\psi\rangle$ is normalized, then*

$$P_\psi^\dagger = P_\psi, \qquad P_\psi^2 = P_\psi, \qquad \mathrm{Tr}(P_\psi) = 1.$$

Proof. The adjoint satisfies

$$P_\psi^\dagger = (|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi| = P_\psi.$$

Also,

$$\begin{aligned} P_\psi^2 &= |\psi\rangle\langle\psi||\psi\rangle\langle\psi| \\ &= \langle\psi \mid \psi\rangle|\psi\rangle\langle\psi| \\ &= P_\psi. \end{aligned}$$

Finally, in an orthonormal basis,

$$\mathrm{Tr}(P_\psi) = \langle\psi \mid \psi\rangle = 1.$$

□

3.4.2 The closure relation

Let $\{|e_j\rangle\}_{j=1}^d$ be an orthonormal basis of a d-dimensional Hilbert space. Then

$$\sum_{j=1}^{d} |e_j\rangle\langle e_j| = \mathbb{I}.$$

This identity is called the closure relation.

Every vector satisfies

$$|\psi\rangle = \sum_{j=1}^{d} \langle e_j \mid \psi\rangle |e_j\rangle.$$

3.4.3 Annotated projectors

Suppose a proposition P_j concerns the reliability of the preparation or identification of the basis state $|e_j\rangle$. An annotated projector may be written

$$\mathfrak{P}_j = \left(|e_j\rangle\langle e_j|, \mathbf{n}_{P_j}\right).$$

The closure relation remains an identity for the operator components. It does not imply that the annotations satisfy a corresponding identity.

For example,

$$\sum_{j=1}^{d} \mathbf{n}_{P_j}$$

need not equal a distinguished neutrosophic triple. A rule for combining annotations must be defined separately if such an aggregate is required.

3.5 The Pauli operators

3.5.1 Definition

The Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Together with the identity matrix, they provide a standard basis for the complex 2×2 matrices when complex coefficients are allowed [2, 8].

Their squares satisfy

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{I}.$$

They are Hermitian:

$$\sigma_j^\dagger = \sigma_j, \qquad j \in \{x, y, z\}.$$

Their products satisfy

$$\sigma_x \sigma_y = i\sigma_z, \qquad \sigma_y \sigma_z = i\sigma_x, \qquad \sigma_z \sigma_x = i\sigma_y.$$

Reversing the order changes the sign.

3.5.2 Pauli commutation relations

The commutator of two operators is

$$[A, B] = AB - BA.$$

For Pauli matrices,

$$[\sigma_x, \sigma_y] = 2i\sigma_z,$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x, \qquad [\sigma_z, \sigma_x] = 2i\sigma_y.$$

The anticommutator is

$$\{A, B\} = AB + BA.$$

For distinct Pauli matrices,

$$\{\sigma_j, \sigma_k\} = 0 \quad (j \neq k),$$

and

$$\{\sigma_j, \sigma_j\} = 2\mathbb{I}.$$

3.5.3 An annotated Pauli operator

Suppose σ_z represents an idealized observable, and P is the proposition that a physical device implements this observable to a specified tolerance. We may write

$$\mathfrak{S}_z = (\sigma_z, \mathbf{n}_P).$$

For example,

$$\mathbf{n}_P = (0.94, 0.04, 0.02)$$

could represent an illustrative assessment of calibration evidence. These numbers are not derived from the Pauli matrix.

The eigenvalues of σ_z remain $+1$ and -1. The annotation does not modify them.

3.6 Hermitian, unitary, and normal operators

3.6.1 Hermitian operators

Definition 3.6.1. An operator A is Hermitian if

$$A^\dagger = A.$$

Hermitian operators have real eigenvalues. In standard quantum mechanics, they represent observables.

Theorem 3.6.2. *If A is Hermitian and*

$$A|\psi\rangle = \lambda|\psi\rangle$$

for a nonzero vector $|\psi\rangle$, then $\lambda \in \mathbb{R}$.

Proof. Taking the inner product with $|\psi\rangle$ gives

$$\langle\psi|A|\psi\rangle = \lambda \||\psi\|^2.$$

Since $A = A^\dagger$,

$$\langle\psi|A|\psi\rangle = \overline{\langle\psi|A|\psi\rangle},$$

so this quantity is real. Because $\||\psi\|^2 > 0$, it follows that λ is real. □

3.6.2 Unitary operators

Definition 3.6.3. An operator U is unitary if

$$U^\dagger U = UU^\dagger = \mathbb{I}.$$

A unitary operator preserves inner products:

$$\langle U\psi \mid U\phi\rangle = \langle\psi|U^\dagger U|\phi\rangle = \langle\psi \mid \phi\rangle.$$

Unitary operators describe closed-system quantum evolution and quantum gates [2].

3.6.3 Normal operators

Definition 3.6.4. An operator A is normal if

$$A^\dagger A = AA^\dagger.$$

Every Hermitian operator and every unitary operator is normal. In finite-dimensional complex Hilbert spaces, normal operators admit an orthonormal basis of eigenvectors.

3.6.4 Annotations of operator properties

Let

$$\mathfrak{A} = (A, \mathbf{n}_P).$$

If P is the proposition that A is Hermitian, then the mathematical question whether

$$A^\dagger = A$$

is decidable from the specified matrix entries. A neutrosophic annotation may instead concern the reliability of those entries, their measurement, or their numerical estimation.

This distinction is essential. If the matrix is exactly specified, its Hermitian property is a mathematical fact. Indeterminacy in an annotation may describe uncertainty about the data or model, not an uncertainty in the truth of a proved algebraic identity.

3.7 Eigenvalues, eigenvectors, and spectral decomposition

3.7.1 Eigenvalues and eigenvectors

A scalar λ is an eigenvalue of A if there is a nonzero vector $|\psi\rangle$ such that

$$A|\psi\rangle = \lambda|\psi\rangle.$$

The vector is an eigenvector associated with λ.

For a finite matrix, eigenvalues are roots of the characteristic equation

$$\det(A - \lambda\mathbb{I}) = 0.$$

3.7.2 Spectral theorem

Theorem 3.7.1. *Let A be a Hermitian operator on a finite-dimensional complex Hilbert space. Then there exist distinct real eigenvalues $\lambda_1, \ldots, \lambda_m$ and orthogonal projectors $P_1, \ldots, P_m$ such that*

$$A = \sum_{j=1}^{m} \lambda_j P_j,$$

where

$$P_j P_k = 0 \quad (j \neq k), \qquad \sum_{j=1}^{m} P_j = \mathbb{I}.$$

This is the spectral decomposition of A [2, 3].

For the Pauli observable σ_z,

$$\sigma_z = P_+ - P_-,$$

where

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

3.7.3 An annotated spectral record

For an annotated observable, one may record

$$\mathfrak{A} = (A, \mathbf{n}_P)$$

along with the spectral data

$$\Big\{(\lambda_j, P_j, \mathbf{n}_{P_j}) : j = 1, \ldots, m\Big\}.$$

The annotations $\mathbf{n}_{P_j}$ may concern the identification of the eigenspaces, the calibration of the corresponding measurement outcomes, or the reliability of the physical implementation.

The spectral theorem itself does not determine these annotations.

Remark 3.7.2. For an exactly known Hermitian matrix, the eigenvalues and spectral projectors are fixed by the matrix. Uncertainty in experimental estimates of a matrix is a different problem from uncertainty in its exact spectral decomposition.

3.8 The trace and its properties

3.8.1 Definition

For a square matrix $A = (a_{jk})$, its trace is

$$\operatorname{Tr}(A) = \sum_j a_{jj}.$$

The trace is linear:

$$\operatorname{Tr}(\alpha A + \beta B) = \alpha \operatorname{Tr}(A) + \beta \operatorname{Tr}(B).$$

For square matrices A and B of the same dimension,

$$\operatorname{Tr}(AB) = \operatorname{Tr}(BA).$$

Consequently, the trace is invariant under a change of basis:

$$\operatorname{Tr}(S^{-1}AS) = \operatorname{Tr}(A)$$

for every invertible matrix S.

3.8.2 Trace of an outer product

For vectors $u, v \in \mathbb{C}^d$,

$$\operatorname{Tr}(|u\rangle\langle v|) = \langle v \mid u\rangle.$$

In particular, for a normalized pure state,

$$\operatorname{Tr}(|\psi\rangle\langle\psi|) = 1.$$

3.8.3 Trace and annotations

An annotated operator may be represented by

$$\mathfrak{A} = (A, \mathbf{n}_P).$$

The ordinary trace remains

$$\operatorname{Tr}(\mathfrak{A}) := \operatorname{Tr}(A)$$

if this notation is adopted as shorthand. It does not include the triple.

There is no canonical matrix trace of a neutrosophic triple. One could define a separate scalar score

$$s(\mathbf{n}) = aT + bI + cF,$$

where $a, b, c \in \mathbb{R}$ are chosen coefficients, but this would be an application-specific statistic rather than the trace of a quantum operator.

3.9 Expectation values

3.9.1 Pure-state expectation values

For a normalized pure state $|\psi\rangle$ and a Hermitian observable A, the expectation value is

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle.$$

It is real because A is Hermitian.

If

$$A = \sum_j \lambda_j P_j,$$

then

$$\langle A \rangle_\psi = \sum_j \lambda_j \langle \psi | P_j | \psi \rangle.$$

The Born rule gives the probability of the outcome associated with P_j:

$$p_j = \langle \psi | P_j | \psi \rangle.$$

Thus the expectation is the probability-weighted average of the eigenvalues.

3.9.2 Density-operator formulation

For a density operator ρ, the expectation value is

$$\langle A \rangle_\rho = \mathrm{Tr}(\rho A).$$

The density operator satisfies

$$\rho^\dagger = \rho, \qquad \rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1.$$

These conditions ensure that ρ represents a valid quantum state [2, 3].

3.9.3 An annotated expectation record

Suppose the proposition P concerns the reliability of the physical implementation of A. We can record

$$\mathfrak{E}_A(\rho) = \left(\mathrm{Tr}(\rho A), \mathbf{n}_P\right).$$

The first component is the ordinary quantum expectation. The second is an annotation about the specified proposition.

Remark 3.9.1. An annotation should not be added to an expectation value without defining the units and the intended interpretation. For example,

$$\mathrm{Tr}(\rho A) + I$$

is not generally a physically meaningful expectation value: I is dimensionless, whereas A may have units of energy or another physical quantity.

3.10 Projection operators and positive operators

3.10.1 Projection operators

Definition 3.10.1. An operator P is an orthogonal projection if

$$P^\dagger = P, \qquad P^2 = P.$$

An orthogonal projection satisfies

$$0 \leq P \leq \mathbb{I}$$

in the operator order. Its eigenvalues are 0 and 1.

For a normalized vector $|\psi\rangle$,

$$\langle \psi | P | \psi \rangle \in [0, 1].$$

This quantity is the probability associated with the projector in a projective measurement.

3.10.2 Positive operators

Definition 3.10.2. An operator A is positive if it is Hermitian and

$$\langle\psi|A|\psi\rangle \geq 0$$

for every vector $|\psi\rangle$.

A positive operator has nonnegative eigenvalues. In quantum measurement theory, positive operators form the building blocks of positive-operator-valued measurements.

A family $\{E_j\}$ defines a positive-operator-valued measurement (POVM) if

$$E_j \geq 0, \qquad \sum_j E_j = \mathbb{I}.$$

For a state ρ, the probability of outcome j is

$$p_j = \mathrm{Tr}(\rho E_j).$$

3.10.3 Annotations of positive operators

Suppose an experimental procedure estimates a positive operator E, and P denotes the proposition that the estimate is compatible with a prescribed calibration model. The annotated record is

$$(E, \mathbf{n}_P).$$

The positivity of the exact mathematical operator is determined by its eigenvalues or quadratic form. An annotation may describe uncertainty in an estimated operator $\widehat{E}$, but it cannot make a nonpositive operator positive by definition.

Proposition 3.10.3. *If ρ is a density operator and $E \geq 0$, then*

$$\mathrm{Tr}(\rho E) \geq 0.$$

Proof. Since $E \geq 0$, its positive square root $E^{1/2}$ exists. By cyclicity of the trace,

$$\mathrm{Tr}(\rho E) = \mathrm{Tr}(E^{1/2}\rho E^{1/2}).$$

The operator $E^{1/2}\rho E^{1/2}$ is positive, because for every vector $|\psi\rangle$,

$$\langle\psi|E^{1/2}\rho E^{1/2}|\psi\rangle = \langle\phi|\rho|\phi\rangle \geq 0, \qquad |\phi\rangle = E^{1/2}|\psi\rangle.$$

Its trace is therefore nonnegative. □

3.11 Commutator algebra

3.11.1 Commutators

For operators A and B, define

$$[A, B] = AB - BA.$$

The commutator measures the failure of the two operators to commute.

The following identities hold:

$$[A, B] = -[B, A],$$

$$[A, B + C] = [A, B] + [A, C],$$

and

$$[A, BC] = [A, B]C + B[A, C].$$

The Jacobi identity is

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$

These relations are important in quantum mechanics because noncommuting observables may have intrinsic measurement limitations [8].

3.11.2 Annotated commutator records

Given annotated operators

$$\mathfrak{A} = (A, \mathbf{n}_P), \qquad \mathfrak{B} = (B, \mathbf{n}_Q),$$

the ordinary commutator is

$$[A, B] = AB - BA.$$

An annotation of the proposition

$$R = \{A \text{ and } B \text{ commute}\}$$

may be written

$$\mathbf{n}(R) = (T_R, I_R, F_R).$$

For exactly specified matrices, the equation

$$[A, B] = 0$$

is an exact algebraic condition. If the matrices are known only through uncertain experimental estimates, the annotation may describe evidence concerning that condition.

Remark 3.11.1. A nonzero commutator is a mathematical property of the specified operators. Neutrosophic indeterminacy should not be interpreted as an automatic modification of the commutator.

3.12 The Heisenberg uncertainty principle

3.12.1 Variances

For a normalized state $|\psi\rangle$ and a Hermitian operator A, define

$$\langle A \rangle = \langle \psi | A | \psi \rangle.$$

The variance is

$$(\Delta A)^2 = \langle \psi | (A - \langle A \rangle \mathbb{I})^2 | \psi \rangle.$$

For two Hermitian operators A and B, the Robertson uncertainty relation states that

$$\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle \psi | [A, B] | \psi \rangle \right|.$$

A stronger form is the Robertson–Schrödinger inequality [9, 10].

3.12.2 Pauli example

For the Pauli operators,

$$[\sigma_x, \sigma_y] = 2i\sigma_z.$$

Hence

$$\Delta\sigma_x \, \Delta\sigma_y \geq |\langle\sigma_z\rangle| \, .$$

For the state $|0\rangle$, one has

$$\langle\sigma_x\rangle = 0, \qquad \langle\sigma_y\rangle = 0, \qquad \langle\sigma_z\rangle = 1.$$

Because $\sigma_x^2 = \sigma_y^2 = \mathbb{I}$,

$$\Delta\sigma_x = 1, \qquad \Delta\sigma_y = 1.$$

Therefore,

$$\Delta\sigma_x \, \Delta\sigma_y = 1,$$

and the lower bound is attained.

3.12.3 Quantum uncertainty versus neutrosophic indeterminacy

The uncertainty relation concerns the mathematical structure of quantum observables and the state in which their variances are calculated. Neutrosophic indeterminacy is a separate annotation.

For instance, the state $|0\rangle$ has a definite σ_z value, but nonzero variances for σ_x and σ_y. An annotation describing complete confidence in the preparation of $|0\rangle$ does not remove these quantum variances.

Conversely, a large indeterminacy score concerning the preparation device does not by itself determine the values of ΔA or ΔB.

3.12.4 A proposed annotation of uncertainty data

A hybrid uncertainty record may be defined by

$$\mathfrak{U}_{A,B}(\psi) = \left(\Delta A, \Delta B, \frac{1}{2}|\langle[A, B]\rangle|, \mathbf{n}_P\right).$$

Here $\mathbf{n}_P$ describes an explicitly stated proposition, such as the reliability of the estimated state preparation.

This record combines ordinary quantum quantities with supplementary information. It does not define a new uncertainty principle.

3.13 Polar decomposition and singular values

3.13.1 Polar decomposition

Every linear operator A on a finite-dimensional complex Hilbert space admits a polar decomposition

$$A = UP,$$

where $P = (A^\dagger A)^{1/2}$ is positive and U is a partial isometry. If A is invertible, U is unitary.

The operator P describes the magnitude component, while U describes the isometric component of the transformation [7, 3].

3.13.2 Singular values

The singular values of A are the nonnegative square roots of the eigenvalues of $A^\dagger A$. They are invariant under unitary changes of basis: if U and V are unitary, then A and UAV have the same singular values.

3.13.3 Annotations of polar data

For an annotated operator

$$\mathfrak{A} = (A, \mathbf{n}_P),$$

one may separately annotate propositions about:

- the accuracy of the estimated singular values;
- the reliability of the estimated polar factor;
- the validity of a numerical approximation;
- the stability of an implementation under perturbations.

These annotations do not replace the polar decomposition. For an exactly known finite-dimensional matrix, the positive factor P is determined by A, and the singular values are determined by $A^\dagger A$.

3.14 Standard quantum-mechanical postulates

The neutrosophic annotation framework must remain consistent with the standard mathematical structure of quantum mechanics if it is intended as an information layer over the conventional theory.

3.14.1 State postulate

A pure quantum state is represented by a unit vector in a complex Hilbert space, with physically equivalent vectors differing only by a global phase. More general states are represented by density operators.

3.14.2 Observable postulate

An observable is represented by a Hermitian operator. Its spectral decomposition identifies its possible measurement outcomes and corresponding projectors.

3.14.3 Measurement postulate

For a projective measurement with projectors $\{P_j\}$, the outcome probabilities are

$$p_j = \mathrm{Tr}(\rho P_j).$$

They satisfy

$$p_j \geq 0, \qquad \sum_j p_j = 1.$$

More generally, a POVM is described by positive operators $\{E_j\}$ summing to the identity, and

$$p_j = \mathrm{Tr}(\rho E_j).$$

3.14.4 Evolution postulate

For a closed quantum system, evolution is represented by a unitary operator:

$$\rho \longmapsto U\rho U^{\dagger}.$$

More general open-system dynamics can be represented by completely positive trace-preserving maps [2, 3].

3.14.5 The role of neutrosophic annotations

A hybrid description can accompany each of these postulates with an annotation about a specified proposition. For example,

$$\mathfrak{S} = (\rho, \mathbf{n}_{P_\rho})$$

may annotate a density operator, and

$$\mathfrak{E} = (E, \mathbf{n}_{P_E})$$

may annotate a measurement effect.

However, the annotation framework must not silently change the normalization of states, the positivity of measurement operators, or the Born-rule probabilities.

If annotations are intended to influence physical predictions rather than merely record information, a new operational model must be specified and tested.

3.15 A worked example: an annotated spin measurement

Consider a spin-$\frac{1}{2}$ system in the state

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The corresponding density operator is

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Consider the observable

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Its expectation is

$$\langle \sigma_z \rangle = \mathrm{Tr}(\rho\sigma_z) = 0.$$

Since $\sigma_z^2 = \mathbb{I}$,

$$\langle \sigma_z^2 \rangle = 1.$$

Therefore,

$$(\Delta\sigma_z)^2 = \langle \sigma_z^2 \rangle - \langle \sigma_z \rangle^2 = 1.$$

The two outcomes have probabilities

$$p_+ = \frac{1}{2}, \qquad p_- = \frac{1}{2}.$$

Now suppose P is the proposition that the measurement apparatus has been calibrated according to a specified protocol. An illustrative annotation is

$$\mathbf{n}_P = (0.91, 0.06, 0.03).$$

The complete annotated record is

$$\mathfrak{M} = \left(\rho, \sigma_z, \left\{\frac{1}{2}, \frac{1}{2}\right\}, \mathbf{n}_P\right).$$

This record distinguishes four items:

1. the prepared quantum state ρ;
2. the measured observable σ_z;
3. the Born-rule outcome probabilities;
4. the supplementary calibration annotation.

The annotation is not used to modify the probabilities. If a calibrated noise model is available, it may be incorporated into a separate physical model of the measurement process. Such a model must be defined explicitly rather than inferred from the annotation alone.

3.16 Limitations and open problems

The proposed framework has several limitations.

First, a neutrosophic annotation depends on a specified proposition and evidence source. There is no universal rule assigning a unique triple to every quantum operator.

Second, the proposed annotated operator is a pair, not a new operator on a Hilbert space. A physical realization of the annotation would require an additional construction.

Third, convex averaging is only one possible aggregation rule. Other rules may be appropriate for different applications, but their properties must be established.

Fourth, the annotations do not automatically define probability distributions, measurement outcomes, or new uncertainty relations.

The following problems provide directions for further research.

Problem 3.16.1. Determine conditions under which an annotation-update map is compatible with operator composition and unitary equivalence.

Problem 3.16.2. Develop a mathematically justified method for annotating estimated Hermitian matrices when the matrix entries are subject to experimental uncertainty.

Problem 3.16.3. Investigate whether annotation rules can be designed to preserve normalized triples under sequential quantum operations.

Problem 3.16.4. Construct a statistically calibrated relationship between experimental evidence and neutrosophic annotations without confusing these annotations with Born-rule probabilities.

Problem 3.16.5. Identify an application in quantum-device diagnostics or quantum-information processing where the annotated framework offers a measurable advantage over standard statistical descriptions.

3.17 Conclusion

We have presented a mathematical introduction to neutrosophically annotated matrices and operators in quantum information. The standard theory of Pauli operators, outer products, Hermitian and unitary operators, spectral decomposition, trace, expectation values, projectors, positive operators, commutators, uncertainty, and polar decomposition has been reviewed.

The central construction is the annotated operator

$$\mathfrak{A} = (A, T, I, F),$$

understood as shorthand for an operator together with a three-component annotation of a specified proposition. This construction separates the algebraic and physical properties of the operator from information concerning its preparation, calibration, or interpretation.

The paper proves elementary results concerning annotation aggregation, operator composition, and positivity, and it illustrates the distinction between quantum uncertainty and neutrosophic indeterminacy.

The proposed framework remains an initial mathematical model. A more complete theory requires operationally defined annotations, justified update rules, and applications whose benefits can be tested. No new quantum-mechanical postulate or computational advantage is asserted here.

Chapter 4

Neutrosophic Tensor Products and Composite Systems

Tensor products are fundamental to the mathematical description of composite quantum systems. They combine the state spaces of individual subsystems, provide a representation of joint states, and describe local operations acting on multipartite systems. This paper develops an explicitly defined neutrosophic annotation framework for tensor products in quantum information. Each subsystem is assigned a three-component annotation representing degrees of truth, indeterminacy, and falsity. A binary aggregation operation is introduced, and its algebraic properties are established. The ordinary Hilbert-space tensor product is retained as the physical state-space construction, while neutrosophic information is carried as additional metadata. Tensor products of annotated states, inner-product records, column vectors, operators, and matrices are examined. Explicit two-qubit calculations illustrate the framework, including the tensor product of Pauli operators and the action of local gates on a composite state. We also discuss the relation between annotations, entanglement, normalization, measurement probabilities, and partial traces. The distinction between established quantum mechanics and the proposed annotation formalism is emphasized throughout. The framework is intended as a mathematically testable starting point for further work, rather than as an established physical extension of quantum theory.

4.1 Introduction

Quantum information theory describes systems whose state spaces are constructed from complex Hilbert spaces. When two subsystems are combined, their joint state space is represented by a tensor product. This construction is essential for describing multi-qubit registers, local quantum gates, correlations, and entanglement [2, 3].

The tensor-product formalism is also an important part of introductory treatments of quantum computing, including the account of composite systems in McMahon's *Quantum Computing Explained* [1]. The present paper takes the same broad mathematical subject as its point of departure but develops an independent neutrosophic annotation model.

Neutrosophic approaches distinguish degrees of truth, indeterminacy, and falsity. In the general neutrosophic literature, these components are used to represent information that is incomplete, uncertain, or inconsistent [6, 6]. Here they are adapted as annotations attached to ordinary quantum states and operators.

The main idea is to keep two mathematical structures separate:

(i) the standard complex Hilbert-space structure, which determines state vectors, operators, inner products, and ordinary quantum probabilities;

(ii) an auxiliary neutrosophic annotation, which records three degrees associated with the information available about a subsystem or an operation.

This distinction is necessary. A triple of degrees does not automatically define a quantum state, a probability distribution, or a density operator. Nor does adding an annotation by itself modify the Born rule or the standard tensor-product construction [2, 3].

The objectives of this paper are:

- to define an algebra of three-component annotations;
- to construct annotated versions of composite quantum states;
- to describe tensor products of vectors, operators, and matrices with annotations;
- to establish basic algebraic properties and provide explicit examples;
- to identify which conclusions follow from standard quantum mechanics and which belong only to the proposed annotation model.

4.2 Mathematical preliminaries

4.2.1 Hilbert spaces and tensor products

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional complex Hilbert spaces with

$$\dim \mathcal{H}_A = n, \qquad \dim \mathcal{H}_B = m.$$

Their composite space is

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B,$$

and it satisfies

$$\dim \mathcal{H}_{AB} = nm.$$

The tensor product is the standard construction used to represent joint quantum systems [1, 2].

If $|\psi\rangle \in \mathcal{H}_A$ and $|\phi\rangle \in \mathcal{H}_B$, then

$$|\psi\rangle \otimes |\phi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B.$$

The vector $|\psi\rangle \otimes |\phi\rangle$ is a product vector. A general vector in the composite space need not be expressible as a single product vector; this distinction underlies the mathematical description of entanglement.

4.2.2 Neutrosophic triples

We use the following bounded version of a neutrosophic triple. The choice of the interval $[0, 1]$ is a modelling convention adopted in this paper, not a universal restriction on all neutrosophic theories.

Definition 4.2.1 (Bounded neutrosophic triple)**.** A bounded neutrosophic triple is an ordered triple

$$\boldsymbol{\nu} = (t, i, f) \in [0, 1]^3.$$

The components t, i, and f are interpreted, respectively, as degrees of truth, indeterminacy, and falsity associated with a specified item of information.

We write

$$\mathcal{T}(\boldsymbol{\nu}) = t, \qquad \mathcal{I}(\boldsymbol{\nu}) = i, \qquad \mathcal{F}(\boldsymbol{\nu}) = f.$$

Remark 4.2.2. The components are not required to sum to one. In particular,

$$t + i + f = 1$$

is not assumed. This convention distinguishes the present annotation from an ordinary probability distribution. A triple such as $(0.8, 0.3, 0.4)$ is allowed because its three components are independent degrees within the chosen interval.

Definition 4.2.3 (Annotation aggregation)**.** For two triples

$$\boldsymbol{\nu}_1 = (t_1, i_1, f_1), \qquad \boldsymbol{\nu}_2 = (t_2, i_2, f_2),$$

define

$$\boxed{\boldsymbol{\nu}_1 \odot \boldsymbol{\nu}_2 = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)\,.}$$

The truth component is combined by multiplication. The indeterminacy and falsity components are combined by the probabilistic-sum operation $x + y - xy$. This is a selected aggregation rule; other applications may require a different one.

Proposition 4.2.4. *The operation $\odot$ is closed on $[0,1]^3$, associative, and commutative. Its identity element is*

$$\boldsymbol{e}_N = (1, 0, 0).$$

Proof. If $x, y \in [0,1]$, then $xy \in [0,1]$. Also,

$$x + y - xy = 1 - (1 - x)(1 - y) \in [0,1].$$

Hence the operation is closed.

Multiplication is associative and commutative. For the other two coordinates, put $s(x,y) = x + y - xy$. Then

$$\begin{aligned} s(s(x,y),z) &= x + y - xy + z - (x + y - xy)z \\ &= x + y + z - xy - xz - yz + xyz \\ &= s(x, s(y,z)). \end{aligned}$$

Thus s is associative; it is also commutative. Associativity and commutativity therefore hold coordinatewise for $\odot$.

Finally,

$$(t, i, f) \odot (1, 0, 0) = (t, i, f),$$

which proves the identity assertion. □

Example 4.2.5. Let

$$\boldsymbol{\nu}_1 = (0.8, 0.2, 0.1), \qquad \boldsymbol{\nu}_2 = (0.9, 0.3, 0.2).$$

Then

$$\begin{aligned} \boldsymbol{\nu}_1 \odot \boldsymbol{\nu}_2 &= \big(0.8(0.9),\ 0.2 + 0.3 - 0.2(0.3), \\ &\qquad\qquad 0.1 + 0.2 - 0.1(0.2)\big) \\ &= (0.72, 0.44, 0.28). \end{aligned}$$

The result is another bounded neutrosophic triple.

4.3 Annotated quantum states

4.3.1 Definition of an annotated state

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space, and let

$$S(\mathcal{H}) = \{|\psi\rangle \in \mathcal{H} : \langle \psi \mid \psi \rangle = 1\}$$

be its unit sphere.

Definition 4.3.1 (Neutrosophically annotated state)**.** A neutrosophically annotated state is a pair

$$\mathsf{NS} = (|\psi\rangle, \boldsymbol{\nu}),$$

where $|\psi\rangle \in S(\mathcal{H})$ is a normalized quantum state and $\boldsymbol{\nu} \in [0,1]^3$ is its annotation.

The vector $|\psi\rangle$ carries the ordinary quantum state information. The triple $\boldsymbol{\nu}$ carries additional descriptive information. In this model, two annotated states can have the same quantum vector and different annotations.

Remark 4.3.2. The annotation is not a replacement for the quantum amplitude. For example, the triple $(0.7, 0.2, 0.1)$ does not mean that the probability of observing a particular outcome is 0.7. The usual measurement probabilities are calculated from the quantum state and the relevant measurement operators, unless an additional, explicitly defined probabilistic model is introduced.

4.3.2 Composite annotated states

Definition 4.3.3 (Tensor product of annotated states)**.** Let

$$\mathsf{NS}_A = (|\psi\rangle, \boldsymbol{\nu}_A), \qquad \mathsf{NS}_B = (|\phi\rangle, \boldsymbol{\nu}_B).$$

Their annotated tensor product is defined by

$$\boxed{\mathsf{NS}_A \otimes_N \mathsf{NS}_B = (|\psi\rangle \otimes |\phi\rangle, \boldsymbol{\nu}_A \odot \boldsymbol{\nu}_B)\,.}$$

The subscript N is used when it is useful to emphasize that the ordinary Hilbert-space tensor product is accompanied by an annotation aggregation rule.

Theorem 4.3.4 (Associativity of annotated composition)**.** *For three annotated states* $\mathsf{NS}_A, \mathsf{NS}_B, \mathsf{NS}_C$*,*

$$(\mathsf{NS}_A \otimes_N \mathsf{NS}_B) \otimes_N \mathsf{NS}_C = \mathsf{NS}_A \otimes_N (\mathsf{NS}_B \otimes_N \mathsf{NS}_C),$$

under the canonical identification of the two associated Hilbert-space tensor products.

Proof. The Hilbert-space tensor product is associative up to its canonical associator. The annotation operation $\odot$ is associative by the preceding proposition. Applying these two facts to the vector and annotation components separately proves the result. □

Remark 4.3.5. Associativity does not imply that every operation on annotated states is linear. The quantum vector component has its usual linear structure, but the annotation component is aggregated by a nonlinear rule. Consequently, the annotated states do not automatically form a complex vector space.

4.3.3 An explicit two-qubit example

Let

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Suppose

$$\mathsf{NS}_A = (|0\rangle, (0.8, 0.2, 0.1)), \qquad \mathsf{NS}_B = (|1\rangle, (0.9, 0.3, 0.2)).$$

The ordinary vector tensor product is

$$|0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The annotation is

$$(0.8, 0.2, 0.1) \odot (0.9, 0.3, 0.2) = (0.72, 0.44, 0.28).$$

Therefore,

$$\boxed{\mathsf{NS}_A \otimes_N \mathsf{NS}_B = \left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, (0.72, 0.44, 0.28) \right).}$$

This example illustrates the division of roles: the four-dimensional vector describes the two-qubit computational basis state, while the triple records the chosen annotation of the composite system.

4.4 Inner products and annotated inner-product records

The ordinary inner product on a composite Hilbert space satisfies

$$\langle \psi \otimes \phi \mid \chi \otimes \eta \rangle = \langle \psi \mid \chi \rangle \langle \phi \mid \eta \rangle.$$

This identity is a standard property of tensor products of Hilbert spaces [7, 3].

For example, because $\langle 0 \mid 1 \rangle = 0$,

$$\langle 0 \otimes 1 \mid 1 \otimes 0 \rangle = \langle 0 \mid 1 \rangle \langle 1 \mid 0 \rangle = 0.$$

Definition 4.4.1 (Annotated inner-product record)**.** For annotated states

$$\mathsf{NS}_A = (|\psi\rangle, \boldsymbol{\nu}_A), \qquad \mathsf{NS}_B = (|\phi\rangle, \boldsymbol{\nu}_B),$$

define the record

$$\mathfrak{I}(\mathsf{NS}_A, \mathsf{NS}_B) = (\langle \psi \mid \phi \rangle, \boldsymbol{\nu}_A \odot \boldsymbol{\nu}_B).$$

Remark 4.4.2. The record $\mathfrak{I}$ is not asserted to be a new Hilbert-space inner product. In particular, it is not a complex-valued map alone, and its annotation component does not satisfy the axioms of an ordinary inner product. It records a standard overlap together with an aggregated annotation.

The inner product remains the correct object for determining orthogonality and amplitudes. The additional triple may be useful for recording the status of the information associated with the two systems, provided the interpretation of the annotation is specified in the application.

4.5 Tensor products of column vectors

4.5.1 Kronecker product

Let

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^n, \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{C}^m.$$

The Kronecker product is the column vector

$$x \otimes y = \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_1 y_m \\ x_2 y_1 \\ \vdots \\ x_n y_m \end{pmatrix} \in \mathbb{C}^{nm}.$$

This is the coordinate representation of the tensor product of the corresponding vectors [1, 2].

If

$$x = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad y = \begin{pmatrix} c \\ d \end{pmatrix},$$

then

$$x \otimes y = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.$$

4.5.2 Annotated column vectors

Definition 4.5.1. An annotated column vector is a pair

$$(x, \boldsymbol{\nu}),$$

where $x \in \mathbb{C}^n$ is a column vector and $\boldsymbol{\nu} \in [0,1]^3$ is its annotation. Its annotated tensor product with $(y, \boldsymbol{\mu})$ is

$$(x, \boldsymbol{\nu}) \otimes_N (y, \boldsymbol{\mu}) = (x \otimes y, \boldsymbol{\nu} \odot \boldsymbol{\mu}).$$

The coordinate calculation is entirely standard; only the auxiliary annotation is combined by $\odot$.

Example 4.5.2. Take

$$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad y = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Then

$$x \otimes y = \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}.$$

If the annotations are

$$\boldsymbol{\nu} = (0.6, 0.2, 0.1), \qquad \boldsymbol{\mu} = (0.7, 0.3, 0.4),$$

their aggregate is

$$\begin{aligned}\boldsymbol{\nu} \odot \boldsymbol{\mu} &= (0.6 \cdot 0.7,\ 0.2 + 0.3 - 0.2 \cdot 0.3,\ 0.1 + 0.4 - 0.1 \cdot 0.4) \\ &= (0.42, 0.44, 0.46).\end{aligned}$$

Thus the annotated column-vector record is

$$\left(\begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}, (0.42, 0.44, 0.46) \right).$$

The vector in this example is not normalized. It is therefore a column-vector example, not a normalized quantum state.

4.6 Operators acting on composite systems

4.6.1 Local operators

Let $A : \mathcal{H}_A \to \mathcal{H}_A$ and $B : \mathcal{H}_B \to \mathcal{H}_B$ be linear operators. Their tensor product is the operator

$$A \otimes B : \mathcal{H}_A \otimes \mathcal{H}_B \longrightarrow \mathcal{H}_A \otimes \mathcal{H}_B$$

defined on product vectors by

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = A|\psi\rangle \otimes B|\phi\rangle,$$

and extended linearly to the entire composite space [2, 3].

The notation $A \otimes I_B$ represents an operation on subsystem A that acts as the identity on subsystem B. Similarly, $I_A \otimes B$ acts on subsystem B alone.

Definition 4.6.1 (Annotated operator)**.** An annotated operator is a pair

$$\mathsf{NO} = (A, \boldsymbol{\alpha}),$$

where A is a linear operator on a finite-dimensional Hilbert space and $\boldsymbol{\alpha} \in [0, 1]^3$ is an annotation associated with the operator.

Definition 4.6.2 (Tensor product of annotated operators)**.** For

$$\mathsf{NO}_A = (A, \boldsymbol{\alpha}), \qquad \mathsf{NO}_B = (B, \boldsymbol{\beta}),$$

define

$$\boxed{\mathsf{NO}_A \otimes_N \mathsf{NO}_B = (A \otimes B, \boldsymbol{\alpha} \odot \boldsymbol{\beta}).}$$

Proposition 4.6.3. *The annotation of an iterated tensor product of annotated operators does not depend on the placement of parentheses.*

Proof. The operator component is associative under the canonical identification of tensor-product spaces. The annotation component is associative by the algebraic properties of $\odot$. Therefore the annotated tensor product is associative in the same sense. □

4.6.2 Action on annotated states

For an annotated state $(|\psi\rangle, \boldsymbol{\nu})$ and an annotated operator $(A, \boldsymbol{\alpha})$, one possible convention is

$$(A, \boldsymbol{\alpha}) \triangleright (|\psi\rangle, \boldsymbol{\nu}) = (A|\psi\rangle, \boldsymbol{\alpha} \odot \boldsymbol{\nu}),$$

provided $A|\psi\rangle \neq 0$.

If a normalized quantum state is required, the quantum component must be normalized separately:

$$|\psi'\rangle = \frac{A|\psi\rangle}{\|A|\psi\rangle\|}.$$

This formula is meaningful when $A|\psi\rangle \neq 0$. The annotation update is an additional modelling rule, not a consequence of the standard quantum operation itself.

Remark 4.6.4. For a general nonunitary operator, applying A to a normalized vector need not produce a normalized vector. In quantum measurement theory, such operators often appear as measurement operators, and the normalization and outcome probability must be treated consistently with the measurement formalism [2, 3]. An annotation should not be used to replace that normalization.

4.7 Tensor products of matrices

4.7.1 The Kronecker product of matrices

Let $A = (a_{ij})$ be an $n \times n$ matrix and B an $m \times m$ matrix. The matrix tensor product is the $nm \times nm$ block matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}.$$

This representation gives a concrete way to compute operations on composite systems [1, 2].

For example, let

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$I_2 \otimes X = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This is a two-qubit operator that applies the bit-flip operator X to the second qubit while leaving the first qubit unchanged.

4.7.2 Algebraic identities

Theorem 4.7.1 (Basic tensor-product identities)**.** *Let A, C be operators on $\mathcal{H}_A$, and let B, D be operators on $\mathcal{H}_B$. Then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD),$$

$$(A \otimes B)^* = A^* \otimes B^*,$$

and

$$\operatorname{Tr}(A \otimes B) = \operatorname{Tr}(A) \operatorname{Tr}(B).$$

Proof. It is enough to verify the identities on product vectors and extend linearly. For the first identity,

$$\begin{aligned}(A \otimes B)(C \otimes D)(|\psi\rangle \otimes |\phi\rangle) &= A(C|\psi\rangle) \otimes B(D|\phi\rangle) \\ &= (AC)|\psi\rangle \otimes (BD)|\phi\rangle.\end{aligned}$$

Since product vectors span the composite space, the first identity follows. The adjoint identity follows from the defining inner-product property of the tensor product. For the trace identity, use product orthonormal bases $\{e_j\}$ and $\{f_k\}$:

$$\begin{aligned}\operatorname{Tr}(A \otimes B) &= \sum_{j,k} \langle e_j \otimes f_k, (A \otimes B)(e_j \otimes f_k)\rangle \\ &= \sum_{j,k} \langle e_j, Ae_j\rangle \langle f_k, Bf_k\rangle \\ &= \operatorname{Tr}(A)\operatorname{Tr}(B).\end{aligned}$$

□

Corollary 4.7.2. *If U and V are unitary operators, then $U \otimes V$ is unitary.*

Proof. By the adjoint and product identities,

$$(U \otimes V)^*(U \otimes V) = (U^*U) \otimes (V^*V) = I_A \otimes I_B.$$

The same computation in the opposite order gives the other unitary identity. □

4.7.3 Annotated matrix products

Suppose

$$\mathsf{NO}_A = (A, \boldsymbol{\alpha}), \qquad \mathsf{NO}_B = (B, \boldsymbol{\beta}).$$

The annotated tensor product is

$$(A \otimes B, \boldsymbol{\alpha} \odot \boldsymbol{\beta}).$$

The ordinary matrix component satisfies the identities above. The annotation is governed by the chosen triple aggregation rule.

In particular, the matrix identities do not imply that the annotations themselves behave like matrices. The annotation is a separate record and is not multiplied as an ordinary complex matrix.

4.8 A two-qubit calculation

Consider the standard single-qubit Hadamard operator

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and the Pauli bit-flip operator

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Their tensor product is

$$H \otimes X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

Because H and X are unitary, $H \otimes X$ is unitary.

For the computational basis state $|00\rangle = |0\rangle \otimes |0\rangle$, we have

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad X|0\rangle = |1\rangle.$$

Consequently,

$$\begin{aligned}(H \otimes X)|00\rangle &= (H|0\rangle) \otimes (X|0\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle \\ &= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle).\end{aligned}$$

This is a normalized superposition of two computational basis states.

Now assign the annotations

$$\boldsymbol{\alpha} = (0.9, 0.1, 0.05), \qquad \boldsymbol{\beta} = (0.8, 0.2, 0.1)$$

to H and X, respectively. Their aggregate is

$$\begin{aligned}\boldsymbol{\alpha} \odot \boldsymbol{\beta} &= \big(0.9(0.8),\ 0.1 + 0.2 - 0.1(0.2), \\ &\qquad\qquad 0.05 + 0.1 - 0.05(0.1)\big) \\ &= (0.72, 0.28, 0.145).\end{aligned}$$

Thus the annotated operator is represented by

$$\left(H \otimes X,\ (0.72, 0.28, 0.145)\right).$$

The resulting quantum vector and the annotation have different interpretations. The vector gives the usual quantum output; the triple summarizes the two operator annotations according to the selected aggregation rule.

4.9 Product states, entanglement, and annotations

4.9.1 Product states and general composite states

A pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a product state if it can be written as

$$|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B.$$

Otherwise, it is entangled. This distinction is a standard feature of quantum information theory [2, 3].

For example, the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

is entangled. It cannot be expressed as a single product of two single-qubit vectors.

The annotation formalism can attach a triple to this state:

$$\mathsf{NS}_{\Phi^+} = \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), (0.85, 0.25, 0.10)\right).$$

The annotation does not prove or disprove entanglement. Entanglement is determined by the quantum vector, or by an appropriate density operator for mixed states, not by the annotation alone.

4.9.2 A limitation of subsystem annotations

Suppose a Bell state is obtained by applying a quantum circuit to initial product states. It may be tempting to interpret its final annotation as a complete description of its correlations. The aggregation operation $\odot$, however, only combines the annotations assigned to the components. It does not encode the full joint amplitude structure of the Bell state.

Therefore, an annotation of the form

$$(|\Phi^+\rangle, \boldsymbol{\nu})$$

should not be understood as a replacement for the state vector or as an entanglement measure. If one wants the annotation to represent correlations, a separate definition involving joint measurements, correlation functions, or reduced density operators is required.

Remark 4.9.1. The present model does not define a new theory of entanglement. It defines a way to attach three-component records to the mathematical objects used in quantum information. A physically meaningful interpretation would require operational definitions and experimental or computational validation.

4.10 Mixed states and partial traces

4.10.1 Density operators

A density operator ρ on a finite-dimensional Hilbert space satisfies

$$\rho = \rho^*, \qquad \rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1.$$

Density operators describe both pure and mixed quantum states [2, 3]. A pure state $|\psi\rangle$ has density operator

$$\rho_\psi = |\psi\rangle\langle\psi|.$$

For two independent systems with density operators ρ_A and ρ_B, the joint density operator is

$$\rho_{AB} = \rho_A \otimes \rho_B.$$

This formula describes a product preparation. A general correlated joint state need not have this form.

4.10.2 Annotated density operators

Definition 4.10.1. An annotated density operator is a pair

$$(\rho, \boldsymbol{\nu}),$$

where ρ is an ordinary density operator and $\boldsymbol{\nu} \in [0,1]^3$ is an auxiliary annotation.

For two annotated density operators, define the product record by

$$(\rho_A, \boldsymbol{\nu}_A) \otimes_N (\rho_B, \boldsymbol{\nu}_B) = (\rho_A \otimes \rho_B, \boldsymbol{\nu}_A \odot \boldsymbol{\nu}_B).$$

Since ρ_A and ρ_B are density operators, $\rho_A \otimes \rho_B$ is positive and has unit trace:

$$\mathrm{Tr}(\rho_A \otimes \rho_B) = \mathrm{Tr}(\rho_A)\,\mathrm{Tr}(\rho_B) = 1.$$

4.10.3 Partial trace

For a joint density operator ρ_{AB}, the reduced state of subsystem A is

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}),$$

where Tr_B denotes the partial trace over subsystem B. The partial trace is central to the description of subsystems and correlations in quantum information [2, 3].

One may define an annotation update rule alongside the partial trace. For example, if a joint record is

$$(\rho_{AB}, (t, i, f)),$$

one could retain the same triple after taking the partial trace:

$$(\mathrm{Tr}_B(\rho_{AB}), (t, i, f)).$$

This is a possible bookkeeping convention, not a uniquely determined neutrosophic law. Another application might aggregate the annotations of several discarded subsystems, or calculate an annotation from measurement data.

Any such rule should be stated explicitly. The standard partial trace acts on the operator ρ_{AB}; it does not determine an auxiliary neutrosophic triple without an additional modelling assumption.

4.11 Consistency questions and possible extensions

The framework above is intentionally conservative: the established quantum structure is unchanged, and neutrosophic information is added as an annotation. Several questions must be addressed before a stronger physical interpretation could be justified.

4.11.1 Choice of aggregation rule

The operation

$$(t_1, i_1, f_1) \odot (t_2, i_2, f_2) = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)$$

is associative and has an identity. These algebraic properties make it suitable for iterated tensor products. Nevertheless, the choice is not unique. A different application might use minimum, maximum, averaging, or a data-dependent rule.

The relevant question is not only whether an aggregation operation is algebraically convenient, but also whether it has a justified interpretation for the information being recorded.

4.11.2 Compatibility with quantum evolution

Suppose U is a unitary operator. Standard quantum evolution sends

$$|\psi\rangle \longmapsto U|\psi\rangle.$$

An annotated model must decide whether the annotation remains fixed, changes according to a rule, or depends on external observations.

For example, one possible convention is

$$(|\psi\rangle, \boldsymbol{\nu}) \longmapsto (U|\psi\rangle, \boldsymbol{\nu}).$$

This rule leaves the annotation unchanged. It is consistent as a bookkeeping convention, but it is not a derivation of neutrosophic dynamics. A nontrivial update rule would need its own axioms and justification.

4.11.3 Compatibility with measurement

In standard quantum mechanics, a measurement is described by measurement operators or, more generally, by a positive operator-valued measure. Outcome probabilities are calculated from the quantum state and these operators [2, 3].

If neutrosophic annotations are intended to describe uncertainty in measurement information, a further map should specify how they relate to observed frequencies or likelihoods. In particular, the truth component should not simply be identified with an outcome probability without a formal definition and a consistency argument.

4.11.4 Multisubsystem systems

For k subsystems with annotations

$$\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_k,$$

the annotation of the composite system can be defined by

$$\boldsymbol{\nu}_{1\cdots k} = \boldsymbol{\nu}_1 \odot \cdots \odot \boldsymbol{\nu}_k.$$

Associativity ensures that the result is independent of the order of parenthesization. Explicitly, the truth component is

$$t_{1\cdots k} = \prod_{j=1}^{k} t_j,$$

while the other components are

$$i_{1\cdots k} = 1 - \prod_{j=1}^{k} (1 - i_j), \qquad f_{1\cdots k} = 1 - \prod_{j=1}^{k} (1 - f_j).$$

These formulas follow directly from the chosen aggregation rule. Their interpretation for large quantum registers remains an open modelling question.

4.12 Further research problems

The framework suggests the following problems.

Problem 1. Alternative annotation algebras. Determine which aggregation operations on $[0, 1]^3$ are associative, have a useful identity, and admit an interpretation appropriate to composite quantum systems.

Problem 2. Annotation and subsystem reduction. Construct an annotation rule for partial traces that is compatible with the interpretation of the annotation and is independent of irrelevant choices of basis.

Problem 3. Operational semantics. Define a procedure for estimating truth, indeterminacy, and falsity from a specified experimental or simulated data set.

Problem 4. Entanglement-sensitive annotations. Investigate whether an annotation can be defined from operational correlations while preserving the distinction between product and entangled states.

Problem 5. Quantum channels. Extend the framework from unitary operators to completely positive trace-preserving maps and determine which annotation update rules remain consistent under composition.

Problem 6. Numerical implementation. Develop algorithms for storing annotated quantum registers and compare their computational overhead with ordinary quantum-state representations.

These questions are open within the particular framework defined here. The algebraic propositions proved above do not, by themselves, resolve the operational or physical questions.

4.13 Conclusion

Tensor products provide the standard mathematical language for composite quantum systems. This paper has introduced an annotated extension in which each quantum state or operator is paired with a bounded neutrosophic triple. A specific aggregation operation was defined, and its closure, associativity, commutativity, and identity properties were established.

The framework was then applied to composite state vectors, inner-product records, column vectors, operators, and matrices. Explicit two-qubit calculations demonstrated how annotations can be combined while the ordinary quantum tensor product remains unchanged. The discussion of entanglement, density operators, partial traces, and measurement identified important limitations of the current construction.

The main contribution is a clearly specified mathematical convention for attaching neutrosophic information to the objects of quantum information theory. It is not claimed that the annotation formalism changes quantum mechanics or has already been experimentally validated. Further research must establish meaningful operational definitions, test the choice of aggregation rule, and determine whether annotations can provide information beyond ordinary quantum-state descriptions.

Chapter 5

Neutrosophic Density Operators

Related literature. An axiomatization of density neutrosophic operators was proposed by Jafari, Rajesh and Nordo [82] (preprint); this proposed notion should be distinguished from the ordinary positive, trace-one quantum density operator.

Density operators provide a unified mathematical description of pure and mixed quantum states. They are indispensable in quantum information theory, particularly when a system is prepared as an ensemble, when information about its preparation is incomplete, or when a subsystem is extracted from a larger composite system. This paper develops a neutrosophic annotation framework for density operators. Each ordinary density operator is paired with a bounded triple representing truth, indeterminacy, and falsity degrees associated with a specified description or preparation record. We distinguish the standard operator-theoretic structure from the additional annotation and define several possible rules for combining annotations in ensembles and under partial trace. The usual characterizations of pure and mixed states, positivity, trace, purity, convex mixtures, reduced states, and the Bloch-vector representation are reviewed and placed in this framework. Explicit examples involving qubits illustrate the calculations. Propositions establish basic properties of the proposed annotation operations, while counterexamples explain why annotations cannot simply be treated as probabilities or inserted into a density matrix without further assumptions. The framework is a mathematical proposal for recording indeterminate information about quantum-state descriptions; it does not modify the Born rule or claim new physical effects. Possible directions for operational definitions and empirical validation are discussed.

5.1 Introduction

A vector in a complex Hilbert space describes a pure quantum state, but it is not always sufficient to describe the information available about a physical preparation. A preparation may produce different states with specified classical probabilities, or a subsystem may be correlated with an environment whose degrees of freedom are not observed. Density operators provide the standard mathematical framework for these situations [1, 2, 3].

For a normalized vector $|\psi\rangle$, the corresponding pure-state density operator is

$$\rho_\psi = |\psi\rangle\langle\psi|.$$

A mixed state is represented by a convex combination

$$\rho = \sum_{j=1}^{m} p_j |\psi_j\rangle\langle\psi_j|, \qquad p_j \geq 0, \quad \sum_{j=1}^{m} p_j = 1.$$

The density-operator formalism makes it possible to calculate measurement probabilities and expectation values without requiring the state to be described by a single vector [2, 3].

Neutrosophic logic introduces truth, indeterminacy, and falsity as distinct informational components [6, 6]. The purpose of this paper is to investigate how such components may be attached to density operators as supplementary information about a state description or preparation.

We emphasize the distinction between two levels:

(i) the ordinary density operator, which obeys the standard conditions of quantum theory and determines the usual quantum probabilities;

(ii) an auxiliary neutrosophic triple, which records additional information according to a chosen annotation convention.

The second component does not automatically constitute a new quantum state. In particular, the three entries of a neutrosophic triple are not assumed to sum to one and are not identified with outcome probabilities.

The main objectives are:

- to define neutrosophically annotated density operators;
- to describe pure and mixed states in the annotated framework;
- to establish the standard density-operator properties that remain unchanged;
- to introduce annotation rules for ensembles and reduced states;
- to examine qubit density operators through the Bloch-vector representation;
- to identify mathematical and operational questions that must be addressed before assigning a physical interpretation to the annotations.

The paper is an original development of these ideas, rather than a reproduction of any textbook chapter. The standard theory of density operators is used as the mathematical foundation [1, 2, 3].

5.2 Preliminaries on density operators

5.2.1 The space of density operators

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. Write $\mathcal{L}(\mathcal{H})$ for the space of linear operators on $\mathcal{H}$.

Definition 5.2.1 (Density operator)**.** An operator $\rho \in \mathcal{L}(\mathcal{H})$ is a density operator if

$$\rho = \rho^*, \qquad \rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1.$$

Here ρ^* denotes the adjoint, and $\rho \geq 0$ means

$$\langle x, \rho x \rangle \geq 0 \quad \text{for every } x \in \mathcal{H}.$$

We denote the set of density operators on $\mathcal{H}$ by $\mathcal{D}(\mathcal{H})$.

The three conditions have distinct roles. Hermiticity ensures that the eigenvalues are real, positivity ensures that they are nonnegative, and unit trace supplies the normalization used in the probability rules of quantum mechanics [2, 3].

Proposition 5.2.2. *If $\rho \in \mathcal{D}(\mathcal{H})$ and*

$$\rho = \sum_{j=1}^{d} \lambda_j |e_j\rangle\langle e_j|$$

is a spectral decomposition in an orthonormal eigenbasis, then

$$\lambda_j \geq 0, \qquad \sum_{j=1}^{d} \lambda_j = 1.$$

Proof. Since ρ is positive, all its eigenvalues are nonnegative. The trace is the sum of eigenvalues, so

$$1 = \mathrm{Tr}(\rho) = \sum_{j=1}^{d} \lambda_j.$$

□

5.2.2 Pure states

Definition 5.2.3 (Pure-state density operator)**.** For a unit vector $|\psi\rangle \in \mathcal{H}$, the operator

$$\rho_\psi = |\psi\rangle\langle\psi|$$

is the density operator of the pure state $|\psi\rangle$.

The operator ρ_ψ is a rank-one orthogonal projection. Indeed,

$$\rho_\psi^* = \rho_\psi$$

and

$$\rho_\psi^2 = |\psi\rangle\langle\psi \mid \psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho_\psi.$$

Moreover,

$$\mathrm{Tr}(\rho_\psi) = \langle\psi \mid \psi\rangle = 1.$$

Proposition 5.2.4. *A density operator ρ represents a pure state if and only if*

$$\rho^2 = \rho.$$

Equivalently, a density operator is pure if and only if

$$\mathrm{Tr}(\rho^2) = 1.$$

Proof. If $\rho = |\psi\rangle\langle\psi|$ for a unit vector, then $\rho^2 = \rho$ and

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}(\rho) = 1.$$

Conversely, suppose ρ is a density operator and $\rho^2 = \rho$. Its eigenvalues satisfy $\lambda_j^2 = \lambda_j$, so each eigenvalue is either 0 or 1. Since their sum is 1, exactly one eigenvalue is 1 and all others are 0. Thus ρ has rank one and is a pure-state density operator.

For the trace characterization, use

$$\mathrm{Tr}(\rho^2) = \sum_j \lambda_j^2, \qquad \sum_j \lambda_j = 1, \qquad \lambda_j \geq 0.$$

The sum of squares equals 1 precisely when one eigenvalue is 1 and all remaining eigenvalues vanish. □

5.2.3 Mixed states

Definition 5.2.5 (Mixed state)**.** A density operator is called mixed if it is not a rank-one projection.

A mixed state can arise from a preparation in which different pure states occur with classical probabilities:

$$\rho = \sum_{j=1}^{m} p_j \rho_j, \qquad \rho_j = |\psi_j\rangle\langle\psi_j|,$$

where

$$p_j \geq 0, \qquad \sum_{j=1}^{m} p_j = 1.$$

This is a convex combination of density operators and is again a density operator [2, 3].

Proposition 5.2.6. *If $\rho_j \in \mathcal{D}(\mathcal{H})$ and $p_j \geq 0$ with $\sum_j p_j = 1$, then*

$$\rho = \sum_j p_j \rho_j$$

belongs to $\mathcal{D}(\mathcal{H})$.

Proof. Hermiticity follows from

$$\rho^* = \sum_j p_j \rho_j^* = \sum_j p_j \rho_j = \rho.$$

For every $x \in \mathcal{H}$,

$$\langle x, \rho x\rangle = \sum_j p_j \langle x, \rho_j x\rangle \geq 0.$$

Finally,

$$\operatorname{Tr}(\rho) = \sum_j p_j \operatorname{Tr}(\rho_j) = \sum_j p_j = 1.$$

Thus ρ is a density operator. □

5.3 Bounded neutrosophic annotations

5.3.1 Definition and interpretation

Definition 5.3.1 (Bounded neutrosophic triple)**.** A bounded neutrosophic triple is an ordered triple

$$\boldsymbol{\nu} = (t, i, f) \in [0, 1]^3.$$

The entries t, i, and f are interpreted as truth, indeterminacy, and falsity degrees associated with a specified description, data record, or preparation claim.

We use the notation

$$\mathcal{T}(\boldsymbol{\nu}) = t, \qquad \mathcal{I}(\boldsymbol{\nu}) = i, \qquad \mathcal{F}(\boldsymbol{\nu}) = f.$$

Remark 5.3.2. The triple is a supplementary information record. Its entries need not sum to one:

$$t + i + f \neq 1$$

is allowed. The triple therefore differs from a probability vector. Its interpretation depends on the application and must be stated explicitly.

Definition 5.3.3 (Neutrosophic aggregation)**.** For

$$\boldsymbol{\nu}_1 = (t_1, i_1, f_1), \qquad \boldsymbol{\nu}_2 = (t_2, i_2, f_2),$$

define

$$\boxed{\boldsymbol{\nu}_1 \odot \boldsymbol{\nu}_2 = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)\,.}$$

This aggregation is one possible modelling choice. It combines truth degrees multiplicatively and combines the other two degrees by the operation $x + y - xy$.

Proposition 5.3.4. *The operation $\odot$ is associative and commutative on $[0,1]^3$, and its identity element is $(1,0,0)$.*

Proof. The truth coordinate is ordinary multiplication, which is associative and commutative. For the other coordinates, define

$$s(x,y) = x + y - xy = 1 - (1-x)(1-y).$$

Then

$$s(s(x,y),z) = 1 - (1-x)(1-y)(1-z) = s(x,s(y,z)).$$

The operation is also symmetric in x and y. Hence $\odot$ is associative and commutative coordinatewise. Finally,

$$(t,i,f) \odot (1,0,0) = (t,i,f).$$

□

5.3.2 Neutrosophically annotated density operators

Definition 5.3.5 (Neutrosophic density operator)**.** A neutrosophic density operator is a pair

$$\boxed{\mathsf{ND} = (\rho, \boldsymbol{\nu}),}$$

where $\rho \in \mathcal{D}(\mathcal{H})$ is an ordinary density operator and $\boldsymbol{\nu} \in [0,1]^3$ is its neutrosophic annotation.

The density-operator component is called the *quantum component*; the triple is called the *annotation component.*

Remark 5.3.6. The pair $(\rho, \boldsymbol{\nu})$ does not define a new density matrix on $\mathcal{H}$. The operator ρ continues to satisfy all standard density-operator conditions. The triple records additional information without changing the matrix entries of ρ.

Example 5.3.7. Let

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \boldsymbol{\nu} = (0.9, 0.2, 0.1).$$

Then

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, (0.9, 0.2, 0.1) \right)$$

is a neutrosophically annotated density operator. Its quantum component is the pure state $|0\rangle\langle 0|$, while the triple is metadata associated with the state description.

5.4 Pure states with neutrosophic annotations

5.4.1 Annotated pure-state operators

For a unit vector $|\psi\rangle$, define

$$\mathsf{ND}_\psi = (|\psi\rangle\langle\psi|, \boldsymbol{\nu}_\psi)\,.$$

The annotation may record information about the preparation procedure, confidence in a state-identification procedure, or indeterminacy in a data record. It does not alter the projector $|\psi\rangle\langle\psi|$.

Proposition 5.4.1. *The quantum component of* ND_ψ *is a rank-one projection, independently of the value of* $\boldsymbol{\nu}_\psi$.

Proof. The result follows from

$$(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi \mid \psi\rangle\langle\psi| = |\psi\rangle\langle\psi|,$$

since $\langle\psi \mid \psi\rangle = 1$. The annotation is not involved in this calculation. □

5.4.2 Example: a superposition state

Consider the qubit

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

Its density matrix is

$$\rho_\psi = \begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix}.$$

For a selected annotation

$$\boldsymbol{\nu}_\psi = (t, i, f),$$

the annotated state is

$$\mathsf{ND}_\psi = \left(\begin{pmatrix} |\alpha|^2 & \alpha\overline{\beta} \\ \overline{\alpha}\beta & |\beta|^2 \end{pmatrix}, (t, i, f) \right).$$

The usual Born probabilities in the computational basis are $|\alpha|^2$ and $|\beta|^2$. They are not replaced by t and f.

5.5 Mixed states and annotated ensembles

5.5.1 An ensemble of annotated preparations

Suppose an ensemble has pure-state preparations

$$\rho_j = |\psi_j\rangle\langle\psi_j|, \qquad j = 1, \ldots, m,$$

with probabilities p_j. The ordinary density operator is

$$\rho = \sum_{j=1}^{m} p_j \rho_j.$$

Assume each preparation also has an annotation

$$\boldsymbol{\nu}_j = (t_j, i_j, f_j).$$

A natural way to summarize the ensemble's annotation is a convex average of the triples.

Definition 5.5.1 (Ensemble-average annotation)**.** For probabilities $p_j \geq 0$ with $\sum_j p_j = 1$, define

$$\boxed{\overline{\boldsymbol{\nu}} = \sum_{j=1}^{m} p_j \boldsymbol{\nu}_j = \left(\sum_j p_j t_j,\ \sum_j p_j i_j,\ \sum_j p_j f_j \right).}$$

The corresponding annotated ensemble is

$$\left(\sum_j p_j \rho_j,\ \sum_j p_j \boldsymbol{\nu}_j \right).$$

Proposition 5.5.2. *The ensemble-average annotation belongs to* $[0,1]^3$*.*

Proof. Each coordinate is a convex combination of numbers in $[0,1]$. Therefore each coordinate also lies in $[0,1]$. □

Remark 5.5.3. The average annotation is not the only possible aggregation rule. It is useful when the triples are numerical scores attached to individual preparations and the p_j are meaningful ensemble weights. The rule $\odot$ defined earlier has a different purpose and should not be substituted for a convex average without an interpretation that justifies doing so.

5.5.2 Example: a two-state mixture

Consider the qubit ensemble

$$\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|.$$

Since

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $\rho = I_2/2$, the maximally mixed qubit state.

Suppose the two preparation records carry annotations

$$\boldsymbol{\nu}_0 = (0.9, 0.1, 0.05), \qquad \boldsymbol{\nu}_1 = (0.7, 0.3, 0.2).$$

The ensemble-average annotation is

$$\begin{aligned} \overline{\boldsymbol{\nu}} &= \frac{1}{2}(0.9, 0.1, 0.05) + \frac{1}{2}(0.7, 0.3, 0.2) \\ &= (0.8, 0.2, 0.125). \end{aligned}$$

The annotated ensemble is therefore

$$\left(\frac{1}{2} I_2,\ (0.8, 0.2, 0.125) \right).$$

The maximally mixed density matrix is determined by the quantum ensemble; the triple summarizes the associated preparation records.

5.6 Key properties and characterizations

5.6.1 Trace, positivity, and normalization

Theorem 5.6.1. *Let $(\rho, \boldsymbol{\nu})$ be a neutrosophically annotated density operator. Then*

$$\rho^* = \rho, \qquad \rho \geq 0, \qquad \operatorname{Tr}(\rho) = 1.$$

These properties are independent of the annotation.

Proof. They follow directly from the definition of a neutrosophically annotated density operator: its quantum component belongs to $\mathcal{D}(\mathcal{H})$. □

This observation is elementary but important. An arbitrary triple cannot correct a nonpositive matrix or normalize an operator whose trace is not one. The quantum component must itself satisfy the standard conditions [2, 3].

5.6.2 Purity

The purity of a density operator is

$$\gamma(\rho) = \operatorname{Tr}(\rho^2).$$

If ρ has eigenvalues $\lambda_1, \ldots, \lambda_d$, then

$$\gamma(\rho) = \sum_{j=1}^{d} \lambda_j^2.$$

Since $\lambda_j \geq 0$ and $\sum_j \lambda_j = 1$,

$$\frac{1}{d} \leq \operatorname{Tr}(\rho^2) \leq 1.$$

The lower bound follows from the Cauchy–Schwarz inequality, and the upper bound follows from

$$\sum_j \lambda_j^2 \leq \left(\sum_j \lambda_j \right)^2 = 1.$$

For an annotated density operator, define its quantum purity by

$$\gamma_Q(\rho, \boldsymbol{\nu}) = \operatorname{Tr}(\rho^2).$$

This notation makes explicit that purity depends only on the quantum component.

Proposition 5.6.2. *Two annotated density operators with the same density matrix have the same quantum purity, even if their annotations differ.*

Proof. If the quantum components are equal to ρ, then both purity values equal $\operatorname{Tr}(\rho^2)$. The annotation does not occur in the definition. □

5.6.3 A limitation of annotation-based purity

One might attempt to define a new purity by combining $\operatorname{Tr}(\rho^2)$ with the truth, indeterminacy, and falsity coordinates. Such a formula would be an additional definition, not a consequence of density-operator theory.

For example, the expression

$$\operatorname{Tr}(\rho^2)t$$

lies in $[0, 1]$ but does not automatically have the properties of quantum purity. Its value depends on the chosen annotation and may change when only the metadata changes. Any proposed new measure would require an explicit operational interpretation and a proof of the properties it is intended to satisfy.

5.7 Characterizing mixed states

5.7.1 Spectral characterization

Every finite-dimensional density operator has a spectral decomposition

$$\rho = \sum_{j=1}^{d} \lambda_j |e_j\rangle\langle e_j|,$$

where $\{e_j\}$ is an orthonormal basis and

$$\lambda_j \geq 0, \qquad \sum_j \lambda_j = 1.$$

The rank of ρ is the number of nonzero eigenvalues. A density operator is pure precisely when its rank is one [2, 3].

For the annotated pair $(\rho, \boldsymbol{\nu})$, the same characterization applies to ρ. The annotation can be used to record additional information about the spectral decomposition, but it does not change the eigenvalues or rank.

5.7.2 Different ensembles can give the same density operator

A density operator need not uniquely specify the ensemble used to prepare it. For example,

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}I_2.$$

The same matrix can also be written as

$$\frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| = \frac{1}{2}I_2,$$

where

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

These are different ensemble descriptions of the same density operator. The nonuniqueness of ensemble decompositions is a standard feature of the density-operator formalism.

Remark 5.7.1. Annotations attached to individual ensemble members may distinguish the two preparation records, but the density operator alone does not select a unique annotation for an ensemble. A convention is needed for deciding whether annotations are attached to individual preparations, to a chosen decomposition, or to the resulting operator.

5.8 Partial trace and reduced density operators

5.8.1 The standard partial trace

Let

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

and let ρ_{AB} be a density operator on the composite system. The reduced density operator on subsystem A is

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}),$$

where Tr_B denotes the partial trace over $\mathcal{H}_B$.

If $\{|b_k\rangle\}$ is an orthonormal basis for $\mathcal{H}_B$, then

$$\mathrm{Tr}_B(\rho_{AB}) = \sum_k (I_A \otimes \langle b_k|)\rho_{AB}(I_A \otimes |b_k\rangle).$$

The resulting operator is a density operator on $\mathcal{H}_A$ [2, 3].

Proposition 5.8.1. *If ρ_{AB} is a density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$, then*

$$\rho_A = \mathrm{Tr}_B(\rho_{AB})$$

is a density operator on $\mathcal{H}_A$.

Proof. The partial trace preserves Hermiticity. For positivity, let $|x\rangle \in \mathcal{H}_A$. Then

$$\begin{aligned}\langle x, \rho_A x\rangle &= \sum_k \langle x \otimes b_k, \rho_{AB}(x \otimes b_k)\rangle \\ &\geq 0,\end{aligned}$$

because ρ_{AB} is positive. Finally, the partial trace preserves the total trace:

$$\mathrm{Tr}(\rho_A) = \mathrm{Tr}(\rho_{AB}) = 1.$$

Thus ρ_A is a density operator. □

5.8.2 An annotation rule for reduction

The standard partial trace determines the reduced quantum state, but it does not determine a neutrosophic triple. We therefore introduce an explicit convention.

Definition 5.8.2 (Annotation-preserving reduction)**.** Let $(\rho_{AB}, \boldsymbol{\nu}_{AB})$ be an annotated density operator on a composite system. Define its annotation-preserving reduction to subsystem A by

$$\boxed{\mathcal{R}_A(\rho_{AB}, \boldsymbol{\nu}_{AB}) = (\mathrm{Tr}_B(\rho_{AB}), \boldsymbol{\nu}_{AB})\,.}$$

This rule retains the annotation unchanged. It is a simple convention for carrying metadata through reduction; it is not forced by quantum mechanics.

Proposition 5.8.3. *Under annotation-preserving reduction, the reduced quantum component is a density operator, and the annotation remains in $[0,1]^3$.*

Proof. The first assertion follows from the partial-trace proposition. The second follows because the original annotation belongs to $[0,1]^3$ and the rule leaves it unchanged. □

5.8.3 An alternative rule for local annotations

In some applications, the composite annotation may be constructed from local annotations:

$$\boldsymbol{\nu}_{AB} = \boldsymbol{\nu}_A \odot \boldsymbol{\nu}_B.$$

If the local annotation $\boldsymbol{\nu}_A$ is available separately, a different reduction rule may retain it:

$$(\rho_{AB}, \boldsymbol{\nu}_A, \boldsymbol{\nu}_B) \longmapsto (\mathrm{Tr}_B(\rho_{AB}), \boldsymbol{\nu}_A).$$

This rule requires the local records to be stored as separate data. In general, one cannot uniquely recover both original triples from their aggregate.

Remark 5.8.4. A useful research problem is to determine conditions under which an annotation of a composite system can be consistently reduced to annotations of its subsystems. Such a problem concerns the annotation model, not the existence of the ordinary reduced density operator.

5.9 The density operator and the Bloch vector

5.9.1 Bloch representation of a qubit

Every qubit density operator can be written as

$$\boxed{\rho = \frac{1}{2}\left(I_2 + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right),}$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The vector

$$\mathbf{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$$

is called the Bloch vector. Positivity of ρ is equivalent to

$$|\mathbf{r}| \leq 1.$$

Pure states satisfy $|\mathbf{r}| = 1$, whereas mixed states satisfy $|\mathbf{r}| < 1$ [2, 3].

5.9.2 Purity in Bloch coordinates

Using the Pauli identities

$$\sigma_j^2 = I_2, \qquad \mathrm{Tr}(\sigma_j) = 0, \qquad \mathrm{Tr}(\sigma_j\sigma_k) = 2\delta_{jk},$$

we obtain

$$\mathrm{Tr}(\rho^2) = \frac{1}{2}(1 + |\mathbf{r}|^2).$$

Proposition 5.9.1. *For a qubit density operator,*

$$\mathrm{Tr}(\rho^2) = \frac{1}{2}(1 + |\mathbf{r}|^2).$$

Consequently, ρ is pure if and only if $|\mathbf{r}| = 1$.

Proof. Let $\mathbf{r} \cdot \boldsymbol{\sigma} = \sum_j r_j\sigma_j$. Then

$$\rho^2 = \frac{1}{4}\left(I_2 + 2\mathbf{r}\cdot\boldsymbol{\sigma} + (\mathbf{r}\cdot\boldsymbol{\sigma})^2\right).$$

The Pauli multiplication rules give

$$(\mathbf{r}\cdot\boldsymbol{\sigma})^2 = |\mathbf{r}|^2 I_2.$$

Taking traces and using $\mathrm{Tr}(I_2) = 2$ and $\mathrm{Tr}(\mathbf{r}\cdot\boldsymbol{\sigma}) = 0$ yields

$$\mathrm{Tr}(\rho^2) = \frac{1}{4}\left(2 + 2|\mathbf{r}|^2\right) = \frac{1}{2}(1 + |\mathbf{r}|^2).$$

The purity characterization follows. □

5.9.3 Annotated Bloch representation

Definition 5.9.2 (Neutrosophically annotated Bloch vector)**.** An annotated Bloch representation is a pair

$$(\mathbf{r}, \boldsymbol{\nu}),$$

where $|\mathbf{r}| \leq 1$ and $\boldsymbol{\nu} \in [0,1]^3$. It represents the annotated density operator

$$\left(\frac{1}{2}(I_2 + \mathbf{r} \cdot \boldsymbol{\sigma}), \boldsymbol{\nu}\right).$$

The Bloch vector determines the quantum density matrix, while the triple supplies an additional record.

Example 5.9.3. Consider the maximally mixed state

$$\rho = \frac{1}{2} I_2.$$

Its Bloch vector is $\mathbf{r} = (0,0,0)$, and

$$\mathrm{Tr}(\rho^2) = \frac{1}{2}.$$

An annotated version may be written as

$$\left(\frac{1}{2} I_2,\ (0.8, 0.25, 0.15)\right).$$

The annotation does not change the Bloch vector or the purity.

5.9.4 Expectation values

For a qubit with density operator

$$\rho = \frac{1}{2}(I_2 + \mathbf{r} \cdot \boldsymbol{\sigma}),$$

the expectation value of an observable

$$A = a_0 I_2 + \mathbf{a} \cdot \boldsymbol{\sigma}$$

is

$$\langle A \rangle_\rho = \mathrm{Tr}(\rho A) = a_0 + \mathbf{r} \cdot \mathbf{a}.$$

This is a standard quantum-mechanical expectation value. An annotation does not alter it unless a separate model explicitly defines a modified statistical interpretation.

For example,

$$\mathrm{Tr}(\rho \sigma_z) = r_z.$$

The value r_z is determined by the quantum density matrix, not by the truth coordinate of the annotation.

5.10 Worked examples

5.10.1 Pure state

Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle).$$

Then

$$\rho_\psi = |\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

A direct computation gives

$$\rho_\psi^2 = \rho_\psi, \qquad \mathrm{Tr}(\rho_\psi) = 1, \qquad \mathrm{Tr}(\rho_\psi^2) = 1.$$

The corresponding Bloch vector is

$$\mathbf{r} = (1, 0, 0).$$

If the annotation is $(0.9, 0.15, 0.05)$, the full record is

$$\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, (0.9, 0.15, 0.05)\right).$$

5.10.2 Mixed state

Consider

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

Then

$$\mathrm{Tr}(\rho) = 1, \qquad \mathrm{Tr}(\rho^2) = \frac{9}{16} + \frac{1}{16} = \frac{5}{8}.$$

The Bloch vector is

$$\mathbf{r} = (0, 0, 1/2),$$

and indeed

$$\frac{1}{2}(1 + |\mathbf{r}|^2) = \frac{1}{2}\left(1 + \frac{1}{4}\right) = \frac{5}{8}.$$

An annotation such as $(0.75, 0.20, 0.10)$ can be attached without changing the density matrix or its purity.

5.10.3 A reduced state from an entangled pure state

Consider the two-qubit Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Its density operator is

$$\rho_{AB} = |\Phi^+\rangle\langle\Phi^+|.$$

Expanding,

$$\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|).$$

Taking the partial trace over the second qubit gives

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \frac{1}{2} I_2.$$

The global state is pure, but the reduced state is mixed:

$$\mathrm{Tr}(\rho_{AB}^2) = 1, \qquad \mathrm{Tr}(\rho_A^2) = \frac{1}{2}.$$

This is a standard feature of entangled states.

If the global state is assigned annotation

$$\boldsymbol{\nu}_{AB} = (0.85, 0.20, 0.10),$$

annotation-preserving reduction gives

$$\left(\frac{1}{2} I_2,\ (0.85, 0.20, 0.10) \right).$$

This last step follows from the proposed annotation convention, whereas the reduced density matrix follows from standard quantum mechanics.

5.11 Limitations and consistency requirements

5.11.1 Annotations are not probabilities

The triple (t, i, f) is not a probability distribution in the usual sense. The sum $t + i + f$ need not equal one, and the components do not automatically correspond to mutually exclusive outcomes. Thus it would be unjustified to replace the probabilities

$$p_k = \mathrm{Tr}(\rho P_k)$$

of a projective measurement by t, i, and f without an additional definition.

5.11.2 Annotations do not repair invalid density operators

Consider

$$A = \begin{pmatrix} 1.2 & 0 \\ 0 & -0.2 \end{pmatrix}.$$

Although $\mathrm{Tr}(A) = 1$, the matrix is not positive because it has a negative eigenvalue. Therefore A is not a density operator. Pairing it with a triple does not make it positive:

$$(A, (0.8, 0.1, 0.1))$$

is not a neutrosophically annotated density operator under our definition.

5.11.3 Nonuniqueness of annotation rules

The ensemble-average operation and the operation $\odot$ have different algebraic purposes. For example,

$$(0.8, 0.2, 0.1) \odot (0.6, 0.3, 0.2) = (0.48, 0.44, 0.28),$$

whereas the equal-weight average is

$$\frac{1}{2}(0.8, 0.2, 0.1) + \frac{1}{2}(0.6, 0.3, 0.2) = (0.7, 0.25, 0.15).$$

Both are well-defined triples, but they encode different rules. The intended meaning of the data must determine which rule, if either, is appropriate.

5.11.4 Operational interpretation

To interpret the annotation physically or experimentally, one would need to specify at least:

(i) the empirical or informational meaning of each component;

(ii) a method for estimating the components from observations;

(iii) a rule for updating annotations after a quantum operation;

(iv) a rule for annotations under partial trace and changes of subsystem;

(v) consistency conditions under equivalent ensemble decompositions of the same density operator.

Without such definitions, the annotation is best treated as metadata rather than as a new physical degree of freedom.

5.12 Further research problems

The framework suggests several questions for future investigation.

Problem 1. Operational semantics. Define truth, indeterminacy, and falsity degrees using specified quantum-state estimation procedures or experimentally recorded data.

Problem 2. Representation independence. Determine when an annotation assigned to a mixed state is independent of the particular ensemble decomposition used to represent it.

Problem 3. Quantum channels. Develop annotation update rules for completely positive trace-preserving maps and test their consistency under composition.

Problem 4. Partial-trace compatibility. Investigate conditions under which the annotation of a composite system can be reduced to local annotations without losing essential information.

Problem 5. Alternative purity measures. If a new annotated purity measure is proposed, characterize its range, invariance properties, and relationship to standard purity.

Problem 6. Qubit and higher-dimensional geometry. Extend the annotated Bloch representation to higher-dimensional quantum systems, where the geometry of density operators is more complicated than the Bloch ball of a qubit.

These are research problems for the proposed annotation model. The standard density-operator results established earlier do not depend on resolving them.

5.13 Conclusion

This paper has presented a neutrosophic annotation framework for density operators. An annotated density operator consists of an ordinary quantum density matrix and a bounded triple recording truth, indeterminacy, and falsity degrees associated with a specified description or preparation record.

The standard theory of pure states, mixed states, convex ensembles, positivity, trace, purity, partial trace, reduced density operators, and the Bloch representation remains unchanged. We introduced an ensemble-average annotation rule and an annotation-preserving reduction rule, while emphasizing that these are modelling conventions rather than consequences of quantum mechanics.

The examples demonstrate that annotations can accompany quantum states without changing their probabilities, eigenvalues, purity, or reduced density matrices. They also show why the annotation must not be confused with a density operator or an ordinary probability distribution.

The framework therefore provides a mathematically explicit starting point for studying supplementary neutrosophic information in quantum state descriptions. Further work is needed to develop operational definitions, test consistency under equivalent ensemble representations, and establish whether the annotations provide useful information in concrete applications.

Chapter 6

Neutrosophic Quantum Measurement

Related literature. Neutrosophic descriptions of measurement and the observer effect have been proposed in [73, 74], while the assignment of uncertainty descriptors to quantum channels is addressed in [79] (preprint).

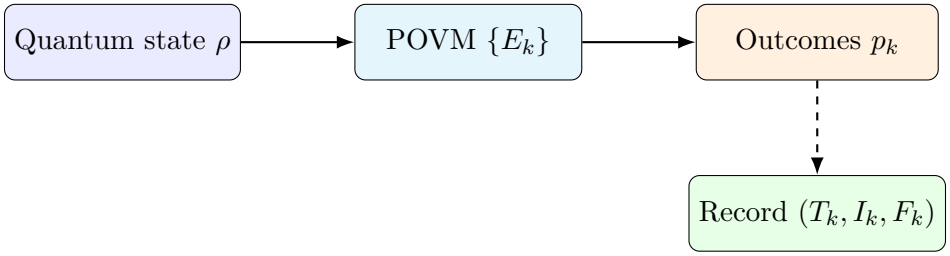


Figure 6.1: Measurement probabilities remain governed by $p_k = \mathrm{Tr}(\rho E_k)$; an optional neutrosophic record is added at the interpretation layer.

Quantum measurement theory specifies the possible outcomes of a measurement, their probabilities, and, when an appropriate measurement instrument is given, the post-measurement state. Projective measurements and positive operator-valued measures (POVMs) provide two central mathematical descriptions of quantum measurements. This paper develops an original neutrosophic annotation framework for recording additional information about measurement outcomes. Each outcome is assigned a bounded triple representing truth, indeterminacy, and falsity degrees, while the ordinary quantum probability is calculated by the standard Born rule. We formulate annotated projective measurements, state discrimination, local measurements on composite systems, generalized measurements, and measurement instruments. Basic propositions establish that the annotation records remain bounded under specified aggregation rules. Worked examples include a computational-basis measurement, a measurement of a superposition state, discrimination of two nonorthogonal states, and a two-qubit local measurement. The paper also distinguishes outcome uncertainty, classical ignorance, quantum indeterminacy in an informational description, and the disturbance associated with measurement. The proposed annotations do not modify quantum probabilities or constitute a new physical measurement law. Their interpretation and usefulness depend on additional operational definitions and empirical validation.

6.1 Introduction

Measurement is the interface between a quantum system and the classical information recorded by an experiment. Quantum theory predicts the possible outcomes of a measurement and their

probabilities. In general, the state following a measurement also depends on the measurement procedure, not merely on the list of outcome probabilities [1, 2, 3].

For a projective measurement, the outcomes are associated with orthogonal projections. More general measurements are described by positive operator-valued measures (POVMs). POVMs are useful when the measurement outcomes cannot be represented by a single orthogonal projective measurement on the original system [11, 12, 2].

Neutrosophic logic distinguishes truth, indeterminacy, and falsity degrees in an informational description [6, 6]. This suggests a supplementary way of recording information about measurement outcomes, especially when the available experimental record is incomplete or when a classification procedure is itself uncertain.

The purpose of this paper is to formulate such a supplementary annotation without changing the standard quantum measurement formalism. The quantum probability of an outcome remains determined by the density operator and the measurement effect. The neutrosophic triple is stored as additional information and is not automatically interpreted as a probability.

The principal objectives are:

(i) to define neutrosophically annotated measurement outcomes;

(ii) to formulate annotation records for projective measurements;

(iii) to examine quantum-state discrimination and nonorthogonal states;

(iv) to extend the annotation framework to composite systems;

(v) to describe POVMs and measurement instruments;

(vi) to identify the consistency requirements for a meaningful operational interpretation.

This manuscript is an independent treatment of these topics, using the general subject of quantum measurement theory as its point of departure. It does not reproduce the exposition or worked examples of McMahon's textbook [1].

6.2 Quantum states and measurement preliminaries

6.2.1 Density operators

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. A density operator is an operator ρ satisfying

$$\rho = \rho^*, \qquad \rho \geq 0, \qquad \operatorname{Tr}(\rho) = 1.$$

We write $\mathcal{D}(\mathcal{H})$ for the set of density operators on $\mathcal{H}$.

A pure state represented by a unit vector $|\psi\rangle$ has density operator

$$\rho_\psi = |\psi\rangle\langle\psi|.$$

Mixed states are represented by density operators that need not have rank one. Density operators provide the standard description of quantum states used in measurement theory [2, 3].

6.2.2 The Born rule

For a projective measurement with orthogonal projections $P_1, \ldots, P_m$ satisfying

$$P_j = P_j^*, \qquad P_j^2 = P_j, \qquad P_j P_k = 0 \quad (j \neq k), \qquad \sum_{j=1}^{m} P_j = I,$$

the probability of outcome j in state ρ is

$$\boxed{p_j = \mathrm{Tr}(\rho P_j).}$$

The probabilities satisfy

$$p_j \geq 0, \qquad \sum_{j=1}^{m} p_j = 1.$$

The Born rule is the standard probability rule of quantum measurement theory. The annotation framework developed below leaves this rule unchanged.

6.2.3 Outcome probabilities and informational records

It is useful to distinguish three objects:

- the quantum state ρ;
- the measurement, described by projections or more general measurement effects;
- the observed outcome and its associated data record.

The first two determine the standard outcome probabilities. The third may include additional information about calibration, classification, missing data, or the reliability of a record. Neutrosophic triples will be attached to this third layer.

6.3 Bounded neutrosophic annotations

6.3.1 Definition

Definition 6.3.1 (Bounded neutrosophic triple)**.** A bounded neutrosophic triple is an ordered triple

$$\boldsymbol{\nu} = (t, i, f) \in [0, 1]^3.$$

The components t, i, and f represent truth, indeterminacy, and falsity degrees associated with a specified informational claim or record.

We write

$$\mathcal{T}(\boldsymbol{\nu}) = t, \qquad \mathcal{I}(\boldsymbol{\nu}) = i, \qquad \mathcal{F}(\boldsymbol{\nu}) = f.$$

Remark 6.3.2. We do not require

$$t + i + f = 1.$$

The triple is not assumed to be a probability distribution. Its components require an application-specific interpretation.

6.3.2 An aggregation operation

For some applications, it is useful to combine two annotation records by a fixed algebraic operation.

Definition 6.3.3 (Annotation aggregation)**.** For

$$\boldsymbol{\nu}_1 = (t_1, i_1, f_1), \qquad \boldsymbol{\nu}_2 = (t_2, i_2, f_2),$$

define

$$\boxed{\boldsymbol{\nu}_1 \odot \boldsymbol{\nu}_2 = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)\,.}$$

Proposition 6.3.4. *The operation $\odot$ is closed on $[0,1]^3$, associative, and commutative. Its identity is $(1,0,0)$.*

Proof. For $x, y \in [0,1]$,

$$xy \in [0,1], \qquad x + y - xy = 1 - (1-x)(1-y) \in [0,1].$$

Thus the operation is closed.

The truth coordinate is combined by multiplication. For the other coordinates, the operation

$$s(x,y) = 1 - (1-x)(1-y)$$

is associative and commutative because multiplication is associative and commutative. Hence $\odot$ has these properties coordinatewise.

Finally,

$$(t,i,f) \odot (1,0,0) = (t,i,f).$$

□

Remark 6.3.5. This operation is a modelling choice. It does not follow from the Born rule, nor does it provide a universal rule for combining measurement records. Other applications may require a weighted average or a different aggregation rule.

6.4 Neutrosophically annotated measurements

6.4.1 Definition

Definition 6.4.1 (Annotated measurement outcome)**.** An annotated measurement outcome is a pair

$$(j, \boldsymbol{\nu}_j),$$

where j is a possible outcome label and $\boldsymbol{\nu}_j \in [0,1]^3$ is an annotation associated with the record for that outcome.

Definition 6.4.2 (Neutrosophically annotated projective measurement)**.** Let $\{P_j\}_{j=1}^m$ be a projective measurement on $\mathcal{H}$. A neutrosophically annotated projective measurement is the collection

$$\mathfrak{M}_N = \{(P_j, \boldsymbol{\nu}_j) : 1 \le j \le m\},$$

where each $\boldsymbol{\nu}_j$ is a bounded neutrosophic triple.

The corresponding quantum outcome probabilities are

$$p_j = \mathrm{Tr}(\rho P_j).$$

The annotation $\boldsymbol{\nu}_j$ is not a replacement for p_j.

Proposition 6.4.3. *For any density operator ρ and any projective measurement $\{P_j\}_{j=1}^m$, the probabilities*

$$p_j = \mathrm{Tr}(\rho P_j)$$

satisfy $p_j \ge 0$ and $\sum_j p_j = 1$, independently of the annotations assigned to the outcomes.

Proof. Since ρ and P_j are positive,

$$\mathrm{Tr}(\rho P_j) = \mathrm{Tr}(P_j \rho P_j) \ge 0.$$

Using $\sum_j P_j = I$, we obtain

$$\sum_j p_j = \mathrm{Tr}\left(\rho \sum_j P_j\right) = \mathrm{Tr}(\rho I) = 1.$$

The annotation triples do not occur in this calculation. □

6.4.2 A probability-weighted annotation summary

When measurement outcomes occur with probabilities p_j, a useful summary of their annotations is the probability-weighted average.

Definition 6.4.4 (Mean annotation of a measurement)**.** For a state ρ and annotated measurement outcomes $\{(P_j, \boldsymbol{\nu}_j)\}_{j=1}^m$, define

$$\boxed{\overline{\boldsymbol{\nu}}_\rho = \sum_{j=1}^m p_j \boldsymbol{\nu}_j, \qquad p_j = \operatorname{Tr}(\rho P_j).}$$

In coordinates,

$$\overline{\boldsymbol{\nu}}_\rho = \left(\sum_j p_j t_j, \sum_j p_j i_j, \sum_j p_j f_j \right).$$

Proposition 6.4.5. *The mean annotation $\overline{\boldsymbol{\nu}}_\rho$ belongs to $[0,1]^3$.*

Proof. Each coordinate is a convex combination of numbers in $[0,1]$, because $p_j \geq 0$ and $\sum_j p_j = 1$. □

Remark 6.4.6. The mean annotation is a numerical summary of the assigned records. It is not an additional measurement probability and need not sum to one across its three coordinates.

6.5 Distinguishing quantum states

6.5.1 The state-discrimination problem

Suppose a system is prepared in one of two possible states ρ_0 and ρ_1. The experimenter must determine which state was prepared by measuring the system. The preparation probabilities are denoted by π_0 and π_1, with

$$\pi_0 + \pi_1 = 1.$$

A measurement intended to distinguish the two states can be described by a two-outcome POVM $\{E_0, E_1\}$, where

$$E_0 \geq 0, \qquad E_1 \geq 0, \qquad E_0 + E_1 = I.$$

If outcome j is interpreted as the decision that state ρ_j was prepared, the average success probability is

$$P_{\text{succ}} = \pi_0 \operatorname{Tr}(\rho_0 E_0) + \pi_1 \operatorname{Tr}(\rho_1 E_1).$$

The optimization of this probability is a standard problem in quantum detection theory [13, 11, 2].

6.5.2 The Helstrom bound

Theorem 6.5.1 (Helstrom bound for binary state discrimination)**.** *For two density operators ρ_0, ρ_1 with prior probabilities π_0, π_1, the maximum success probability over all binary quantum measurements is*

$$\boxed{P_{\text{succ}}^{\max} = \frac{1}{2}\left(1 + \|\pi_0\rho_0 - \pi_1\rho_1\|_1\right),}$$

where $\|A\|_1 = \operatorname{Tr}\sqrt{A^*A}$ is the trace norm.

Remark 6.5.2. The Helstrom bound is a standard result in quantum detection theory. The neutrosophic annotation introduced here does not change the bound. A separate model would be needed to determine whether an annotation contains useful information for choosing or evaluating a measurement.

6.5.3 Example: two nonorthogonal pure states

Consider the equally likely states

$$|\psi_0\rangle = |0\rangle, \qquad |\psi_1\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

Their overlap is

$$|\langle\psi_0 \mid \psi_1\rangle| = \frac{1}{\sqrt{2}} \neq 0.$$

They are therefore not orthogonal and cannot be distinguished perfectly in a single measurement.

For two equally likely pure states, the optimal success probability is

$$P_{\text{succ}}^{\max} = \frac{1}{2}\left(1 + \sqrt{1 - |\langle\psi_0 \mid \psi_1\rangle|^2}\right).$$

Thus

$$\boxed{P_{\text{succ}}^{\max} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right).}$$

Suppose the two possible decision records are assigned

$$\boldsymbol{\nu}_0 = (0.9, 0.1, 0.05), \qquad \boldsymbol{\nu}_1 = (0.7, 0.25, 0.2).$$

These triples may summarize the reliability or uncertainty of the associated records. They do not make the two quantum states orthogonal and do not change the Helstrom bound.

6.6 Projective measurements

6.6.1 Measurement of an observable

Let A be a Hermitian operator with spectral decomposition

$$A = \sum_{j=1}^{m} a_j P_j,$$

where the a_j are distinct eigenvalues and the P_j are orthogonal spectral projections.

A measurement of A yields the value a_j with probability

$$p_j = \operatorname{Tr}(\rho P_j).$$

The expectation value is

$$\mathbb{E}_\rho[A] = \sum_j a_j p_j = \operatorname{Tr}(\rho A).$$

These formulas are standard in quantum mechanics [1, 2, 3].

An annotated measurement can attach a triple $\boldsymbol{\nu}_j$ to each outcome a_j. The mean annotation is then

$$\overline{\boldsymbol{\nu}}_\rho = \sum_j \operatorname{Tr}(\rho P_j)\boldsymbol{\nu}_j.$$

The observable's expectation value remains $\operatorname{Tr}(\rho A)$.

6.6.2 Post-measurement state

For a projective measurement with outcome projection P_j, the selective post-measurement state, when $p_j > 0$, is

$$\boxed{\rho'_j = \frac{P_j \rho P_j}{\mathrm{Tr}(\rho P_j)}.}$$

If the outcome is not recorded, the post-measurement state is

$$\rho' = \sum_j P_j \rho P_j.$$

This is the usual Lüders measurement rule for projective measurements [14, 11, 2].

Definition 6.6.1 (Annotated Lüders update)**.** Let $(\rho, \boldsymbol{\nu})$ be an annotated density operator and let $\boldsymbol{\mu}_j$ be an annotation assigned to outcome j. For $p_j > 0$, define

$$\mathcal{L}_{j,N}(\rho, \boldsymbol{\nu}) = \left(\frac{P_j \rho P_j}{p_j}, \boldsymbol{\mu}_j \right), \qquad p_j = \mathrm{Tr}(\rho P_j).$$

This rule replaces the incoming annotation by an outcome-specific annotation. It is a proposed data-update convention; the quantum component is the standard selective Lüders update.

Remark 6.6.2. An annotation update need not be determined by the projection operator. In an application, $\boldsymbol{\mu}_j$ might depend on the detector record, calibration, or classification procedure. Such dependencies must be defined explicitly.

6.6.3 Example: computational-basis measurement

Consider

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

The computational-basis measurement uses

$$P_0 = |0\rangle\langle 0|, \qquad P_1 = |1\rangle\langle 1|.$$

The outcome probabilities are

$$p_0 = |\alpha|^2, \qquad p_1 = |\beta|^2.$$

If outcome 0 occurs and $p_0 > 0$, the post-measurement state is $|0\rangle\langle 0|$. If outcome 1 occurs and $p_1 > 0$, it is $|1\rangle\langle 1|$.

Assign the outcome annotations

$$\boldsymbol{\nu}_0 = (0.95, 0.10, 0.02), \qquad \boldsymbol{\nu}_1 = (0.80, 0.25, 0.10).$$

The mean annotation is

$$\begin{aligned} \overline{\boldsymbol{\nu}}_\rho = {} & |\alpha|^2 (0.95, 0.10, 0.02) \\ & + |\beta|^2 (0.80, 0.25, 0.10). \end{aligned}$$

This weighted triple summarizes the outcome annotations according to the Born probabilities.

6.7 Measurements on composite systems

6.7.1 Local measurements

Let

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

Suppose subsystem A is measured with POVM effects $\{E_a\}$ and subsystem B with POVM effects $\{F_b\}$. The joint local measurement has effects

$$E_a \otimes F_b.$$

For a joint density operator ρ_{AB}, the joint probability is

$$\boxed{p(a,b) = \mathrm{Tr}(\rho_{AB}(E_a \otimes F_b)).}$$

This expression applies whether or not the joint state is a product state [2, 3].

6.7.2 Annotated joint outcomes

Definition 6.7.1. An annotated joint measurement is a family

$$\{(E_a \otimes F_b, \boldsymbol{\nu}_{ab})\}_{a,b},$$

where $\boldsymbol{\nu}_{ab} \in [0,1]^3$ is associated with the joint outcome (a,b).

The mean joint annotation is

$$\overline{\boldsymbol{\nu}}_{\rho_{AB}} = \sum_{a,b} p(a,b)\boldsymbol{\nu}_{ab}.$$

Proposition 6.7.2. *If $\{E_a\}$ and $\{F_b\}$ are POVMs, then*

$$p(a,b) = \mathrm{Tr}(\rho_{AB}(E_a \otimes F_b))$$

is a joint probability distribution. Consequently, the mean joint annotation belongs to $[0,1]^3$.

Proof. Each $E_a \otimes F_b$ is positive, so $p(a,b) \geq 0$. Also,

$$\begin{aligned}\sum_{a,b} p(a,b) &= \mathrm{Tr}\left(\rho_{AB}\left(\sum_a E_a\right) \otimes \left(\sum_b F_b\right)\right) \\ &= \mathrm{Tr}(\rho_{AB}(I_A \otimes I_B)) \\ &= 1.\end{aligned}$$

The mean annotation is therefore a convex combination of triples in $[0,1]^3$. □

6.7.3 Example: measuring two qubits

Consider the product state

$$\rho_{AB} = |0\rangle\langle 0| \otimes |1\rangle\langle 1|.$$

Measure both qubits in the computational basis. The only outcome with nonzero probability is $(0,1)$:

$$p(0,1) = 1, \qquad p(0,0) = p(1,0) = p(1,1) = 0.$$

Suppose the joint outcome $(0, 1)$ is assigned

$$\boldsymbol{\nu}_{01} = (0.9, 0.15, 0.05).$$

The mean annotation is

$$\overline{\boldsymbol{\nu}}_{\rho_{AB}} = (0.9, 0.15, 0.05),$$

because the other outcomes have zero probability.

For an entangled state, the joint probabilities need not factor as $p_A(a)p_B(b)$. The same annotation formula applies, but its weights must be the actual joint probabilities.

6.8 Generalized measurements and POVMs

6.8.1 Definition of a POVM

Definition 6.8.1 (Positive operator-valued measure)**.** A finite POVM on a Hilbert space $\mathcal{H}$ is a collection of positive operators $\{E_j\}_{j=1}^{m}$ such that

$$E_j \geq 0, \qquad \sum_{j=1}^{m} E_j = I.$$

For a density operator ρ, the probability of outcome j is

$$\boxed{p_j = \operatorname{Tr}(\rho E_j).}$$

Since each effect is positive and the effects sum to the identity, the probabilities are nonnegative and sum to one [11, 12, 2].

A POVM effect is not necessarily a projection. Consequently, POVMs can describe measurements more general than projective measurements on the original system.

6.8.2 Neutrosophically annotated POVMs

Definition 6.8.2 (Annotated POVM)**.** An annotated POVM is a collection

$$\mathfrak{E}_N = \{(E_j, \boldsymbol{\nu}_j) : j = 1, \ldots, m\},$$

where $\{E_j\}_{j=1}^{m}$ is a POVM and $\boldsymbol{\nu}_j \in [0, 1]^3$.

The probability of outcome j remains

$$p_j = \operatorname{Tr}(\rho E_j),$$

and the mean annotation is

$$\overline{\boldsymbol{\nu}}_\rho = \sum_j \operatorname{Tr}(\rho E_j)\boldsymbol{\nu}_j.$$

Proposition 6.8.3. *For every annotated POVM and every density operator ρ, the mean annotation belongs to $[0, 1]^3$.*

Proof. The Born rule gives $p_j \geq 0$ and $\sum_j p_j = 1$. The mean annotation is a convex combination of bounded triples. □

6.8.3 An unsharp qubit measurement

Let σ_z be the Pauli z operator:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For a parameter $\eta \in [0,1]$, define

$$E_+ = \frac{1}{2}(I + \eta\sigma_z), \qquad E_- = \frac{1}{2}(I - \eta\sigma_z).$$

Their matrices are

$$E_+ = \frac{1}{2}\begin{pmatrix} 1+\eta & 0 \\ 0 & 1-\eta \end{pmatrix}, \qquad E_- = \frac{1}{2}\begin{pmatrix} 1-\eta & 0 \\ 0 & 1+\eta \end{pmatrix}.$$

Both are positive and

$$E_+ + E_- = I.$$

Thus $\{E_+, E_-\}$ is a POVM.

For a qubit density operator

$$\rho = \frac{1}{2}(I + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z),$$

the outcome probabilities are

$$p_+ = \frac{1}{2}(1 + \eta r_z), \qquad p_- = \frac{1}{2}(1 - \eta r_z).$$

When $\eta = 1$, this is the projective measurement of σ_z. When $\eta = 0$, both effects equal $I/2$, so the outcomes carry no information about the state through these effects.

If the outcomes receive annotations $\boldsymbol{\nu}_+$ and $\boldsymbol{\nu}_-$, then

$$\overline{\boldsymbol{\nu}}_\rho = p_+\boldsymbol{\nu}_+ + p_-\boldsymbol{\nu}_-.$$

The annotation summary depends on the state through the standard outcome probabilities, but it does not change those probabilities.

6.9 Measurement operators and state update

6.9.1 Measurement instruments

A POVM determines outcome probabilities but does not, by itself, uniquely specify the post-measurement state. To describe state changes, one may use measurement operators M_j satisfying

$$\sum_j M_j^* M_j = I.$$

The effects are

$$E_j = M_j^* M_j,$$

and the outcome probabilities are

$$p_j = \mathrm{Tr}(M_j \rho M_j^*) = \mathrm{Tr}(\rho E_j).$$

When $p_j > 0$, the conditional post-measurement state is

$$\boxed{\rho_j' = \frac{M_j \rho M_j^*}{p_j}.}$$

This is the standard measurement-operator formalism [2, 3].

More generally, an instrument associates a completely positive operation with each outcome. The total operation obtained by summing over outcomes is trace-preserving when the measurement is complete.

6.9.2 An annotated measurement instrument

Definition 6.9.1. Let $\{M_j\}_{j=1}^m$ be measurement operators and let $\boldsymbol{\mu}_j \in [0,1]^3$ be outcome annotations. For an annotated input $(\rho, \boldsymbol{\nu})$, define the conditional annotated update

$$\boxed{\mathcal{M}_{j,N}(\rho, \boldsymbol{\nu}) = \left(\frac{M_j \rho M_j^*}{p_j}, \boldsymbol{\mu}_j \right),}$$

whenever $p_j = \mathrm{Tr}(M_j \rho M_j^*) > 0$.

This definition separates the standard conditional quantum state from the proposed annotation assigned to the observed outcome.

If no outcome is recorded, the quantum state becomes

$$\rho' = \sum_j M_j \rho M_j^*.$$

An annotation for this unconditioned state must be specified separately. One possible convention is to use the probability-weighted mean

$$\overline{\boldsymbol{\mu}}_\rho = \sum_j p_j \boldsymbol{\mu}_j.$$

This convention summarizes the outcome records but does not replace the quantum operation.

Remark 6.9.2. Different measurement instruments can have the same POVM effects while producing different conditional post-measurement states. Therefore, a neutrosophic update rule based only on the POVM effects cannot in general determine the full post-measurement quantum state.

6.10 Distinguishing quantum probabilities from annotations

The framework requires a careful distinction between probability, uncertainty, and annotation.

6.10.1 Quantum probability

For a state ρ and effect E_j, the outcome probability is

$$p_j = \mathrm{Tr}(\rho E_j).$$

This value is determined by the standard quantum state and measurement effect.

6.10.2 Indeterminacy in an information record

The annotation component i_j can be used to describe an application-specific form of indeterminacy, such as an incomplete classification or an uncertain detector record. It is not automatically equal to p_j, $1 - p_j$, or the variance of an observable.

6.10.3 Falsity and alternative outcomes

The falsity component f_j may describe the degree to which a specified claim about an outcome is rejected under a chosen information model. It should not automatically be identified with the probability that another outcome occurs.

6.10.4 Truth and successful classification

The truth component t_j may be used as a score for the reliability of an outcome label, provided its operational meaning is defined. It is not automatically equal to the probability of that outcome.

Remark 6.10.1. An application may choose to calibrate one of these components against empirical frequencies. Such a calibration would be an additional statistical model, not a consequence of neutrosophic notation alone.

6.11 Limitations and consistency conditions

6.11.1 Dependence on the annotation convention

The same quantum measurement can be equipped with different annotation assignments. Consequently, the annotation is not uniquely determined by the quantum state and the measurement operators.

A mathematical application should specify:

(i) what claim or record each triple annotates;

(ii) how the components are estimated or assigned;

(iii) whether annotations depend on the state, detector, or outcome label;

(iv) how annotations are combined across repeated trials;

(v) how annotations change under post-measurement updates.

6.11.2 No automatic change to the Born rule

Replacing

$$p_j = \mathrm{Tr}(\rho E_j)$$

by one of the components t_j, i_j, or f_j would change the measurement model and would require a separate theory. The framework presented here does not make that replacement.

6.11.3 Consistency under repeated measurements

Repeated measurements may produce correlated outcomes, and a measurement may change the state on which a later measurement acts. The quantum calculation must therefore update the density operator according to the specified measurement instrument.

An annotation update should be coordinated with that quantum update if it is intended to describe a sequence of measurements. For example, an annotation assigned to the second outcome may depend on the first outcome's record, but such dependence must be explicitly defined.

6.11.4 Operational and empirical questions

To establish a useful application, future work should identify datasets or experimental settings in which the annotations have a clear interpretation and provide information not already captured by the ordinary state and measurement record.

6.12 Further research problems

The proposed framework leads to several questions.

Problem 1. Operational definition. Define truth, indeterminacy, and falsity degrees for a specified quantum measurement task, using an explicit procedure for assigning or estimating them.

Problem 2. Calibration. Investigate whether annotations can be calibrated against detector errors, missing outcomes, or uncertainty in outcome classification without confusing them with Born probabilities.

Problem 3. Measurement instruments. Construct annotation update rules compatible with the composition of completely positive operations and the conditioning on observed outcomes.

Problem 4. State discrimination. Determine whether annotated experimental records can improve the selection or evaluation of measurements in a well-defined state-discrimination problem.

Problem 5. Composite measurements. Study how local and joint annotations behave when the underlying quantum state is correlated or entangled.

Problem 6. Repeated measurements. Develop a formalism for sequences of annotated measurement records that preserves the standard quantum probability rules.

Problem 7. Comparison with established methods. Compare the proposed annotations with established statistical methods for detector calibration, Bayesian inference, and quantum state estimation.

These are open directions for the annotation model, rather than claims that the standard theory of quantum measurement is incomplete.

6.13 Conclusion

This paper has developed a neutrosophic annotation framework for quantum measurement theory. The framework attaches bounded triples to measurement outcomes while retaining the ordinary quantum formalism for probabilities and state changes.

Projective measurements, state discrimination, measurements on composite systems, POVMs, and measurement instruments were described in the annotated setting. The mathematical examples show how annotations can be combined by a probability-weighted average while the Born rule remains unchanged. The paper also distinguished outcome probabilities from the informational meanings that might be assigned to truth, indeterminacy, and falsity.

The central contribution is a clearly specified separation between quantum measurement data and supplementary neutrosophic annotations. The proposed definitions do not establish a new physical measurement law. Further work is required to provide operational definitions, prove consistency results for sequential measurements, and assess whether the annotations are useful in concrete quantum-information applications.

Chapter 7

Neutrosophic Entanglement

Related literature. Smarandache's treatment of partial entanglement [74] and principles involving partial locality and indeterminacy [76] provide motivations for annotating entanglement evidence; neither substitutes for standard entanglement criteria.

Entanglement is a fundamental feature of quantum theory and a central resource in quantum information processing. It describes correlations that cannot be explained by a convex mixture of product states and is closely related to Bell inequalities, quantum communication, and quantum computation. This paper develops a mathematical framework for attaching neutrosophic annotations to entanglement-related quantities. An annotation is a triple consisting of a degree of support, a degree of indeterminacy, and a degree of opposition. The standard quantum mechanical formalism is retained: states are represented by density operators, separability is defined through convex combinations of product states, and Bell correlations are evaluated using the usual quantum expectation values. The proposed annotations are supplementary descriptors rather than replacements for quantum probabilities or entanglement measures.

The paper reviews bipartite states, the Bell basis, Schmidt decomposition, partial traces, Bell–CHSH inequalities, concurrence, entanglement fidelity, and purification. It then introduces consistency conditions for annotated entanglement assessments, derives elementary properties of annotation aggregation, and presents worked examples involving Bell states, product states, and partially entangled two-qubit states. Finally, it identifies mathematical and experimental questions that must be resolved before such annotations can be used as a validated scientific methodology. The framework is intended as a precise starting point for investigating uncertainty and incomplete information in the assessment of quantum correlations.

7.1 Introduction

Quantum entanglement is one of the defining characteristics of quantum mechanics. Entangled states exhibit correlations that cannot be represented as classical mixtures of independent subsystem states. The subject originates in foundational discussions of quantum theory, including the Einstein–Podolsky–Rosen argument [18], and was developed into a quantitative framework by Bell's theorem [15]. Subsequent work has established entanglement as a resource for quantum communication, quantum cryptography, and quantum information processing [19, 2].

McMahon's *Quantum Computing Explained* includes a chapter on entanglement that discusses Bell's theorem, bipartite systems, the Bell basis, Schmidt decomposition, and entanglement fidelity [1]. The present paper follows a comparable broad educational progression but does not reproduce that chapter. Instead, it develops an independent mathematical proposal for supplementing standard entanglement analysis with neutrosophic annotations.

Neutrosophic theory studies descriptions involving degrees of truth, indeterminacy, and falsity or opposition. In a basic formulation, an object is associated with a triple

$$(t, i, f) \in [0, 1]^3,$$

where the components represent three distinct assessments [6, 22]. They need not sum to one. The interpretation of these components depends on the application and must be specified carefully.

The main idea here is to associate an annotation

$$\nu(\rho_{AB}) = (t, i, f)$$

with a bipartite quantum state ρ_{AB}. Informally, the components may describe the support for an entanglement assessment, the indeterminacy in that assessment, and the opposition to it. These quantities do not replace the density operator, the Born rule, or established entanglement measures.

The distinction is important. Quantum mechanics already provides precise probability assignments and operationally defined observables. A neutrosophic annotation is useful only if it records additional information, such as uncertainty in a model, conflicting diagnostic criteria, or incomplete experimental evidence. Without an explicit operational interpretation, the triple is a mathematical label rather than a new physical observable.

The objectives of this paper are:

(1) to review the mathematical foundations of bipartite entanglement;

(2) to formulate a consistent annotation space for entanglement-related assessments;

(3) to give examples illustrating how annotations may accompany standard quantum calculations;

(4) to identify limitations and directions for future research.

7.2 Mathematical preliminaries

7.2.1 Quantum states and composite systems

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional complex Hilbert spaces describing two quantum subsystems. Their composite system is described by

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B.$$

A pure state is represented by a normalized vector

$$|\psi\rangle \in \mathcal{H}_{AB}, \qquad \langle\psi \mid \psi\rangle = 1.$$

The corresponding density operator is

$$\rho_{AB} = |\psi\rangle\langle\psi|.$$

More generally, a mixed state is a positive semidefinite operator satisfying

$$\rho_{AB} \geq 0, \qquad \mathrm{Tr}(\rho_{AB}) = 1.$$

Density operators provide a common representation for pure states, statistical mixtures, and reduced states of subsystems [2, 3].

For an observable A, the expectation value is

$$\langle A\rangle_\rho = \mathrm{Tr}(\rho A).$$

If $\{E_j\}$ is a positive operator-valued measurement (POVM), then

$$p_j = \mathrm{Tr}(\rho E_j), \qquad E_j \geq 0, \qquad \sum_j E_j = \mathbb{I},$$

gives the standard quantum probability of outcome j.

All probabilities in this paper retain their ordinary probabilistic meaning.

7.2.2 Product states and separability

Definition 7.2.1 (Product state)**.** A bipartite pure state $|\psi\rangle_{AB}$ is a product state if there exist normalized vectors $|a\rangle \in \mathcal{H}_A$ and $|b\rangle \in \mathcal{H}_B$ such that

$$|\psi\rangle_{AB} = |a\rangle_A \otimes |b\rangle_B.$$

Otherwise, it is entangled.

Definition 7.2.2 (Separable mixed state)**.** A bipartite density operator is separable if it can be written as

$$\rho_{AB} = \sum_{k=1}^{m} p_k \, \rho_A^{(k)} \otimes \rho_B^{(k)},$$

where $p_k \geq 0$, $\sum_k p_k = 1$, and each $\rho_A^{(k)}$ and $\rho_B^{(k)}$ is a density operator. A state that has no such representation is entangled.

This definition distinguishes entanglement from ordinary classical correlation. A separable state can possess nontrivial correlations; what it lacks is entanglement in the sense of nonseparability [24, 19].

Remark 7.2.3. For mixed states, the distinction between separability and entanglement cannot generally be determined by inspecting a single state vector, since a mixed state need not have a unique ensemble decomposition.

7.3 Bipartite qubits and the Bell basis

7.3.1 Two-qubit states

A qubit has computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The computational basis for two qubits is

$$|00\rangle, \quad |01\rangle, \quad |10\rangle, \quad |11\rangle.$$

A general pure two-qubit state has the form

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle,$$

where

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1.$$

A pure state of this form is a product state precisely when

$$ad - bc = 0.$$

Indeed, if the state is a product

$$(\alpha|0\rangle + \beta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle),$$

then

$$a = \alpha\gamma, \quad b = \alpha\delta, \quad c = \beta\gamma, \quad d = \beta\delta,$$

and consequently $ad - bc = 0$. Conversely, a nonzero coefficient matrix of determinant zero has rank one and admits such a factorization.

Thus, for a pure two-qubit state,

$$|\psi\rangle \text{ is entangled} \quad \Longleftrightarrow \quad ad - bc \neq 0.$$

7.3.2 The Bell states

The four Bell states are

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{7.1}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle), \tag{7.2}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \tag{7.3}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \tag{7.4}$$

They form an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$, called the Bell basis [16, 2].

For example, the density operator of $|\Phi^+\rangle$ is

$$\rho_{\Phi^+} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right).$$

The cross terms are essential: they encode coherence between the two components of the superposition.

Tracing out either subsystem gives

$$\rho_A = \mathrm{Tr}_B(\rho_{\Phi^+}) = \frac{\mathbb{I}_2}{2}, \qquad \rho_B = \mathrm{Tr}_A(\rho_{\Phi^+}) = \frac{\mathbb{I}_2}{2}.$$

Although the composite state is pure, each subsystem is maximally mixed. This is a standard signature of entanglement for a pure bipartite state.

7.4 Schmidt decomposition

Theorem 7.4.1 (Schmidt decomposition)**.** *Let $|\psi\rangle$ be a pure state in a finite-dimensional bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Then there are orthonormal families $\{|u_j\rangle\}$ and $\{|v_j\rangle\}$ and nonnegative numbers λ_j such that*

$$|\psi\rangle = \sum_{j=1}^{r} \sqrt{\lambda_j}\, |u_j\rangle_A \otimes |v_j\rangle_B,$$

where

$$\lambda_j > 0, \qquad \sum_{j=1}^{r} \lambda_j = 1.$$

The integer r is the Schmidt rank.

Proof. Choose orthonormal bases in the two Hilbert spaces and write

$$|\psi\rangle = \sum_{j,k} c_{jk} |j\rangle_A |k\rangle_B.$$

The coefficient array $C = (c_{jk})$ is a complex matrix. By singular-value decomposition,

$$C = U \Sigma V^*,$$

where U, V are unitary and Σ is diagonal apart from possible rectangular padding. The nonzero diagonal entries s_j are singular values. Transforming to the corresponding orthonormal bases gives

$$|\psi\rangle = \sum_{j} s_j |u_j\rangle_A |v_j\rangle_B.$$

Normalization implies

$$\sum_{j} s_j^2 = 1.$$

Setting $\lambda_j = s_j^2$ yields the required representation. □

Corollary 7.4.2. *A pure bipartite state is entangled if and only if its Schmidt rank is greater than one.*

Proof. Schmidt rank one gives a product vector. Conversely, a product vector has only one nonzero Schmidt coefficient. □

The reduced density operators are

$$\rho_A = \sum_{j=1}^{r} \lambda_j |u_j\rangle\langle u_j|, \qquad \rho_B = \sum_{j=1}^{r} \lambda_j |v_j\rangle\langle v_j|.$$

They have the same nonzero eigenvalues. This permits pure-state entanglement to be quantified using the entropy of either reduced state [20, 2].

7.4.1 An entropy-based entanglement measure

For a pure bipartite state, define the entanglement entropy by

$$E(|\psi\rangle) = S(\rho_A) = -\operatorname{Tr}(\rho_A \log_2 \rho_A).$$

In terms of Schmidt coefficients,

$$E(|\psi\rangle) = -\sum_{j=1}^{r} \lambda_j \log_2 \lambda_j,$$

with $0 \log_2 0 = 0$.

For a product state, one Schmidt coefficient equals one and the others vanish, so $E = 0$. For a maximally entangled state of two qubits, the Schmidt coefficients are $1/2, 1/2$, giving

$$E(|\Phi^+\rangle) = 1 \text{ bit}.$$

For mixed states, entanglement entropy of a reduced subsystem is not by itself a general entanglement measure, since classical mixing can also make reduced states mixed. Mixed-state entanglement requires additional tools, such as entanglement of formation, negativity, or suitable entanglement witnesses [19].

7.5 Bell's theorem and CHSH correlations

7.5.1 Local hidden-variable inequality

Bell's theorem establishes that quantum predictions cannot in general be reproduced by local hidden-variable models satisfying the relevant assumptions [15]. A widely used test is the Clauser–Horne–Shimony–Holt (CHSH) inequality [17].

Let Alice choose between observables A_0, A_1 and Bob between B_0, B_1, each with outcomes $+1$ or -1. In a local hidden-variable model, the CHSH expression satisfies

$$|S| \leq 2,$$

where

$$S = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle.$$

Quantum mechanics permits values up to $2\sqrt{2}$ for two-qubit systems, the Tsirelson bound [23].

7.5.2 The CHSH operator

For observables A_0, A_1 on subsystem A and B_0, B_1 on subsystem B, define

$$\mathcal{B}_{\mathrm{CHSH}} = A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1).$$

The quantum value is

$$S = \mathrm{Tr}(\rho_{AB} \mathcal{B}_{\mathrm{CHSH}}).$$

For the Bell state $|\Phi^+\rangle$, choose

$$A_0 = \sigma_z, \qquad A_1 = \sigma_x,$$

and

$$B_0 = \frac{\sigma_z + \sigma_x}{\sqrt{2}}, \qquad B_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}},$$

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The relevant correlations of $|\Phi^+\rangle$ yield

$$S = 2\sqrt{2}.$$

Thus the state violates the CHSH inequality for these settings.

Remark 7.5.1. A violation of a Bell inequality is a strong certificate of nonclassical correlations, but not every entangled state violates the CHSH inequality for every measurement choice. In particular, some mixed entangled states admit no CHSH violation under the standard two-setting test.

7.6 Concurrence and two-qubit entanglement

7.6.1 Pure-state concurrence

For a normalized pure two-qubit state

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle,$$

the concurrence is

$$C(|\psi\rangle) = 2|ad - bc|.$$

It satisfies

$$0 \leq C(|\psi\rangle) \leq 1.$$

A product state has concurrence zero, while every Bell state has concurrence one.

For a Schmidt-form state

$$|\psi\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle, \qquad 0 \leq \lambda \leq 1,$$

we obtain

$$C(|\psi\rangle) = 2\sqrt{\lambda(1-\lambda)}.$$

The concurrence is zero at $\lambda = 0, 1$ and reaches one at $\lambda = 1/2$.

7.6.2 Mixed-state concurrence

For a two-qubit density operator ρ, define the spin-flipped operator

$$\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),$$

where complex conjugation is taken in the computational basis and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

Let r_1, r_2, r_3, r_4 be the square roots of the eigenvalues of $\rho\widetilde{\rho}$, arranged in nonincreasing order. Then Wootters' formula is

$$C(\rho) = \max\{0, r_1 - r_2 - r_3 - r_4\}.$$

This gives a computable entanglement measure for arbitrary two-qubit mixed states [25].

The formula is specific to two-qubit systems. It should not be applied without modification to arbitrary dimensions or to multipartite entanglement.

7.7 A neutrosophic annotation framework

7.7.1 Annotation triples

We now introduce the proposed supplementary description. The definition is mathematical and does not alter the standard quantum-mechanical state space.

Definition 7.7.1 (Neutrosophic annotation)**.** A neutrosophic annotation is an ordered triple

$$\nu = (t, i, f) \in [0, 1]^3,$$

where:

- t represents the degree of support for a specified entanglement assessment;
- i represents the degree of indeterminacy associated with that assessment;
- f represents the degree of opposition to that assessment.

The meaning of each coordinate must be fixed for the application under consideration. For example, t could encode the degree of support supplied by an agreed collection of diagnostic criteria, while i could encode unresolved uncertainty in the data.

No identity such as

$$t + i + f = 1$$

is imposed. In particular, the triple is not automatically a probability distribution.

Definition 7.7.2 (Annotated quantum state)**.** An annotated bipartite quantum state is a pair

$$(\rho_{AB}, \nu),$$

where ρ_{AB} is a density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\nu \in [0,1]^3$ is an annotation associated with a stated assessment of ρ_{AB}.

The density operator remains the physical state description in the standard formalism. The annotation stores supplementary information about the assessment, model, or evidence.

7.7.2 Basic consistency conditions

An annotation should be interpreted relative to a declared decision rule. For instance, an assessment might concern whether a state is entangled, whether a measured correlation violates CHSH, or whether a state reconstruction has sufficient quality.

Definition 7.7.3 (Entanglement-assessment annotation)**.** Let $\mathcal{D}$ be a specified diagnostic procedure for assessing entanglement. An entanglement-assessment annotation is a triple

$$\nu_{\mathcal{D}}(\rho) = (t_{\mathcal{D}}(\rho), i_{\mathcal{D}}(\rho), f_{\mathcal{D}}(\rho)) \in [0,1]^3$$

assigned according to a documented rule involving the output of $\mathcal{D}$.

This definition deliberately does not prescribe a universal assignment rule. Different diagnostic procedures may use different evidence, thresholds, and calibration methods.

A mathematically meaningful application should state:

(a) the assessment being annotated;

(b) the data or mathematical quantities used to assign the triple;

(c) how the three coordinates are calibrated;

(d) how annotations are compared or aggregated;

(e) how the assignment is validated.

7.7.3 A simple aggregation operation

For bookkeeping, one may use a coordinatewise operation that combines independent annotation records. Define

$$(t_1, i_1, f_1) \odot (t_2, i_2, f_2) = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)\,.$$

This operation is a proposed mathematical convention, not a law of quantum mechanics.

Proposition 7.7.4. *The operation $\odot$ is closed on $[0,1]^3$, commutative, associative, and has identity $(1,0,0)$.*

Proof. For $x, y \in [0,1]$, both xy and

$$x + y - xy = 1 - (1-x)(1-y)$$

belong to $[0,1]$. Thus the operation is closed.

Commutativity follows from the symmetry of each coordinate formula. Associativity of the first coordinate follows from

$$(t_1 t_2)t_3 = t_1(t_2 t_3).$$

For the second coordinate, observe that

$$1 - (i_1 + i_2 - i_1 i_2) = (1 - i_1)(1 - i_2).$$

Therefore combining three second coordinates gives

$$1 - (1 - i_1)(1 - i_2)(1 - i_3),$$

independently of the grouping. The same argument applies to the third coordinate.

Finally,

$$(t, i, f) \odot (1, 0, 0) = (t, i, f).$$

Hence $(1, 0, 0)$ is an identity. □

Remark 7.7.5. The operation above should only be used when its interpretation is suitable for the application. For example, multiplying support scores may be inappropriate if the evidence sources are dependent. The operation does not combine quantum states and does not replace the tensor product or any quantum channel.

7.7.4 Averaging annotations

Suppose an experiment produces outcomes or records indexed by $j = 1, \ldots, m$, and each record is assigned an annotation

$$\nu_j = (t_j, i_j, f_j).$$

Let $w_j \geq 0$ with $\sum_j w_j = 1$. Define the weighted mean annotation by

$$\overline{\nu} = \sum_{j=1}^{m} w_j \nu_j = \left(\sum_j w_j t_j,\ \sum_j w_j i_j,\ \sum_j w_j f_j \right).$$

Proposition 7.7.6. *The weighted mean annotation belongs to* $[0,1]^3$.

Proof. Each coordinate is a convex combination of numbers in $[0,1]$. Therefore each coordinate also belongs to $[0,1]$. □

The weights might represent relative frequencies of records or another explicitly justified averaging scheme. They must not be confused with quantum probabilities unless the annotation procedure specifically establishes that relationship.

7.8 Annotated entanglement examples

7.8.1 A product state

Consider

$$|\psi_{\text{prod}}\rangle = |0\rangle \otimes |1\rangle = |01\rangle.$$

Its density operator is

$$\rho_{\text{prod}} = |01\rangle\langle 01|.$$

The state is separable and has Schmidt rank one. Consequently,

$$E(|\psi_{\text{prod}}\rangle) = 0, \qquad C(|\psi_{\text{prod}}\rangle) = 0.$$

For illustration, suppose a specified diagnostic procedure assigns

$$\nu_{\text{prod}} = (0.05, 0.10, 0.95).$$

The high third coordinate is intended to record strong opposition to the assessment that the state is entangled. The triple is an illustrative annotation, not a quantity derived from the density operator alone.

7.8.2 A Bell state

Consider

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

As shown earlier, its reduced density operators are maximally mixed. Its Schmidt coefficients are $1/\sqrt{2}, 1/\sqrt{2}$, so

$$E(|\Phi^+\rangle) = 1, \qquad C(|\Phi^+\rangle) = 1.$$

With the CHSH measurement settings described in Section 5,

$$|S| = 2\sqrt{2} > 2.$$

An illustrative annotation might be

$$\nu_{\Phi^+} = (0.98, 0.02, 0.01).$$

This could represent strong support for an entanglement assessment with little reported uncertainty or opposition under a chosen diagnostic protocol. It is not a universal annotation for all experiments preparing a Bell state.

7.8.3 A partially entangled state

Let

$$|\psi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle, \qquad 0 < \lambda < 1.$$

Its reduced state is

$$\rho_A = \lambda|0\rangle\langle 0| + (1-\lambda)|1\rangle\langle 1|.$$

The entanglement entropy is

$$E(|\psi_\lambda\rangle) = -\lambda \log_2 \lambda - (1-\lambda)\log_2(1-\lambda),$$

and its concurrence is

$$C(|\psi_\lambda\rangle) = 2\sqrt{\lambda(1-\lambda)}.$$

For $\lambda = 0.8$,

$$C(|\psi_{0.8}\rangle) = 2\sqrt{0.8 \cdot 0.2} = 0.8,$$

and

$$E(|\psi_{0.8}\rangle) = -0.8\log_2(0.8) - 0.2\log_2(0.2) \approx 0.722.$$

The state is entangled because both Schmidt coefficients are nonzero.

An illustrative annotation such as

$$\nu_{\psi_{0.8}} = (0.80, 0.15, 0.10)$$

may be used to record a chosen assessment, but the numerical coordinates do not follow from concurrence or entropy without an additional calibration rule.

7.8.4 Comparison of the examples

State	Schmidt rank	Concurrence	Entropy (bits)
$\|01\rangle$	1	0	0
$\|\psi_{0.8}\rangle$	2	0.8	≈ 0.722
$\|\Phi^+\rangle$	2	1	1

The table contains standard quantum quantities. An annotation table could be added separately if the assignment procedure were defined and justified. Keeping these two kinds of information separate prevents a supplementary descriptive score from being mistaken for an established entanglement measure.

7.9 Entanglement fidelity

Entanglement fidelity measures how well a quantum operation preserves a state when that state is viewed as part of a larger purified system. It is a standard concept in quantum information theory [21, 2].

Let ρ be a state on a system A, and let $|\Psi\rangle_{RA}$ be a purification, so that

$$\mathrm{Tr}_R(|\Psi\rangle\langle\Psi|_{RA}) = \rho.$$

Let $\mathcal{E}$ be a completely positive trace-preserving quantum channel acting on A. The entanglement fidelity is

$$F_e(\rho, \mathcal{E}) = \langle\Psi|(\mathrm{id}_R \otimes \mathcal{E})(|\Psi\rangle\langle\Psi|)|\Psi\rangle.$$

It satisfies

$$0 \leq F_e(\rho, \mathcal{E}) \leq 1.$$

For the identity channel,

$$F_e(\rho, \mathrm{id}) = 1.$$

If $\{K_\alpha\}$ are Kraus operators for $\mathcal{E}$, then

$$\mathcal{E}(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger, \qquad \sum_\alpha K_\alpha^\dagger K_\alpha = \mathbb{I},$$

and

$$F_e(\rho, \mathcal{E}) = \sum_\alpha |\mathrm{Tr}(\rho K_\alpha)|^2 .$$

7.9.1 A possible annotation of fidelity

One may associate a neutrosophic triple with the assessment of whether a channel preserves entanglement adequately:

$$\nu_F(\rho, \mathcal{E}) = (t_F, i_F, f_F).$$

For example, a research protocol could define a fidelity threshold F_0 and set the support coordinate according to a calibrated rule based on F_e. However, such a rule must specify how uncertainty in the estimated fidelity affects the other coordinates.

A possible deterministic convention is

$$t_F = F_e, \qquad i_F = 0, \qquad f_F = 1 - F_e.$$

This is mathematically well-defined, but it is only a reparameterization of a single scalar. It does not provide additional information beyond F_e. A genuinely useful neutrosophic description would need independent, defensible information for its indeterminacy coordinate, for example an uncertainty estimate from finite-sample reconstruction.

Remark 7.9.1. The previous observation is an important limitation. A triple does not automatically improve a scalar measure. Its usefulness must be demonstrated through a clear operational interpretation, empirical validation, or a mathematical result that cannot be obtained from the original measure alone.

7.10 Purification and the role of an auxiliary system

Theorem 7.10.1 (Purification)**.** *Every finite-dimensional density operator ρ_A admits a pure-state purification on a larger Hilbert space $\mathcal{H}_R \otimes \mathcal{H}_A$.*

Proof. Write the spectral decomposition

$$\rho_A = \sum_{j=1}^{r} \lambda_j |u_j\rangle\langle u_j|,$$

where $\lambda_j > 0$ and $\sum_j \lambda_j = 1$. Choose orthonormal vectors $|j\rangle_R$ in an auxiliary Hilbert space of dimension at least r, and define

$$|\Psi\rangle_{RA} = \sum_{j=1}^{r} \sqrt{\lambda_j}\, |j\rangle_R \otimes |u_j\rangle_A.$$

The vector is normalized because

$$\langle\Psi|\Psi\rangle = \sum_j \lambda_j = 1.$$

Taking the partial trace over R gives

$$\mathrm{Tr}_R(|\Psi\rangle\langle\Psi|_{RA}) = \sum_j \lambda_j |u_j\rangle\langle u_j| = \rho_A.$$

Thus $|\Psi\rangle_{RA}$ is a purification. □

Purification reveals that a mixed state can be understood as the reduced state of a pure state on a larger system. This construction is central to quantum information theory and to the study of entanglement [2, 3].

An annotation can be attached either to the reduced state or to a specific assessment of its purification. These are not necessarily the same assessment: the reduced state alone

does not specify a unique purification vector, and different purifications can be related by transformations on the auxiliary system.

Accordingly, a proposed annotated framework should specify whether its annotation is invariant under such transformations. If the annotation is intended to describe only the physical reduced state, invariance under changes of purification is a natural consistency requirement.

7.11 Propositions for annotated entanglement assessments

The following statements clarify what can be established from the definitions and what would require additional assumptions.

Proposition 7.11.1 (Independence of the annotation from the state)**.** *Let ρ be a fixed density operator and let $\nu_1, \nu_2 \in [0,1]^3$. If annotations are not constrained by an assignment rule, then both (ρ, ν_1) and (ρ, ν_2) satisfy the definition of an annotated quantum state.*

Proof. The definition requires only that ρ be a density operator and that the annotation belong to $[0,1]^3$. Both conditions hold for each pair. □

This proposition shows that the bare definition does not determine an annotation from a quantum state. A scientifically useful application must add a rule linking the annotation to specified data or evidence.

Proposition 7.11.2 (Convex closure of mean annotations)**.** *Let $\nu_1, \ldots, \nu_m \in [0,1]^3$ and let $w_j \geq 0$ with $\sum_j w_j = 1$. Then*

$$\overline{\nu} = \sum_{j=1}^{m} w_j \nu_j$$

belongs to $[0,1]^3$.

Proof. Each coordinate of $\overline{\nu}$ is a convex combination of real numbers in the interval $[0,1]$, and hence lies in that interval. □

Proposition 7.11.3 (A deterministic annotation adds no information)**.** *Suppose an annotation is defined by a deterministic function*

$$\nu(\rho) = g(q(\rho)),$$

where $q(\rho)$ is an already available quantum quantity and g is a fixed function. Then the annotation is determined by $q(\rho)$ and provides no additional independent information about ρ once $q(\rho)$ is known.

Proof. Knowing $q(\rho)$ determines $g(q(\rho))$ uniquely. Therefore the annotation can be calculated from $q(\rho)$ without any additional data. □

Remark 7.11.4. The last proposition does not rule out the use of annotations. It identifies a design criterion: if an annotation is intended to provide information beyond a conventional entanglement measure, its assignment should incorporate independently justified information, such as uncertainty, model disagreement, or measurement limitations.

7.12 Potential applications

7.12.1 Incomplete state tomography

Quantum state tomography reconstructs an estimate of a density operator from measurement data. Finite samples, calibration errors, and model assumptions may produce uncertainty in the reconstructed state. Statistical and operational approaches to quantum state estimation are discussed in the quantum information literature [2, 3].

A proposed annotation could record:

- support for an entanglement conclusion across a specified family of admissible reconstructed states;
- indeterminacy caused by finite data or model ambiguity;
- opposition arising from admissible reconstructions that are separable.

To make these ideas precise, one would first define a confidence region $\mathcal{R}$ of density operators and then state exactly how the three coordinates are computed from $\mathcal{R}$. For example, one could report the fraction of sampled admissible models that are entangled, together with a separate uncertainty summary. Such a procedure requires a specified sampling measure and should not be described as a universal neutrosophic law.

7.12.2 Comparison of entanglement diagnostics

Different diagnostics answer different questions. A positive concurrence, a negative partial transpose, and a Bell-inequality violation are not interchangeable statements. For example, Bell violation certifies a particular form of nonclassical correlation, whereas failure to violate a chosen Bell inequality does not establish separability.

An annotation system could record the degree to which the available diagnostics agree. To avoid circularity, its assignment rule should distinguish:

(i) disagreement between diagnostic procedures;

(ii) statistical uncertainty within one procedure;

(iii) a genuine difference between the mathematical properties tested by those procedures.

7.12.3 Quantum communication

Entanglement fidelity is relevant to quantum communication, where noise can reduce the quality of transmitted quantum information. Supplementary annotations might be used to document uncertainty in estimates of channel performance or disagreements between models of the communication channel.

Any resulting performance claims should still be stated in terms of established quantities, such as fidelity, error rates, or certified entanglement bounds. An annotation alone does not establish a communication advantage.

7.13 Limitations and open problems

The framework proposed here is preliminary. Several questions must be resolved before it can be regarded as a mature theory.

Problem 1. Operational semantics. Construct a reproducible assignment rule for (t, i, f) from experimental data or a mathematically defined uncertainty model.

Problem 2. Local-unitary invariance. Determine conditions under which an annotation remains unchanged under transformations

$$\rho \longmapsto (U_A \otimes U_B)\rho(U_A^\dagger \otimes U_B^\dagger).$$

Such invariance is natural when the annotation is intended to describe an intrinsic entanglement property.

Problem 3. Monotonicity under local operations. Investigate whether a proposed support score is monotone under local operations and classical communication (LOCC), when this is appropriate to its intended interpretation. A general annotation triple need not satisfy entanglement-monotone conditions.

Problem 4. Consistency under purification. Determine whether annotations intended to describe a reduced state are independent of the choice of its purification.

Problem 5. Statistical calibration. Develop procedures for calibrating indeterminacy and opposition using finite-sample data, uncertainty regions, or validated statistical models.

Problem 6. Comparison with established measures. Determine whether annotations add predictive or descriptive value beyond concurrence, negativity, entanglement entropy, and entanglement fidelity.

Problem 7. Multipartite extension. Investigate how annotations should distinguish bipartite entanglement, genuine multipartite entanglement, and different forms of multipartite correlation.

These problems are mathematical research directions rather than claims that the corresponding results have already been obtained.

7.14 Conclusion

This paper has presented an introductory mathematical framework for attaching neutrosophic annotations to quantum entanglement assessments. The standard theory of bipartite entanglement was reviewed through product and separable states, the Bell basis, Schmidt decomposition, Bell–CHSH correlations, concurrence, entanglement fidelity, and purification.

The proposed annotation associates a triple

$$\nu = (t, i, f) \in [0, 1]^3$$

with a specified entanglement assessment. Its coordinates are intended to record support, indeterminacy, and opposition under an explicitly declared interpretation. Elementary aggregation properties were established, and examples illustrated how such annotations may accompany standard quantum calculations.

The central limitation is that an arbitrary annotation is not determined by the quantum state. Without an operational assignment rule, calibration, and validation, it cannot be interpreted as a new physical quantity or as an improvement over existing entanglement measures. Future work should therefore prioritize invariant definitions, statistically meaningful uncertainty assignments, and comparisons with established entanglement diagnostics.

Chapter 8

Neutrosophic Quantum Gates and Circuits

Related literature. Neutrosophic transformation gates were considered in [73]. Connections with realizable hardware implementations remain exploratory [81] (preprint).

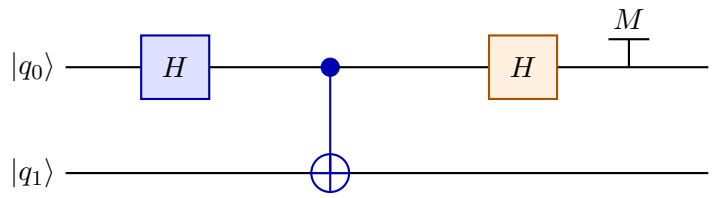


Figure 8.1: Example two-qubit gate circuit: a Hadamard gate, controlled-NOT, and subsequent readout. Annotations can be attached to the resulting classical records.

Quantum gates and circuits provide the operational foundation of gate-based quantum computation. Single-qubit gates implement unitary transformations on individual qubits, while controlled and other multiqubit gates generate correlations and enable the construction of quantum algorithms. This paper develops a proposed mathematical framework for attaching neutrosophic annotations to quantum gates and circuits. An annotation is a triple consisting of support, indeterminacy, and opposition coordinates in the unit interval. The usual quantum formalism is retained: gates act as unitary operators, composite systems are described by tensor products, and measurement probabilities are calculated using the Born rule.

The paper reviews the Pauli, Hadamard, phase, rotation, controlled-NOT, and Toffoli gates; develops the matrix description of circuits; and gives worked examples involving superposition, entanglement, and reversible computation. It then defines annotated gates and circuits, introduces an aggregation rule for supplementary annotation records, and discusses gate-level and circuit-level reliability assessment. The distinction between mathematical annotation and physical noise is emphasized. A neutrosophic triple does not replace a quantum channel, a process matrix, a fidelity measure, or a probability distribution. Its usefulness depends on an explicit assignment rule and independent validation. Several research problems are formulated concerning composition, invariance, calibration, and the relation between annotations and established quantum-information measures.

8.1 Introduction

Quantum computation uses quantum states and quantum operations to process information. In the gate-based model, a computation is represented by a sequence of elementary transformations

followed by measurements. Single-qubit gates manipulate amplitudes and relative phases, while multiqubit gates allow interactions and entanglement. A quantum circuit specifies how these operations are arranged and connected [26, 2, 1].

Unlike a classical Boolean gate, a quantum gate acting on a closed system must preserve inner products. Consequently, its matrix representation is unitary. This condition ensures that normalized quantum states remain normalized and that quantum evolution is reversible at the level of an isolated system [2, 3].

In practical implementations, however, the intended operation may differ from the operation realized by the hardware. Sources of imperfection include calibration uncertainty, control errors, decoherence, crosstalk, leakage, and imperfect readout. Quantum information theory describes physical noise using quantum channels and related operational tools [2, 3].

Neutrosophic theory provides a different descriptive language based on three components commonly interpreted as truth, indeterminacy, and falsity or opposition [6, 22]. This paper investigates whether such triples can be used as supplementary annotations for quantum gates and circuits.

The proposed annotation does not change the quantum gate itself. Instead, it records an assessment associated with a gate, a circuit, a calibration procedure, or an available body of evidence. For example, an annotation might summarize support for a claim that a gate meets a specified performance target, indeterminacy arising from incomplete calibration, and opposition arising from observed deviations.

The paper has five main objectives:

(1) to review the mathematical representation of elementary quantum gates and circuits;

(2) to define annotated gates and annotated circuits;

(3) to establish elementary properties of an annotation aggregation rule;

(4) to demonstrate the framework using standard circuit calculations;

(5) to identify conditions needed for a scientifically meaningful application.

The discussion is related to the introductory treatment of quantum gates and circuits in McMahon's *Quantum Computing Explained* [1], but the definitions and proposed annotation framework below are developed independently.

8.2 Preliminaries: qubits and unitary operations

8.2.1 The state of a qubit

The state space of a qubit is the two-dimensional complex Hilbert space $\mathbb{C}^2$. Its computational basis is

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A normalized pure state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1,$$

where $\alpha, \beta \in \mathbb{C}$.

A pure state is represented by the density operator

$$\rho = |\psi\rangle\langle\psi|.$$

More generally, a mixed state is a positive semidefinite operator satisfying

$$\rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1.$$

Density operators are useful when the system is not prepared in a single known pure state or when only part of a larger system is accessible [2, 3].

8.2.2 Unitary gates

Definition 8.2.1 (Quantum gate)**.** A unitary quantum gate on a finite-dimensional Hilbert space $\mathcal{H}$ is a linear operator

$$U : \mathcal{H} \longrightarrow \mathcal{H}$$

satisfying

$$U^\dagger U = U U^\dagger = \mathbb{I}_{\mathcal{H}}.$$

If the input state is $|\psi\rangle$, the output state is

$$|\psi'\rangle = U|\psi\rangle.$$

The norm is preserved:

$$\langle\psi'|\psi'\rangle = \langle\psi|U^\dagger U|\psi\rangle = \langle\psi|\psi\rangle.$$

For a density operator,

$$\rho' = U\rho U^\dagger.$$

Proposition 8.2.2. *If U is unitary and ρ is a density operator, then $U\rho U^\dagger$ is also a density operator.*

Proof. Positivity follows because, for every vector $|v\rangle$,

$$\langle v|U\rho U^\dagger|v\rangle = \langle U^\dagger v|\rho|U^\dagger v\rangle \geq 0.$$

Moreover, by cyclicity of the trace,

$$\mathrm{Tr}(U\rho U^\dagger) = \mathrm{Tr}(U^\dagger U\rho) = \mathrm{Tr}(\rho) = 1.$$

Thus the transformed operator is positive semidefinite and has unit trace. □

8.2.3 Global phase

The vectors $|\psi\rangle$ and $e^{i\phi}|\psi\rangle$ represent the same physical pure state for any real ϕ, since

$$|e^{i\phi}\psi\rangle\langle e^{i\phi}\psi| = |\psi\rangle\langle\psi|.$$

A global phase applied to an entire state is therefore unobservable. By contrast, a relative phase between components can change measurement statistics and interference.

This distinction becomes important when defining annotations: an annotation intended to describe physical behavior should not change merely because a matrix representation is multiplied by an irrelevant global phase.

8.3 Elementary single-qubit gates

8.3.1 Pauli gates

The Pauli matrices are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are unitary and Hermitian:

$$X^\dagger = X, \quad Y^\dagger = Y, \quad Z^\dagger = Z,$$

and

$$X^2 = Y^2 = Z^2 = \mathbb{I}_2.$$

The X gate exchanges the computational basis states:

$$X|0\rangle = |1\rangle, \qquad X|1\rangle = |0\rangle.$$

The Z gate changes the relative phase:

$$Z|0\rangle = |0\rangle, \qquad Z|1\rangle = -|1\rangle.$$

The Y gate combines a bit flip and a phase change:

$$Y|0\rangle = i|1\rangle, \qquad Y|1\rangle = -i|0\rangle.$$

8.3.2 Hadamard gate

The Hadamard gate is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It satisfies

$$H^\dagger = H, \qquad H^2 = \mathbb{I}_2,$$

and

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

It converts computational-basis states into equal-amplitude superpositions.

For example, if the initial state is $|0\rangle$, then after the Hadamard gate the state is

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

A computational-basis measurement yields either outcome with probability $1/2$.

8.3.3 Phase gates

For a real parameter ϕ, define

$$P(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}.$$

This gate acts by

$$P(\phi)|0\rangle = |0\rangle, \qquad P(\phi)|1\rangle = e^{i\phi}|1\rangle.$$

The commonly used phase gate S and T gate are

$$S = P\left(\frac{\pi}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

and

$$T = P\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Phase gates are central to many circuit constructions [2].

8.3.4 Rotation gates

Using the Pauli matrices, define

$$R_x(\theta) = e^{-i\theta X/2}, \quad R_y(\theta) = e^{-i\theta Y/2}, \quad R_z(\theta) = e^{-i\theta Z/2}.$$

For example,

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

Each rotation is unitary for real θ.

The angle θ is a control parameter. An imperfectly calibrated angle may be modeled by $\theta + \delta\theta$. The resulting operation is still unitary in an idealized closed-system model, but it differs from the intended gate. Such deviations motivate a distinction between the ideal gate and an annotation describing uncertainty in its implementation.

8.4 Multiqubit gates

8.4.1 Tensor products

For two qubits, the composite state space is

$$\mathcal{H}_{AB} = \mathbb{C}^2 \otimes \mathbb{C}^2.$$

The tensor product of two operators U and V is denoted $U \otimes V$. If U and V are unitary, then

$$(U \otimes V)^\dagger (U \otimes V) = (U^\dagger U) \otimes (V^\dagger V) = \mathbb{I}_2 \otimes \mathbb{I}_2.$$

Thus $U \otimes V$ is unitary.

A gate acting only on the first qubit of a two-qubit system is represented by $U \otimes \mathbb{I}_2$, while a gate acting only on the second is represented by $\mathbb{I}_2 \otimes V$.

8.4.2 Controlled-NOT gate

The controlled-NOT (CNOT) gate flips the target qubit if the control qubit is in state $|1\rangle$. In the ordered basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, its matrix is

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Its action on the computational basis is

$$|00\rangle \longmapsto |00\rangle, \qquad |01\rangle \longmapsto |01\rangle,$$
$$|10\rangle \longmapsto |11\rangle, \qquad |11\rangle \longmapsto |10\rangle.$$

Since it permutes an orthonormal basis, CNOT is unitary.

8.4.3 Controlled phase

The controlled-phase gate is

$$\mathrm{CP}(\phi) = \mathrm{diag}(1, 1, 1, e^{i\phi}).$$

Only the basis state $|11\rangle$ receives the phase factor $e^{i\phi}$. For $\phi = \pi$, this is the controlled-Z gate:

$$\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1).$$

8.4.4 Toffoli gate

The Toffoli gate, also called the controlled-controlled-NOT gate, has two control qubits and one target qubit. It flips the target only when both controls are in state $|1\rangle$:

$$|a, b, c\rangle \longmapsto |a, b, c \oplus (ab)\rangle,$$

where $a, b, c \in \{0, 1\}$ and $\oplus$ denotes addition modulo two.

The Toffoli gate is a reversible classical gate and is also a valid quantum gate because its matrix is a permutation matrix. Reversible classical computation and universal quantum circuits are discussed in standard references such as [26, 2].

8.5 Quantum circuits as matrix products

8.5.1 Circuit composition

Suppose a circuit consists of gates $U_1, U_2, \ldots, U_n$, applied in that order. The total transformation is

$$U_{\mathrm{tot}} = U_n U_{n-1} \cdots U_2 U_1.$$

The rightmost operator acts first. If all the gates are unitary, then the total transformation is unitary.

Theorem 8.5.1. *A finite composition of unitary gates is unitary.*

Proof. For two unitary operators U and V,

$$(UV)^\dagger (UV) = V^\dagger U^\dagger U V = V^\dagger V = \mathbb{I}.$$

Similarly, $(UV)(UV)^\dagger = \mathbb{I}$. Thus UV is unitary. Induction on the number of factors proves the result for every finite composition. □

8.5.2 A circuit creating a Bell state

Consider two qubits initially in state $|00\rangle$. Apply a Hadamard gate to the first qubit and then apply CNOT:

$$|00\rangle \xrightarrow{H \otimes \mathbb{I}_2} \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\mathrm{CNOT}} \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The output is the Bell state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The total gate is

$$U_{\mathrm{Bell}} = \mathrm{CNOT}(H \otimes \mathbb{I}_2).$$

The state is entangled and cannot be written as a tensor product of single-qubit states.

8.5.3 Circuit equivalence

Two circuits implementing operators U and V are physically equivalent on a closed system if they differ only by a global phase:

$$U = e^{i\phi}V.$$

They then produce the same output density operator for every input density operator:

$$U\rho U^\dagger = V\rho V^\dagger.$$

A proposed annotation for physical circuit behavior should respect this equivalence. An annotation intended instead to describe hardware implementation records may distinguish two implementations even if their ideal operators are equivalent. The intended interpretation must therefore be specified.

8.6 Neutrosophic annotation space

8.6.1 Definition

Definition 8.6.1 (Neutrosophic triple)**.** A neutrosophic triple is an ordered triple

$$\nu = (t, i, f) \in [0, 1]^3.$$

The coordinates are interpreted as support, indeterminacy, and opposition relative to a stated assessment.

The components are not required to satisfy

$$t + i + f = 1.$$

In particular, they are not automatically the probabilities of three mutually exclusive events. Their interpretation is application-dependent and must be specified [6, 22].

Definition 8.6.2 (Annotated quantum gate)**.** An annotated quantum gate is a pair

$$\widehat{U} = (U, \nu_U),$$

where U is a unitary operator on a finite-dimensional Hilbert space and

$$\nu_U = (t_U, i_U, f_U) \in [0, 1]^3$$

is a supplementary annotation associated with a declared assessment of the gate.

The unitary operator specifies the ideal transformation. The annotation may describe evidence concerning implementation quality, calibration, model uncertainty, or another explicitly defined property.

Remark 8.6.3. The triple is not a replacement for the gate matrix. A triple alone does not specify how a quantum state is transformed.

8.6.2 Annotated circuits

Definition 8.6.4 (Annotated circuit)**.** Let $U_1, \ldots, U_n$ be unitary gates whose dimensions and wiring are compatible. An annotated circuit is a sequence

$$\widehat{\mathsf{C}} = ((U_1, \nu_1), \ldots, (U_n, \nu_n))$$

together with the circuit structure specifying the order and location of the gates.

The ideal circuit operator is

$$U_{\mathsf{C}} = U_n \cdots U_2 U_1.$$

The sequence of annotations is additional information. To obtain a single circuit-level triple, an aggregation rule must be chosen and justified.

8.6.3 A possible aggregation rule

For two triples, define

$$(t_1, i_1, f_1) \odot (t_2, i_2, f_2) = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)\,.$$

This is a proposed convention for aggregating annotations. It is not a standard quantum operation.

Proposition 8.6.5. *The operation $\odot$ maps $[0,1]^3 \times [0,1]^3$ into $[0,1]^3$, is commutative and associative, and has identity $(1,0,0)$.*

Proof. The first coordinate is a product of numbers in $[0,1]$. For the other coordinates, use

$$x + y - xy = 1 - (1-x)(1-y).$$

If $x, y \in [0,1]$, this expression belongs to $[0,1]$. This proves closure.

Commutativity follows from symmetry. Associativity of the first coordinate follows from associativity of multiplication. For the second coordinate, combining three inputs gives

$$1 - (1-i_1)(1-i_2)(1-i_3),$$

independently of grouping. The same formula holds for the third coordinate with f_j in place of i_j. Thus the operation is associative. Finally,

$$(t, i, f) \odot (1, 0, 0) = (t, i, f).$$

□

8.6.4 Circuit-level annotation

For a circuit with gate annotations $\nu_1, \ldots, \nu_n$, define the aggregated annotation

$$\nu_{\mathsf{C}} = \nu_1 \odot \nu_2 \odot \cdots \odot \nu_n.$$

By associativity, this does not depend on the placement of parentheses.

Writing

$$\nu_j = (t_j, i_j, f_j),$$

the aggregate is

$$\nu_{\mathsf{C}} = \left(\prod_{j=1}^{n} t_j,\ 1 - \prod_{j=1}^{n} (1 - i_j),\ 1 - \prod_{j=1}^{n} (1 - f_j) \right).$$

Remark 8.6.6. This formula is a mathematical aggregation rule, not a derived formula for the probability that a physical circuit succeeds. In particular, multiplying support coordinates is not justified as a reliability calculation unless the assignment rule and assumptions about dependence have been validated.

8.7 Examples of annotated gates and circuits

8.7.1 An annotated Hadamard gate

Consider the ideal Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Suppose a calibration protocol assigns the illustrative triple

$$\nu_H = (0.96, 0.03, 0.04).$$

The annotated gate is

$$\widehat{H} = (H, \nu_H).$$

The triple is not obtained from the matrix H alone. It is assumed to summarize a separately specified assessment.

Applying H to $|0\rangle$ still gives

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

The annotation does not change the output state or its measurement probabilities.

8.7.2 An annotated CNOT gate

Let

$$C = \mathrm{CNOT}$$

and suppose its annotation is

$$\nu_C = (0.92, 0.06, 0.08).$$

Then

$$\widehat{C} = (C, \nu_C).$$

The ideal action remains

$$C|10\rangle = |11\rangle.$$

The annotation records a separate assessment of the gate.

8.7.3 A two-gate circuit

Consider the circuit

$$|00\rangle \xrightarrow{H\otimes\mathbb{I}_2} \xrightarrow{C} |\Phi^+\rangle.$$

Suppose the annotations are

$$\nu_{H\otimes\mathbb{I}_2} = (0.96, 0.03, 0.04), \qquad \nu_C = (0.92, 0.06, 0.08).$$

Using the aggregation rule gives

$$t_{\mathsf{C}} = 0.96(0.92) = 0.8832,$$

$$i_{\mathsf{C}} = 1 - (1 - 0.03)(1 - 0.06) = 0.0882,$$

and

$$f_{\mathsf{C}} = 1 - (1 - 0.04)(1 - 0.08) = 0.1168.$$

Therefore,

$$\nu_{\mathsf{C}} = (0.8832, 0.0882, 0.1168).$$

These values are illustrative. They do not imply that the probability of successful Bell-state preparation is 0.8832. To make that claim, the first coordinate would have to be defined and experimentally calibrated as a success probability, which is not assumed here.

8.7.4 A three-gate circuit

Consider

$$U_{\mathsf{C}} = R_z(\theta) H R_x(\phi),$$

where the rightmost gate acts first. Suppose the annotations are

$$\nu_{R_x} = (t_x, i_x, f_x), \quad \nu_H = (t_H, i_H, f_H), \quad \nu_{R_z} = (t_z, i_z, f_z).$$

The proposed aggregate is

$$\nu_{\mathsf{C}} = \left(t_x t_H t_z,\, 1-(1-i_x)(1-i_H)(1-i_z),\, 1-(1-f_x)(1-f_H)(1-f_z)\right).$$

The formula makes the bookkeeping explicit, but the physical meaning of each coordinate still depends on the annotation procedure.

8.8 Measurement and output statistics

8.8.1 The Born rule

A quantum circuit typically ends with a measurement. For a computational-basis measurement of a state

$$|\psi\rangle = \sum_{x=0}^{2^n-1} a_x |x\rangle,$$

the probability of obtaining outcome x is

$$p(x) = |a_x|^2.$$

The normalization condition gives

$$\sum_x p(x) = 1.$$

For a density operator ρ, the corresponding probability is

$$p(x) = \mathrm{Tr}(\rho |x\rangle\langle x|).$$

These probabilities are determined by the quantum state and measurement operators, not by a neutrosophic annotation [2, 3].

8.8.2 Measurement after Bell-state preparation

For the state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

a computational-basis measurement gives

$$\mathbb{P}(00) = \frac{1}{2}, \qquad \mathbb{P}(11) = \frac{1}{2},$$

and

$$\mathbb{P}(01) = \mathbb{P}(10) = 0.$$

The annotation of the gates may accompany the report of these probabilities, but it does not modify them.

For example, a report may list:

- the ideal circuit operator;

- the theoretical output state;
- the Born-rule probabilities;
- the measured frequencies;
- the annotation and the rule used to obtain it.

Separating these quantities prevents descriptive assessments from being confused with physical predictions.

8.9 Ideal gates and noisy implementations

8.9.1 Quantum channels

An ideal gate is described by a unitary operator U. A general noisy implementation is more naturally described by a completely positive trace-preserving map

$$\mathcal{E}(\rho) = \sum_k K_k \rho K_k^\dagger, \qquad \sum_k K_k^\dagger K_k = \mathbb{I}.$$

Such a map is called a quantum channel [2, 3].

The distinction is fundamental:

- U describes an ideal unitary transformation;
- $\mathcal{E}$ describes a physical operation that may include noise and information loss;
- ν is a supplementary annotation under a chosen interpretation.

An annotation cannot replace the channel because a triple does not determine the action of $\mathcal{E}$ on arbitrary input states.

8.9.2 Example: a depolarizing channel

For a single qubit, one convention for the depolarizing channel is

$$\mathcal{D}_p(\rho) = (1-p)\rho + p\frac{\mathbb{I}_2}{2}, \qquad 0 \le p \le 1.$$

Here p is the depolarizing parameter. At $p = 0$, the channel is the identity; at $p = 1$, it outputs the maximally mixed state.

An annotation might be associated with the estimated parameter or with uncertainty in that estimate:

$$\nu_{\mathcal{D}_p} = (t_p, i_p, f_p).$$

The three coordinates must be calculated by an explicitly defined procedure. They are not automatically equal to $(1-p, p, 0)$, nor are they required to sum to one.

8.9.3 Gate fidelity

The average gate fidelity compares an implemented operation with a target unitary under a specified averaging convention. It is one of several standard ways to characterize gate performance; process fidelity and other channel measures are also used [27, 2].

A possible annotation is

$$\nu_{\text{gate}} = (t_{\text{gate}}, i_{\text{gate}}, f_{\text{gate}}),$$

where the assignment rule may use the estimated fidelity, its uncertainty, and an agreed performance threshold.

For example, a protocol might define support from the fraction of a confidence region that satisfies a performance criterion. The precise definition would depend on the statistical model and the geometry of the confidence region. No universal conversion from gate fidelity to a neutrosophic triple is assumed in this paper.

8.10 Consistency properties and propositions

8.10.1 Closure of circuit annotations

Proposition 8.10.1. *If every gate annotation belongs to* $[0,1]^3$*, then the aggregate circuit annotation defined using* $\odot$ *belongs to* $[0,1]^3$*.*

Proof. The result follows by repeated application of the closure property of $\odot$. Equivalently, the three coordinates of the aggregate are

$$\prod_{j=1}^{n} t_j, \qquad 1-\prod_{j=1}^{n}(1-i_j), \qquad 1-\prod_{j=1}^{n}(1-f_j).$$

Each product lies in $[0,1]$, so all three displayed coordinates belong to $[0,1]$. □

8.10.2 Dependence on the chosen annotation rule

Proposition 8.10.2. *The ideal circuit operator* U_{C} *and the annotation* ν_{C} *are mathematically distinct pieces of data. The operator alone does not determine the annotation under the definitions given here.*

Proof. The definition of an annotated circuit requires both the gate sequence and the annotations assigned to its gates. No assignment function from unitary operators to triples has been imposed. Therefore, for a fixed unitary circuit, different triples are permitted until an assignment rule is specified. □

Remark 8.10.3. This proposition identifies a limitation rather than a physical discovery. To make the framework predictive, one must specify how annotations are obtained and what information they add.

8.10.3 Global-phase invariance

Proposition 8.10.4. *Suppose a circuit annotation is intended to characterize only the ideal physical transformation. If*

$$U_{\mathsf{C}} = e^{i\phi}V_{\mathsf{C}},$$

then a physically invariant annotation rule should assign the same annotation to the two circuits.

Proof. For every density operator ρ,

$$U_{\mathsf{C}}\rho U_{\mathsf{C}}^{\dagger} = e^{i\phi}V_{\mathsf{C}}\rho e^{-i\phi}V_{\mathsf{C}}^{\dagger} = V_{\mathsf{C}}\rho V_{\mathsf{C}}^{\dagger}.$$

The two operators therefore induce the same transformation on density operators. An annotation intended to depend only on that transformation should consequently be equal for the two circuits. □

Remark 8.10.5. If annotations record implementation-specific facts, such as different control pulses or calibration procedures, equal ideal operators need not imply equal implementation annotations. The required invariance depends on what the annotation is intended to describe.

8.11 Potential applications

8.11.1 Calibration records

Calibration procedures estimate control parameters and characterize deviations from target operations. An annotation could summarize the status of a calibration assessment, provided that the support, indeterminacy, and opposition coordinates are explicitly defined.

For example, the annotation might incorporate:

(a) uncertainty in an estimated rotation angle;

(b) disagreement between calibration runs;

(c) the degree to which the estimated gate meets a predeclared specification.

These sources of information should be recorded separately before an aggregation rule is applied.

8.11.2 Circuit design and comparison

Suppose two circuits implement the same logical operation but use different numbers or types of physical gates. A comparison could report:

- circuit depth and gate count;
- the ideal circuit operator;
- estimated noise or fidelity;
- uncertainty in those estimates;
- an optional neutrosophic annotation.

An annotation may be useful for organizing qualitative or uncertain evidence, but it should not replace quantitative comparisons using gate counts, error rates, resource estimates, or validated channel metrics.

8.11.3 Educational applications

The annotation framework may also be used in teaching. Students can distinguish the exact algebraic transformation of an ideal gate from uncertainty in whether a laboratory implementation meets a stated target. This separation can help explain why unitarity is a mathematical property of the ideal gate, while physical performance requires additional experimental evidence.

8.12 Limitations and open research problems

The framework presented here is preliminary. The following problems indicate the work needed to make it more rigorous.

Problem 1. Operational assignment. Define a reproducible procedure that assigns (t, i, f) from calibration data, channel estimates, or a specified statistical model.

Problem 2. Circuit-composition validity. Determine when the proposed operation $\odot$ is justified. In particular, investigate dependence between errors at different gates and the effect of correlations on a circuit-level assessment.

Problem 3. Global-phase invariance. Construct annotation rules invariant under global phase when the annotation is intended to describe only the ideal physical operation.

Problem 4. Relation to channel metrics. Study whether annotations add useful information beyond average gate fidelity, process fidelity, diamond distance, or other established measures of quantum-channel performance.

Problem 5. Uncertainty calibration. Develop statistically principled methods for assigning the indeterminacy coordinate from finite experimental samples.

Problem 6. Compilation and circuit equivalence. Investigate whether equivalent circuit decompositions receive consistent annotations and identify cases in which implementation-specific differences should be preserved.

Problem 7. Experimental validation. Test whether annotation-based reports improve the interpretability, reproducibility, or comparison of real quantum-device experiments.

Problem 8. Extension to adaptive circuits. Extend the framework to circuits in which later operations depend on earlier measurement outcomes, taking care to distinguish classical control, quantum probabilities, and annotation updates.

These are research questions, not claims that the proposed framework has already solved the corresponding problems.

8.13 Conclusion

This paper has developed a proposed framework for attaching neutrosophic annotations to quantum gates and circuits. The standard mathematical theory of quantum computation was reviewed through unitary operators, Pauli gates, the Hadamard and phase gates, rotations, controlled gates, tensor products, and circuit composition. Worked examples illustrated superposition, Bell-state preparation, measurement, and the distinction between ideal gates and noisy implementations.

An annotated gate was defined as a pair consisting of a unitary operator and a supplementary triple

$$\nu = (t, i, f) \in [0, 1]^3.$$

An associative aggregation rule was introduced for combining such triples along a circuit. Its algebraic properties were proved, while its limitations as a physical reliability formula were explicitly identified.

The main conclusion is that a neutrosophic annotation should be treated as supplementary descriptive information until a reproducible operational interpretation has been established. A triple does not replace a unitary operator, quantum channel, measurement probability, or fidelity measure. Future research should concentrate on calibration, invariance, statistical meaning, and experimental validation.

Chapter 9

Neutrosophic Quantum Algorithms

Related literature. Proposed neutrosophic variants of Shor's and Grover's algorithms [77] and convergence results for neutrosophic amplitude amplification [78] are available as preprints. These information-theoretic extensions do not by themselves establish new physical speedups.

Quantum algorithms exploit superposition, interference, entanglement, and structured unitary transformations to solve certain computational problems with advantages over known classical procedures. This paper develops a neutrosophic annotation framework for describing quantum algorithms when their mathematical specification, oracle assumptions, input information, or implementation reliability is incomplete or uncertain. Each annotated algorithmic component is associated with a triple (t, i, f), where the coordinates represent degrees of support, indeterminacy, and opposition relative to a declared assessment criterion. The triple supplements, but does not replace, the complex amplitudes and probability distributions of quantum mechanics.

The framework is applied to Hadamard transformations and interference, quantum parallelism, the Deutsch–Jozsa algorithm, the quantum Fourier transform, phase estimation, Shor's factoring procedure, and Grover's search algorithm. We distinguish exact mathematical statements from annotations of model adequacy and hardware reliability. A composition rule for annotations is introduced and its algebraic properties are proved. A separate reliability calculation is provided for serial components under an explicit independence assumption. Worked examples illustrate how an annotation can record uncertainty about an oracle promise or a circuit implementation without changing the standard quantum probabilities. Finally, limitations and possible research directions are discussed. The proposed framework is a descriptive extension rather than a new quantum speedup theorem.

9.1 Introduction

Quantum computation studies information processing using quantum systems and quantum-mechanical transformations. The basic mathematical language consists of complex Hilbert spaces, unitary operators, measurement operators, and probability distributions determined by the Born rule. Standard introductions include Nielsen and Chuang [2], McMahon [1], and the quantum information text of Watrous [3].

Quantum algorithms are not simply classical algorithms with random choices. Their distinctive mechanisms include coherent superposition and interference. A quantum circuit can transform amplitudes so that some measurement outcomes become more likely while others are suppressed. This mechanism underlies algorithms such as Deutsch–Jozsa, phase estimation, Shor's factoring algorithm, and Grover's search algorithm [28, 31, 29].

The purpose of this paper is to introduce a neutrosophic annotation layer for discussing such algorithms. Neutrosophic logic and set theory provide a language for representing truth, indeterminacy, and falsity as distinct assessment coordinates [32, 6]. Here, these coordinates

are used to describe the status of a specified claim about an algorithm, such as whether an oracle promise has been verified, whether a circuit specification is complete, or whether a hardware implementation is considered reliable.

A crucial distinction is necessary. A quantum state is not replaced by a neutrosophic triple, and quantum amplitudes are not interpreted as truth degrees. Instead, the quantum state remains an ordinary mathematical object. The neutrosophic triple records supplementary information about the knowledge, assumptions, or implementation associated with that object.

The central questions are:

1. How can a neutrosophic annotation be attached to a quantum algorithm without altering its standard mathematical semantics?

2. How can annotations be propagated through a sequence of algorithmic components?

3. How can the framework describe incomplete oracle promises, uncertain circuit specifications, and implementation reliability?

4. How can it be applied to representative quantum algorithms while preserving their established complexity and correctness claims?

The paper is organized as follows. Section 9.2 recalls the mathematical framework of quantum algorithms. Section 9.3 introduces neutrosophic annotations. Section 9.4 studies their composition. Sections 9.5–9.10 apply the framework to quantum interference, Deutsch–Jozsa, Fourier analysis, phase estimation, Shor's algorithm, and Grover's search. The final sections discuss limitations and future directions.

9.2 Mathematical preliminaries for quantum algorithms

9.2.1 Computational basis and quantum registers

For a single qubit, the state space is $\mathbb{C}^2$, with computational basis

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A pure qubit state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad \alpha, \beta \in \mathbb{C}, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

The Born rule assigns probabilities $|\alpha|^2$ and $|\beta|^2$ to the outcomes 0 and 1, respectively [2, 3].

For an n-qubit register, the state space is

$$\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}, \qquad \dim \mathcal{H}_n = 2^n.$$

A normalized pure state is

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} a_x |x\rangle, \qquad \sum_{x \in \{0,1\}^n} |a_x|^2 = 1.$$

Measurement in the computational basis yields x with probability $|a_x|^2$.

9.2.2 Quantum circuits and unitary transformations

An ideal quantum algorithm can be represented by a circuit composed of unitary transformations and measurements. If the input state is $|\psi_0\rangle$ and the successive unitary gates are $U_1, \ldots, U_m$, then the final state before measurement is

$$|\psi_m\rangle = U_m U_{m-1} \cdots U_1 |\psi_0\rangle.$$

Each gate satisfies $U_j^\dagger U_j = \mathbb{I}$.

The order of composition matters: in general, $U_j U_k \neq U_k U_j$. Moreover, amplitudes must be added before their squared moduli are calculated. This distinction is essential to quantum interference.

9.2.3 Oracle model

Many quantum algorithms are analyzed using an oracle, or black-box, model. An oracle represents a function or operation whose internal implementation is not directly counted as part of the algorithm. For a Boolean function $f : \{0,1\}^n \to \{0,1\}$, a standard reversible oracle is

$$U_f |x, y\rangle = |x, y \oplus f(x)\rangle,$$

where $\oplus$ denotes addition modulo two.

The query complexity of an algorithm counts how often it invokes the oracle. This does not automatically account for the physical cost of implementing the oracle, loading data, or correcting errors. Such distinctions are important when interpreting quantum speedups [2, 30].

9.3 Neutrosophic annotation framework

9.3.1 Annotation triples

We use the three coordinates truth, indeterminacy, and falsity as descriptive degrees relative to a specified claim or assessment criterion. The terminology follows the broad neutrosophic approach of distinguishing support, indeterminacy, and opposition [32, 6].

Definition 9.3.1 (Neutrosophic annotation)**.** A neutrosophic annotation is a triple

$$\nu = (t, i, f) \in [0,1]^3,$$

where:

- t is the degree of support for a declared claim;
- i is the degree of indeterminacy associated with that claim;
- f is the degree of opposition to that claim.

The meaning of each coordinate must be fixed by the application.

The triple is not required to satisfy $t + i + f = 1$. In particular, the coordinates are not automatically probabilities and need not form a probability distribution.

Remark 9.3.2. An annotation is meaningful only relative to a declared proposition. For example, an annotation for the claim "the oracle satisfies the promised function property" is different from an annotation for the claim "the hardware implements the oracle with low error." These claims should not be conflated.

9.3.2 Annotated algorithmic components

Definition 9.3.3 (Neutrosophically annotated component)**.** An annotated algorithmic component is a pair

$$\mathfrak{C} = (C, \nu_C),$$

where C is a standard mathematical component, such as a unitary gate, oracle, subroutine, or measurement procedure, and $\nu_C = (t_C, i_C, f_C)$ is an annotation relative to a specified assessment criterion.

The underlying component retains its ordinary semantics. For instance, if C is a unitary matrix, then the condition $C^\dagger C = \mathbb{I}$ is unchanged by attaching an annotation.

We distinguish three types of annotation throughout the paper:

(i) *Specification annotation*: completeness or uncertainty concerning the mathematical description of a component.

(ii) *Promise annotation*: support or uncertainty concerning assumptions imposed on an input or oracle.

(iii) *Implementation annotation*: evidence about how closely a physical device realizes an ideal operation.

9.3.3 A convenient annotation space

Let

$$\mathfrak{N} = [0, 1]^3.$$

For $\nu = (t, i, f)$, define

$$s(\nu) = t - f$$

as a signed support index. It is not a probability of success. It merely records the balance between support and opposition under the chosen annotation convention.

Definition 9.3.4 (Complete-support annotation)**.** An annotation $\nu = (t, i, f)$ is called fully supportive, relative to its declared criterion, if

$$t = 1, \qquad i = 0, \qquad f = 0.$$

It is called maximally indeterminate if $i = 1$.

These terms concern the annotation, not the quantum state itself. For example, a maximally mixed density operator is not thereby "maximally indeterminate" in the present annotation sense; the two notions belong to different mathematical layers.

9.3.4 Annotation and quantum probability

Suppose a circuit produces a quantum state

$$|\psi\rangle = \sum_x a_x |x\rangle.$$

The standard measurement distribution is

$$p(x) = |a_x|^2, \qquad \sum_x p(x) = 1.$$

An annotation ν_{model} can be attached to the claim that the circuit model adequately describes the intended experiment. It does not change $p(x)$.

Proposition 9.3.5 (Separation of semantics)**.** *Let C be a fully specified ideal quantum circuit and let $\nu_C \in \mathfrak{N}$ be an annotation attached to a declared meta-level claim about C. If the annotation does not modify the mathematical action of C, then the ideal output distribution computed from C is independent of ν_C.*

Proof. The ideal output distribution is obtained by applying the operators in C to the input state and then applying the Born rule. The annotation is additional information and, by hypothesis, is not an operator acting on the state. Therefore it does not enter the state transformation or the Born-rule calculation. □

Remark 9.3.6. If the annotation is used to select among different physical implementations, or to define a probability distribution over possible circuits, then it may influence a subsequent model-selection or risk calculation. That is a separate construction and must be specified explicitly.

9.4 Composition of neutrosophic annotations

A quantum algorithm is usually built by composing many gates and subroutines. We now introduce one possible rule for propagating annotations through a serial chain. The rule is a modeling convention, not a consequence of quantum mechanics.

Definition 9.4.1 (Serial annotation product)**.** For $\nu_1 = (t_1, i_1, f_1)$ and $\nu_2 = (t_2, i_2, f_2)$, define

$$\nu_1 \odot \nu_2 = (t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2)\,.$$

The truth coordinate uses a product rule, while the indeterminacy and falsity coordinates use the probabilistic-sum expression $x + y - xy$. These choices are useful for illustration, but they are not uniquely determined by neutrosophic theory.

Proposition 9.4.2 (Closure and associativity)**.** *The operation $\odot$ maps $\mathfrak{N} \times \mathfrak{N}$ into $\mathfrak{N}$ and is associative.*

Proof. For $x, y \in [0,1]$, one has $xy \in [0,1]$ and

$$x + y - xy = 1 - (1 - x)(1 - y) \in [0,1].$$

Thus each coordinate of $\nu_1 \odot \nu_2$ belongs to $[0,1]$.

Associativity of the truth coordinate follows from $(t_1 t_2)t_3 = t_1(t_2 t_3)$. For the other two coordinates, use

$$1 - (x + y - xy) = (1 - x)(1 - y).$$

It follows that the complement of the iterated probabilistic sum is $(1 - x_1)(1 - x_2)(1 - x_3)$, independently of grouping. Hence $\odot$ is associative. □

Remark 9.4.3. The operation is not a physical law. In particular, its truth coordinate may be interpreted as a serial reliability product only when the truth values have been calibrated as appropriate success probabilities and the independence assumptions required for the product are justified. The three coordinates should not automatically be interpreted as independent probabilities.

9.4.1 A separate reliability calculation

For comparison, suppose $R_1, \ldots, R_m$ are independent events that the corresponding hardware components operate successfully, with $\mathbb{P}(R_j) = r_j$. Then

$$\mathbb{P}\left(\bigcap_{j=1}^{m} R_j\right) = \prod_{j=1}^{m} r_j.$$

This is a standard probability identity under independence, not a general neutrosophic rule.

If independence is not justified, the product formula cannot be used without additional assumptions. Correlated noise, shared control errors, and common calibration failures can invalidate it.

9.5 Hadamard transformations and quantum interference

9.5.1 The Hadamard gate

The Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It satisfies $H^\dagger H = \mathbb{I}$ and acts on computational basis states by

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

Applying H to $|0\rangle$ and then measuring produces either outcome with probability $1/2$. Applying H a second time returns the original state because $H^2 = \mathbb{I}$.

Example 9.5.1 (Interference after two Hadamard gates)**.** Starting from $|0\rangle$,

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}.$$

A second Hadamard gives

$$H\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{1}{2}\left[(|0\rangle + |1\rangle) + (|0\rangle - |1\rangle)\right] = |0\rangle.$$

The amplitudes for $|1\rangle$ cancel. This is destructive interference.

9.5.2 Neutrosophic annotation of interference claims

An annotation can record the status of a claim such as

$$P: \quad \text{“the intended phase relation is correctly realized.”}$$

For an ideal mathematical circuit, the phase relation can be proved exactly. For a physical device, its realization may be affected by calibration error, decoherence, or imperfect control.

For illustration, suppose the assessment is

$$\nu_P = (0.85, 0.10, 0.05).$$

This triple is a declared assessment of the evidence supporting P, not a Born-rule distribution. It must not be used to replace the calculated measurement probabilities.

A useful reporting practice is to state the criterion and source of the annotation. For example:

- criterion: fidelity of the implemented Hadamard gate to its ideal target;
- evidence: a specified calibration experiment;
- annotation: a triple derived by a declared scoring procedure.

Without such a procedure, numerical annotation values are illustrative rather than experimentally established.

9.5.3 Quantum parallelism and its limitation

For a Boolean function f, the oracle U_f acts linearly. If the input register is in a uniform superposition, then

$$U_f\left(\frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle|0\rangle\right)=\frac{1}{\sqrt{2^n}}\sum_x|x\rangle|f(x)\rangle.$$

This expression contains coherent contributions associated with all basis inputs. However, a single measurement does not reveal the full list of values $f(x)$. Quantum parallelism becomes algorithmically useful only when later operations exploit interference or other quantum correlations to concentrate information in measurable outcomes [2, 28].

Remark 9.5.2. An annotation for oracle completeness can express uncertainty about whether the physical device implements the promised function. It cannot, by itself, supply the missing function values or turn a single measurement into access to the entire superposition.

9.6 The Deutsch–Jozsa algorithm

9.6.1 Problem statement

Let $f:\{0,1\}^n\to\{0,1\}$ be promised to be either:

(i) constant, meaning $f(x)$ has the same value for every x; or

(ii) balanced, meaning $f(x)=0$ for exactly half of all inputs and $f(x)=1$ for the other half.

The Deutsch–Jozsa algorithm distinguishes these cases with one oracle query in the ideal quantum query model [28, 2].

9.6.2 Derivation

Prepare the state

$$|0\rangle^{\otimes n}|1\rangle.$$

Apply Hadamard gates to all $n+1$ qubits. The state becomes

$$\frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}.$$

Using the oracle and phase kickback, one obtains

$$\frac{1}{\sqrt{2^n}}\sum_x(-1)^{f(x)}|x\rangle\otimes\frac{|0\rangle-|1\rangle}{\sqrt{2}}.$$

Applying Hadamard gates to the first register gives amplitude

$$a_y = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)+x\cdot y}$$

for outcome y, where $x \cdot y$ is the bitwise inner product modulo two.

For $y = 0^n$,

$$a_{0^n} = \frac{1}{2^n} \sum_x (-1)^{f(x)}.$$

If f is constant, this amplitude is 1 or -1. If f is balanced, the positive and negative terms cancel and the amplitude is zero. Therefore:

$$\mathbb{P}(0^n) = \begin{cases} 1, & f \text{ is constant,} \\ 0, & f \text{ is balanced.} \end{cases}$$

9.6.3 Neutrosophic treatment of the promise

The algorithm's exact conclusion depends on the promise that the function belongs to one of the two specified classes. If the promise is not known to hold, the standard conclusion does not automatically apply.

Let P_f denote the proposition that the oracle function satisfies the promise. An annotation

$$\nu(P_f) = (t_f, i_f, f_f)$$

can summarize an external assessment of this proposition. Here the coordinate f_f is the falsity coordinate, not the Boolean function f.

If the promise has not been verified, a large indeterminacy coordinate may be appropriate. If independent tests or a trusted specification provide evidence, the annotation may be updated according to a predefined assessment method.

Proposition 9.6.1 (Conditional correctness)**.** *Suppose the oracle implements a Boolean function satisfying the constant-or-balanced promise and all gates are ideal. Then the Deutsch–Jozsa circuit classifies the two cases exactly. An annotation of uncertainty about the promise does not alter this conditional mathematical statement.*

Proof. The amplitude calculation above shows that the all-zero outcome has probability one in the constant case and zero in the balanced case. The proof uses the promise and the ideal oracle action. The annotation does not enter the calculation. If the promise fails, the hypotheses of the proposition are absent, so its conclusion is not guaranteed. □

Remark 9.6.2. The result is a statement about an ideal oracle model. The query advantage should not be confused with a complete comparison of physical runtime, oracle construction cost, or hardware reliability.

9.7 The quantum Fourier transform

9.7.1 Definition

For $N = 2^n$, the quantum Fourier transform is the unitary operator

$$F_N|x\rangle = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} e^{2\pi i x y/N} |y\rangle, \qquad 0 \le x < N.$$

Its matrix entries are

$$(F_N)_{y,x} = \frac{1}{\sqrt{N}} e^{2\pi i x y/N}.$$

The quantum Fourier transform is central to phase estimation and period-finding methods [2, 31].

9.7.2 A basic example

For $N = 2$,

$$F_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H.$$

For $N = 4$,

$$F_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

The transform is unitary because the columns are orthonormal. In particular, for distinct integers x, x' modulo N,

$$\frac{1}{N} \sum_{y=0}^{N-1} e^{2\pi i (x-x')y/N} = 0.$$

9.7.3 Annotated Fourier transform

Suppose F_N is approximated by a circuit $\widetilde{F}_N$. The ideal operator and its implementation should be distinguished:

$$F_N^\dagger F_N = \mathbb{I}, \qquad \widetilde{F}_N^\dagger \widetilde{F}_N = \mathbb{I}$$

only when the implementation is itself an exact unitary circuit. In a real experiment, the effective process may be noisy and may require a quantum-channel description.

An annotation

$$\nu_{\mathrm{QFT}} = (t_{\mathrm{QFT}}, i_{\mathrm{QFT}}, f_{\mathrm{QFT}})$$

may record the evidence supporting a claim that the chosen circuit approximates the desired transform within a stated tolerance. A quantitative statement requires a metric, such as operator norm, average gate fidelity, or a channel distance, together with a specified tolerance.

Definition 9.7.1 (Tolerance-based implementation claim)**.** Let U be an ideal unitary and $\widetilde{U}$ an implemented unitary. For a tolerance $\delta > 0$, define the claim

$$P_\delta : \quad \left\| U - \widetilde{U} \right\| \leq \delta,$$

where $\|\cdot\|$ denotes the operator norm. A neutrosophic annotation may be attached to the evidence for P_δ.

The annotation does not replace the norm inequality. If a verified bound is available, it should be reported directly; the annotation is supplementary.

9.8 Quantum phase estimation

9.8.1 Basic formulation

Let U be a unitary operator with an eigenvector $|u\rangle$ satisfying

$$U|u\rangle = e^{2\pi i\theta}|u\rangle, \qquad 0 \le \theta < 1.$$

The phase-estimation algorithm aims to estimate θ using controlled powers of U and an inverse quantum Fourier transform [2, 3].

With m control qubits, an ideal circuit prepares a superposition whose phase-dependent component is

$$\frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} e^{2\pi ik\theta}|k\rangle.$$

Applying the inverse Fourier transform produces a distribution concentrated near integers representing the binary approximation to θ. If θ has an exact m-bit binary representation, the ideal procedure recovers it exactly.

9.8.2 Uncertainty in the eigenphase model

In applications, the assumptions can be imperfect. The supplied state may not be an exact eigenvector, controlled powers may have implementation errors, and the physical operation may drift during the experiment.

Let P_{eig} denote the claim that the supplied state is an eigenstate of the intended operation to the required accuracy. Let P_{ctrl} denote the claim that the controlled powers are correctly implemented. An annotation can be assigned separately to each:

$$\nu_{\text{eig}} = (t_{\text{eig}}, i_{\text{eig}}, f_{\text{eig}}), \qquad \nu_{\text{ctrl}} = (t_{\text{ctrl}}, i_{\text{ctrl}}, f_{\text{ctrl}}).$$

Keeping these claims separate is useful because uncertainty about the input state is different from uncertainty about the gates.

Remark 9.8.1. A triple does not by itself give a bound on phase-estimation error. To obtain such a bound, the coordinates must be linked to a calibrated statistical model or to verified operator estimates. Otherwise, the annotation remains a structured qualitative or semiquantitative assessment.

9.9 Shor's factoring algorithm

9.9.1 Order finding

Shor's algorithm reduces integer factoring to an order-finding problem [31]. Given an integer N to factor and an integer a relatively prime to N, the order of a modulo N is the smallest positive integer r such that

$$a^r \equiv 1 \pmod{N}.$$

A quantum period-finding procedure uses phase estimation or an equivalent Fourier-analysis construction to obtain information about r. Classical post-processing, typically using continued fractions, then attempts to recover the order.

If r is even and

$$a^{r/2} \not\equiv -1 \pmod{N},$$

then nontrivial factors may be obtained from

$$\gcd(a^{r/2} - 1, N), \qquad \gcd(a^{r/2} + 1, N).$$

The algorithm is probabilistic overall, and its success depends on the choice of a and the successful recovery of useful period information.

9.9.2 Sources of uncertainty

A practical implementation may involve uncertainty in:

(a) the calibration and fidelity of controlled modular arithmetic;

(b) the reliability of the phase-estimation subroutine;

(c) the interpretation of measured Fourier outcomes;

(d) the classical post-processing and verification of factors.

We may associate annotations with these stages:

$$\nu_{\mathrm{mod}}, \quad \nu_{\mathrm{phase}}, \quad \nu_{\mathrm{meas}}, \quad \nu_{\mathrm{post}}.$$

Such annotations can help organize a risk report, but they must not be mistaken for a proof that the algorithm succeeds.

9.9.3 Verification as a separate stage

A proposed factor d of N can be checked by exact integer arithmetic:

$$1 < d < N, \qquad N \bmod d = 0.$$

This verification step is classical and deterministic. An annotation may express the confidence associated with the input data or software environment, but the divisibility test itself is an exact mathematical calculation for given integers.

Proposition 9.9.1 (Verified factor output)**.** *Suppose an algorithm outputs integers N and d and the exact conditions*

$$1 < d < N, \qquad d \mid N$$

are checked successfully. Then d is a nontrivial factor of N, independently of any neutrosophic annotation.

Proof. By definition, $d \mid N$ means that $N = dk$ for some integer k. Since $1 < d < N$, both d and k are nontrivial positive factors when N is a positive composite integer and the exact divisibility condition holds. □

Remark 9.9.2. Neutrosophic annotations are potentially useful for reporting uncertainty in the route by which a candidate factor was found. They do not weaken or replace the exact arithmetic verification.

9.10 Grover's quantum search algorithm

9.10.1 Search problem

Suppose there are N candidate items and M marked items. An oracle identifies marked items, and Grover's algorithm amplifies their amplitude [29, 2]. Define

$$\sin^2 \theta = \frac{M}{N}, \qquad 0 < \theta \leq \frac{\pi}{2}.$$

Starting from the uniform superposition, each Grover iteration rotates the state in the two-dimensional space spanned by the uniform superpositions of marked and unmarked items. After k iterations, the probability of measuring a marked item is

$$p_k = \sin^2((2k+1)\theta).$$

For $M \ll N$, an iteration count of order

$$k = O\left(\sqrt{\frac{N}{M}}\right)$$

can yield a high success probability. This is a quadratic improvement in query complexity over unstructured classical search, not an exponential speedup.

9.10.2 Example

Let $N = 16$ and $M = 1$. Then

$$\sin\theta = \frac{1}{4}, \qquad \theta = \arcsin\left(\frac{1}{4}\right).$$

For $k = 2$ iterations,

$$p_2 = \sin^2\left(5\arcsin\frac{1}{4}\right).$$

This is a high probability of finding the unique marked item in the ideal model, though it is not exactly one. The exact value is determined by the displayed expression.

9.10.3 Neutrosophic annotation of the oracle

The oracle is central to the search problem. If its marking rule is incorrect, the algorithm may amplify an unintended set. Let

$$P_{\text{oracle}}: \quad \text{“the oracle marks precisely the intended items.”}$$

An annotation

$$\nu_{\text{oracle}} = (t_{\text{oracle}}, i_{\text{oracle}}, f_{\text{oracle}})$$

can record evidence for this claim.

A separate annotation can be used for the implementation of the diffusion operator. This separation prevents a common conceptual error: an accurately implemented search circuit can still solve the wrong problem if the oracle specification is wrong.

Proposition 9.10.1 (Ideal Grover probability)**.** *Under the standard ideal oracle and diffusion operations, with M marked items among N items, the probability after k iterations is*

$$p_k = \sin^2((2k+1)\theta), \qquad \sin^2\theta = M/N.$$

This probability is determined by the quantum circuit and is independent of supplementary annotations.

Proof. The initial uniform state has amplitude $\sin\theta$ in the normalized uniform superposition of marked states and amplitude $\cos\theta$ in the corresponding unmarked subspace. The Grover iteration performs a rotation by angle 2θ in this plane. After k iterations, the angle from the unmarked direction is $(2k+1)\theta$, giving marked-state probability $\sin^2((2k+1)\theta)$. □

9.10.4 A limitation of annotation-based reliability claims

If the annotation coordinates have not been calibrated to observed hardware behavior, no numerical success probability follows from them. For example, the triple

$$(0.9, 0.05, 0.05)$$

does not establish a 90% probability of success. To make a probabilistic reliability statement, one must define the event, specify the data and estimation method, and account for statistical uncertainty.

9.11 An illustrative annotated algorithm

Consider a simplified workflow containing four components:

$$C_1 = \text{state preparation}, \quad C_2 = \text{oracle}, \quad C_3 = \text{interference circuit}, \quad C_4 = \text{measurement}.$$

Suppose an assessment protocol assigns the illustrative annotations

$$\begin{aligned}\nu_1 &= (0.95, 0.03, 0.02),\\ \nu_2 &= (0.80, 0.15, 0.05),\\ \nu_3 &= (0.90, 0.07, 0.03),\\ \nu_4 &= (0.92, 0.05, 0.03).\end{aligned}$$

These values are hypothetical and are not based on experimental data.

Using the serial annotation product from Section 9.4, the combined annotation is

$$\nu_{\text{chain}} = \nu_1 \odot \nu_2 \odot \nu_3 \odot \nu_4.$$

Its truth coordinate is

$$t_{\text{chain}} = 0.95 \cdot 0.80 \cdot 0.90 \cdot 0.92.$$

Its indeterminacy coordinate is

$$i_{\text{chain}} = 1 - (1 - 0.03)(1 - 0.15)(1 - 0.07)(1 - 0.05),$$

and its falsity coordinate is

$$f_{\text{chain}} = 1 - (1 - 0.02)(1 - 0.05)(1 - 0.03)(1 - 0.03).$$

These expressions are direct consequences of the chosen annotation rule. They are not automatically the probability of a successful quantum computation.

The example demonstrates a useful reporting principle: the annotation rule and its inputs should be made explicit so that the result can be reproduced. If the coordinates are intended to represent probabilities of component success, the protocol must justify that interpretation and its dependence assumptions.

9.12 From qualitative annotations to calibrated models

The annotation framework becomes scientifically useful only when its coordinates have operational meanings. Several approaches are possible.

9.12.1 Expert assessment

In an exploratory study, experts may assign coordinates based on available documentation, calibration records, or a structured checklist. Such values should be reported as judgments, with the assessment criterion and source of evidence stated.

9.12.2 Empirical calibration

Suppose a collection of trials yields data about a declared event E. A statistical procedure can estimate $\mathbb{P}(E)$ and quantify uncertainty in that estimate. A neutrosophic reporting layer could record, separately:

- support from the observed data;
- indeterminacy due to limited data or model uncertainty;
- evidence against the claim or evidence of failure.

A mapping from these statistical quantities to (t, i, f) must be specified before numerical comparisons are meaningful. Different mappings may lead to different triples.

9.12.3 Formal verification

For an ideal circuit, properties such as unitarity, equivalence of two circuit descriptions, or correctness under a formal oracle promise may be established mathematically. Such results should be stated as theorems or verified assertions, rather than represented only by numerical annotation scores.

For a physical implementation, formal verification of the circuit description does not by itself prove that the device realizes the description. Hardware calibration and experimental validation remain separate tasks.

9.13 Comparison with standard quantum analysis

The proposed approach supplements rather than competes with the standard mathematical tools of quantum information theory.

Question	**Standard tool**	**Possible annotation role**
What is the output distribution?	Born rule and quantum state	Record uncertainty about whether the model applies
Is the oracle promise satisfied?	Mathematical specification or verification	Summarize the status of the available evidence
How accurate is a gate?	Norm bounds, fidelity, or channel metrics	Record evidence and unresolved uncertainty
Does an algorithm succeed?	Correctness proof and success probability	Separate ideal correctness from implementation concerns
What is the query complexity?	Oracle-query analysis	Record assumptions that may affect interpretation

Table 9.1: Standard quantum analysis and the supplementary role of neutrosophic annotations.

The main advantage of this separation is conceptual clarity. A quantum algorithm can be mathematically correct under stated assumptions while the physical implementation, input specification, or experimental evidence remains uncertain. A single scalar “confidence” score

may hide these distinctions; a triple makes them explicit, provided the coordinates are properly defined.

However, the triple also introduces modeling choices. Without calibration, it does not supply an objective measure of uncertainty, and it does not automatically improve algorithmic complexity or computational performance.

9.14 Limitations and open problems

The framework is preliminary and has several limitations.

9.14.1 Non-uniqueness of the composition rule

The operation $\odot$ is one possible rule. Other applications may require different aggregation operators, especially when components are parallel, correlated, adaptive, or conditionally dependent. A principled comparison of such operators remains open.

9.14.2 No automatic probabilistic interpretation

The coordinates t, i, f do not automatically sum to one and are not Born-rule probabilities. A statistical interpretation requires an explicit mapping and empirical or theoretical justification.

9.14.3 Quantum channels and correlated noise

A realistic quantum process is often represented by a completely positive trace-preserving map rather than an ideal unitary. Extending the annotation scheme to quantum channels requires care: a scalar triple cannot replace a channel representation, its Kraus operators, or a quantitative error metric.

9.14.4 Complexity analysis

A neutrosophic annotation does not, by itself, establish a quantum speedup. Complexity statements require precise models of input size, oracle access, gate costs, error tolerance, and classical post-processing. The established analyses of Grover's search and Shor's algorithm remain the appropriate basis for their known query and computational complexity claims [29, 31].

9.14.5 Research questions

Several directions merit further investigation:

1. Develop calibrated annotation maps from experimental data to neutrosophic triples.
2. Compare serial and parallel aggregation rules under explicit dependence assumptions.
3. Study annotations for quantum channels and fault-tolerant circuits.
4. Define formal semantics for annotations of oracle promises.
5. Investigate whether annotation-based reporting improves reproducibility and communication of experimental limitations.
6. Test the framework on benchmark circuits and publish the underlying data and calibration procedure.

These questions define a research program; they are not claims that the present paper has already solved the corresponding problems.

9.15 Conclusion

We have proposed a neutrosophic annotation framework for describing quantum algorithms in the presence of incomplete specifications, uncertain oracle promises, and implementation limitations. The framework was illustrated using Hadamard transformations and interference, the Deutsch–Jozsa algorithm, the quantum Fourier transform, phase estimation, Shor's factoring algorithm, and Grover's search algorithm.

The principal methodological distinction is between the mathematical semantics of a quantum algorithm and the additional information associated with its specification or realization. Quantum states, unitary transformations, measurement probabilities, and correctness proofs remain standard quantum objects. Neutrosophic triples provide a supplementary vocabulary for reporting support, indeterminacy, and opposition relative to explicitly declared claims.

A composition operation was introduced and proved associative, but its interpretation was deliberately limited: it is an annotation convention, not a physical law. Similarly, numerical annotations cannot be treated as probabilities unless a calibration procedure establishes that meaning.

The next step toward a scientifically testable framework is to define operational annotation criteria, calibrate them against reproducible data, and compare their usefulness with established metrics of quantum circuit accuracy and reliability.

Chapter 10

Neutrosophic Teleportation and Superdense Coding

Related literature. A neutrosophic discussion of partial entanglement and teleportation is given in [74], building on earlier proposals [72].

Quantum teleportation and superdense coding are fundamental applications of entanglement in quantum information theory. This paper develops a mathematical framework for studying these protocols in a neutrosophic information setting. The central idea is to distinguish the quantum state itself from an additional triple-valued description of the reliability, indeterminacy, and falsity associated with reported classical information.

We first review the mathematical structure of standard teleportation and superdense coding. We then introduce neutrosophic annotations for measurement outcomes, classical communication records, and protocol-level reliability. A normalized special case is developed in which the three components are nonnegative and sum to one. We formulate propositions concerning annotation propagation, deterministic correction, and information capacity. We also discuss entanglement swapping and the Peres partial-transposition criterion, explaining why uncertainty annotations cannot replace the positivity and trace conditions of quantum mechanics.

The proposed framework is an information-layer extension rather than a new physical law. In particular, its operational value depends on a specified measurement model, a calibration procedure, and experimentally testable definitions of the neutrosophic components.

10.1 Introduction

Quantum information theory studies the representation, transformation, and transmission of information by quantum systems. The qubit, entanglement, quantum measurement, and unitary evolution provide the basic mathematical ingredients of this theory [2, 3].

Two important protocols are quantum teleportation and superdense coding. Teleportation transfers an unknown quantum state by combining shared entanglement, a joint measurement, classical communication, and a conditional quantum operation [33]. Superdense coding uses shared entanglement to communicate two classical bits by transmitting one qubit [34].

Neutrosophic logic introduces truth, indeterminacy, and falsity as separate descriptive components [6]. In applications involving imperfect information, these components can represent support for a statement, unresolved information, and evidence against the statement. Their interpretation depends on the particular application.

The purpose of this paper is to investigate how such descriptive components may be attached to records in quantum communication protocols. We do not assume that a neutrosophic triple

is itself a quantum state, nor do we replace Born probabilities with arbitrary triples. Instead, we distinguish the physical quantum state from an additional information layer.

The main objectives are:

(i) to formulate a precise notation for neutrosophic annotations of quantum communication records;

(ii) to describe teleportation with annotated classical outcomes;

(iii) to formulate an analogous description of superdense coding;

(iv) to examine entanglement swapping and partial transposition in the proposed framework; and

(v) to identify mathematical conditions required for a physically meaningful implementation.

The presentation of standard quantum protocols follows the established theory of quantum information [2, 3]. The additional neutrosophic constructions are proposed definitions whose empirical interpretation requires further study.

10.2 Preliminaries

10.2.1 Qubits and quantum states

A qubit is represented by a unit vector in

$$\mathcal{H}_2 = \mathbb{C}^2.$$

Relative to the computational basis $\{|0\rangle, |1\rangle\}$, a pure qubit state has the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1, \tag{10.1}$$

where $\alpha, \beta \in \mathbb{C}$.

A mixed quantum state is represented by a density operator ρ satisfying

$$\rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1. \tag{10.2}$$

For a pure state, $\rho = |\psi\rangle\langle\psi|$.

A measurement described by positive operators $\{E_j\}_{j=1}^m$ satisfies

$$E_j \geq 0, \qquad \sum_{j=1}^{m} E_j = \mathrm{I}.$$

The Born probability of outcome j is

$$p_j = \mathrm{Tr}(\rho E_j). \tag{10.3}$$

These probabilities remain the operational probabilities in the framework developed below.

10.2.2 Bell states

The four Bell states form an orthonormal basis of $\mathcal{H}_2 \otimes \mathcal{H}_2$:

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \tag{10.4}$$

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \tag{10.5}$$

$$|\Psi^+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \tag{10.6}$$

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \tag{10.7}$$

These states are maximally entangled and are central to both teleportation and superdense coding [2].

10.2.3 Neutrosophic information triples

Definition 10.2.1. A neutrosophic information triple is an ordered triple

$$n = (t, i, f),$$

where $t, i, f \in [0, 1]$. The components represent, respectively, truth-support, indeterminacy, and falsity-support for a specified proposition or record.

In the general neutrosophic tradition, the components need not sum to one [6]. For certain probabilistic applications, however, it is useful to consider the restricted class

$$\mathcal{S}_3 = \left\{(t, i, f) \in [0, 1]^3 : t + i + f = 1\right\}.$$

Definition 10.2.2. A normalized neutrosophic record is an element of $\mathcal{S}_3$. Its components are interpreted as three mutually exclusive descriptive categories: accepted, unresolved, and rejected.

Remark 10.2.3. The normalization $t + i + f = 1$ is an additional modeling assumption, not a universal axiom of neutrosophic logic. Moreover, the components are not automatically Born probabilities. Their probabilistic interpretation must be justified by the protocol's measurement and calibration model.

Definition 10.2.4. Let Ω be a finite set of classical records. A neutrosophic annotation is a map

$$\nu : \Omega \longrightarrow [0, 1]^3.$$

If $\nu(\omega) = (t_\omega, i_\omega, f_\omega)$, then the triple describes the available information about record ω.

For a normalized annotation, one imposes

$$t_\omega + i_\omega + f_\omega = 1 \quad \text{for every } \omega \in \Omega.$$

The annotation map is additional data. It does not change the Hilbert space, the quantum amplitudes, or the measurement probabilities unless an explicit physical model says otherwise.

10.2.4 Annotation propagation

Suppose a protocol produces a record ω and a deterministic processing rule transforms it into $g(\omega)$. One natural rule is to preserve the annotation:

$$\nu_{\text{out}}(g(\omega)) = \nu_{\text{in}}(\omega), \tag{10.8}$$

provided that g is injective on the relevant records or that the annotation is retained separately for each input record.

If several input records are merged into one output record, an aggregation rule is needed. Such a rule is not uniquely determined by the neutrosophic definitions and must be specified by the application.

10.3 Quantum teleportation

10.3.1 Standard teleportation protocol

Suppose Alice possesses an unknown qubit

$$|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$$

and Alice and Bob share the Bell state $|\Phi^+\rangle_{BC}$. Alice holds systems A and B, while Bob holds C.

The initial state is

$$|\Psi_0\rangle = |\psi\rangle_A \otimes |\Phi^+\rangle_{BC}. \tag{10.9}$$

Expanding gives

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}\left[\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right].$$

Expressing Alice's two qubits in the Bell basis yields

$$\begin{aligned}|\Psi_0\rangle = \frac{1}{2}\big[&|\Phi^+\rangle_{AB}(\alpha|0\rangle + \beta|1\rangle)_C \\ &+|\Phi^-\rangle_{AB}(\alpha|0\rangle - \beta|1\rangle)_C \\ &+|\Psi^+\rangle_{AB}(\beta|0\rangle + \alpha|1\rangle)_C \\ &+|\Psi^-\rangle_{AB}(\beta|0\rangle - \alpha|1\rangle)_C\big].\end{aligned} \tag{10.10}$$

Alice measures her two qubits in the Bell basis. Each of the four outcomes has probability $1/4$. She communicates two classical bits to Bob, who performs the corresponding unitary correction.

The standard correction table is

Alice's outcome	Classical bits	Bob's correction
$\|\Phi^+\rangle$	00	I
$\|\Phi^-\rangle$	01	Z
$\|\Psi^+\rangle$	10	X
$\|\Psi^-\rangle$	11	XZ

Here

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The correction associated with the final row is XZ for the displayed convention; global phases do not affect the resulting physical state.

Theorem 10.3.1 (Correctness of teleportation)**.** *In the ideal noiseless protocol, Bob recovers the unknown input state exactly after receiving Alice's classical outcome and applying the corresponding correction.*

Proof. Identity (10.10) gives the conditional state at Bob's site for each Bell outcome. The correction table applies the inverse of the corresponding conditional transformation, up to an irrelevant global phase. Therefore, for every possible outcome, Bob's corrected state is $\alpha|0\rangle + \beta|1\rangle$. □

This is the standard teleportation result [33, 2].

10.3.2 Neutrosophic annotation of teleportation

Let the outcome set be

$$\Omega_T = \{00, 01, 10, 11\}.$$

Associate with each classical outcome a neutrosophic annotation

$$\nu_T : \Omega_T \longrightarrow [0,1]^3, \qquad \nu_T(m) = (t_m, i_m, f_m). \tag{10.11}$$

The interpretation may be:

- t_m: support for the correctness of the reported measurement record;
- i_m: unresolved or ambiguous information about the record;
- f_m: support for the proposition that the record is incorrect.

These meanings refer to the classical record, not to the truth, indeterminacy, and falsity of the quantum state itself.

Definition 10.3.2. A neutrosophically annotated teleportation protocol consists of the standard teleportation protocol together with an annotation map ν_T on the classical outcome set and a specified rule for updating annotations after communication and correction.

For example, consider the normalized annotation

$$\nu_T(00) = (0.90, 0.08, 0.02).$$

This expresses a model-dependent assessment of the record 00. It does not imply that the quantum measurement outcome 00 has Born probability 0.90. For the ideal Bell measurement, its Born probability remains 1/4.

Proposition 10.3.3. *Suppose that the Bell measurement and correction operations are ideal, the classical message is received without alteration, and the annotation is passive, meaning that it does not modify the quantum operations. Then the quantum output state of the annotated protocol equals the quantum output state of the corresponding standard teleportation protocol.*

Proof. The assumptions leave the initial state, measurement operators, classical correction rule, and conditional unitary transformations unchanged. The annotation adds only a label to the classical record. Since no quantum operation depends on that label, the conditional output states remain the same as those of the standard protocol. Hence the output state is unchanged. □

Remark 10.3.4. This proposition is deliberately limited. If indeterminacy is intended to represent an actual classical bit-flip probability, detector failure rate, or loss of a message, it must be connected to a stochastic or quantum channel. In that case, the physical output need not be identical to the ideal output.

10.3.3 A reliability model for classical messages

A possible normalized model interprets the three components as probabilities of correct delivery, unresolved delivery, and incorrect delivery:

$$t + i + f = 1.$$

Suppose a binary classical message is transmitted through a symmetric bit-flip channel with error probability p. One may set

$$t = 1 - p, \qquad i = 0, \qquad f = p.$$

This is a conventional probability model embedded into a three-component notation. It does not introduce a new physical channel.

If the error event cannot be reliably classified, one may instead assign positive weight to i. The method for estimating that weight must be determined from the communication system.

10.3.4 Fidelity under a classical correction error

Let U_m denote the correction that should be applied for outcome m, and suppose Bob instead applies $U_{\widehat{m}}$. The resulting state is

$$|\psi_{\text{out}}\rangle = U_{\widehat{m}} U_m^\dagger |\psi\rangle.$$

Its fidelity with the desired state is

$$F(\psi) = \left| \langle \psi | U_{\widehat{m}} U_m^\dagger | \psi \rangle \right|^2. \tag{10.12}$$

For the Pauli correction set, a nontrivial mismatch can cause a state-dependent fidelity loss. The neutrosophic annotation can describe uncertainty about the message, but the physical fidelity is computed from the actual operation and input state.

10.4 Superdense coding

10.4.1 Standard protocol

Superdense coding allows Alice to communicate one of four classical messages by sending one qubit, provided Alice and Bob initially share a Bell pair [34].

Suppose the shared state is

$$|\Phi^+\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

Alice encodes two classical bits by applying one of four unitary operators to her qubit:

$$00 \mapsto \mathrm{I}, \quad 01 \mapsto Z, \quad 10 \mapsto X, \quad 11 \mapsto XZ.$$

The resulting states are, up to global phase,

Message	Shared state
00	$\|\Phi^+\rangle$
01	$\|\Phi^-\rangle$
10	$\|\Psi^+\rangle$
11	$\|\Psi^-\rangle$

Alice sends her qubit to Bob. Bob performs a Bell-basis measurement on the two qubits and identifies the message.

Theorem 10.4.1 (Ideal superdense coding)**.** *With a shared maximally entangled qubit pair, noiseless transmission of Alice's qubit, and an ideal Bell measurement, the protocol transmits two classical bits by sending one qubit.*

Proof. The four encoding operations transform the shared Bell state into four mutually orthogonal Bell states. An ideal measurement in this basis identifies the encoded message without error. Thus Alice's choice among four messages is recovered from one transmitted qubit together with the preshared entanglement. □

The result is an established consequence of quantum entanglement [34, 2].

10.4.2 Neutrosophic encoding records

Let

$$\Omega_S = \{00, 01, 10, 11\}$$

be the set of possible messages, and define

$$\nu_S : \Omega_S \longrightarrow [0,1]^3.$$

For each message m, the triple

$$\nu_S(m) = (t_m, i_m, f_m)$$

describes the assessment of the associated message record.

For example, suppose the received message is 10 and its annotation is

$$\nu_S(10) = (0.96, 0.03, 0.01).$$

The triple may represent a high-confidence record under a specified calibration procedure. It does not change the orthogonality of the Bell states.

Proposition 10.4.2. *If neutrosophic annotations are passive and the encoding, transmission, and decoding operations are ideal, then adding the annotations does not alter the two-bit decoding capacity of the standard superdense-coding protocol.*

Proof. The four encoded quantum states remain the same orthogonal Bell states. The decoder therefore distinguishes the four messages exactly as in the standard protocol. Since the annotation layer does not alter the states or measurement, the decoding statistics remain unchanged. □

Remark 10.4.3. This result should not be interpreted as an increase in quantum channel capacity. A new capacity theorem would require a precise definition of the physical channel, the allowed resources, and the operational meaning of the neutrosophic information.

10.4.3 A noisy-message example

Suppose a classical message used in a protocol passes through a binary symmetric channel with bit-flip probability p. Under independent errors on the two bits, the probability that both bits arrive correctly is

$$P(\text{correct message}) = (1-p)^2.$$

The probability of exactly one bit error is

$$P(\text{one error}) = 2p(1-p),$$

and the probability of two bit errors is

$$P(\text{two errors}) = p^2.$$

These three probabilities sum to one. They can be used as a normalized three-category record if the categories are explicitly defined as correct, one-bit error, and two-bit error. However, this interpretation differs from assigning truth, indeterminacy, and falsity to a single proposition. The categories must not be conflated without an explicit modeling rule.

10.5 Comparison of teleportation and superdense coding

The two protocols use shared entanglement in different ways.

Feature	Teleportation	Superdense coding
Primary objective	Transfer an unknown quantum state	Transmit classical information
Shared resource	Entangled qubit pair	Entangled qubit pair
Classical communication	Two classical bits	No classical message required for decoding
Quantum transmission	No transmission of the unknown qubit itself	One qubit sent to Bob
Output	Reconstructed quantum state	Decoded classical message
Neutrosophic annotation	Measurement and correction records	Encoding and decoding records

In both protocols, the physical role of entanglement is independent of whether one adds a separate neutrosophic description of the classical records. This separation makes it possible to investigate imperfect information without changing the established quantum-mechanical formalism.

10.6 Entanglement swapping

10.6.1 Standard entanglement swapping

Entanglement swapping extends the teleportation idea. Suppose Alice and Bob share $|\Phi^+\rangle_{AB}$ and Charlie and David share $|\Phi^+\rangle_{CD}$. The joint state can be written

$$\begin{aligned}|\Phi^+\rangle_{AB}|\Phi^+\rangle_{CD} = \frac{1}{2}\big[&|\Phi^+\rangle_{BC}|\Phi^+\rangle_{AD} + |\Phi^-\rangle_{BC}|\Phi^-\rangle_{AD} \\ &+|\Psi^+\rangle_{BC}|\Psi^+\rangle_{AD} + |\Psi^-\rangle_{BC}|\Psi^-\rangle_{AD}\big].\end{aligned} \tag{10.13}$$

The Bell states on the right are arranged with the outer systems A, D and middle systems B, C.

If Bob and Charlie perform a Bell measurement on B, C, the outer systems A, D are projected onto a corresponding Bell state, conditional on the measurement outcome. Local corrections can convert the conditional state into a chosen Bell state [37].

10.6.2 Neutrosophic description of swapping

Let Ω_E denote the four Bell-measurement outcomes in the swapping experiment. An annotation

$$\nu_E : \Omega_E \longrightarrow [0,1]^3$$

can record the confidence, ambiguity, and error assessment associated with each reported outcome.

For example, a detector record may be annotated by

$$\nu_E(\Phi^+) = (t, i, f).$$

The triple does not, by itself, specify the conditional density matrix of the outer systems. That matrix must be calculated from the measurement instrument and the actual outcome.

Proposition 10.6.1. *A passive annotation of the Bell-measurement record does not alter the conditional outer-pair state in an ideal entanglement-swapping experiment.*

Proof. Equation (10.13) determines the conditional states under the standard Bell measurement. If the annotation does not change the measurement operators, the selected outcome, or the correction operations, the conditional state is unchanged. □

Remark 10.6.2. If the annotation instead represents detector misidentification, then it should be combined with a detector confusion matrix. In that situation, the state assigned to a reported outcome can differ from the state assigned to the actual outcome.

10.7 The Peres partial-transposition condition

10.7.1 Partial transpose

Let ρ be a bipartite density operator on $\mathcal{H}_A \otimes \mathcal{H}_B$. In product bases, write

$$\rho = \sum_{i,j,k,l} \rho_{ij,kl} |i\rangle\langle j| \otimes |k\rangle\langle l|.$$

The partial transpose with respect to subsystem B is defined by

$$\rho^{T_B} = \sum_{i,j,k,l} \rho_{ij,kl} |i\rangle\langle j| \otimes |l\rangle\langle k|. \tag{10.14}$$

The Peres criterion states that if ρ is separable, then

$$\rho^{T_B} \geq 0. \tag{10.15}$$

Thus a negative eigenvalue of ρ^{T_B} proves that the state is entangled [36].

For bipartite systems of dimensions 2×2 and 2×3, positivity under partial transpose is also sufficient for separability [35]. In larger dimensions it is necessary but not sufficient.

10.7.2 Bell-state example

Consider the Bell state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad \rho_{\Phi^+} = |\Phi^+\rangle\langle\Phi^+|.$$

Its density matrix in the computational basis is

$$\rho_{\Phi^+} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Taking the partial transpose on the second subsystem gives

$$\rho_{\Phi^+}^{T_B} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues are

$$\frac{1}{2}, \quad \frac{1}{2}, \quad \frac{1}{2}, \quad -\frac{1}{2}.$$

The negative eigenvalue proves that the Bell state is entangled.

10.7.3 Neutrosophic annotations and positivity

It is tempting to attach a triple

$$\nu(\rho) = (t, i, f)$$

to a density operator. Such notation is permissible as an external descriptive label, but a triple alone does not determine a density operator or its entanglement properties.

Proposition 10.7.1. *Suppose a density operator ρ is accompanied by a passive neutrosophic annotation $\nu(\rho)$. The annotation does not change the eigenvalues of ρ^{T_B} and therefore does not change the conclusion of the Peres criterion.*

Proof. The partial transpose is computed from the matrix entries of ρ. A passive annotation leaves all those entries unchanged. Hence ρ^{T_B} and its eigenvalues remain unchanged. □

Remark 10.7.2. A different theory could attempt to define neutrosophic density operators. Such a theory would need to specify the state space, positivity conditions, trace normalization, measurement probabilities, and the behavior of composite systems. These requirements cannot be replaced by assigning three numbers to an ordinary density operator.

10.8 A channel-based formulation

A more operational approach is to model imperfections through a quantum channel. A completely positive, trace-preserving map has a Kraus representation

$$\mathcal{E}(\rho) = \sum_k K_k \rho K_k^\dagger, \qquad \sum_k K_k^\dagger K_k = \mathrm{I}. \tag{10.16}$$

This formalism provides a standard mathematical description of noise and information loss [2, 3].

The neutrosophic annotation can be attached to the channel's classical metadata, detector outcomes, or estimated error categories. For instance, an experimental system might store:

- a measured quantum output state;
- an estimated probability of a correct record;

- an uncertainty score for the calibration;
- an estimated probability of a wrong record.

If these quantities are normalized and calibrated, they can be represented by a triple (t, i, f). If the three quantities have overlapping meanings or are not exhaustive, normalization need not be appropriate.

Definition 10.8.1. A channel-annotated neutrosophic quantum protocol is a tuple

$$\mathcal{P} = (\rho, \mathcal{E}, \mathcal{M}, \nu),$$

where ρ is an input quantum state, $\mathcal{E}$ is a specified quantum channel, $\mathcal{M}$ is a measurement procedure, and ν assigns neutrosophic annotations to the resulting classical records.

This definition explicitly separates the physical components $(\rho, \mathcal{E}, \mathcal{M})$ from the descriptive component ν.

Proposition 10.8.2. *For a fixed physical protocol $(\rho, \mathcal{E}, \mathcal{M})$, changing only the annotation map ν does not change the distribution of physical measurement outcomes.*

Proof. The outcome probabilities are determined by the input state, the channel, and the measurement operators. Since these quantities are fixed, the Born probabilities remain fixed. The map ν changes only the annotations assigned after the physical outcomes are obtained. □

Remark 10.8.3. If the annotation is used to control subsequent operations, then the complete protocol must include that feedback rule. The preceding proposition applies only when the annotation is passive.

10.9 Information-theoretic interpretation

Let M be a classical message with probability distribution $p(m)$. Its Shannon entropy is

$$H(M) = -\sum_m p(m) \log_2 p(m).$$

For four equiprobable messages, $H(M) = 2$ bits.

Suppose an annotation assigns to each received message a normalized triple

$$\nu(m) = (t_m, i_m, f_m).$$

The triples do not, on their own, determine the conditional distribution of the actual message given the received record. Consequently, they do not determine mutual information or channel capacity without additional assumptions.

If t_m, i_m, f_m are explicitly defined as probabilities of three mutually exclusive outcomes, the resulting probability model can be analyzed using ordinary information theory. For example, its entropy can be calculated as

$$H(\nu(m)) = -t_m \log_2 t_m - i_m \log_2 i_m - f_m \log_2 f_m,$$

with the convention $0 \log_2 0 = 0$.

This quantity is the entropy of the three-category model, not automatically the entropy of the quantum state or the transmitted message.

Remark 10.9.1. A proposed neutrosophic capacity theory should define the encoder, decoder, resource constraints, and error criterion. It should then prove a capacity theorem rather than infer a new capacity from the presence of an additional descriptive triple.

10.10 Worked comparative example

Consider an ideal teleportation experiment repeated N times. Alice's Bell measurement produces four outcomes, each with probability $1/4$. Suppose a calibration procedure assigns the same normalized annotation to each record:

$$\nu(m) = (0.97, 0.02, 0.01).$$

The expected number of occurrences of each outcome is $N/4$. The annotation may be interpreted as a separate assessment of record reliability, but it does not alter this expected count.

Now suppose an independently calibrated classical channel has bit-flip probability $p = 0.01$. Under a symmetric bit-flip model, one may instead assign

$$(t, i, f) = (0.99, 0, 0.01).$$

If the two classical bits are independently transmitted, the probability that both are correct is

$$(0.99)^2 = 0.9801.$$

These two triples have different meanings: the first is a proposed annotation of a measurement record, whereas the second represents a specific classical error model. A rigorous application must state which interpretation is intended.

10.11 Limitations and research directions

The framework presented here is preliminary and should be developed further before being regarded as a distinct physical theory.

10.11.1 Operational meaning

The terms truth, indeterminacy, and falsity must be defined for a particular experiment. A practical definition could use independently calibrated detector confidence, unresolved events, and identified errors. Different experiments may require different interpretations.

10.11.2 Consistency with quantum mechanics

The density operator must remain positive semidefinite and trace one. Quantum channels must be completely positive and trace preserving when they describe deterministic physical evolution. A neutrosophic extension must explain how these constraints are preserved.

10.11.3 Composition of systems

A multi-qubit theory requires a rule for combining annotations under tensor products and measurements. Simple componentwise multiplication or addition is not automatically justified. Any proposed rule should be checked against independence, correlated errors, and entangled inputs.

10.11.4 Experimental validation

A meaningful test should compare predictions from the annotated model with measured frequencies, calibration data, and established noise models. The additional parameters should improve a specified prediction or uncertainty assessment without double-counting ordinary probabilities.

10.11.5 Potential research questions

The following questions may guide further work:

1. Can a normalized neutrosophic annotation be derived from a well-defined detector confusion model?
2. Which annotation-composition rules preserve normalization under sequential communication?
3. Can uncertainty about classical records be incorporated into a Bayesian estimate of teleportation fidelity?
4. What operational task, if any, is improved by the annotation layer?
5. Can a channel-based formulation be developed that has experimentally testable predictions beyond those of standard quantum information theory?

10.12 Conclusion

We have developed a preliminary framework for attaching neutrosophic information triples to records arising in quantum teleportation, superdense coding, and entanglement swapping. We have also examined the relation between such annotations and the Peres partial-transposition criterion.

The central mathematical distinction is between a quantum state and a description of information about that state or its associated classical records. In the passive setting studied here, the annotation does not alter quantum probabilities, teleportation fidelity, Bell-state orthogonality, or the partial-transpose criterion. If the annotation is intended to represent physical uncertainty, it must be connected to an explicit stochastic or quantum channel.

Accordingly, the present construction is best regarded as a proposed information-layer formalism. Its development into a substantive neutrosophic quantum information theory requires precise operational definitions, composition rules, and experimentally testable results.

Chapter 11

Neutrosophic Quantum Cryptography

Related literature. The distinction between logical indeterminacy and quantum measurement probability is emphasized in [75]. Proposed neutrosophic quantum-state formalisms are explored in [73].

Quantum cryptography uses quantum-mechanical principles to establish and protect cryptographic information. Quantum key distribution (QKD), in particular, permits two legitimate parties to establish a shared secret key while testing for disturbances that may be caused by an eavesdropper. This paper develops a proposed neutrosophic information layer for describing confidence, indeterminacy, and suspected error in quantum cryptographic records.

We distinguish ordinary quantum states and Born probabilities from neutrosophic annotations attached to preparation records, measurement outcomes, authenticated classical messages, and parameter estimates. The framework is applied to the BB84 and B92 protocols, an entanglement-based E91 protocol, and a simplified ancilla-based attack model. We also discuss the relationship between classical RSA cryptography and quantum computation.

Several elementary propositions establish that passive annotations do not change standard quantum statistics. A channel-based model is then used to show how indeterminacy can be connected to experimentally measurable error and inconclusive-outcome rates. The paper emphasizes that a neutrosophic annotation does not, by itself, establish cryptographic security. Security claims require explicit assumptions, authentication, parameter estimation, and a suitable security proof.

11.1 Introduction

Cryptography studies methods for protecting information against unauthorized access and manipulation. Classical cryptographic systems use mathematical problems, secret keys, and computational assumptions. RSA, for example, is based on modular arithmetic and the computational difficulty of factoring suitably chosen large integers [45].

Quantum cryptography introduces a different resource: the physical behavior of quantum systems. In quantum key distribution, two parties prepare, transmit, and measure quantum states to establish a shared key. The protocol includes public communication and statistical tests designed to detect disturbances [39, 41].

The subject of this paper is the possible use of neutrosophic information triples to represent uncertainty in the records generated by these protocols. Neutrosophic logic distinguishes truth, indeterminacy, and falsity as separate descriptive components [6]. In a cryptographic setting, such components might describe confidence in a measurement record, an unresolved event, and evidence of a faulty or inconsistent record.

The distinction between physical probabilities and descriptive annotations is essential. The probability of a quantum measurement outcome is determined by the quantum state and the measurement operators. A neutrosophic triple does not replace the Born rule or automatically define a new quantum state.

The principal objectives are:

(i) to introduce a precise notation for neutrosophic annotations of cryptographic records;

(ii) to formulate annotated descriptions of BB84, B92, and E91 quantum key distribution;

(iii) to investigate how annotations can represent inconclusive measurements, noise, and suspected eavesdropping;

(iv) to examine the limitations of such annotations in a simplified controlled-NOT attack model; and

(v) to identify the conditions needed for a cryptographically meaningful implementation.

The standard quantum protocols discussed below are established results in quantum information theory [2, 46]. The neutrosophic constructions are proposed modeling extensions and are not claimed to constitute a new unconditional security theorem.

11.2 Classical and quantum cryptographic preliminaries

11.2.1 Classical encryption and RSA

A classical public-key cryptosystem has an encryption algorithm Enc, a decryption algorithm Dec, a public key, and a private key. RSA was introduced by Rivest, Shamir, and Adleman [45].

Choose two large primes p and q and define

$$n = pq, \qquad \varphi(n) = (p-1)(q-1).$$

Choose an integer e relatively prime to $\varphi(n)$ and an integer d satisfying

$$ed \equiv 1 \pmod{\varphi(n)}.$$

For a message representative m with $0 \leq m < n$, the textbook RSA operations are

$$c \equiv m^e \pmod{n}, \qquad m \equiv c^d \pmod{n}.$$

This elementary description illustrates the arithmetic structure of RSA. Practical RSA encryption requires appropriate encoding and padding; textbook RSA alone is not a secure encryption scheme.

Shor's quantum algorithm can factor integers in polynomial time in their bit length in the standard quantum circuit model [47]. Consequently, a sufficiently capable fault-tolerant quantum computer would threaten RSA implementations whose security depends on factoring. This does not mean that ordinary quantum key distribution is a direct replacement for all public-key cryptography: QKD and public-key encryption solve different operational problems.

11.2.2 Quantum states and measurements

A qubit is a unit vector in $\mathbb{C}^2$:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1.$$

The computational basis is $\{|0\rangle, |1\rangle\}$. The diagonal basis is

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

For a density operator ρ and a measurement effect E_j, the Born probability is

$$p_j = \mathrm{Tr}(\rho E_j). \tag{11.1}$$

A measurement described by effects $\{E_j\}$ satisfies

$$E_j \geq 0, \qquad \sum_j E_j = \mathrm{I}.$$

In cryptographic applications, measurement outcomes are recorded as classical data. Those records may contain detector errors, missing events, or uncertain basis information. Such imperfections motivate an additional descriptive layer.

11.3 Neutrosophic annotations

Definition 11.3.1. A neutrosophic triple is an ordered triple

$$\nu = (t, i, f), \qquad t, i, f \in [0, 1],$$

where t, i, and f denote truth-support, indeterminacy, and falsity-support, respectively, for a specified proposition or record.

The general neutrosophic framework does not require the three components to sum to one [6]. In a restricted probabilistic model, it is useful to consider

$$\mathcal{N}_3 = \{(t, i, f) \in [0, 1]^3 : t + i + f = 1\}.$$

Definition 11.3.2. A normalized neutrosophic annotation is an element of $\mathcal{N}_3$. It can be interpreted probabilistically only when the three categories are defined as mutually exclusive and exhaustive events.

Remark 11.3.3. The words "truth" and "falsity" do not automatically have operational probability meanings. In this paper, their interpretation must be specified for each cryptographic application. For example, t may denote support for the correctness of a reported detector record, i unresolved status, and f evidence of a recording error.

Definition 11.3.4. Let Ω be a finite set of classical records. A neutrosophic annotation map is a function

$$\nu : \Omega \longrightarrow [0, 1]^3.$$

For $\omega \in \Omega$, write

$$\nu(\omega) = (t_\omega, i_\omega, f_\omega).$$

A record may contain a preparation label, a measurement result, a basis label, a time stamp, or an authenticated message. An annotation is additional metadata; it is not itself a quantum state.

11.3.1 Passive and operational annotations

Definition 11.3.5. An annotation is *passive* if it does not change the state preparation, quantum channel, measurement, classical communication, or decision rule of a protocol.

Definition 11.3.6. An annotation is *operational* if it affects a subsequent decision, such as whether a detection event is accepted, discarded, or subjected to additional testing.

These two notions must be distinguished. Passive annotations preserve the standard physical protocol. Operational annotations may change its accepted data and therefore require a new analysis of its statistics and security.

Proposition 11.3.7. *Let a quantum cryptographic protocol have fixed input states, channels, and measurements. If a neutrosophic annotation is passive, then the Born probability of each physical measurement outcome is unchanged.*

Proof. For each measurement effect E_j, the outcome probability is given by (11.1). A passive annotation changes neither ρ nor E_j. Hence

$$\operatorname{Tr}(\rho E_j)$$

is unchanged for every outcome j. □

Remark 11.3.8. The proposition does not establish that the protocol remains secure under arbitrary operational use of the annotations. Security also depends on authentication, device assumptions, information leakage, and the post-processing procedure.

11.4 The BB84 protocol

11.4.1 Standard protocol

The BB84 protocol was introduced by Bennett and Brassard [39]. Alice chooses a random bit and a random basis, computational or diagonal, and sends the corresponding qubit to Bob. Bob independently chooses a measurement basis.

The encoding table is

Bit	Computational	Diagonal
0	$\lvert 0\rangle$	$\lvert +\rangle$
1	$\lvert 1\rangle$	$\lvert -\rangle$

After transmission, Alice and Bob publicly compare their basis choices over an authenticated classical channel. They retain positions where their bases agree. These retained positions form the sifted key. They disclose a sample of sifted bits to estimate the quantum bit error rate (QBER), then perform suitable error correction and privacy amplification.

The public discussion must be authenticated. Without authentication, an active adversary may impersonate the parties or modify the classical communication.

11.4.2 Why basis mismatch matters

The computational and diagonal bases are mutually unbiased. For example,

$$|\langle 0 \mid + \rangle|^2 = |\langle 0 \mid - \rangle|^2 = \frac{1}{2}.$$

Therefore, a measurement in the wrong basis produces a random bit in the ideal model.

This property is central to detecting certain eavesdropping attacks. An eavesdropper who measures unknown states in a randomly chosen basis may disturb the states later measured by Bob.

11.4.3 Neutrosophic annotation of BB84 records

Let

$$\Omega_{\mathrm{BB84}} = \{(a, b, x, y) : a, b, x, y \in \{0, 1\}\},$$

where a is Alice's basis label, b is Bob's basis label, x is Alice's encoded bit, and y is Bob's measurement outcome.

A neutrosophic map

$$\nu_{\mathrm{BB84}} : \Omega_{\mathrm{BB84}} \longrightarrow [0, 1]^3$$

may annotate each recorded event.

For example,

$$\nu_{\mathrm{BB84}}(a, b, x, y) = (0.96, 0.03, 0.01)$$

might describe a detector record considered reliable with a small unresolved component and a small estimated error component.

This triple does not replace the conditional quantum probabilities. In the ideal BB84 model, if the bases agree and there is no noise or eavesdropping, Bob obtains Alice's bit with probability one. The annotation instead describes an external assessment of the recorded event.

11.4.4 QBER as an operational statistic

For N disclosed sifted-key positions, define

$$\widehat{q} = \frac{N_{\mathrm{err}}}{N},$$

where N_{err} is the number of positions where the disclosed bits disagree.

The observed QBER is an empirical statistic. It is not identical to a neutrosophic indeterminacy score. A high unresolved-record rate and a high bit-error rate can have different causes and must be measured separately.

Proposition 11.4.1. *Suppose an annotation is passive and the accepted sifted-key positions are fixed independently of the annotation. Then adding the annotation does not change the observed QBER.*

Proof. The QBER depends only on the number of accepted positions and the number of disagreements among them. A passive annotation changes neither the accepted positions nor the recorded bit values. Therefore, the ratio defining $\widehat{q}$ is unchanged. □

11.4.5 An ideal intercept-resend example

Consider an intercept-resend attacker who chooses between the two BB84 bases uniformly at random, measures each intercepted qubit, and resends the measured state.

For a sifted-key position, Alice and Bob used the same basis. The attacker chooses the wrong basis with probability 1/2. Conditional on this event, the resent state gives Bob the wrong bit with probability 1/2. Thus, under the idealized attack,

$$q_{\mathrm{IR}} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

The expected QBER introduced by this attack is therefore 25% in the ideal model. Real implementations may have additional detector errors and other imperfections.

A neutrosophic annotation may record uncertainty about a measured QBER or the reliability of a detector event. It does not alter the derivation above.

11.5 A neutrosophic treatment of inconclusive records

Some quantum measurements can produce an inconclusive outcome. This is especially relevant to protocols using nonorthogonal states, such as B92.

Definition 11.5.1. Let a measurement record take values in

$$\Omega = \{\text{accepted}, \text{inconclusive}, \text{rejected}\}.$$

A normalized record model assigns probabilities

$$t = \Pr(\text{accepted}), \quad i = \Pr(\text{inconclusive}), \quad f = \Pr(\text{rejected}),$$

with

$$t + i + f = 1.$$

Remark 11.5.2. This is a three-category probability model using neutrosophic notation. It is not the only possible interpretation of a neutrosophic triple. Here, indeterminacy is explicitly identified with an inconclusive outcome.

Suppose a measurement has an acceptance probability p_{acc} and an inconclusive probability p_{inc}. If all remaining events are classified as rejected, then

$$t = p_{\text{acc}}, \qquad i = p_{\text{inc}}, \qquad f = 1 - p_{\text{acc}} - p_{\text{inc}}.$$

This interpretation is valid only if the three categories are mutually exclusive and exhaustive.

Proposition 11.5.3. *If $p_{\text{acc}} \geq 0$, $p_{\text{inc}} \geq 0$, and $p_{\text{acc}} + p_{\text{inc}} \leq 1$, the triple defined above belongs to $\mathcal{N}_3$.*

Proof. All three components are nonnegative. Moreover,

$$t + i + f = p_{\text{acc}} + p_{\text{inc}} + 1 - p_{\text{acc}} - p_{\text{inc}} = 1.$$

Each component is at most one, so the triple belongs to $\mathcal{N}_3$. □

This elementary result gives a mathematically precise case in which a normalized triple can summarize experimental categories. It does not establish a new quantum-security property.

11.6 The B92 protocol

11.6.1 Nonorthogonal signal states

Bennett introduced the B92 protocol using two nonorthogonal quantum states [38]. A simple realization uses

$$|\psi_0\rangle = |0\rangle, \qquad |\psi_1\rangle = |+\rangle.$$

Their overlap is

$$|\langle\psi_0 \mid \psi_1\rangle| = \frac{1}{\sqrt{2}},$$

so the states are not orthogonal.

Unlike orthogonal states, these two states cannot be perfectly distinguished in a single measurement without an inconclusive result for all possible inputs. B92 uses measurement outcomes to identify certain transmitted bits and discards inconclusive events.

A complete implementation must specify the measurement operators, the treatment of loss, and the security assumptions.

11.6.2 A simple unambiguous-discrimination model

For two equally likely pure states with overlap $c = |\langle\psi_0 \mid \psi_1\rangle|$, the optimal unambiguous state-discrimination measurement has average inconclusive probability c under the standard ideal assumptions [42, 40, 44].

For the two states above, $c = 1/\sqrt{2}$. Hence the optimal unambiguous measurement has

$$p_{\text{inc}} = \frac{1}{\sqrt{2}}, \qquad p_{\text{acc}} = 1 - \frac{1}{\sqrt{2}}.$$

Numerically,

$$p_{\text{inc}} \approx 0.7071, \qquad p_{\text{acc}} \approx 0.2929.$$

These probabilities concern this idealized unambiguous-discrimination task. They should not be mistaken for the key-generation rate of every B92 implementation, since practical protocols depend on their exact preparation, measurement, loss, and post-processing procedures.

11.6.3 Neutrosophic record for B92

The three-category model gives

$$\nu_{\text{B92}} = \left(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right),$$

if acceptance, inconclusive detection, and rejection are taken as the three exhaustive categories and no rejection category is present in the idealized measurement.

The interpretation is:

- t: probability of a conclusive accepted event;
- i: probability of an inconclusive event;
- f: probability of a rejected event.

Here the second component has a direct operational meaning. In other contexts, a neutrosophic indeterminacy score may instead represent uncertainty in the calibration or the classification of a record.

Remark 11.6.1. An inconclusive measurement is not the same as a false measurement. It can be a valid, explicitly predicted outcome of the measurement procedure. Consequently, it should not automatically be classified as evidence of eavesdropping.

11.7 The E91 protocol and entanglement

11.7.1 Entanglement-based quantum key distribution

Ekert proposed an entanglement-based approach to quantum cryptography using correlated measurements on shared entangled states [41].

Consider the singlet state

$$|\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

For spin measurements along unit vectors $\mathbf{a}$ and $\mathbf{b}$, the ideal quantum correlation is

$$E(\mathbf{a}, \mathbf{b}) = \langle(\mathbf{a} \cdot \boldsymbol{\sigma}) \otimes (\mathbf{b} \cdot \boldsymbol{\sigma})\rangle = -\mathbf{a} \cdot \mathbf{b},$$

where $\boldsymbol{\sigma}$ denotes the vector of Pauli operators.

Selected measurement settings can be used to estimate a Bell-inequality expression. A violation of an appropriate Bell inequality rules out local hidden-variable models satisfying the assumptions of that inequality. In quantum cryptography, observed correlations can contribute to assessing the security of a key, provided that the protocol and its security proof justify the inference.

11.7.2 Neutrosophic annotations of correlations

Let the measurement setting pair be (a, b) and the outcomes be $(x, y) \in \{-1, +1\}^2$. A record is

$$\omega = (a, b, x, y).$$

An annotation map

$$\nu_{\mathrm{E91}}(\omega) = (t_\omega, i_\omega, f_\omega)$$

may summarize the reliability of the record.

For a finite sample, the empirical correlation for settings (a, b) is

$$\widehat{E}(a, b) = \frac{1}{N_{ab}} \sum_{k=1}^{N_{ab}} x_k y_k,$$

where N_{ab} is the number of valid records for that setting pair.

The empirical correlation is calculated from observed outcomes. It is distinct from an annotation describing confidence in those outcomes.

Proposition 11.7.1. *If the annotations are passive and do not change which measurement records enter the correlation estimate, then the empirical correlation $\widehat{E}(a, b)$ is unchanged.*

Proof. The expression for $\widehat{E}(a, b)$ depends on the observed values x_k, y_k and the sample size N_{ab}. Passive annotations change none of these quantities. The sum and its denominator therefore remain the same. □

11.7.3 A limitation of the annotation approach

A high-confidence annotation does not establish that a Bell test is loophole-free or that the key is secure. For example, detection losses, setting dependence, imperfect random-number generation, and classical communication assumptions can affect the interpretation of observed correlations.

A security analysis must state the assumptions and derive a bound on the information available to an adversary. Neutrosophic triples may supplement the experimental record but cannot replace that derivation.

11.8 A controlled-NOT attack model

11.8.1 An ancilla interaction

An eavesdropper may attempt to acquire information by coupling a transmitted qubit to an ancillary system. A simple controlled-NOT (CNOT) interaction is

$$U_{\mathrm{CNOT}}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle,$$

where $x, y \in \{0, 1\}$ and $\oplus$ denotes addition modulo two.

If the ancilla begins in $|0\rangle$, then

$$U_{\mathrm{CNOT}}|0\rangle|0\rangle = |0\rangle|0\rangle, \qquad U_{\mathrm{CNOT}}|1\rangle|0\rangle = |1\rangle|1\rangle.$$

The ancilla therefore acquires information about a computational-basis bit.

For a diagonal-basis input,

$$|+\rangle|0\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}},$$

the interaction gives

$$U_{\mathrm{CNOT}}|+\rangle|0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

The output is entangled. The reduced state of the transmitted qubit is

$$\rho_A = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{\mathrm{I}}{2}.$$

Thus the diagonal-basis input has been disturbed: the transmitted qubit is no longer in the pure state $|+\rangle$.

This calculation illustrates the information-disturbance trade-off in a simple model. It is not a complete description of every possible attack against BB84, B92, or E91.

11.8.2 A neutrosophic attack record

Let ω describe an observed event, including the preparation basis, detection result, and any calibration metadata. An annotation

$$\nu_{\mathrm{attack}}(\omega) = (t_\omega, i_\omega, f_\omega)$$

may be used to record:

- support for the hypothesis that the record is consistent with the calibrated apparatus;
- unresolved uncertainty about the record or attack classification;
- support for the hypothesis that the record contains an error or anomaly.

The interpretation of f_ω must be chosen carefully. Evidence of a faulty record is not automatically evidence of a malicious eavesdropper. Ordinary device noise can also produce errors.

Proposition 11.8.1. *A passive neutrosophic annotation cannot distinguish an eavesdropping attack from ordinary channel noise when both hypotheses induce exactly the same probability distribution on the observed records.*

Proof. Suppose the attack hypothesis and the noise hypothesis induce the same distribution P on all recorded data. Any decision procedure using only those data has the same distribution of outputs under both hypotheses. A passive annotation computed solely from the same records cannot change this equality. Therefore, it cannot distinguish the two hypotheses without additional information or assumptions. □

This proposition emphasizes that an annotation cannot create information that is absent from the observations.

11.9 From neutrosophic records to statistical inference

A practical implementation needs a method for estimating its annotation components.

Suppose a calibration experiment classifies each record into exactly one of three categories:

$$C_1 = \text{accepted}, \quad C_2 = \text{unresolved}, \quad C_3 = \text{identified error}.$$

Let n_j be the number of observations in category C_j, and let

$$N = n_1 + n_2 + n_3.$$

The empirical frequency estimator is

$$\widehat{t} = \frac{n_1}{N}, \qquad \widehat{i} = \frac{n_2}{N}, \qquad \widehat{f} = \frac{n_3}{N}.$$

For $N > 0$, these estimates satisfy

$$\widehat{t} + \widehat{i} + \widehat{f} = 1.$$

This is an ordinary categorical probability model. Its benefit is that it gives a clear experimental interpretation to the three numbers. Its limitation is that the categories must be defined before collecting the data, and the estimates have sampling uncertainty.

11.9.1 A simple confidence interval

If n_{err} errors are observed in N independent trials, the empirical error rate is

$$\widehat{q} = \frac{n_{\text{err}}}{N}.$$

A statistical confidence interval can be calculated using an appropriate binomial method, such as the Wilson interval [48]. The choice of interval and confidence level should be stated in an experimental report.

A neutrosophic indeterminacy component should not be substituted for a confidence interval. The former is a model component; the latter quantifies sampling uncertainty under specified statistical assumptions.

Remark 11.9.1. If the three annotation components are elicited from experts rather than estimated as category frequencies, their interpretation is different. Expert assessments should not be presented as calibrated probabilities without supporting validation.

11.10 Security interpretation

11.10.1 What a QKD security claim requires

A complete QKD security analysis generally addresses:

1. the physical preparation and measurement model;
2. the quantum channel and possible adversarial interactions;
3. authenticated classical communication;
4. parameter estimation from disclosed test data;
5. error correction and information reconciliation;
6. privacy amplification;
7. finite-key effects and implementation assumptions.

The BB84 protocol and subsequent security analyses have established rigorous approaches to these issues under specified assumptions [39, 43, 46]. In the bibliography below, the Mayers citation is given without a space in its key.

11.10.2 Passive annotations

When an annotation is passive, it can be useful for logging detector confidence, calibration status, and unresolved records. Since it does not affect the physical data or protocol decisions, the standard security analysis remains applicable to the same physical implementation under its original assumptions.

This conclusion does not mean that the annotation improves security. It means only that passive metadata does not modify the protocol.

11.10.3 Operational annotations

Suppose a protocol accepts a record only when

$$t_\omega \geq \theta$$

for a fixed threshold $\theta \in [0, 1]$. This is an operational rule: the annotation determines which records are retained.

The accepted sample may then have a different error rate and different detection statistics. An adversary may also influence the annotation if its inputs depend on untrusted measurements or device reports.

Therefore, the original security proof cannot simply be reused without checking that the filtering rule is covered by the proof's assumptions. In particular, discarding records according to a score correlated with the adversary's information can create selection effects.

Proposition 11.10.1. *Let a QKD protocol apply a record-selection rule based on a neutrosophic annotation. If the rule changes the accepted data set, then the empirical QBER and the resulting key rate need not equal those of the original protocol.*

Proof. Both the empirical QBER and the key rate depend on the accepted data. Changing the acceptance rule can change the numerator and denominator of the QBER, as well as the number of retained signals. Therefore, neither statistic is generally invariant under such selection. □

11.10.4 No automatic security enhancement

The following implications do not hold without additional hypotheses:

$$\text{high truth-support} \not\Rightarrow \text{secure key},$$

$$\text{high indeterminacy} \not\Rightarrow \text{eavesdropping},$$

$$\text{low falsity-support} \not\Rightarrow \text{absence of an attack}.$$

These statements reflect the difference between record-level metadata and cryptographic security. A security claim must be based on an appropriate model and proof.

11.11 Worked example

Suppose a laboratory performs $N = 1000$ calibration trials. The records are classified as follows:

$$n_1 = 970, \qquad n_2 = 20, \qquad n_3 = 10,$$

where n_1 denotes accepted records, n_2 unresolved records, and n_3 identified errors.

The normalized empirical annotation is

$$(\widehat{t}, \widehat{i}, \widehat{f}) = (0.97, 0.02, 0.01).$$

The three components sum to one and have an explicit frequency interpretation.

Now suppose a separate BB84 test reveals that 40 of 500 disclosed sifted bits disagree. Its observed QBER is

$$\widehat{q} = \frac{40}{500} = 0.08.$$

The QBER is 8%. It is not correct to infer that the QBER equals $\widehat{f} = 1\%$, because the two statistics describe different events and may have different denominators.

If the laboratory uses the annotation to discard unresolved records, it must recompute the relevant error and detection statistics on the retained sample. It must also verify that this data-dependent selection is compatible with its security proof.

This example illustrates a useful role for the proposed framework: keeping record reliability, unresolved classification, and measured bit errors separate instead of treating them as interchangeable quantities.

11.12 Research directions

The proposed framework suggests several questions.

11.12.1 Axioms for annotation propagation

A future theory should specify how annotations behave under:

- sequential measurements;
- independent and correlated detector errors;
- classical communication and message authentication;
- discarding inconclusive events;
- error correction and privacy amplification.

The rules should be mathematically consistent and should not duplicate uncertainty already represented in the physical noise model.

11.12.2 Finite-key security

In practical QKD, the number of transmitted signals is finite. A possible research direction is to investigate whether a carefully defined annotation model can improve the reporting or estimation of detector uncertainty while retaining a rigorous finite-key proof.

Such a claim would require a precise definition of the annotation, an analysis of its dependence on observed data, and a proof that the resulting key satisfies the desired security definition.

11.12.3 Device imperfections

Neutrosophic metadata could be investigated as a structured representation of:

- uncertain detector calibration;
- inconclusive measurements;
- missing or delayed detection events;
- disagreement between independent diagnostics;

- uncertainty in an estimated channel parameter.

The usefulness of the approach should be tested against established statistical and quantum-channel models.

11.12.4 Comparison with conventional models

A meaningful comparison should evaluate whether the annotation provides information beyond standard error rates, confidence intervals, likelihood models, or Bayesian parameter estimates. If the triple is merely a re-expression of three ordinary category probabilities, it may offer convenient bookkeeping but does not automatically provide a new security principle.

11.13 Conclusion

This paper has proposed a neutrosophic information layer for describing records arising in quantum cryptography. The framework has been discussed in connection with RSA, BB84, B92, E91, and a simple controlled-NOT attack model.

The principal distinction is between quantum probabilities and annotations of classical records. Passive annotations leave the physical measurement statistics unchanged. Operational annotations can alter data selection and must therefore be included in the protocol's statistical and security analysis.

A normalized triple can be useful when its components represent mutually exclusive and exhaustive outcomes, such as accepted, inconclusive, and rejected records. Other interpretations require their own operational definitions. In particular, uncertainty in a record does not automatically imply an eavesdropping attack, and a high-confidence annotation does not establish the security of a key.

The proposed approach is therefore a preliminary mathematical framework rather than a new cryptographic security theorem. Its further development requires precise annotation rules, empirical calibration, comparison with established models, and a rigorous security analysis of any operational use.

Chapter 12

Neutrosophic Quantum Noise and Error Correction

Related literature. Neutrosophic descriptors for noisy channels and syndrome information have been proposed in the preprints [79, 80]. The physical recovery map must still satisfy the standard quantum-channel conditions.

Quantum information processing is vulnerable to noise, decoherence, imperfect control, and measurement errors. Quantum error correction addresses these difficulties by encoding logical information into a larger physical system and detecting error syndromes without directly measuring the encoded logical state. Standard descriptions of quantum noise use completely positive, trace-preserving maps, Kraus operators, and physically specified error models.

This paper develops a proposed neutrosophic information layer for describing quantum noise and error-correction records. Each relevant record may be assigned a triple representing truth-support, indeterminacy, and falsity-support. We distinguish these descriptive components from the density operator and from the probabilities defining a quantum channel. We review bit-flip, phase-flip, depolarizing, amplitude-damping, and phase-damping channels, and then examine the three-qubit repetition code and general quantum error-correction conditions.

Several propositions establish when passive neutrosophic annotations leave quantum statistics unchanged and explain why such annotations cannot substitute for physical noise models. We also introduce a normalized three-category model for accepted, unresolved, and identified-error records. The resulting framework provides a possible language for recording uncertainty in quantum experiments, while emphasizing that any claimed improvement in error correction must be demonstrated by a specified channel model, a valid decoding procedure, and quantitative performance analysis.

12.1 Introduction

Quantum computation stores and processes information using quantum states and quantum operations. Unlike ideal classical bits, qubits are affected by coherent control errors, environmental interactions, energy relaxation, and loss of phase coherence. These effects must be described and controlled if a quantum computation is to remain reliable [2, 3].

Quantum error correction provides a systematic method for protecting logical information against a specified class of physical errors. The three-qubit repetition code illustrates the basic idea of detecting bit-flip errors, while more sophisticated codes protect against general single-qubit errors [51, 52].

Neutrosophic logic distinguishes truth, indeterminacy, and falsity as separate descriptive components [6]. In the context of quantum experiments, such components might be used to annotate the reliability of a measurement record, uncertainty in a detector classification, or evidence that an event has been incorrectly recorded.

The purpose of this paper is to investigate how neutrosophic annotations can be incorporated into the description of quantum noise and error-correction procedures without confusing them with the underlying quantum mechanics.

The main objectives are:

(i) to formulate a mathematical representation of neutrosophic annotations for quantum records;

(ii) to review standard single-qubit noise channels using Kraus operators;

(iii) to study bit-flip correction through the three-qubit repetition code;

(iv) to relate annotation uncertainty to syndrome records and decoding decisions;

(v) to state the limitations of passive annotations and the requirements for an operational extension.

The physical noise channels and error-correction principles used here are standard results [50, 2, 49]. The neutrosophic constructions are proposed extensions for information recording and uncertainty assessment. They are not presented as established new quantum channels or as a new quantum-error-correction theorem.

12.2 Quantum states and quantum operations

12.2.1 Qubits and density operators

The state space of a qubit is

$$\mathcal{H}_2 = \mathbb{C}^2.$$

A pure state can be written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1. \tag{12.1}$$

A general quantum state is represented by a density operator ρ satisfying

$$\rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1. \tag{12.2}$$

For a pure state, $\rho = |\psi\rangle\langle\psi|$.

The Pauli operators are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The operator X interchanges the computational basis states, while Z changes the relative phase of $|1\rangle$.

12.2.2 Quantum channels

A quantum operation describing a deterministic, trace-preserving physical process is represented by a completely positive, trace-preserving (CPTP) map $\mathcal{E}$ [50, 2].

Definition 12.2.1. A linear map

$$\mathcal{E} : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H}')$$

is completely positive and trace preserving if it admits a Kraus representation

$$\mathcal{E}(\rho) = \sum_k K_k \rho K_k^\dagger, \qquad \sum_k K_k^\dagger K_k = \mathrm{I}. \tag{12.3}$$

Here $\mathcal{L}(\mathcal{H})$ denotes the space of linear operators on $\mathcal{H}$. The trace-preserving condition ensures that the output density operator has trace one whenever the input density operator does.

A quantum operation conditioned on a particular measurement outcome need not be trace preserving. Its probability is obtained from the trace of the corresponding unnormalized output state. Conditional operations must therefore be distinguished from deterministic channels.

12.3 Neutrosophic annotations for quantum records

12.3.1 Basic definition

Definition 12.3.1. A neutrosophic information triple is

$$\nu = (t, i, f), \qquad t, i, f \in [0, 1],$$

where t, i, and f represent truth-support, indeterminacy, and falsity-support for a specified proposition or information record.

The general neutrosophic framework does not require the three components to sum to one [6]. A restricted normalized class is

$$\mathcal{N}_3 = \{(t, i, f) \in [0, 1]^3 : t + i + f = 1\}.$$

Definition 12.3.2. A normalized neutrosophic annotation is a triple in $\mathcal{N}_3$. A probabilistic interpretation is appropriate when the three components correspond to mutually exclusive and exhaustive categories.

For quantum experiments, the interpretation of the components must be stated explicitly. One possible choice is:

- t: support for the validity of a recorded event;
- i: unresolved or ambiguous status of the event;
- f: support for the conclusion that the event was incorrectly recorded or classified.

These quantities do not automatically describe the quantum state itself.

Definition 12.3.3. Let Ω be a set of experimental records. A neutrosophic annotation map is a function

$$\nu : \Omega \longrightarrow [0, 1]^3.$$

For $\omega \in \Omega$, write

$$\nu(\omega) = (t_\omega, i_\omega, f_\omega).$$

A record may include a measurement outcome, a detector identifier, a calibration status, or an error syndrome.

12.3.2 Passive annotations

Definition 12.3.4. An annotation is called *passive* if it does not modify the input state, the physical quantum channel, the measurement operators, or the decoding procedure.

Proposition 12.3.5. *Suppose that a quantum experiment uses a density operator ρ, a quantum channel $\mathcal{E}$, and a measurement described by effects $\{E_j\}$. If a neutrosophic annotation is passive, then the probability of every physical measurement outcome remains unchanged.*

Proof. The outcome probability is

$$p_j = \operatorname{Tr}(E_j \mathcal{E}(\rho)).$$

A passive annotation changes neither ρ, $\mathcal{E}$, nor E_j. Therefore the expression on the right is unchanged for every outcome j. □

Remark 12.3.6. The proposition establishes invariance of physical statistics, not an improvement in noise resistance. If annotations are used to discard events or select decoding operations, the resulting procedure is operational and requires a separate analysis.

12.4 Single-qubit errors

12.4.1 Bit-flip errors

A bit-flip error applies the Pauli operator X. A simple bit-flip channel is

$$\mathcal{E}_X(\rho) = (1-p)\rho + pX\rho X, \qquad 0 \le p \le 1. \tag{12.4}$$

Its Kraus operators are

$$K_0 = \sqrt{1-p}\,\mathrm{I}, \qquad K_1 = \sqrt{p}\,X.$$

They satisfy

$$K_0^\dagger K_0 + K_1^\dagger K_1 = \mathrm{I}.$$

For a computational-basis input, this channel flips the bit with probability p. For a general superposition, the channel also changes the off-diagonal entries of the density matrix.

12.4.2 Phase-flip errors

A phase-flip error applies Z:

$$\mathcal{E}_Z(\rho) = (1-p)\rho + pZ\rho Z. \tag{12.5}$$

A phase flip leaves the computational-basis measurement probabilities unchanged but reverses the sign of the off-diagonal terms of a pure superposition.

The distinction between bit flips and phase flips is central to quantum error correction [2, 51].

12.4.3 Combined Pauli noise

A general single-qubit Pauli channel can be written

$$\mathcal{E}_P(\rho) = p_I\rho + p_X X\rho X + p_Y Y\rho Y + p_Z Z\rho Z, \tag{12.6}$$

where

$$p_I, p_X, p_Y, p_Z \ge 0, \qquad p_I + p_X + p_Y + p_Z = 1.$$

The identity term corresponds to no Pauli error. The other terms represent bit, combined bit-and-phase, and phase errors.

The parameters are physical channel probabilities. They should not be identified automatically with neutrosophic truth, indeterminacy, and falsity.

12.4.4 Neutrosophic annotation of an error event

Suppose an experimental procedure records whether a bit flip occurred. The record may be annotated by

$$\nu_X = (t_X, i_X, f_X).$$

For example, if the three categories are defined as “no detected flip”, “unresolved event”, and “detected flip”, then a normalized triple can be estimated from their observed frequencies.

However, this is a classification of records, not necessarily the underlying physical bit-flip channel. A detector can fail to identify an error, and a reported error can itself be incorrect.

Proposition 12.4.1. *The physical bit-flip probability p in (12.4) cannot, in general, be recovered from a neutrosophic annotation unless a measurement and calibration model relating the annotation to the actual channel is specified.*

Proof. The annotation describes a record, while p is a parameter of the physical channel. Different physical error probabilities and different detector misclassification rates can produce the same distribution of recorded labels. Without assumptions or calibration data linking the two, the annotation does not uniquely determine p. □

12.5 Quantum operations and the Kraus representation

The Kraus representation provides a general framework for describing physical quantum noise [50]. It also allows different physical mechanisms to be compared within one mathematical formalism.

Theorem 12.5.1. *Let $\{K_k\}_{k=1}^m$ be operators satisfying*

$$\sum_{k=1}^{m} K_k^\dagger K_k = \mathrm{I}.$$

Then the map

$$\mathcal{E}(\rho) = \sum_{k=1}^{m} K_k \rho K_k^\dagger$$

is trace preserving.

Proof. By linearity and cyclicity of the trace,

$$\begin{aligned}
\mathrm{Tr}(\mathcal{E}(\rho)) &= \sum_k \mathrm{Tr}(K_k \rho K_k^\dagger) \\
&= \sum_k \mathrm{Tr}(K_k^\dagger K_k \rho) \\
&= \mathrm{Tr}\left(\left(\sum_k K_k^\dagger K_k\right)\rho\right) \\
&= \mathrm{Tr}(\rho).
\end{aligned}$$

Thus the trace is preserved. □

Complete positivity follows from the operator-sum representation. The trace condition alone, without complete positivity, would not be sufficient for a valid quantum channel.

12.5.1 Annotations of Kraus events

In some implementations, a detector or environment monitor produces a classical record associated with an error event. If such records are available, a neutrosophic annotation may describe uncertainty in their classification.

Nevertheless, a Kraus decomposition is not generally a unique physical list of distinguishable events. Different Kraus representations can describe the same channel. Therefore, an annotation assigned to a particular Kraus index has a direct physical interpretation only when the experimental setup supports that interpretation.

Remark 12.5.2. One should not define a neutrosophic quantum channel merely by replacing the Kraus probabilities with three unrelated numbers. A valid extension must preserve positivity, normalization, and the operational meaning of measurement probabilities.

12.6 The depolarizing channel

12.6.1 Definition

For a qubit, consider the depolarizing channel

$$\mathcal{E}_{\mathrm{dep}}(\rho) = (1-p)\rho + p\frac{\mathrm{I}}{2}, \qquad 0 \le p \le 1. \tag{12.7}$$

This convention means that with probability p the state is replaced by the maximally mixed state $\mathrm{I}/2$. Other conventions for the depolarizing parameter are also common.

A Kraus representation is

$$K_0 = \sqrt{1 - \frac{3p}{4}}\,\mathrm{I},$$

and

$$K_1 = \sqrt{\frac{p}{4}}\,X, \qquad K_2 = \sqrt{\frac{p}{4}}\,Y, \qquad K_3 = \sqrt{\frac{p}{4}}\,Z.$$

Indeed,

$$\sum_{j=0}^{3} K_j^{\dagger} K_j = \mathrm{I}.$$

To verify the channel formula, use

$$X\rho X + Y\rho Y + Z\rho Z = 2\,\mathrm{Tr}(\rho)\mathrm{I} - \rho.$$

For a density operator with $\mathrm{Tr}(\rho) = 1$, the result is exactly (12.7).

12.6.2 Action on a pure state

For $\rho = |\psi\rangle\langle\psi|$,

$$\mathcal{E}_{\mathrm{dep}}(\rho) = (1-p)|\psi\rangle\langle\psi| + p\frac{\mathrm{I}}{2}.$$

The output is a mixture of the original state and the maximally mixed state. As p increases, the output becomes less dependent on the input state.

12.6.3 Neutrosophic description

One may associate an annotation

$$\nu_{\mathrm{dep}} = (t, i, f)$$

with a calibration record for the depolarizing experiment. For example, i might represent uncertainty in the estimate of p, while f might represent support for an identified calibration failure.

The parameter p and the annotation i are different quantities: p describes the physical channel, whereas i describes uncertainty about a specified record or proposition.

Proposition 12.6.1. *If the physical depolarizing parameter p is fixed and only a passive annotation is changed, the output density operator in (12.7) is unchanged.*

Proof. The output density operator depends only on ρ and p. A passive annotation modifies neither of these quantities. Therefore the output operator remains the same. □

12.7 Amplitude damping

12.7.1 Physical interpretation

Amplitude damping models energy relaxation in a two-level system. In the simplest model, the excited state $|1\rangle$ decays to the ground state $|0\rangle$ with probability γ.

The Kraus operators are

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \tag{12.8}$$

where $0 \le \gamma \le 1$.

They satisfy

$$K_0^\dagger K_0 + K_1^\dagger K_1 = \mathrm{I}.$$

For a general density matrix

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix},$$

the channel gives

$$\mathcal{E}_{\mathrm{AD}}(\rho) = \begin{pmatrix} \rho_{00} + \gamma\rho_{11} & \sqrt{1-\gamma}\rho_{01} \\ \sqrt{1-\gamma}\rho_{10} & (1-\gamma)\rho_{11} \end{pmatrix}. \tag{12.9}$$

The excited-state population decreases, while the ground-state population increases.

12.7.2 Example

For the excited state $\rho = |1\rangle\langle 1|$, equation (12.9) yields

$$\mathcal{E}_{\mathrm{AD}}(|1\rangle\langle 1|) = \gamma|0\rangle\langle 0| + (1-\gamma)|1\rangle\langle 1|.$$

For $\gamma = 1$, the output is $|0\rangle\langle 0|$; for $\gamma = 0$, the state is unchanged.

12.7.3 Neutrosophic annotation of relaxation

Suppose an experiment records whether a relaxation event occurred. If the possible record categories are “no relaxation”, “unresolved”, and “relaxation detected”, a normalized annotation can summarize their observed frequencies.

It is important to distinguish the probability γ of physical relaxation from uncertainty about whether the detector correctly identified that event. The detector may have false positives, false negatives, and missing records.

A meaningful model should therefore include both the quantum channel and a classical observation model.

12.8 Phase damping

12.8.1 A phase-damping channel

Phase damping suppresses coherence without directly changing computational-basis populations. One Kraus representation is

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{pmatrix}, \qquad K_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix},$$

where $0 \leq \lambda \leq 1$.

These operators satisfy

$$K_0^\dagger K_0 + K_1^\dagger K_1 = \mathrm{I}.$$

The resulting channel is

$$\mathcal{E}_{\mathrm{PD}}(\rho) = \begin{pmatrix} \rho_{00} & \sqrt{1-\lambda}\rho_{01} \\ \sqrt{1-\lambda}\rho_{10} & \rho_{11} \end{pmatrix}. \tag{12.10}$$

The diagonal entries remain fixed, whereas the off-diagonal entries are reduced.

Remark 12.8.1. The parameterization above is a particular phase-damping convention. Other texts use a different parameter or a related phase-flip model. The physical interpretation should always be tied to the displayed channel.

12.8.2 Comparison with amplitude damping

Amplitude damping changes populations and coherence. Phase damping preserves computational-basis populations and changes coherence. These mechanisms have different physical origins and may require different mitigation or correction strategies.

A neutrosophic annotation may represent uncertainty in the estimated channel parameters, but it cannot identify the physical mechanism from the triple alone. Additional measurements or a model-selection procedure are required.

12.9 The three-qubit bit-flip code

12.9.1 Encoding a logical qubit

The three-qubit repetition code encodes a logical qubit as

$$|0_L\rangle = |000\rangle, \qquad |1_L\rangle = |111\rangle. \tag{12.11}$$

An arbitrary logical state is

$$|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle.$$

The code is designed to correct a single bit flip. It does not correct an arbitrary single-qubit error without further encoding or a different code [51, 2].

12.9.2 Syndrome measurements

The stabilizer generators are

$$S_1 = Z_1 Z_2, \qquad S_2 = Z_2 Z_3.$$

The encoded states have eigenvalue $+1$ for both generators. Measuring these operators reveals parity information without distinguishing $|0_L\rangle$ from $|1_L\rangle$.

The syndrome table for a single bit flip is

Error	S_1	S_2
I	$+1$	$+1$
X_1	-1	$+1$
X_2	-1	-1
X_3	$+1$	-1

Thus the four cases have distinct syndromes.

The correction procedure applies X_j when the syndrome identifies a bit flip on qubit j.

12.9.3 Correction of one bit flip

Theorem 12.9.1. *The three-qubit repetition code corrects any single bit-flip error X_j, $j = 1, 2, 3$, on an encoded logical state.*

Proof. Consider

$$|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle.$$

A bit flip on qubit 1 gives

$$X_1|\psi_L\rangle = \alpha|100\rangle + \beta|011\rangle.$$

The syndrome is $(-1, +1)$, which uniquely identifies X_1. Applying X_1 again restores the original state because $X_1^2 = \mathrm{I}$.

The same reasoning applies to X_2 and X_3: their syndromes are distinct, and the corresponding Pauli operator reverses each error. Hence the encoded state is recovered for every single bit-flip error. □

12.9.4 Limitations of the code

The repetition code does not correct two arbitrary bit flips. For example, X_1X_2 produces the same syndrome as a single X_3 error, so the usual decoder may apply X_3 and transform the error into $X_1X_2X_3$ relative to the original codeword.

Furthermore, the repetition code alone does not correct arbitrary phase errors. A phase flip on one physical qubit changes the relative phase between the two components of the logical superposition. Protection against general single-qubit errors requires a code satisfying stronger conditions, such as the Shor or Steane code [51, 52].

12.10 Neutrosophic syndrome records

12.10.1 Annotation of syndrome outcomes

Let the syndrome set be

$$\Omega_S = \{(+,+), (-,+), (-,-), (+,-)\}.$$

An annotation map

$$\nu_S : \Omega_S \longrightarrow [0,1]^3$$

assigns a triple to each reported syndrome.

For example,

$$\nu_S(-,+) = (0.95, 0.04, 0.01)$$

may represent a record with high support for its validity, a small unresolved component, and a small estimated probability of misclassification.

This triple does not mean that the physical syndrome has Born probability 0.95. The syndrome probability is determined by the encoded state, the error channel, and the syndrome measurement.

Definition 12.10.1. A neutrosophically annotated error-correction procedure consists of:

1. a physical encoding and noise model;
2. a syndrome measurement;
3. a decoding and correction rule;
4. an annotation map for syndrome and detector records; and
5. an explicit rule, if any, by which annotations affect decoding.

12.10.2 Passive syndrome annotations

Proposition 12.10.2. *If the three-qubit repetition code uses its standard syndrome decoder and the neutrosophic annotations are passive, then the encoded quantum output is identical to that of the standard decoder.*

Proof. The encoder, noise channel, syndrome measurement, and correction operations are unchanged. The annotations add metadata to the classical syndrome records but do not alter the selected correction. Therefore, the quantum output state is unchanged. □

12.10.3 Operational syndrome annotations

An operational decoder might reject a syndrome record when its annotation is considered too uncertain, or request a repeated measurement when the apparatus supports one.

Such a rule can change the decoding performance. It may also introduce selection bias if the probability of rejecting a record depends on the actual error or on information available to an adversary.

Consequently, an operational neutrosophic decoder must be assessed using a fully specified noise model and a quantitative performance criterion.

12.11 General quantum error-correction conditions

The theory of quantum error correction provides conditions for determining whether a code can correct a specified collection of errors [49].

Theorem 12.11.1 (Knill–Laflamme condition)**.** *Let $\mathcal{Q}$ be a code subspace with projector P, and let $\{E_a\}$ be a collection of error operators. The code can perfectly correct the errors in this collection if and only if there exist complex numbers c_{ab} such that*

$$PE_a^\dagger E_b P = c_{ab} P \qquad \text{for all } a, b. \tag{12.12}$$

The condition ensures that the error process does not reveal information about the encoded logical state through distinctions that cannot be corrected.

12.11.1 What annotations can and cannot do

A neutrosophic triple attached to an error label does not establish condition (12.12). The condition concerns operators acting on the code subspace, not confidence labels assigned to records.

Annotations may be useful for describing uncertainty about which error occurred, but the existence of a correcting recovery operation must be established from the physical error operators.

Proposition 12.11.2. *Suppose a code and an error set satisfy* (12.12). *Attaching passive neutrosophic annotations to classical records does not invalidate the algebraic Knill–Laflamme condition.*

Proof. The condition depends only on the code projector P and the error operators E_a. Passive annotations modify none of these objects. Hence the operator identities in (12.12) remain valid. □

This proposition is an invariance statement, not a new error-correction criterion.

12.12 A probabilistic model of indeterminate error records

12.12.1 Three-category observation model

Suppose each detected event is classified into exactly one of the following categories:

$$C_1 = \text{correctly classified},$$

$$C_2 = \text{unresolved},$$

$$C_3 = \text{identified misclassification}.$$

If the corresponding probabilities are t, i, f, then

$$t + i + f = 1, \qquad t, i, f \geq 0.$$

The triple (t, i, f) is a normalized representation of the observation model.

This interpretation differs from a general neutrosophic triple whose components need not sum to one. It is adopted here because the categories are mutually exclusive and exhaustive.

Proposition 12.12.1. *Suppose $N > 0$ independent records are classified into the three categories, with counts n_1, n_2, n_3. The empirical estimates*

$$\widehat{t} = \frac{n_1}{N}, \qquad \widehat{i} = \frac{n_2}{N}, \qquad \widehat{f} = \frac{n_3}{N}$$

belong to $\mathcal{N}_3$.

Proof. Each count is nonnegative and at most N, so each estimate lies in $[0, 1]$. Since the categories are exhaustive,

$$n_1 + n_2 + n_3 = N.$$

Dividing by N gives

$$\widehat{t} + \widehat{i} + \widehat{f} = 1.$$

Thus the estimated triple belongs to $\mathcal{N}_3$. □

12.12.2 Separating physical errors from record errors

Consider a bit-flip channel with physical error probability p. Suppose a detector identifies a true bit flip with sensitivity s and incorrectly reports a bit flip when none occurred with probability r. The observed probability of a reported flip is

$$p_{\text{obs}} = sp + r(1-p). \tag{12.13}$$

Equation (12.13) shows that the observed error label is not generally the physical error probability. Even if p_{obs} is known, p cannot be determined uniquely unless the detector parameters are known or additional data are available.

This illustrates why a neutrosophic annotation requires a specified relationship to the physical system before it can be used to estimate noise.

12.13 An annotated quantum-channel model

A possible formal structure is the tuple

$$\mathfrak{Q} = (\rho, \mathcal{E}, \mathcal{M}, \mathcal{D}, \nu), \tag{12.14}$$

where:

- ρ is the input density operator;
- $\mathcal{E}$ is the physical noise channel;
- $\mathcal{M}$ is the measurement procedure;
- $\mathcal{D}$ is the decoder or recovery rule;
- ν annotates classical experimental records.

The physical part of the model is $(\rho, \mathcal{E}, \mathcal{M}, \mathcal{D})$. The annotation map ν supplies additional descriptive information.

Proposition 12.13.1. *For a fixed physical model and a fixed decoder, changing only a passive annotation map does not change the physical output state or the outcome distribution.*

Proof. The output state and measurement distribution are determined by the physical model and decoder. Since neither is changed by a passive annotation, the output state and all outcome probabilities remain the same. □

12.13.1 When annotations influence recovery

If the recovery operation is selected as a function of the annotation, the decoder becomes a function of the measured record and its annotation. The overall procedure must then be described as a measurement-conditioned recovery process.

For instance, a decoder might choose a recovery operation R_ω based on a syndrome record ω and its annotation $\nu(\omega)$. To analyze such a procedure, one must specify:

1. how $\nu(\omega)$ is obtained;
2. whether it depends on trusted or untrusted data;
3. the recovery operation selected for each record;
4. the treatment of unresolved and discarded events;
5. the resulting logical error probability.

Without these specifications, no general improvement in fidelity or error-correction performance follows from the use of neutrosophic terminology.

12.14 Logical error rate and fidelity

The performance of an error-correction procedure should be evaluated by a physical criterion.

For an input state ρ and an effective logical channel $\mathcal{E}_L$, one may study the output fidelity

$$F(\rho, \mathcal{E}_L(\rho)) = \left(\operatorname{Tr} \sqrt{\sqrt{\rho}\mathcal{E}_L(\rho)\sqrt{\rho}} \right)^2 .$$

For a pure input $\rho = |\psi\rangle\langle\psi|$, this reduces to

$$F = \langle\psi|\mathcal{E}_L(|\psi\rangle\langle\psi|)|\psi\rangle.$$

One may also estimate the logical error rate from repeated trials using a specified logical measurement or a suitable process-characterization method.

A neutrosophic annotation could supplement these statistics by recording uncertainty in calibration, syndrome identification, or data processing. It should not be confused with fidelity itself.

Remark 12.14.1. A claim that a neutrosophic procedure improves quantum error correction should compare its logical error rate or fidelity against a specified baseline under the same physical noise assumptions. It should also account for the cost of additional measurements, discarded events, and classical processing.

12.15 Worked example

Suppose a three-qubit repetition-code experiment produces 1000 syndrome records. A calibration procedure classifies:

$$n_1 = 960, \qquad n_2 = 25, \qquad n_3 = 15,$$

as correctly classified, unresolved, and identified misclassification records, respectively.

The empirical normalized annotation is

$$(\widehat{t}, \widehat{i}, \widehat{f}) = (0.960, 0.025, 0.015).$$

These values summarize the classification frequencies.

Suppose a separate logical test estimates a logical error rate of

$$q_L = \frac{12}{1000} = 0.012.$$

The logical error rate is 1.2%. It is not equal to the annotation component $\widehat{f} = 1.5\%$, because the quantities measure different events.

To determine whether the annotation is useful, one would need to test whether it improves the estimation of logical performance, supports a better recovery rule, or identifies calibration failures more accurately than an ordinary statistical model.

The numerical example is illustrative; it does not represent an experimental result.

12.16 Limitations and research directions

12.16.1 Operational definitions

The components t, i, and f must be defined for a particular experiment. Truth-support might refer to a verified detector record, indeterminacy to an unresolved event, and falsity-support to an identified misclassification. Different applications may require different categories.

12.16.2 Consistency with quantum mechanics

The quantum state must remain positive semidefinite and trace one. A deterministic physical channel must be completely positive and trace preserving. An annotation framework must not violate these conditions or substitute descriptive labels for physical probabilities.

12.16.3 Composition and correlated errors

A useful theory should explain how annotations behave when several qubits interact, when errors are correlated, and when multiple syndrome measurements are combined. Componentwise addition or multiplication is not automatically a valid composition law.

12.16.4 Comparison with established methods

The framework should be compared with conventional probability estimates, detector confusion matrices, Bayesian models, confidence intervals, and fault-tolerant decoding methods. If the annotation merely restates ordinary category frequencies, its contribution may be organizational rather than mathematically new.

12.16.5 Possible research questions

The following questions merit further investigation:

1. Can a calibrated neutrosophic annotation improve the estimation of logical error rates?
2. Under what conditions can uncertain syndrome records be used without invalidating a code's recovery guarantees?
3. Can a decoder using annotation metadata achieve a lower logical error rate than a standard decoder under a specified physical noise model?
4. How should unresolved events be treated in finite-sample performance estimates?
5. Can a rigorous operational framework preserve the standard quantum-channel conditions while providing additional experimentally useful information?

Any affirmative result should include formal assumptions and a proof, or reproducible numerical or experimental evidence where appropriate.

12.17 Conclusion

This paper has developed a preliminary framework for describing quantum noise and error-correction records using neutrosophic annotations. Standard bit-flip, phase-flip, depolarizing, amplitude-damping, and phase-damping channels were reviewed, followed by the three-qubit repetition code and the Knill–Laflamme error-correction condition.

The central distinction is between the physical quantum channel and the additional description of experimental records. Passive annotations leave quantum statistics unchanged. Operational annotations may influence event selection or decoding and therefore require a separate performance and security analysis.

Normalized triples are useful when their components represent mutually exclusive and exhaustive categories. More general neutrosophic triples require an explicit interpretation and should not be automatically treated as probabilities.

The framework is intended as a starting point for research into uncertainty-aware quantum information processing. A substantive extension of quantum error correction would require

precise annotation rules, a physical noise model, a well-defined recovery procedure, and a demonstrable improvement in a quantitative performance measure.

Chapter 13

Neutrosophic Quantum Information Theory

Related literature. For perspectives on indeterminacy and quantum information models, see [75, 72]; for a proposed operator-theoretic formulation, see [79] (preprint).

We develop a mathematical framework for studying selected tools of quantum information theory from a neutrosophic perspective. The standard theory of quantum states, trace distance, fidelity, no-cloning, entanglement, concurrence, and quantum entropy is reviewed and then supplemented by a three-valued annotation formalism representing truth, indeterminacy, and falsity of information associated with a quantum-information record. We emphasize the distinction between physical uncertainty encoded by a density operator and epistemic or observational indeterminacy represented by an annotation. This distinction is essential: a neutrosophic annotation does not, by itself, change the underlying quantum state or its measurable statistics. We establish elementary properties of the annotation space, construct examples, and formulate a cautious extension of state-comparison and information-reporting procedures. We also identify conditions under which the proposed formalism is mathematically meaningful and discuss limitations, including the need for operational definitions and experimental validation before neutrosophic quantities can be interpreted as physical measures.

13.1 Introduction

Quantum information theory studies how information is encoded, transformed, transmitted, and recovered by quantum systems. Its mathematical tools include density operators, trace distance, fidelity, entanglement measures, and several notions of entropy. These concepts are central to quantum communication, quantum cryptography, quantum computation, and quantum-state estimation [2, 3].

McMahon's *Quantum Computing Explained* presents an introductory treatment of these tools, including the no-cloning theorem, trace distance, fidelity, entanglement of formation, concurrence, and information content and entropy [1]. The present paper considers these themes independently and asks how a neutrosophic representation can supplement the reporting and interpretation of quantum-information records.

Neutrosophic theory introduces degrees associated with truth, indeterminacy, and falsity. In its general form, these components need not be constrained to sum to one, although normalized subclasses can be useful in particular applications [6]. In a quantum-information setting, a possible use is to annotate whether a statement about a quantum experiment is supported, unresolved, or contradicted by available evidence.

Such annotations must not be confused with the Born probabilities of quantum mechanics. A density operator already represents a quantum state, including the statistical description of a preparation or ensemble. A neutrosophic triple, as used here, records an additional assessment of information about that state. It is not a new quantum state unless a separate physical and mathematical model explicitly defines it as one.

The main aims of this paper are:

(i) to present the standard quantum-information quantities needed for state comparison and entanglement analysis;

(ii) to define a neutrosophic annotation space for quantum-information records;

(iii) to formulate annotation-aware reporting rules without altering established quantum probabilities;

(iv) to provide examples and identify open questions for further mathematical and experimental research.

13.2 Quantum states and neutrosophic annotations

13.2.1 Density operators

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. A quantum state is represented by a density operator $\rho : \mathcal{H} \to \mathcal{H}$ satisfying

$$\rho \geq 0, \qquad \operatorname{Tr}(\rho) = 1. \tag{13.1}$$

The set of all such states is denoted by $\mathcal{D}(\mathcal{H})$.

A pure state has the form

$$\rho = |\psi\rangle\langle\psi|, \qquad \langle\psi \mid \psi\rangle = 1, \tag{13.2}$$

whereas a mixed state can be represented as

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \qquad p_j \geq 0, \qquad \sum_j p_j = 1. \tag{13.3}$$

The density-operator formalism is standard in quantum mechanics and quantum information theory [2, 3].

For a measurement described by positive operators $\{M_x\}_x$ satisfying $\sum_x M_x = I$, the probability of outcome x is

$$p(x \mid \rho) = \operatorname{Tr}(M_x \rho). \tag{13.4}$$

These probabilities are fixed by the state and measurement model. A separate annotation does not change them.

13.2.2 Neutrosophic triples

Definition 13.2.1. A neutrosophic annotation is a triple

$$\nu = (t, i, f) \in [0, 1]^3, \tag{13.5}$$

where t, i, and f denote, respectively, the recorded degrees of support, indeterminacy, and opposition for a specified information claim.

The triple is associated with a claim and a specified evidential procedure. For example, a claim might be that an estimated state is close to a target state, or that a detector outcome has been reliably classified.

In general, we do not impose

$$t + i + f = 1. \tag{13.6}$$

The three components describe different assessments and are not automatically mutually exclusive probabilities. If a particular application requires a normalized model, it must state the additional assumptions that justify normalization.

Definition 13.2.2. An annotated quantum-information record is a pair

$$\mathcal{R} = (\rho, \nu), \tag{13.7}$$

where $\rho \in \mathcal{D}(\mathcal{H})$ is a quantum state and $\nu = (t, i, f)$ is an annotation concerning a specified claim about that state, its preparation, its measurement, or its estimated properties.

Remark 13.2.3. The pair (ρ, ν) is an information record, not necessarily a density operator on a larger Hilbert space. The annotation cannot be substituted for ρ in the Born rule. To give the annotation a physical interpretation, one would need an explicit model connecting it to preparations, measurements, or statistical inference.

13.2.3 A basic annotation space

Let

$$\mathfrak{N} = [0, 1]^3. \tag{13.8}$$

For two annotations $\nu_1 = (t_1, i_1, f_1)$ and $\nu_2 = (t_2, i_2, f_2)$, define the componentwise distance

$$d_{\mathfrak{N}}(\nu_1, \nu_2) = \sqrt{(t_1 - t_2)^2 + (i_1 - i_2)^2 + (f_1 - f_2)^2}. \tag{13.9}$$

Proposition 13.2.4. *The function $d_{\mathfrak{N}}$ is a metric on $\mathfrak{N}$.*

Proof. Non-negativity follows from the non-negativity of squares. The distance is zero precisely when all three coordinates agree, and symmetry follows from $(a - b)^2 = (b - a)^2$. The triangle inequality is the ordinary Euclidean triangle inequality in $\mathbb{R}^3$. □

This metric measures the numerical difference between annotations. It is not a quantum-state distance and does not quantify physical distinguishability.

13.3 The no-cloning theorem

13.3.1 Statement of the theorem

The no-cloning theorem states that no single fixed quantum operation can perfectly copy every arbitrary unknown pure quantum state [58, 54].

Theorem 13.3.1 (No-cloning theorem)**.** *Let $\mathcal{H}$ contain at least two linearly independent unit vectors. There is no unitary operator U and fixed normalized blank state $|b\rangle$ such that*

$$U(|\psi\rangle \otimes |b\rangle) = |\psi\rangle \otimes |\psi\rangle \tag{13.10}$$

for every normalized $|\psi\rangle \in \mathcal{H}$.

Proof. Suppose such a unitary existed for two normalized states $|\psi\rangle$ and $|\phi\rangle$. Unitarity preserves inner products, so

$$\langle\psi \mid \phi\rangle = \big(\langle\psi \mid \phi\rangle\big)^2. \tag{13.11}$$

Writing $z = \langle\psi \mid \phi\rangle$ gives $z = z^2$, hence $z \in \{0, 1\}$. This is impossible for a pair of distinct, nonorthogonal states, for which $0 < |z| < 1$. Therefore no such universal cloning operation exists. □

The theorem does not prohibit copying known classical descriptions, nor does it prohibit perfect copying of a set of mutually orthogonal states. Its restriction concerns universal perfect copying of arbitrary unknown quantum states.

13.3.2 Neutrosophic interpretation

Suppose a laboratory record contains an estimated state ρ and an annotation

$$\nu = (t, i, f). \tag{13.12}$$

A large value of i might represent uncertainty about the preparation record, incomplete calibration, or unresolved evidence about the state estimate. Such an annotation does not remove the no-cloning restriction.

Proposition 13.3.2. *Adding an annotation $\nu \in \mathfrak{N}$ to a state record does not, by itself, provide a physical operation that clones the underlying state.*

Proof. The annotation changes the record from ρ to (ρ, ν) but leaves the quantum state ρ unchanged. Any proposed physical cloning procedure acting on ρ would still have to satisfy the ordinary quantum-mechanical requirements. In particular, the no-cloning theorem continues to apply to universal perfect cloning of arbitrary unknown pure states. □

Remark 13.3.3. An annotation may help distinguish a well-documented preparation from a poorly documented one. It cannot make an otherwise impossible quantum operation possible.

13.4 Trace distance and state distinguishability

13.4.1 Definition and properties

For density operators ρ and σ, the trace distance is defined by

$$\mathsf{D}(\rho, \sigma) = \frac{1}{2}\|\rho - \sigma\|_1, \tag{13.13}$$

where the trace norm is

$$\|A\|_1 = \operatorname{Tr}\sqrt{A^\dagger A}. \tag{13.14}$$

For quantum states,

$$0 \le \mathsf{D}(\rho, \sigma) \le 1. \tag{13.15}$$

Trace distance is a metric on the space of density operators and has an operational interpretation in terms of optimal state discrimination [2, 3].

For any two states ρ, σ and any measurement event represented by an operator M satisfying $0 \le M \le I$,

$$|\operatorname{Tr}(M\rho) - \operatorname{Tr}(M\sigma)| \le \mathsf{D}(\rho, \sigma). \tag{13.16}$$

Thus, a small trace distance guarantees that no single measurement event can have substantially different probabilities under the two states.

13.4.2 Pure-state formula

For pure states $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, one obtains

$$\mathsf{D}(\rho, \sigma) = \sqrt{1 - |\langle\psi \mid \phi\rangle|^2}. \tag{13.17}$$

Example 13.4.1. Consider

$$|\psi\rangle = |0\rangle, \qquad |\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle, \tag{13.18}$$

where $0 \leq \theta \leq \pi/2$. Since $|\langle\psi \mid \phi\rangle|^2 = \cos^2\theta$, the trace distance is

$$\mathsf{D}(\rho, \sigma) = \sin\theta. \tag{13.19}$$

For $\theta = 0$, the states coincide and the distance is zero. For $\theta = \pi/2$, they are orthogonal and perfectly distinguishable, with distance one.

13.4.3 Annotation-aware reporting

Let the two records be

$$\mathcal{R}_1 = (\rho, \nu_1), \qquad \mathcal{R}_2 = (\sigma, \nu_2). \tag{13.20}$$

The physical state distance remains $\mathsf{D}(\rho, \sigma)$. For a combined record, one may define the descriptive quantity

$$\Delta_\lambda(\mathcal{R}_1, \mathcal{R}_2) = \mathsf{D}(\rho, \sigma) + \lambda d_{\mathfrak{N}}(\nu_1, \nu_2), \qquad \lambda \geq 0. \tag{13.21}$$

This is a record-comparison score, not a quantum trace distance.

Remark 13.4.2. The parameter λ is a user-selected scale factor. The quantity Δ_λ has no established operational meaning in quantum mechanics unless a particular application defines and validates one. It must not be used as a replacement for trace distance in quantum-state discrimination theorems.

13.5 Fidelity

13.5.1 Definition

For density operators ρ and σ, we use the squared fidelity convention

$$\mathsf{F}(\rho, \sigma) = \left(\operatorname{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2. \tag{13.22}$$

This convention gives $0 \leq \mathsf{F}(\rho, \sigma) \leq 1$. Some references define fidelity without the outer square, so the convention should always be stated [57, 2].

Fidelity equals one if and only if the two density operators are identical. For pure states,

$$\mathsf{F}(|\psi\rangle, |\phi\rangle) = |\langle\psi \mid \phi\rangle|^2. \tag{13.23}$$

If $\rho = |\psi\rangle\langle\psi|$ is pure, then

$$\mathsf{F}(\rho, \sigma) = \langle\psi|\sigma|\psi\rangle. \tag{13.24}$$

13.5.2 Relation to trace distance

The Fuchs–van de Graaf inequalities relate trace distance and squared fidelity:

$$1 - \sqrt{\mathsf{F}(\rho,\sigma)} \le \mathsf{D}(\rho,\sigma) \le \sqrt{1 - \mathsf{F}(\rho,\sigma)}. \tag{13.25}$$

These inequalities show that high fidelity implies small trace distance, while small fidelity indicates that the states are far apart in an operationally relevant sense [55].

Example 13.5.1. Let $|\psi\rangle = |0\rangle$ and $|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$. Then

$$\mathsf{F}(\rho,\sigma) = \cos^2\theta, \qquad \mathsf{D}(\rho,\sigma) = \sin\theta. \tag{13.26}$$

Consequently,

$$\mathsf{D}(\rho,\sigma) = \sqrt{1 - \mathsf{F}(\rho,\sigma)} \tag{13.27}$$

for these pure states.

13.5.3 Fidelity with a neutrosophic annotation

A state-estimation report may include a fidelity estimate and an annotation concerning the reliability of that estimate:

$$\mathcal{R} = (\rho, \nu), \qquad \nu = (t, i, f). \tag{13.28}$$

For example, the fidelity calculation may be numerically precise while the state-preparation record is incomplete. The annotation can record that distinction, provided its meaning and assignment procedure are explicitly specified.

It is important to distinguish fidelity itself from confidence in a fidelity estimate. The first is a function of density operators; the second depends on an estimation procedure, sampling uncertainty, calibration, and assumptions about the experiment.

13.6 Entanglement of formation and concurrence

13.6.1 Bipartite pure states

Let a bipartite system have Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. A pure state $|\psi\rangle_{AB}$ has reduced state

$$\rho_A = \mathrm{Tr}_B\big(|\psi\rangle\langle\psi|\big). \tag{13.29}$$

The entanglement entropy of the pure bipartite state is

$$E(|\psi\rangle_{AB}) = \mathsf{S}(\rho_A), \tag{13.30}$$

where S is the von Neumann entropy defined by

$$\mathsf{S}(\rho) = -\,\mathrm{Tr}(\rho \log_2 \rho). \tag{13.31}$$

For a pure bipartite state, the reduced states on A and B have the same nonzero eigenvalues, so either reduced state gives the same entanglement entropy [2].

13.6.2 Entanglement of formation

For a mixed bipartite state ρ_{AB}, entanglement of formation is defined by the convex-roof construction

$$\mathsf{E}_{\mathrm{F}}(\rho_{AB}) = \inf_{\{p_j, |\psi_j\rangle\}} \sum_j p_j \mathsf{S}\left(\mathrm{Tr}_B |\psi_j\rangle\langle\psi_j|\right), \tag{13.32}$$

where the infimum is taken over all pure-state ensembles satisfying

$$\rho_{AB} = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \qquad p_j \geq 0, \quad \sum_j p_j = 1. \tag{13.33}$$

The convex roof captures the minimum average pure-state entanglement over all ensemble decompositions of the mixed state [53].

13.6.3 Concurrence for two qubits

For a normalized pure two-qubit state

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle, \tag{13.34}$$

the concurrence is

$$\mathsf{C}(|\psi\rangle) = 2|ad - bc|. \tag{13.35}$$

It ranges from zero for product states to one for maximally entangled two-qubit pure states.

For a mixed two-qubit state ρ, define the spin-flipped operator

$$\widetilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{13.36}$$

where complex conjugation is taken in the computational basis and

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{13.37}$$

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ be the nonnegative square roots of the eigenvalues of $\rho\widetilde{\rho}$. Wootters' formula is

$$\mathsf{C}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}. \tag{13.38}$$

For two-qubit states, the entanglement of formation is

$$\mathsf{E}_{\mathrm{F}}(\rho) = h_2\left(\frac{1 + \sqrt{1 - \mathsf{C}(\rho)^2}}{2}\right), \tag{13.39}$$

where the binary entropy function is

$$h_2(p) = -p\log_2 p - (1-p)\log_2(1-p), \tag{13.40}$$

with the convention $0\log_2 0 = 0$ [25].

13.6.4 Example: a maximally entangled state

Consider the Bell state

$$|\Phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{13.41}$$

Its reduced state is

$$\rho_A = \mathrm{Tr}_B |\Phi^+\rangle\langle\Phi^+| = \frac{I}{2}. \tag{13.42}$$

Therefore,

$$\mathsf{S}(\rho_A) = 1, \qquad \mathsf{C}(|\Phi^+\rangle) = 1, \qquad \mathsf{E}_{\mathrm{F}}(|\Phi^+\rangle) = 1. \tag{13.43}$$

The state contains one bit of entanglement under these standard measures.

13.6.5 Neutrosophic annotations for entanglement reports

Suppose a tomography procedure estimates a two-qubit state $\widehat{\rho}$ and calculates $\mathsf{C}(\widehat{\rho})$. The report may be augmented by

$$\nu_{\mathrm{ent}} = (t_{\mathrm{ent}}, i_{\mathrm{ent}}, f_{\mathrm{ent}}). \tag{13.44}$$

Here the components may be assigned according to a stated protocol for evaluating the evidence for an entanglement claim. For instance, indeterminacy might record an unresolved classification due to finite data or incomplete calibration.

However, a numerical concurrence and a neutrosophic annotation are different objects. The concurrence is computed from a density matrix; the annotation describes the status of an information claim about that computation or experiment.

Remark 13.6.1. A reliable implementation should specify how the annotation is derived from data, how it responds to statistical error, and whether different observers or laboratories can reproduce the assignment. Without these rules, the triple is descriptive rather than a new entanglement measure.

13.7 Information content and quantum entropy

13.7.1 Classical Shannon entropy

For a discrete random variable X with probabilities $p_1, \dots, p_n$, Shannon entropy is

$$H(X) = -\sum_{j=1}^{n} p_j \log_2 p_j, \tag{13.45}$$

where $0 \log_2 0$ is interpreted as zero [4]. The entropy is measured in bits when base-two logarithms are used.

For a fair binary variable,

$$H(X) = -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2} = 1. \tag{13.46}$$

For a deterministic variable, entropy is zero.

13.7.2 Von Neumann entropy

The quantum analogue of Shannon entropy is the von Neumann entropy

$$\mathsf{S}(\rho) = -\operatorname{Tr}(\rho \log_2 \rho). \tag{13.47}$$

If the eigenvalues of ρ are $\lambda_1, \dots, \lambda_d$, then

$$\mathsf{S}(\rho) = -\sum_{j=1}^{d} \lambda_j \log_2 \lambda_j. \tag{13.48}$$

For a pure state, the spectrum consists of one eigenvalue equal to one and the rest zero, giving

$$\mathsf{S}(\rho) = 0. \tag{13.49}$$

For the maximally mixed state $\rho = I/d$,

$$\mathsf{S}(I/d) = \log_2 d. \tag{13.50}$$

These statements concern the entropy of the state, not automatically the entanglement between subsystems.

13.7.3 The Holevo quantity

Consider an ensemble of quantum states $\{p_x, \rho_x\}$ with average state

$$\overline{\rho} = \sum_x p_x \rho_x. \tag{13.51}$$

The Holevo quantity is

$$\chi = \mathsf{S}(\overline{\rho}) - \sum_x p_x \mathsf{S}(\rho_x). \tag{13.52}$$

The Holevo bound states that the accessible classical mutual information between the ensemble label and the outcome of a measurement is at most χ [56, 2].

Thus,

$$I(X:Y) \leq \chi. \tag{13.53}$$

This inequality is fundamental to the theory of classical information encoded in quantum states.

13.7.4 A neutrosophic information annotation

An information record may include a triple

$$\nu_{\text{info}} = (t, i, f) \tag{13.54}$$

to describe the status of a claim about the estimated entropy or accessible information. For example, the claim may concern whether the available measurements adequately determine the spectrum of an estimated density operator.

The annotation is not itself a Shannon or von Neumann entropy. In particular, the expression

$$-\left(t \log_2 t + i \log_2 i + f \log_2 f\right) \tag{13.55}$$

can be calculated when the components are in $[0, 1]$, with the usual zero convention, but it should not automatically be called a neutrosophic entropy. In general, $t + i + f$ need not equal one, and the components need not be probabilities of mutually exclusive outcomes.

If an application deliberately imposes

$$t + i + f = 1, \tag{13.56}$$

then the expression above is the Shannon entropy of a three-outcome probability distribution. That interpretation depends on the normalization and probability assumptions, not merely on the use of a neutrosophic triple.

13.7.5 A separate annotation-uncertainty functional

For applications needing a scalar summary of an annotation, one possible descriptive functional is

$$U(\nu) = i. \tag{13.57}$$

It reports the indeterminacy component directly. Another possible functional is the Euclidean distance from a chosen reference annotation ν_0:

$$U_{\nu_0}(\nu) = d_{\mathfrak{N}}(\nu, \nu_0). \tag{13.58}$$

These quantities summarize annotation data only. They do not replace von Neumann entropy or measure physical uncertainty unless a validated model establishes such a relationship.

13.8 An annotation-aware framework

13.8.1 Record structure

A general quantum-information record may be written as

$$\mathcal{R} = (\rho, \mathcal{C}, \nu), \tag{13.59}$$

where:

- ρ is the quantum state or estimated density operator;
- $\mathcal{C}$ is a specified claim, such as a state-identification, fidelity, or entanglement claim;
- $\nu = (t, i, f)$ is an annotation associated with the evidence for $\mathcal{C}$.

This structure separates the mathematical object being studied from a claim about that object and an assessment of the available evidence.

13.8.2 A reporting protocol

An annotation-aware study can follow these steps:

1. Specify the quantum system and the state space.
2. Describe the preparation and measurement procedures.
3. Estimate the density operator from the observed data.
4. Compute the standard quantum-information quantity, such as trace distance, fidelity, concurrence, or entropy.
5. Define the claim being evaluated and the rule for assigning the three annotation components.
6. Report the standard quantity and the annotation separately, including statistical and systematic limitations.

This procedure prevents an annotation from being mistaken for a replacement for quantum mechanics. It also allows another researcher to reproduce the state estimate and the annotation assignment independently.

13.8.3 A simple reporting example

Suppose a protocol estimates the fidelity between a prepared state and a target state as

$$\mathsf{F}(\widehat{\rho}, \rho_{\text{target}}) = 0.96. \tag{13.60}$$

A report might associate the annotation

$$\nu = (0.85, 0.10, 0.05) \tag{13.61}$$

with the claim that the experiment supports a high-fidelity preparation.

These numbers are illustrative only. The fidelity is the computed state-comparison quantity, while the triple is an annotation whose assignment rule would need to be specified. The triple must not be interpreted as a confidence interval or as probabilities unless its construction justifies that interpretation.

13.8.4 Monotonicity and physical consistency

A central question for future work is whether an annotation rule respects the behavior of standard quantum quantities under physical operations. A quantum channel is a completely positive, trace-preserving linear map Φ. Trace distance satisfies

$$\mathsf{D}(\Phi(\rho), \Phi(\sigma)) \leq \mathsf{D}(\rho, \sigma). \tag{13.62}$$

This contractivity expresses the fact that quantum processing cannot increase the distinguishability of two states as measured by trace distance.

An annotation rule may or may not satisfy a corresponding monotonicity property. Such a property cannot be assumed from the triple notation alone. It must be stated as an axiom or derived from an explicit model of how annotations are updated.

For example, if a proposed annotation-update map is

$$\mathcal{U}_\Phi : \mathfrak{N} \to \mathfrak{N}, \tag{13.63}$$

one might investigate conditions under which

$$d_{\mathfrak{N}}(\mathcal{U}_\Phi(\nu_1), \mathcal{U}_\Phi(\nu_2)) \leq d_{\mathfrak{N}}(\nu_1, \nu_2). \tag{13.64}$$

This is a possible mathematical requirement for an annotation model; it is not a general law of neutrosophic theory or quantum mechanics.

13.9 Limitations and open problems

The framework presented here is deliberately conservative. It provides an annotation layer but does not claim that neutrosophic triples constitute a new physical theory of quantum information.

Several questions remain open.

Problem 1. Operational meaning. Define an experimental procedure that assigns t, i, and f to a stated quantum-information claim. The assignment should be reproducible and distinguish missing evidence from contradictory evidence.

Problem 2. Statistical calibration. Determine whether annotations can be calibrated against finite-sample error, state-estimation uncertainty, detector inefficiency, and systematic errors.

Problem 3. Channel compatibility. Identify useful conditions on annotation-update maps under completely positive, trace-preserving quantum channels.

Problem 4. Entanglement analysis. Determine whether an annotation rule can provide additional information about the reliability of an entanglement estimate without being confused with concurrence or entanglement of formation.

Problem 5. Information-theoretic inequalities. Investigate whether suitably defined annotation quantities obey analogues of entropy inequalities or data-processing relations. Such results would require explicit definitions and proofs.

Problem 6. Experimental validation. Compare annotation-based reporting with established methods for uncertainty quantification and confidence assessment in quantum-state tomography.

In particular, a proposed neutrosophic quantity should not be claimed to improve quantum communication, state discrimination, or entanglement detection merely because it introduces a third component. Any such claim requires a clearly defined task, a baseline method, and mathematical or experimental evidence.

13.10 Conclusion

We have presented an introductory mathematical framework for viewing selected tools of quantum information theory through neutrosophic annotations. The standard no-cloning theorem, trace distance, fidelity, concurrence, entanglement of formation, and quantum entropy remain defined by their usual quantum-information formulas.

The proposed extension associates a triple (t, i, f) with a specified claim about a quantum-information record. This can be useful for distinguishing the underlying quantum state from the status of the evidence, documentation, or classification attached to it. The framework therefore offers a possible language for recording indeterminacy in quantum-information workflows.

The principal limitation is also clear: the triple alone does not define a new physical observable, a new quantum state, or a new entropy. A stronger theory would require operational semantics, update rules, consistency properties, and experimental validation. These requirements provide a concrete starting point for further research on neutrosophic methods in quantum information theory.

Chapter 14

Neutrosophic Adiabatic Quantum Computation

Adiabatic quantum computation encodes a computational problem in a Hamiltonian whose ground state represents a desired solution. A quantum system is initialized in the ground state of an easily prepared Hamiltonian and evolves under a time-dependent Hamiltonian toward a problem Hamiltonian. The success of the method depends on the spectral structure of the interpolation, the minimum energy gap, the evolution schedule, and the physical quality of the implementation. This paper develops an annotation-based neutrosophic perspective on adiabatic quantum computation. A triple (t, i, f) is associated with specified claims about Hamiltonian construction, spectral-gap estimates, model adequacy, and the interpretation of computational outputs. We formulate an annotated Hamiltonian record, discuss uncertainty in gap estimation, present a binary optimization example, and identify conditions needed for a mathematically consistent implementation. The proposed annotations are kept distinct from quantum probabilities, energy eigenvalues, and the physical state of the system. Consequently, the framework does not alter the standard adiabatic evolution equation or establish a speedup by itself. Its purpose is to provide a structured language for documenting support, indeterminacy, and opposition in the information used to design and evaluate an adiabatic computation.

14.1 Introduction

Quantum computation uses quantum-mechanical systems to represent and process information. In the circuit model, computation is described by sequences of quantum gates. Adiabatic quantum computation provides a different formulation: a computational problem is encoded in a Hamiltonian, and the system is guided toward the ground state of that Hamiltonian [62, 1].

The general idea is to start from a Hamiltonian H_B whose ground state is easy to prepare and gradually transform it into a problem Hamiltonian H_P. If the evolution is sufficiently slow and appropriate spectral conditions hold, the quantum adiabatic principle provides a route for the state to remain close to the instantaneous ground state [61, 64, 63].

The efficiency of this approach depends crucially on the minimum spectral gap along the interpolation. A small gap may require a long runtime, although the precise scaling depends on the Hamiltonian path, the schedule, and the theorem or performance criterion being used. The existence of an adiabatic algorithm therefore does not automatically imply that it outperforms a classical algorithm [60, 59].

Neutrosophic theory introduces truth, indeterminacy, and falsity as distinct components of an information assessment [6]. In computational science, a related annotation may be useful when an algorithm relies on approximate models, incomplete calibration, uncertain Hamiltonian coefficients, or unresolved estimates of a minimum spectral gap.

The purpose of this paper is to investigate an annotation framework for such information. The principal contributions are:

(i) a formal distinction between a physical Hamiltonian and an annotation about the adequacy of its specification;

(ii) an annotation scheme for spectral-gap estimates and computational claims;

(iii) a worked example of encoding a binary optimization problem in a problem Hamiltonian;

(iv) a discussion of how model uncertainty, measurement uncertainty, and neutrosophic annotations should be distinguished;

(v) a list of mathematical and computational problems needed to develop the framework further.

The paper is an original subject-adjacent development inspired by the topic of adiabatic quantum computation in McMahon [1]; it does not reproduce the exposition of that chapter.

14.2 Preliminaries on quantum Hamiltonians

14.2.1 Hamiltonians and quantum states

Let $\mathcal{H}$ be a finite-dimensional complex Hilbert space. A Hamiltonian is a self-adjoint operator

$$H = H^\dagger : \mathcal{H} \longrightarrow \mathcal{H}. \tag{14.1}$$

Its eigenvalues are real. We write them in nondecreasing order, including multiplicities, as

$$E_0(H) \leq E_1(H) \leq \cdots \leq E_{d-1}(H), \tag{14.2}$$

where $d = \dim \mathcal{H}$.

The ground-state energy is $E_0(H)$. If the ground state is nondegenerate, there is a normalized vector $|\phi_0\rangle$ such that

$$H|\phi_0\rangle = E_0(H)|\phi_0\rangle. \tag{14.3}$$

The ground state is the state associated with the smallest energy eigenvalue.

For a nondegenerate ground state, the first spectral gap is

$$\Delta(H) = E_1(H) - E_0(H). \tag{14.4}$$

When the ground state is degenerate, one must specify the relevant gap above the ground-state eigenspace. In this paper, the symbol Δ will be used for the gap between the ground-state sector and the next energy level.

Definition 14.2.1. A Hamiltonian is called gapped above its ground-state sector if there is a positive number Δ such that every eigenvalue outside that sector is at least Δ above the ground-state energy.

14.2.2 Time-dependent evolution

Let $H(t)$ be a continuous, time-dependent Hamiltonian. The state vector $|\psi(t)\rangle$ evolves according to the Schrödinger equation

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{14.5}$$

Here $\hbar$ is the reduced Planck constant.

If $H(t)$ is self-adjoint at each time, the ideal evolution is unitary. The system's state changes according to the Hamiltonian path. The annotation framework introduced below does not modify equation (14.5).

14.2.3 The interpolation Hamiltonian

Adiabatic quantum computation typically uses an interpolation of the form

$$H(s) = (1 - s)H_B + sH_P, \qquad 0 \leq s \leq 1, \tag{14.6}$$

where:

- H_B is the initial or driver Hamiltonian;
- H_P is the problem Hamiltonian;
- s is a dimensionless interpolation parameter.

At $s = 0$, the Hamiltonian is H_B, and at $s = 1$ it is H_P. A common choice for H_B is a transverse-field Hamiltonian whose ground state is a tensor product of single-qubit superpositions. The problem Hamiltonian is constructed so that its ground state encodes an optimal solution of the problem [62, 60].

A more general interpolation allows a schedule $s = s(t)$:

$$H(t) = H(s(t)), \qquad s(0) = 0, \quad s(T) = 1. \tag{14.7}$$

The schedule may be chosen to allocate more runtime to regions where the gap is small or where the Hamiltonian changes rapidly.

14.3 Adiabatic evolution and the spectral gap

14.3.1 The adiabatic principle

Suppose $H(s)$ has a nondegenerate ground state $|\phi_0(s)\rangle$ for every $s \in [0, 1]$. If the system starts in $|\phi_0(0)\rangle$ and the Hamiltonian changes sufficiently slowly, the evolved state can remain close to the instantaneous ground state.

A basic condition for a useful adiabatic approximation is that the relevant spectral gap does not close:

$$\Delta_{\min} = \min_{s \in [0,1]} \Delta(H(s)) > 0. \tag{14.8}$$

For finite-dimensional smooth Hamiltonian families, a positive minimum gap is a key ingredient in standard adiabatic theorems. The required runtime also depends on derivatives of $H(s)$ and on the precise error bound being used [63].

Theorem 14.3.1 (Qualitative adiabatic principle)**.** *Let $H(s)$ be a sufficiently smooth family of finite-dimensional self-adjoint operators with a nondegenerate ground state separated from the rest of the spectrum by a uniform positive gap on $[0, 1]$. Under the hypotheses of an applicable adiabatic theorem, evolution over a sufficiently long runtime starting in the ground state of $H(0)$ keeps the state close to the instantaneous ground-state subspace.*

Remark 14.3.2. The theorem is stated qualitatively because different quantitative adiabatic theorems impose different regularity conditions and produce different error bounds. One should not infer a universal runtime formula from the gap alone. The minimum gap is important, but it is not the only quantity governing runtime [63, 60].

14.3.2 Why the gap matters

A common schematic estimate for an adiabatic runtime has the form

$$T \gtrsim \frac{C}{\varepsilon} \frac{\mathcal{K}}{\Delta_{\min}^{p}}, \tag{14.9}$$

where ε is a target error tolerance, C and $\mathcal{K}$ depend on the chosen theorem and Hamiltonian path, and p is a positive exponent determined by the particular bound and schedule.

Equation (14.9) is schematic, not a universal theorem. It illustrates that a small gap can make adiabatic evolution expensive. A rigorous runtime analysis must use a specific bound with all of its hypotheses and constants.

If the minimum gap decreases exponentially with the input size, a simple adiabatic strategy may require exponential runtime. If the gap and relevant derivative terms have favorable scaling, a polynomial runtime may be possible. Neither outcome follows from the word "adiabatic" alone.

14.3.3 Schedule selection

The linear schedule $s(t) = t/T$ is convenient but not always optimal. A schedule may be adapted to the local gap. For example, a formal local rule might choose

$$\frac{ds}{dt} = c\,\Delta(H(s))^{q}, \tag{14.10}$$

where $c > 0$ and $q > 0$ are selected for a particular analysis. Such a rule slows the interpolation when the gap becomes small.

Equation (14.10) is only a schematic example of local scheduling. An implementable schedule must respect the relevant adiabatic theorem, hardware constraints, and the method by which the gap is estimated.

14.4 Neutrosophic annotations

14.4.1 Definition of an annotation

Definition 14.4.1. A neutrosophic annotation is a triple

$$\nu = (t, i, f) \in [0, 1]^3, \tag{14.11}$$

where t, i, and f denote the recorded degrees of support, indeterminacy, and opposition for a specified claim.

In general, the components are not required to sum to one. They are assessments associated with a claim, not necessarily probabilities of mutually exclusive events. This distinction is important when applying neutrosophic terminology to a quantum algorithm [6].

Definition 14.4.2. An annotated Hamiltonian record is a tuple

$$\mathcal{H}_{\text{rec}} = (H, \mathcal{C}, \nu), \tag{14.12}$$

where H is a specified Hamiltonian, $\mathcal{C}$ is a claim concerning the Hamiltonian or its computational use, and $\nu = (t, i, f)$ is an annotation about the evidence for that claim.

Examples of claims include:

- the coefficients in H_P accurately encode the intended optimization objective;

- a numerical estimate of $\Delta_{\min}$ is reliable within a stated tolerance;
- the implemented driver Hamiltonian is sufficiently close to the intended mathematical model;
- the final measurement procedure identifies a ground state with a specified success rate.

Remark 14.4.3. The tuple $\mathcal{H}_{\text{rec}}$ is an information record. The annotation ν is not an extra term in H, does not change the energy spectrum, and does not modify the Schrödinger equation. A different physical model would be needed to interpret the triple as part of the system's dynamics.

14.4.2 Annotation of the interpolation path

For an interpolation $H(s)$, one may associate a parameterized annotation

$$\nu(s) = \big(t(s), i(s), f(s)\big), \qquad 0 \le s \le 1. \tag{14.13}$$

This annotation can describe the evidential status of a claim about the model at each point along the path.

For example:

- $t(s)$ may record support for the adequacy of the Hamiltonian specification;
- $i(s)$ may record unresolved uncertainty in the model or in numerical calculations;
- $f(s)$ may record evidence contradicting a specified adequacy claim.

The meanings of these quantities must be fixed by a protocol. They should not be inferred from the energy eigenvalues alone.

14.4.3 A metric on annotation triples

For $\nu_1 = (t_1, i_1, f_1)$ and $\nu_2 = (t_2, i_2, f_2)$, define

$$d_{\mathcal{N}}(\nu_1, \nu_2) = \sqrt{(t_1 - t_2)^2 + (i_1 - i_2)^2 + (f_1 - f_2)^2}. \tag{14.14}$$

Proposition 14.4.4. *The function $d_{\mathcal{N}}$ is a metric on $[0,1]^3$.*

Proof. The assertion follows from the standard Euclidean metric on $\mathbb{R}^3$. Non-negativity, symmetry, and definiteness follow coordinatewise, while the triangle inequality follows from the Euclidean triangle inequality. □

For a continuous annotation path $\nu(s)$, its variation may be measured by

$$L_{\mathcal{N}}(\nu) = \int_0^1 \sqrt{\left(\frac{dt}{ds}\right)^2 + \left(\frac{di}{ds}\right)^2 + \left(\frac{df}{ds}\right)^2}\, ds, \tag{14.15}$$

provided the component functions are differentiable. This is a geometric measure of how the annotation changes. It is not a physical action or a quantum runtime.

14.5 An annotated adiabatic computational model

14.5.1 The four components of the model

We describe an annotation-aware adiabatic computation by

$$\mathfrak{A} = (H_B, H_P, s(t), \mathcal{M}; \nu), \tag{14.16}$$

where:

- H_B is the driver Hamiltonian;
- H_P is the problem Hamiltonian;
- $s(t)$ is the interpolation schedule;
- $\mathcal{M}$ is the final measurement procedure;
- ν is an annotation of a specified computational claim or record.

The physical algorithm is specified by the Hamiltonians, schedule, initial state, and measurement procedure. The annotation records additional information about the specification, evidence, or interpretation of that algorithm.

14.5.2 Separating physical and informational uncertainty

At least three distinct forms of uncertainty may occur:

(a) **Quantum measurement uncertainty:** outcomes are governed by the Born rule and the quantum state.

(b) **Statistical uncertainty:** finite data produce uncertainty in estimated quantities, such as energies or gaps.

(c) **Model uncertainty:** the implemented Hamiltonian or noise model may differ from the intended mathematical description.

A neutrosophic annotation may be used to summarize an explicitly defined assessment of the second or third type. It must not be confused with the first type, which is part of the physical theory.

14.5.3 A possible annotation-aware report

Suppose numerical calculations produce an estimate $\widehat{\Delta}_{\min}$ of the minimum gap. An associated record might be

$$\mathcal{R}_\Delta = \left(\widehat{\Delta}_{\min}, [\Delta_-, \Delta_+], \nu_\Delta\right), \tag{14.17}$$

where $[\Delta_-, \Delta_+]$ is a stated uncertainty interval and ν_Δ describes a claim about the adequacy of the estimate.

The interval and annotation have distinct meanings. The interval should arise from a specified numerical or statistical method, whereas the annotation is assigned under a separately documented rule.

Remark 14.5.1. A triple such as $(0.8, 0.15, 0.05)$ has no automatic statistical meaning. It is not a confidence level, a probability distribution, or a measurement result unless the application provides an appropriate calibration procedure.

14.6 Encoding an optimization problem

14.6.1 Binary objective functions

Many combinatorial optimization problems can be expressed as minimizing a function

$$C : \{0,1\}^n \longrightarrow \mathbb{R}. \tag{14.18}$$

The goal is to find

$$x^* \in \arg\min_{x \in \{0,1\}^n} C(x). \tag{14.19}$$

To encode the problem in a quantum system, associate each binary variable x_j with a qubit. The computational basis state

$$|x_1 x_2 \cdots x_n\rangle \tag{14.20}$$

represents a candidate binary string. A diagonal problem Hamiltonian can be defined by

$$H_P|x\rangle = C(x)|x\rangle. \tag{14.21}$$

Its ground-state basis vectors correspond to minimizers of C, provided the encoding is implemented correctly.

For the driver Hamiltonian, a common choice is

$$H_B = -\sum_{j=1}^{n} X_j, \tag{14.22}$$

where X_j denotes the Pauli-X operator acting on qubit j and the identity on the remaining qubits. The ground state of this driver is

$$|+\rangle^{\otimes n}, \qquad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}. \tag{14.23}$$

The interpolation is then

$$H(s) = (1-s)\left(-\sum_{j=1}^{n} X_j\right) + sH_P. \tag{14.24}$$

14.6.2 A two-variable example

Consider the objective

$$C(x_1, x_2) = x_1 + x_2 - 2x_1x_2, \qquad x_1, x_2 \in \{0,1\}. \tag{14.25}$$

Its values are

x_1	x_2	$C(x_1, x_2)$
0	0	0
0	1	1
1	0	1
1	1	0

The minimizing strings are 00 and 11.

Introduce the computational-basis number operators

$$n_j = \frac{I - Z_j}{2}, \tag{14.26}$$

where Z_j is the Pauli-Z operator on qubit j. The problem Hamiltonian corresponding to (14.25) is

$$H_P = n_1 + n_2 - 2n_1n_2. \tag{14.27}$$

Equivalently,

$$H_P = \frac{I - Z_1}{2} + \frac{I - Z_2}{2} - \frac{1}{2}(I - Z_1)(I - Z_2) \tag{14.28}$$

$$= \frac{1}{2}\left(I + Z_1Z_2\right). \tag{14.29}$$

The energies of $|00\rangle$ and $|11\rangle$ are zero, and the energies of $|01\rangle$ and $|10\rangle$ are one. Hence the ground-state subspace is two-dimensional.

This example illustrates why the ground-state degeneracy must be handled carefully. The relevant spectral gap is the difference between the ground energy and the first energy above the ground-state subspace, not the difference between two vectors within the degenerate ground-state subspace.

14.6.3 A neutrosophic record for the objective

Suppose the cost function is derived from an application whose coefficients are not fully verified. One may record the model as

$$\mathcal{R}_C = (C, \mathcal{C}_{\text{model}}, \nu_C), \qquad \nu_C = (t_C, i_C, f_C). \tag{14.30}$$

Here $\mathcal{C}_{\text{model}}$ is the claim that the cost function accurately represents the intended objective.

For example, a large indeterminacy component may be used to flag unresolved questions about the input data or penalty coefficients. Such an annotation does not change the mathematical minimizers of the explicitly specified function C. If the coefficients change, then the cost function and its problem Hamiltonian must be changed explicitly.

14.6.4 A necessary distinction

An annotation cannot repair an incorrect problem encoding. If the intended cost function is C but the implemented Hamiltonian encodes a different function $\widetilde{C}$, the ground state of the implemented Hamiltonian solves the encoded problem, not necessarily the intended one.

Consequently, a robust implementation should separately check:

1. correctness of the objective function;
2. correctness of the mapping from the objective to H_P;
3. accuracy of the implemented Hamiltonian coefficients;
4. validity of the initial-state preparation;
5. suitability of the interpolation schedule;
6. reliability of the final measurement and decoding.

An annotation scheme can document the status of these checks, but it does not replace the checks themselves.

14.7 Ground-state success and output interpretation

14.7.1 Success probability

Let $|\psi(T)\rangle$ be the final state produced by ideal evolution. Let P_* be the orthogonal projector onto the ground-state subspace of H_P. The probability of obtaining a ground-state result in a projective energy measurement is

$$p_{\mathrm{gs}} = \langle\psi(T)|P_*|\psi(T)\rangle. \tag{14.31}$$

If the ground state is nondegenerate, $P_* = |\phi_0\rangle\langle\phi_0|$ and

$$p_{\mathrm{gs}} = |\langle\phi_0 \mid \psi(T)\rangle|^2. \tag{14.32}$$

A high success probability means that the final state has substantial weight in the ground-state subspace. It does not, by itself, establish that the algorithm is computationally efficient, nor does it establish superiority over a classical method.

14.7.2 Annotated output record

An output record may be written as

$$\mathcal{R}_{\mathrm{out}} = (\widehat{x}, \widehat{p}_{\mathrm{gs}}, \nu_{\mathrm{out}}), \tag{14.33}$$

where $\widehat{x}$ is the reported solution and $\widehat{p}_{\mathrm{gs}}$ is an estimate of the ground-state success probability.

The annotation ν_{out} can be associated with a claim such as "the measured string is a valid minimizer of the intended objective." It should reflect a documented assessment procedure, rather than being assigned solely by inspection.

If the problem Hamiltonian has multiple ground states, the algorithm need only produce a member of the ground-state subspace to succeed under the usual ground-state criterion. If the application requires one particular minimizer, the success criterion must specify that additional requirement.

14.8 Model perturbations and annotation updates

14.8.1 Perturbed Hamiltonians

In a real device, the implemented Hamiltonian may differ from the intended Hamiltonian. Let

$$\widetilde{H}(s) = H(s) + V(s), \tag{14.34}$$

where $V(s)$ is a self-adjoint perturbation.

The operator norm

$$\|V(s)\| = \sup_{\|\psi\|=1} \|V(s)\psi\| \tag{14.35}$$

provides a measure of the perturbation magnitude. Standard spectral perturbation results relate changes in eigenvalues to this norm; in finite dimensions, Weyl's eigenvalue inequality gives

$$|\widetilde{E}_j(s) - E_j(s)| \le \|V(s)\| \tag{14.36}$$

when the eigenvalues are ordered consistently.

For a nondegenerate ground state, this implies the bound

$$|\widetilde{\Delta}(s) - \Delta(s)| \le 2\|V(s)\|, \tag{14.37}$$

where $\widetilde{\Delta}(s)$ and $\Delta(s)$ denote the corresponding first gaps, assuming the same ordered-eigenvalue indexing and compatible ground-state structure.

Thus, if $\|V(s)\|$ is small relative to the gap, the perturbation bound may certify that the gap remains positive at that value of s. Uniform certification over the whole path requires a uniform bound on $V(s)$ and a positive lower bound on the ideal gap.

Remark 14.8.1. The estimate (14.37) is a mathematical perturbation bound. It does not give a neutrosophic annotation automatically. An annotation would require an additional rule connecting perturbation information to the support, indeterminacy, and opposition components.

14.8.2 A possible update rule

Suppose a laboratory produces a new model assessment ν_{new} after calibration. A simple record update is

$$\mathcal{R}_{\text{old}} = (H, \mathcal{C}, \nu_{\text{old}}) \quad \longmapsto \quad \mathcal{R}_{\text{new}} = (\widetilde{H}, \mathcal{C}, \nu_{\text{new}}). \tag{14.38}$$

This notation records both a change in the physical model and a change in the assessment.

One must not assume that the three annotation components increase or decrease monotonically under calibration. Such behavior depends on the assignment protocol. New evidence may increase support, reveal previously hidden indeterminacy, or reveal that an earlier claim was false.

A future theory could specify an update map

$$\mathcal{U} : \mathfrak{N} \times \mathcal{D} \longrightarrow \mathfrak{N}, \tag{14.39}$$

where $\mathcal{D}$ is a space of calibration or experimental data. The properties of $\mathcal{U}$ would need to be studied mathematically and evaluated against real data.

14.9 Comparison with standard adiabatic computation

The following table summarizes the distinction between the standard physical model and the proposed annotation layer.

Quantity	**Standard meaning**	**Role of annotation**
Hamiltonian $H(s)$	Self-adjoint operator governing ideal evolution	Record whether the model specification is adequately supported
Ground-state energy	Lowest energy eigenvalue	Record the reliability of an energy estimate
Spectral gap $\Delta(s)$	Energy difference separating spectral levels	Record uncertainty in a gap estimate or its model assumptions
Success probability	Probability of a specified measurement outcome	Document the evidential status of the success claim
Runtime T	Duration of the evolution	Record uncertainty in a runtime estimate or implementation

The annotation layer is supplementary. It should not be described as improving the algorithm unless a concrete procedure uses the annotations to alter decisions and a theoretical or experimental analysis establishes a benefit.

14.10 Limitations and research problems

The framework in this paper is a proposal for structured information reporting, not a demonstrated improvement to adiabatic quantum computation. Several questions remain open.

Problem 1. Operational semantics. Develop a reproducible procedure for assigning (t, i, f) to a claim about a Hamiltonian, a spectral gap, or an output state.

Problem 2. Gap-estimation uncertainty. Determine how annotation components should be related to certified numerical bounds, statistical estimates, and finite-precision computations.

Problem 3. Perturbation sensitivity. Investigate annotation-update rules that remain consistent with rigorous spectral perturbation bounds.

Problem 4. Runtime certification. Define how annotations can summarize the assumptions and uncertainty of a particular adiabatic runtime theorem without replacing its mathematical hypotheses.

Problem 5. Optimization correctness. Establish methods for recording the distinction between uncertainty in input data, errors in Hamiltonian encoding, and uncertainty in the final measurement.

Problem 6. Comparative evaluation. Test whether annotation-aware reporting improves the reproducibility, interpretability, or reliability of quantum-optimization studies.

Problem 7. Computational usefulness. If annotations are used to choose schedules or allocate computational resources, determine whether that use yields measurable benefits over standard methods.

A particularly important question is whether a well-defined annotation protocol can improve decision-making when the Hamiltonian is known only approximately. For example, a protocol might flag regions where gap estimates are poorly supported and trigger further numerical analysis. Such a procedure would be useful only if the flagging rule is specified and its effectiveness is demonstrated.

14.11 Conclusion

Adiabatic quantum computation provides a framework in which a computational problem is encoded in a Hamiltonian and the system is evolved from an easily prepared initial ground state toward the ground state of the problem Hamiltonian. The minimum spectral gap, the interpolation schedule, the regularity of the Hamiltonian path, and the physical implementation all influence performance.

This paper has proposed a neutrosophic annotation framework for recording support, indeterminacy, and opposition associated with claims about the Hamiltonian, its spectral properties, and the computational output. A binary optimization example illustrates how a problem Hamiltonian can be specified independently of the annotation.

The mathematical distinction between physical dynamics and information about those dynamics is essential. The Schrödinger equation, the ground-state spectrum, and the standard adiabatic principle remain unchanged by the introduction of an annotation triple. Likewise, a neutrosophic record does not by itself certify a spectral gap, reduce runtime, or provide a quantum speedup.

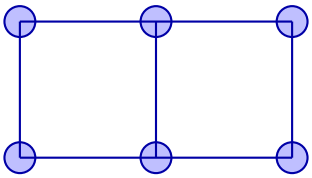
Further work should focus on operational definitions, reproducible annotation rules, compatibility with spectral perturbation theory, and experimental evaluation. These requirements

provide a concrete research program for investigating whether neutrosophic information can become a useful supplementary tool in the analysis and reporting of adiabatic quantum computations.

Chapter 15

Neutrosophic Cluster-State Quantum Computing

Related literature. Neutrosophic quantum-state proposals [73] and the quantum error annotation approach [80] (preprint) offer possible interpretations of incomplete graph-state and stabilizer records.



Edges correspond to controlled-Z entanglement

Figure 15.1: A 2×3 graph-state lattice, an elementary resource for measurement-based quantum computation.

Cluster-state quantum computing is a model of quantum computation in which an entangled multiqubit resource state is prepared before computation proceeds through appropriately chosen single-qubit measurements and classical feed-forward. This paper develops an original neutrosophic perspective on cluster-state computation. We review graph states, adjacency matrices, stabilizer operators, entanglement witnesses, and measurement-based quantum processing. We then introduce truth–indeterminacy–falsity annotations for records concerning graph specification, state preparation, stabilizer measurements, entanglement certification, and measurement outcomes. Elementary properties of the annotation space are established, and examples illustrate how the annotations can document incomplete or conflicting experimental evidence without changing the underlying quantum formalism. A path-graph cluster state is used to demonstrate the graph-state construction and stabilizer structure. The distinction between physical quantum uncertainty, statistical estimation error, and informational indeterminacy is maintained throughout. The proposed framework is descriptive rather than a new physical theory: its annotations do not themselves establish entanglement, alter Born probabilities, or improve computational performance. We conclude with research problems concerning calibration, measurement-based computation, and operational validation of neutrosophic annotations.

15.1 Introduction

Quantum computation can be formulated in terms of quantum circuits or through measurement-based models. In the circuit model, quantum gates are applied sequentially to a collection of

qubits. In measurement-based quantum computing, a highly entangled resource state is first prepared, and computation is then performed by measuring individual qubits in appropriately selected bases, with classical processing of measurement outcomes [69, 65].

Cluster states are important examples of graph states used as universal resources for measurement-based quantum computation. Their mathematical structure can be described using a graph, controlled-phase operations, and stabilizer generators. This graph-based description provides a useful connection between quantum information, graph theory, and the algebra of Pauli operators [68, 65].

McMahon's *Quantum Computing Explained* discusses cluster-state quantum computing, adjacency matrices, stabilizer states, entanglement witnesses, and cluster-state processing [1]. The present paper develops these topics independently and considers how neutrosophic annotations might supplement the documentation and interpretation of cluster-state experiments.

Neutrosophic theory distinguishes degrees of truth, indeterminacy, and falsity associated with an information claim [6]. In a cluster-state experiment, such annotations might record the evidential status of a claim that a particular entangling gate was implemented correctly, that a stabilizer expectation value is reliable, or that the prepared state satisfies a specified entanglement criterion.

The central methodological point is that these annotations are not quantum states and are not automatically probabilities. The quantum state, its stabilizers, and the probabilities of measurement outcomes remain defined by standard quantum mechanics.

The goals of this paper are:

(i) to formulate graph states and cluster states using adjacency matrices and controlled-phase gates;

(ii) to present the stabilizer description of graph states;

(iii) to discuss entanglement witnesses and their experimental interpretation;

(iv) to define neutrosophic annotations for cluster-state records;

(v) to provide explicit examples and formulate research questions for an operational neutrosophic framework.

This is an original mathematical development inspired by the subject of cluster-state quantum computing. It is not a reproduction of McMahon's chapter.

15.2 Preliminaries

15.2.1 Qubits and Pauli operators

The Hilbert space of a single qubit is $\mathbb{C}^2$, with computational basis $\{|0\rangle, |1\rangle\}$. The Pauli operators are

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15.1}$$

The Hadamard operator is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{15.2}$$

It maps computational-basis states to superpositions:

$$H|0\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad H|1\rangle = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{15.3}$$

For an n-qubit system, the Hilbert space is

$$\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}. \tag{15.4}$$

An operator such as X_j means that X acts on qubit j and the identity acts on all other qubits.

15.2.2 Controlled-phase operations

The controlled-Z operation on qubits j and k is

$$\mathrm{CZ}_{jk} = \mathrm{diag}(1, 1, 1, -1) \tag{15.5}$$

in the two-qubit computational basis. Its action on a basis state is

$$\mathrm{CZ}_{jk}|x_j x_k\rangle = (-1)^{x_j x_k}|x_j x_k\rangle, \qquad x_j, x_k \in \{0, 1\}. \tag{15.6}$$

The controlled-Z gate is symmetric in its two qubits and is entangling when applied to suitable product states.

Controlled-phase gates commute with one another when acting on computational-basis systems. This property makes it convenient to specify graph states by a collection of edges.

15.3 Graphs and adjacency matrices

15.3.1 Graph definition

Definition 15.3.1. A finite simple undirected graph is a pair

$$G = (V, E), \tag{15.7}$$

where $V = \{1, \ldots, n\}$ is a finite vertex set and E is a set of unordered pairs $\{j, k\}$ with $j \neq k$.

In a graph-state construction, each vertex represents a qubit and each edge specifies a controlled-phase operation.

Definition 15.3.2. The adjacency matrix of a finite simple undirected graph $G = (V, E)$ is the matrix

$$A(G) = (a_{jk})_{j,k=1}^{n} \tag{15.8}$$

defined by

$$a_{jk} = \begin{cases} 1, & \{j, k\} \in E, \\ 0, & \{j, k\} \notin E. \end{cases} \tag{15.9}$$

Because the graph is simple and undirected,

$$a_{jk} = a_{kj}, \qquad a_{jj} = 0. \tag{15.10}$$

Thus $A(G)$ is a symmetric binary matrix with zero diagonal.

15.3.2 Neighborhoods

For a vertex $j \in V$, its neighborhood is

$$N(j) = \{k \in V : \{j, k\} \in E\}. \tag{15.11}$$

The degree of vertex j is $|N(j)|$.

The adjacency matrix describes the same neighborhood relation:

$$N(j) = \{k : a_{jk} = 1\}. \tag{15.12}$$

This information determines the stabilizer generators of the corresponding graph state.

15.3.3 A path graph

Consider the path graph on three vertices,

$$1 \; - \; 2 \; - \; 3. \tag{15.13}$$

Its edge set is

$$E = \{\{1,2\},\{2,3\}\}, \tag{15.14}$$

and its adjacency matrix is

$$A(G) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{15.15}$$

The neighborhoods are

$$N(1) = \{2\}, \qquad N(2) = \{1,3\}, \qquad N(3) = \{2\}. \tag{15.16}$$

This graph provides a simple example for constructing a cluster state and identifying its stabilizer operators.

15.4 Graph states and cluster states

15.4.1 Definition of a graph state

For a graph $G = (V, E)$ with n vertices, define the graph state by

$$|G\rangle = \prod_{\{j,k\}\in E} \mathrm{CZ}_{jk} |+\rangle^{\otimes n}. \tag{15.17}$$

The order of the controlled-Z gates does not matter because they commute.

The construction has two stages:

1. prepare each qubit in the state $|+\rangle$;
2. apply a controlled-Z gate for every edge of G.

The resulting state is generally entangled whenever the graph contains edges.

Definition 15.4.1. A cluster state is a graph state associated with a graph chosen to support a cluster-state quantum-computing protocol. Common examples use one-dimensional chains or two-dimensional lattices.

The term “cluster state” is often used particularly for graph states associated with lattice-like structures, while “graph state” is the more general construction [68, 65].

15.4.2 Computational-basis expression

The graph state can be written as

$$|G\rangle = \frac{1}{2^{n/2}} \sum_{x\in\{0,1\}^n} (-1)^{\sum_{\{j,k\}\in E} x_j x_k} |x_1 x_2 \cdots x_n\rangle. \tag{15.18}$$

The exponent only matters modulo two. Each edge contributes a phase factor $(-1)^{x_j x_k}$.

Example 15.4.2. For the three-vertex path graph,

$$|G\rangle = \mathrm{CZ}_{12}\,\mathrm{CZ}_{23}|+\rangle^{\otimes 3}. \tag{15.19}$$

Using the computational basis,

$$|G\rangle = \frac{1}{\sqrt{8}}\big(|000\rangle + |001\rangle + |010\rangle - |011\rangle \tag{15.20}$$

$$+ |100\rangle + |101\rangle - |110\rangle + |111\rangle\big). \tag{15.21}$$

The minus signs occur for strings for which at least one edge has both endpoint bits equal to one.

15.4.3 Preparation and physical implementation

The mathematical definition in equation (15.17) describes an ideal state. A physical implementation may involve imperfect gates, state-preparation errors, decoherence, and measurement errors. The resulting state may therefore be represented by a density operator ρ rather than the ideal pure-state projector $|G\rangle\langle G|$.

An experimental report should distinguish:

- the intended graph G;
- the ideal graph state $|G\rangle$;
- the actual prepared state ρ;
- the evidence supporting the claim that ρ is close to $|G\rangle\langle G|$.

The neutrosophic annotations introduced later will be attached to specified claims about these objects, not substituted for the objects themselves.

15.5 Stabilizer states

15.5.1 Stabilizer operators

For each vertex j, define

$$K_j = X_j \prod_{k \in N(j)} Z_k. \tag{15.22}$$

These operators are called the graph-state stabilizer generators [68, 66].

Each K_j is Hermitian and satisfies

$$K_j^2 = I. \tag{15.23}$$

The graph-state stabilizers commute:

$$[K_j, K_\ell] = 0 \qquad \text{for all } j, \ell. \tag{15.24}$$

The graph state satisfies

$$K_j|G\rangle = |G\rangle \qquad \text{for every } j \in V. \tag{15.25}$$

Proposition 15.5.1. *For the graph state defined by equation (15.17), each generator K_j in equation (15.22) stabilizes $|G\rangle$.*

Proof. Let

$$U_G = \prod_{\{j,k\}\in E} \mathrm{CZ}_{jk}\,. \tag{15.26}$$

Then

$$|G\rangle = U_G|+\rangle^{\otimes n}. \tag{15.27}$$

The controlled-Z conjugation identities give

$$U_G X_j U_G^\dagger = X_j \prod_{k\in N(j)} Z_k = K_j. \tag{15.28}$$

Therefore,

$$\begin{aligned} K_j|G\rangle &= U_G X_j U_G^\dagger U_G|+\rangle^{\otimes n} && (15.29)\\ &= U_G X_j|+\rangle^{\otimes n} && (15.30)\\ &= U_G|+\rangle^{\otimes n} && (15.31)\\ &= |G\rangle, && (15.32) \end{aligned}$$

because $X|+\rangle = |+\rangle$. This proves the claim. □

15.5.2 The stabilizer group

The stabilizer group of a graph state is generated by the operators $K_1, \ldots, K_n$:

$$\mathcal{S}(G) = \langle K_1, \ldots, K_n\rangle. \tag{15.33}$$

For an n-qubit graph state, these independent commuting generators define a stabilizer state.

For the three-vertex path graph,

$$K_1 = X_1 Z_2, \qquad K_2 = Z_1 X_2 Z_3, \qquad K_3 = Z_2 X_3. \tag{15.34}$$

The ideal state satisfies

$$K_1|G\rangle = K_2|G\rangle = K_3|G\rangle = |G\rangle. \tag{15.35}$$

15.5.3 Stabilizer expectation values

For a density operator ρ, the expectation value of a stabilizer generator is

$$\langle K_j\rangle_\rho = \mathrm{Tr}(\rho K_j). \tag{15.36}$$

Since K_j is Hermitian and has eigenvalues ± 1,

$$-1 \le \mathrm{Tr}(\rho K_j) \le 1. \tag{15.37}$$

For the ideal graph state,

$$\mathrm{Tr}(|G\rangle\langle G|K_j) = 1. \tag{15.38}$$

A value close to one is consistent with the expected stabilizer relation. However, individual stabilizer expectation values should be interpreted in the context of the complete test being performed; one or a few expectation values do not automatically certify every property of a prepared state.

15.6 Entanglement and entanglement witnesses

15.6.1 Entanglement

A pure multipartite state is fully separable if it can be written as a tensor product of single-party states. A mixed state is fully separable if it is a convex combination of such product-state projectors.

For a bipartite system, a pure state is entangled if it cannot be expressed as

$$|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B. \tag{15.39}$$

Graph states associated with connected graphs are entangled across every nontrivial bipartition, although the exact pattern of multipartite entanglement depends on the graph [68].

15.6.2 Entanglement witnesses

Definition 15.6.1. An entanglement witness is a Hermitian operator W such that

$$\mathrm{Tr}(W\sigma) \geq 0 \tag{15.40}$$

for every state σ in a specified separable-state class, while

$$\mathrm{Tr}(W\rho) < 0 \tag{15.41}$$

for at least one entangled state ρ.

The separable-state class must be specified: for example, bipartite separability across a chosen cut or full separability among all parties. Entanglement witnesses are useful because their expectation values can sometimes be estimated without reconstructing the full density operator [70, 67].

15.6.3 A graph-state witness

For a graph with n vertices and stabilizer generators K_j, consider

$$W_G = \frac{n}{2} I - \frac{1}{2} \sum_{j=1}^{n} K_j. \tag{15.42}$$

This operator is a standard fully-separable-state witness for graph states in the graph-state stabilizer setting [71].

For the ideal graph state,

$$\langle W_G \rangle_{|G\rangle} = \frac{n}{2} - \frac{1}{2} n = 0. \tag{15.43}$$

For a state with sufficiently large stabilizer expectation values, the witness expectation can be negative, providing evidence of entanglement under the applicable witness criterion.

Remark 15.6.2. The precise witness criterion depends on the witness and the separability class being tested. A witness does not detect every entangled state. Its experimental use requires reliable estimates of the relevant expectation values and their uncertainties.

15.6.4 Annotation of an entanglement claim

Let $\mathcal{C}_{\text{ent}}$ denote the claim that an experimentally prepared state is entangled according to a specified criterion. An annotation

$$\nu_{\text{ent}} = (t_{\text{ent}}, i_{\text{ent}}, f_{\text{ent}}) \tag{15.44}$$

may record the evidential status of that claim.

For example, i_{ent} might be assigned a high value when the witness expectation has a statistical uncertainty interval crossing the certification threshold. The precise assignment rule must be declared in advance.

The annotation is not itself an entanglement witness. The witness is an operator with a mathematical property concerning separable and entangled states, whereas the triple is a separate assessment of a claim about experimental evidence.

15.7 Cluster-state processing

15.7.1 Measurement-based quantum computation

In measurement-based quantum computation, computation proceeds by performing single-qubit measurements on an entangled resource state. The measurement bases and the interpretation of later outcomes can depend on earlier classical outcomes. This adaptive procedure is often called one-way quantum computation [69].

The general stages are:

1. prepare a graph or cluster state;
2. choose a measurement pattern determined by the desired computation;
3. measure selected qubits in specified bases;
4. process the classical measurement outcomes;
5. apply any required feed-forward corrections or interpret the output using the measurement record.

The resource state contains entanglement used by the measurement pattern. Measurements consume parts of this resource and transform the remaining quantum state.

15.7.2 Single-qubit measurement bases

A measurement in the equatorial basis at angle α can be described by the orthonormal states

$$|+_\alpha\rangle = \frac{|0\rangle + e^{i\alpha}|1\rangle}{\sqrt{2}}, \qquad |-_\alpha\rangle = \frac{|0\rangle - e^{i\alpha}|1\rangle}{\sqrt{2}}. \tag{15.45}$$

The corresponding projectors are

$$M_{0|\alpha} = |+_\alpha\rangle\langle+_\alpha|, \qquad M_{1|\alpha} = |-_\alpha\rangle\langle-_\alpha|. \tag{15.46}$$

For a single-qubit state ρ, the Born probabilities are

$$p(m|\rho,\alpha) = \operatorname{Tr}(M_{m|\alpha}\rho), \qquad m \in \{0,1\}. \tag{15.47}$$

They satisfy

$$p(0|\rho,\alpha) + p(1|\rho,\alpha) = 1. \tag{15.48}$$

In a cluster-state computation, the relevant measurement acts on one subsystem of an entangled multiqubit state. The conditional state of the remaining qubits depends on the measurement outcome.

15.7.3 Classical feed-forward

Measurement outcomes may determine later measurement angles or final correction operators. A general description is a conditional rule

$$\alpha_j = F_j(m_1, \ldots, m_{j-1}), \tag{15.49}$$

where F_j is a classical control rule and $m_1, \ldots, m_{j-1}$ are previous measurement outcomes.

The correction pattern is part of the computational protocol. If the classical record is incomplete or a measurement outcome is misclassified, the implemented procedure may differ from the intended one.

A neutrosophic annotation may be attached to a record of the feed-forward decision, but it must not replace the actual measurement outcome or the classical rule.

15.7.4 An annotation-aware processing record

We may represent a cluster-state computation by

$$\mathcal{P} = (G, \rho, \mathcal{M}, \mathcal{F}, \nu), \tag{15.50}$$

where:

- G is the intended graph;
- ρ is the prepared resource state;
- $\mathcal{M}$ specifies the measurement pattern;
- $\mathcal{F}$ specifies the classical feed-forward rule;
- ν annotates a specified claim about the record.

This tuple makes the roles of the physical state, graph specification, measurement protocol, classical processing, and informational assessment explicit.

15.8 Neutrosophic annotation framework

15.8.1 The annotation space

Definition 15.8.1. The neutrosophic annotation space is

$$\mathfrak{N} = [0, 1]^3. \tag{15.51}$$

An element

$$\nu = (t, i, f) \in \mathfrak{N} \tag{15.52}$$

records the degrees of support, indeterminacy, and opposition associated with a specified information claim.

No normalization condition

$$t + i + f = 1 \tag{15.53}$$

is imposed in the general model. The components are not automatically probabilities of mutually exclusive outcomes.

Definition 15.8.2. An annotated cluster-state record is a tuple

$$\mathcal{R} = (G, \rho, \mathcal{C}, \nu), \tag{15.54}$$

where G is the intended graph, ρ is the actual or estimated quantum state, $\mathcal{C}$ is a specified claim, and ν is an annotation about the evidence for that claim.

Examples of claims include:

- the implemented graph matches the intended adjacency matrix;
- the prepared state has the expected stabilizer expectation values;
- a selected entanglement witness certifies entanglement;
- the measurement record has been decoded correctly;
- the reported output agrees with the specified measurement-based protocol.

15.8.2 A metric on annotations

For $\nu_1 = (t_1, i_1, f_1)$ and $\nu_2 = (t_2, i_2, f_2)$, define

$$d_{\mathfrak{N}}(\nu_1, \nu_2) = \sqrt{(t_1 - t_2)^2 + (i_1 - i_2)^2 + (f_1 - f_2)^2}. \tag{15.55}$$

Proposition 15.8.3. *The function $d_{\mathfrak{N}}$ is a metric on $\mathfrak{N}$.*

Proof. It is the restriction of the Euclidean metric on $\mathbb{R}^3$ to the cube $[0,1]^3$. Therefore it is nonnegative, symmetric, zero exactly when the arguments are equal, and satisfies the triangle inequality. □

This metric compares annotation triples. It is not a measure of quantum-state distinguishability, entanglement, or computational performance.

15.8.3 Annotation of stabilizer data

Suppose an experiment estimates the expectation value

$$\widehat{k}_j \approx \mathrm{Tr}(\rho K_j) \tag{15.56}$$

from repeated measurements. The experimental report may contain a tuple

$$\mathcal{R}_j = (\widehat{k}_j, \epsilon_j, \nu_j), \tag{15.57}$$

where ϵ_j is an uncertainty estimate produced by a specified statistical method and ν_j annotates a claim about the reliability of the estimate.

The uncertainty estimate and annotation have different roles. The former should be computed from the measurement model and observed data. The latter is assigned by an additional protocol.

For example, a high indeterminacy component may be used when the number of measurements is small, the calibration is incomplete, or the statistical interval is too wide to support the intended claim. This interpretation is only meaningful once a reproducible assignment rule is supplied.

15.8.4 Annotation of graph specification

The intended graph may be represented by its adjacency matrix $A(G)$, while the implemented entangling pattern may correspond to a different matrix $A(\widetilde{G})$.

A graph-specification record can be written as

$$\mathcal{R}_G = (A(G), A(\widetilde{G}), \nu_G). \tag{15.58}$$

The annotation ν_G concerns the claim that the implemented entangling pattern matches the intended graph.

If

$$A(G) = A(\widetilde{G}), \tag{15.59}$$

the two simple undirected graphs have the same labeled edge set. If the matrices differ, then at least one edge is missing or added in the implementation.

An annotation does not repair a graph mismatch. The graph specification or physical preparation procedure must be corrected explicitly.

15.9 Worked example: a three-qubit path cluster

15.9.1 Ideal state and stabilizers

Consider again the path graph

$$1 \; - \; 2 \; - \; 3. \tag{15.60}$$

Its ideal graph state is

$$|G\rangle = \mathrm{CZ}_{12}\,\mathrm{CZ}_{23}|+\rangle^{\otimes 3}. \tag{15.61}$$

The stabilizer generators are

$$K_1 = X_1 Z_2, \qquad K_2 = Z_1 X_2 Z_3, \qquad K_3 = Z_2 X_3. \tag{15.62}$$

For the ideal state,

$$\langle K_1 \rangle = \langle K_2 \rangle = \langle K_3 \rangle = 1. \tag{15.63}$$

15.9.2 A simple witness value

For $n = 3$, define

$$W_G = \frac{3}{2} I - \frac{1}{2}(K_1 + K_2 + K_3). \tag{15.64}$$

Suppose the measured expectation values are

$$\langle K_1 \rangle = 0.92, \qquad \langle K_2 \rangle = 0.88, \qquad \langle K_3 \rangle = 0.90. \tag{15.65}$$

Then the estimated witness value is

$$\begin{aligned} \langle W_G \rangle &= \frac{3}{2} - \frac{1}{2}(0.92 + 0.88 + 0.90) && (15.66)\\ &= \frac{3}{2} - \frac{1}{2}(2.70) && (15.67)\\ &= 0.15. && (15.68) \end{aligned}$$

This value is not negative, so these illustrative numbers do not provide entanglement certification through this witness criterion. This does not prove that the state is separable; a witness can fail to detect entanglement that is present.

Now suppose a different experiment gives

$$\langle K_1\rangle = 0.99, \qquad \langle K_2\rangle = 0.98, \qquad \langle K_3\rangle = 0.99. \tag{15.69}$$

The corresponding estimated value is

$$\langle W_G\rangle = \frac{3}{2} - \frac{1}{2}(2.96) = 0.02. \tag{15.70}$$

It is again nonnegative. The example demonstrates why a witness criterion must be applied to the actual mathematical expression and why high stabilizer expectation values alone do not guarantee a negative value for every witness.

15.9.3 A stronger stabilizer-based witness

For this particular graph state, consider the fidelity

$$F_G = \langle G|\rho|G\rangle. \tag{15.71}$$

For a pure target state, F_G is the overlap with the target projector. The operator

$$W_F = \frac{1}{2}I - |G\rangle\langle G| \tag{15.72}$$

is an entanglement witness for the three-qubit path graph with respect to full separability: fully separable states have overlap with $|G\rangle$ at most $1/2$, whereas

$$F_G > \frac{1}{2} \tag{15.73}$$

implies that the state is not fully separable. This fidelity witness is useful only if the overlap is estimated or bounded reliably.

The witness W_F and the stabilizer-based expression W_G above are distinct operators. Their detection properties must not be conflated.

15.9.4 An annotation record

Suppose an experiment reports a fidelity estimate $\widehat{F}_G$ with a statistical uncertainty interval. The associated annotation might be

$$\nu_F = (t_F, i_F, f_F). \tag{15.74}$$

The triple could document the status of the claim that the prepared state is sufficiently close to the target cluster state. Its assignment would need a rule based on the experimental data, calibration, and a chosen threshold.

For example, if the confidence interval for F_G lies entirely above $1/2$, that evidence supports the full-separability witness test described above. If the interval crosses $1/2$, the test may be inconclusive. A neutrosophic annotation can be used to record this distinction, but it cannot replace the confidence interval or the witness theorem.

15.10 Physical uncertainty and informational indeterminacy

Three concepts should be kept separate.

15.10.1 Quantum measurement uncertainty

Measurement outcomes are governed by the Born rule. For a state ρ and a measurement described by operators $\{M_m\}$,

$$p(m|\rho) = \mathrm{Tr}(M_m \rho). \tag{15.75}$$

These probabilities describe the quantum measurement statistics under the stated model.

15.10.2 Statistical uncertainty

Finite samples lead to uncertainty in estimates of expectation values, fidelity, and other quantities. This uncertainty should be quantified by appropriate statistical methods, such as confidence intervals or other justified uncertainty estimates.

15.10.3 Model uncertainty

The physical implementation may differ from the ideal graph-state preparation. For example, a controlled-Z gate may have an error, a qubit may experience decoherence, or a detector may be miscalibrated.

A neutrosophic annotation may be attached to a claim about statistical or model uncertainty. It should not be used to replace the Born probabilities or to disguise a lack of quantitative uncertainty analysis.

15.11 Limitations and research directions

The framework presented here is an annotation proposal. Its mathematical usefulness depends on giving operational meaning to the triple (t, i, f) and showing that the annotation adds information beyond standard experimental reporting.

The following problems are natural directions for future work.

Problem 1. Operational assignment. Define a reproducible algorithm for assigning annotations to graph-specification claims, stabilizer estimates, and entanglement certifications.

Problem 2. Statistical calibration. Determine how the annotation components should relate to sample size, confidence intervals, systematic error, and uncertainty in state reconstruction.

Problem 3. Stabilizer consistency. Investigate whether annotation rules can summarize the reliability of a collection of commuting stabilizer measurements without changing their quantum-mechanical meaning.

Problem 4. Graph reconstruction. Develop methods for recording uncertainty about an implemented graph and distinguishing missing edges from measurement errors or state-preparation errors.

Problem 5. Measurement-based computation. Study whether annotations of measurement outcomes and feed-forward decisions improve the diagnosis of failures in a specified measurement-based protocol.

Problem 6. Entanglement certification. Compare annotation-based reporting with established witness methods and quantify whether the annotations improve reproducibility or interpretation.

Problem 7. Experimental validation. Test the framework on simulated or physical cluster-state data and compare it with conventional uncertainty reporting.

A central challenge is to avoid simply renaming known uncertainty measures. A substantive contribution would require a precise assignment rule, a mathematical property or operational advantage, and evidence that the proposed annotation improves the analysis of a well-defined task.

15.12 Conclusion

Cluster-state quantum computing represents computation through an entangled resource state and a pattern of single-qubit measurements with classical processing. Graph states provide a compact mathematical description through adjacency matrices, controlled-phase gates, and stabilizer generators. Entanglement witnesses can then provide experimentally accessible tests of selected entanglement properties.

This paper has proposed a neutrosophic annotation framework for recording the evidential status of claims about graph specifications, state preparation, stabilizer measurements, entanglement witnesses, and measurement-based processing. The annotations distinguish support, indeterminacy, and opposition, while leaving the standard quantum state and measurement rules unchanged.

The framework is not, by itself, a new physical model or a new entanglement measure. Its value must be established through explicit operational definitions, reproducible assignment rules, and comparison with established statistical and experimental methods. These requirements provide a concrete program for further study of neutrosophic information in cluster-state quantum computing.

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