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Minimal resolutions and algebraic properties of some classes of graded ideals

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Declaration of Authorship

I, Ernesto LAX, declare that this thesis titled "Minimal resolutions and algebraic properties of some classes of graded ideals" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: Ernesto Lax

Date: 12/01/2026

*"Mathematics knows no races or geographical boundaries;
for mathematics, the cultural world is one country."*

David Hilbert

UNIVERSITY OF MESSINA

Abstract

Department of Mathematical and Computer Sciences,
Physical Sciences and Earth Sciences

Doctor of Philosophy

Minimal resolutions and algebraic properties of some classes of graded ideals

by Ernesto LAX

This thesis investigates the interaction between Combinatorics and Commutative Algebra, in order to study minimal resolutions and homological properties of graded ideals. In the exterior algebra, we construct minimal graded resolutions of monomial ideals with linear quotients via iterated mapping cones, describing their Betti numbers and differentials. In the polynomial ring, we develop the theory of vector–spread strongly stable ideals, generalizing both Macaulay and Kruskal–Katona theorems and describe an upper bound for their Betti numbers. For the class of squarefree principal vector–spread Borel ideals, we describe the primary decomposition and characterize the sequentially Cohen–Macaulay property. Moreover, we completely classify the ideals in this class having the property that their ordinary and symbolic powers coincide. For edge ideals of forests, we apply Betti splitting techniques to their squarefree powers, proving the non–increasingness of the normalized depth function and computing regularity in terms of matching numbers, thus confirming a conjecture of Erey and Hibi. We further study binomial edge ideals, establishing the sequentially Cohen–Macaulayness of cycles, wheels, block graphs, and cones. Finally, we present the *Macaulay2* package `SCMAlgebras`, which provides computational tools for testing the sequentially Cohen–Macaulayness.

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(**Italian version**)

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To my parents, to whom I owe everything.

To my brother, for his warm trust.

To the memory of my beloved abuela Emilia.

Introduction

The study of graded ideals and their minimal free resolutions is a central theme in Commutative Algebra, with deep historical roots and wide-reaching connections to Geometry, Combinatorics, and Computational Algebra. Over the last century, the development of homological methods has shaped much of the modern theory. A foundational milestone was Macaulay's 1927 theorem [88], which characterized the possible Hilbert functions of graded ideals in a polynomial ring. In this context, he introduced *lexicographic ideals*, a distinguished class of monomial ideals that remain central in extremal problems involving Hilbert functions and Betti numbers. This insight established monomial ideals as a natural testing ground for homological questions: their combinatorial structure often makes algebraic invariants explicitly computable. Building on this viewpoint, explicit constructions such as the Taylor resolution [113], the Eliahou–Kervaire resolution of stable ideals [41], whose role is played in the exterior algebra framework by the Cartan complex [10], and cellular resolutions of squarefree monomial ideals [12, 13] revealed profound links between free resolutions, polyhedral geometry, and topological combinatorics.

In recent decades, these connections have expanded into new research directions. A landmark contribution came from Stanley [110, 112], who associated to any simplicial complex a squarefree monomial ideal, namely the *Stanley–Reisner ideal*, leading to the solution of the Upper Bound Conjecture for simplicial spheres, where the theory of Cohen–Macaulay rings played a decisive role. This work firmly established a deep and prolific bridge between Combinatorics and Commutative Algebra. In this perspective, Villarreal's introduction of *edge ideals* [118, 119] created a fertile interaction between graph theory and commutative algebra, where constructions such as the minimal primary decomposition and invariants such as projective dimension, depth, and regularity can be interpreted through the combinatorics of the underlying graphs. More recently, binomial ideals, and in particular *binomial edge ideals* introduced by Herzog et al. [69] and Ohtani [95], have further enriched this dialogue by encoding determinantal conditions attached to graphs. These objects have opened new perspectives on the study of Cohen–Macaulayness [21, 22, 42], Gorensteinness [59, 99], and sequential Cohen–Macaulayness [45, 106], all investigated via combinatorial techniques.

The present dissertation aims to contribute to a deeper understanding of these topics, with particular regard to minimal graded resolutions and algebraic properties of significant classes of graded ideals, under the unifying principle that the combinatorial structure underlying an ideal governs its homological behaviour. Throughout the thesis, a computational approach also intervenes through the use of the *Macaulay2* software [62] and of its packages, to generate examples confirming the produced results.

We now proceed to describe the structure of the thesis.

In Chapter 1 we collect some fundamental notions and results regarding commutative algebra and exterior algebra. In Section 1.1, we recall the notions of Krull dimension and depth, whose interplay leads to the notion of Cohen–Macaulay modules. We then review graded algebras as the setting for important constructions such as Hilbert function, minimal free resolutions, and many key invariants: Betti numbers, projective dimension, and Castelnuovo–Mumford regularity. Standard homological tools like the Koszul complex and local cohomology are also presented. The section closes with an overview of the sequentially Cohen–Macaulay property, together with its main characterizations. In Section 1.2, we recall the basic facts on the exterior algebra and its role in studying monomial ideals, with particular focus on stable and strongly stable ideals. The Cartan complex is also recalled, serving as the analogue of the Koszul complex in the exterior algebra framework. Moreover, we introduce monomial ideals with linear quotients, illustrating their key properties. It is well known that, even if the exterior algebra is not commutative, it behaves like a commutative local ring [24] in many cases. Hence, many notions and results that hold in one context can be translated to the other one with some suitable modifications.

In Chapter 2 we study minimal graded free resolutions of monomial ideals in the exterior algebra $E = K\langle e_1, \dots, e_n\rangle$ over a field K. The motivation of this study lies in the work of Herzog and Takayama [75], in which they showed that in polynomial rings a free resolution can be constructed for monomial ideals by applying the mapping cone construction iteratively. This procedure is particularly effective for ideals with linear quotients. Analogously to the polynomial case [75], we determine a minimal graded free resolution F of a monomial ideal in E with linear quotients by iterated mapping cones and we describe explicitly the free E-modules which appear in F. Here the Cartan complex plays the role of the Koszul complex. Moreover, we are also able to get informations on the graded Betti numbers and to provide an explicit description of the differentials in F for ideals with linear quotients satisfying an additional condition. A significant example of monomial ideals of E with linear quotients is the class of strongly stable ideals whose minimal graded free resolution has been completely described by Aramova, Herzog and Hibi in [10] (see, also, [9]). More recently, minimal graded free resolutions in the setting of skew–commutative algebras have been investigated in [51, 89]. Furthermore, new results concerning the iterated mapping cone method were established in [117], where the polynomial ring was replaced by an arbitrary commutative strongly Koszul algebra.

For a monomial ideal I with linear quotients, consider the short exact sequences

$$0 \longrightarrow E/L_j \xrightarrow{\alpha} E/I_j \xrightarrow{\beta} E/I_{j+1} \longrightarrow 0,$$

where $I_j = (u_1, \dots, u_j)$ and $L_j = (u_1, \dots, u_j) : (u_{j+1})$, $j = 1, \dots, r$. In each step, since I has linear quotients, the ideal L_j is generated by variables $\{e_{k_1}, \dots, e_{k_\ell}\}$. Hence, if $C^{(j)} = C.(e_{k_1}, \dots, e_{k_\ell}; E)$ is the Cartan complex for the sequence $e_{k_1}, \dots, e_{k_\ell}$, one has that $C^{(j)}$ is a minimal graded free resolution of E/L_j [10]. If one consider $F^{(j)}$ to be a minimal graded free resolution of E/I_j, since the modules of $C^{(j)}$ are free, one has a complex homomorphism $\psi^{(j)} : C^{(j)} \to F^{(j)}$, giving rise to an acyclic complex $C(\psi^{(j)})$, called the *cone complex*, which is a free resolution of E/I_{j+1}. Thus, by iterating this construction, adding one generator of I at time, one obtains a graded free resolution of E/I.

Theorem 2.1.1. (Crupi–Ficarra–Lax, 2024 [32, Theorem 3.1]) *Let I be a monomial ideal of E with linear quotients with respect to the order $u_1, \dots, u_r$ of the minimal generators of I and let F be the iterated mapping cone derived from the sequence $u_1, \dots, u_r$. Then F is a minimal graded free resolution of E/I and the free E-module F_i which appears in the ith step* *of F has homogeneous basis*

$$\{f(a;u),\ \textit{with}\ u \in G(I),\ a \in \mathbb{N}^n,\ \mathrm{supp}(a) \subset \mathrm{set}(u),\ |a| = i-1\},$$

with $\deg f(a;u) = |a| + \deg u$, *for all* $i > 0$.

By the above result we obtain formulas for computing the graded Betti numbers (Corollary 2.1.3), the Poincaré series (Corollary 2.1.5) and the complexity (thus the depth *via* a result of Aramova, Avramov and Herzog [9, Theorem 3.2], an exterior algebra analogous to the Auslander–Buchsbaum formula) of a monomial ideal with linear quotients.

We then work on the problem of describing the differentials of F explicitly (Section 2.2). For this aim, we introduce the *regular decomposition function* of an ideal with linear quotients. Our main result is the following.

Theorem 2.2.5. (Crupi–Ficarra–Lax, 2024 [32, Theorem 4.5]) *Let I be a monomial ideal of E with linear quotients, and F the graded minimal free resolution of E/I. Suppose that the decomposition function $g : M(I) \to G(I)$ is regular. Then the maps in the resolution F are given by $d(f(0;u)) = (-1)^{\deg(u)}u$, for $u \in G(I)$, and for $a \in \mathbb{N}^n$, $a \neq 0$, $u \in G(I)$ and* $\mathrm{supp}(a) \subset \mathrm{set}(u)$,

$$\begin{aligned} d(f(a;u)) \;=\; & -\sum_{t\in\mathrm{supp}(a)} (-1)^{\deg(u)} e_t f(a-\varepsilon_t;u) \\ & + \sum_{t\in\mathrm{supp}(a)\setminus\mathrm{supp}(u)} (-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} f(a-\varepsilon_t; g(e_t u)). \end{aligned}$$

Finally, as an application, in Section 2.3, we introduce the class of vector–spread strongly stable ideals [55] in E, for which we compute their graded Betti numbers (Corollary 2.3.5) generalizing the one stated in [10] for the ordinary strongly stable ideal in E.

In Chapter 3, we work in the standard graded polynomial ring $S = K[x_1, \dots, x_n]$, over a field K and consider the class of $\mathbf{t}$-*spread monomial ideals* for a fixed vector $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$, introduced first in [55]. A monomial $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in S$, with $\ell \leq d$, is said to be $\mathbf{t}$-*spread* if it satisfies the inequality $j_{k+1} - j_k \geq t_k$ for all $k = 1, \dots, \ell-1$. A monomial ideal $I \subset S$ is called a $\mathbf{t}$-*spread monomial ideal* if it is generated by $\mathbf{t}$-spread monomials.

A crucial notion is the one of $\mathbf{t}$-*spread strongly stable ideal*, generalizing the classical notions: in fact, for $\mathbf{t} = \mathbf{0} = (0, \dots, 0)$, one recovers the standard definition of a strongly stable ideal, while for $\mathbf{t} = \mathbf{1} = (1, \dots, 1)$ one obtains the notion of squarefree strongly stable ideal.

The study of strongly stable ideals is classical and extensively developed, a deep review of this topic can be found in [68]. In characteristic zero, it is known that the generic initial ideal of any homogeneous ideal in S is strongly stable [68, Proposition 4.6.2]. Their structure and minimal graded free resolutions were explicitly determined by Eliahou and Kervaire in their groundbreaking work [41], a cornerstone for many advances in Combinatorial Commutative Algebra.

Likewise, the theory of vector–spread strongly stable ideals has undergone significant advances in recent years (see, for instance, [5, 7, 28, 36, 93] and the references therein).

To a $\mathbf{t}$-spread ideal I in S, we associate a unique $f_{\mathbf{t}}$-vector and we prove that if I is $\mathbf{t}$-spread strongly stable, then there exists a unique *t-spread lex ideal* $I^{\mathbf{t},\text{lex}}$ sharing the same $f_{\mathbf{t}}$-vector of I. Moreover, we characterize the possible $f_{\mathbf{t}}$-vectors of $\mathbf{t}$-spread strongly stable ideals generalizing the well–known theorems of Macaulay and Kruskal–Katona.

Theorem 3.3.8. (Crupi–Ficarra–Lax, 2024 [31, Theorem 4.8]) *Let $f = (f_{-1}, f_0, \dots, f_{d-1})$ be a sequence of non-negative integers. The following conditions are equivalent:*

1. *there exists a $\mathbf{t}$-spread strongly stable ideal $I \subset S = K[x_1, \dots, x_n]$ such that*
$$f_{\mathbf{t}}(I) = f;$$
2. *$f_{-1} = 1$ and $f_{\ell+1} \le f_{\ell}^{(\ell+1,\mathbf{t})}$, for all $\ell = -1, \dots, d-2$.*

This result is achieved, *via* the introduction of a new combinatorial operator, $a \mapsto a^{(\ell,\mathbf{t})}$, obtained by the binomial expansion of a with respect to ℓ.

As an application, in Section 3.4, we prove the vector–spread version of the well–known result proved by Bigatti [15] and Hulett [76], independently. In particular, we prove that among all $\mathbf{t}$-spread strongly stable ideals with the same $f_{\mathbf{t}}$-vector, the t-spread lex ideals have the largest Betti numbers.

Theorem 3.4.1. (Crupi–Ficarra–Lax, 2024 [31, Theorem 5.1]) *Let $I \subset S = K[x_1, \dots, x_n]$ be a $\mathbf{t}$-spread strongly stable ideal. Then,*
$$\beta_{i,j}(I) \ \le\ \beta_{i,j}(I^{\mathbf{t},\text{lex}}), \quad \textit{for all } i \textit{ and } j.$$

In Chapter 4, we continue the study of vector–spread ideal. Specifically, given $\mathbf{t}$-spread monomials $u_1, \dots, u_m \in S$, we denote by $B_{\mathbf{t}}(u_1, \dots, u_m)$ the smallest $\mathbf{t}$-spread strongly stable ideal of S containing all u_i, with respect to the inclusion order. The monomials $u_1, \dots, u_m$ are referred to as the $\mathbf{t}$-*spread Borel generators* of $B_{\mathbf{t}}(u_1, \dots, u_m)$. In the case where $m = 1$, *i.e.* $I = B_{\mathbf{t}}(u)$, the ideal I is called a *principal* $\mathbf{t}$-*spread Borel ideal*, with $\mathbf{t}$-spread Borel generator u.

For a principal $\mathbf{t}$-spread Borel ideal we compute the minimal primary decomposition (Section 4.1). Our main result is the following, involving the key notion of $\mathbf{t}$-spread support.

Theorem 4.1.2. (Crupi–Ficarra–Lax, 2025 [34, Theorem 1.2]) *The minimal primary decomposition of $B_{\mathbf{t}}(u)$ is*
$$B_{\mathbf{t}}(u) = \Big(\bigcap_{G \in \mathcal{G}} P_{[n]\setminus G}\Big) \cap \Big(\bigcap_{v \in G(B_{\mathbf{t}}(u))} P_{[n]\setminus \text{supp}_{\mathbf{t}}(v)}\Big),$$
where $\mathcal{G}$ is the family of all subsets of $[n]$ of the form
$$\text{supp}_{\mathbf{t}}(x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{s-1}}x_{j_s}) \cup [j_s+1, n],$$
such that $x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{s-1}}x_{j_s} \in G(B_{\mathbf{t}}(x_{j_1}x_{j_2}\cdots x_{j_{s-1}}x_{j_s}))$, for some $s = 1, \dots, d$.

As a consequence we compute explicitly the height of the ideal $B_{\mathbf{t}}(u)$ as $\min(u)$ (Corollary 4.1.3).

Then, in Section 4.2, we show that squarefree principal vector–spread Borel ideals are sequentially Cohen–Macaulay (Theorem 4.2.1). To this end, we prove that their

Alexander duals are vertex splittable, in the sense of Moradi and Khosh–Ahang [92]. Whether this result holds in the non–squarefree setting is an open question.

Finally, in Section 4.3, we give a complete characterization of the squarefree principal vector–spread Borel ideals whose ordinary and symbolic powers coincide, with the following result involving the notion of normally torsionfree ideals.

Theorem 4.3.1. (Crupi–Ficarra–Lax, 2025 [34, Theorem 3.1]) *Let* $I = B_{\mathbf{t}}(u)$. *The following statements are equivalent.*

(a) *I is normally torsionfree.*

(b) *$I^{(k)} = I^k$, for all integers $k \geq 1$.*

(c) *$j_i \leq \sum_{s=1}^{i} t_s$ for all $i = 1, \dots, d-1$.*

The proof of the above result mainly follows the one outlined in [93], with some major difference for the proof of the implication (b) $\Rightarrow$ (c).

In Chapter 5 we present the results of [33]. For a finite simple graph G, with vertex set $V(G) = \{1, \dots, n\}$ and edge set $E(G)$, we consider the *edge ideal* [118, 119] attached to it as the ideal of the polynomial ring $S = K[x_1, \dots, x_n]$, with K a field, generated by all monomials $x_i x_j$, such that i, j is an edge of G.

A crucial notion is the one of *squarefree power* [16] of a squarefree monomial ideal $I \subset S$, *i.e.* the ideal $I^{[k]}$ generated by all squarefree generators of I^k.

Such a concept was recently related to the notion of k-matching [16, 47], namely a subset M of $E(G)$ of size k such that $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$. In case $I = I(G)$ is the edge ideal of a graph G, $I(G)^{[k]}$ is the squarefree monomial ideal generated by $\mathbf{x}_{V(M)}$, where is the vertex set of the k-matching M.

We deeply investigate the squarefree powers of edge ideals, with particular regard to forests. Our results are based on the technique of *Betti splitting*, introduced by Francisco, Hà and Van Tuyl in [58]. Let $I \subset S$ be a monomial ideal. By $G(I)$ we denote the unique minimal set of monomial generators of I. Let $I_1, I_2 \subset S$ monomial ideals such that $G(I)$ is the disjoint union of $G(I_1)$ and $G(I_2)$. We say that $I = I_1 + I_2$ is a *Betti splitting* [58, Definition 1.1] if

$$\beta_{i,j}(I) = \beta_{i,j}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j}(I_1 \cap I_2), \quad \text{for all } i, j \geq 0. \tag{1}$$

When $I = I_1 + I_2$ is a Betti splitting one can deduce important homological invariants for I. More in detail one has the following result.

Proposition 5.1.1. (Francisco–Hà–Van Tuyl, 2009 [58, Corollary 2.2]) *Let $I = I_1 + I_2$ be a Betti splitting. Then*

$$\begin{aligned} \operatorname{reg}(I) &= \max\{\operatorname{reg}(I_1), \operatorname{reg}(I_2), \operatorname{reg}(I_1 \cap I_2) - 1\}, \\ \operatorname{pd}(I) &= \max\{\operatorname{pd}(I_1), \operatorname{pd}(I_2), \operatorname{pd}(I_1 \cap I_2) + 1\}. \end{aligned}$$

where $\operatorname{reg}(-)$ *is the regularity, and* $\operatorname{pd}(-)$ *is the projective dimension.*

In Section 5.1, we consider Betti splittings of squarefree powers. Theorem 5.1.2 is a technical but powerful criterion that can be used to establish if a certain decomposition $I = I_1 + I_2$ is a Betti splitting. This fact is used in Lemma 5.1.5 to reprove a result in [56] under weaker assumptions (Proposition 5.1.6).

Section 5.2 deeply investigates some Betti splittings of squarefree powers of edge ideals of forests. These Betti splittings will serve as important tools throughout the chapter. A leaf $v \in V(G)$, with unique neighbor w, is called a *distant leaf* if at most one of the neighbors of w is not a leaf. In Proposition 5.2.1 we prove that every forest has a distant leaf (see also [78, Proposition 9.1.1], [49, Lemma 20]). After assuming Setup 5.2.2, we provide a natural Betti splitting for $I(G)^{[k]}$ (Proposition 5.2.5)

$$I(G)^{[k]} = I(G_1)^{[k]} + x_n x_{n-1} I(G_2)^{[k-1]},$$

where G_1 and G_2 are the induced subgraphs of G on the vertex sets $\{1, \ldots, n-1\}$ and $\{1, \ldots, n-2\}$, respectively.

Let $I \subset S$ be a squarefree monomial ideal and let d_k denote the minimum degree of monomials belonging to $G(I^{[k]})$. In [48, Proposition 1.1] the authors proved that

$$\operatorname{depth}\left(S/I^{[k]}\right) \geq d_k - 1.$$

Based on this, they introduced the *normalized depth function* of I as

$$g_I(k) = \operatorname{depth}\left(S/I^{[k]}\right) - (d_k - 1)$$

In contrast to the behaviour of the depth function of ordinary powers [23, 67, 63], in [48] the authors made the following prediction.

Conjecture. *For any squarefree ideal $I \subset S$, the function $g_I(k)$ is a non–increasing function.*

Such a conjecture was recently proved to not be true for any squarefree monomial ideal by Sayed Fakhari in [107]. It is still an open problem if the conjecture is true for edge ideals.

In Section 5.3, we investigate the *normalized depth function* of edge ideals, especially of forests. Our main result in this section is the following.

Theorem 5.3.4. (Crupi–Ficarra–Lax, 2025 [33, Theorem 3.4]) *Let G be a forest. Then $g_{I(G)}$ is non–increasing.*

We also compute the normalized depth function of the edge ideal of a path and use this result to obtain a general upper bound for the normalized depth function of the edge ideal of any graph.

In the last section, we compute the Castelnuovo–Mumford regularity $\operatorname{reg}(I(G)^{[k]})$ in terms of the combinatorics of the forest G, namely the *k-admissible matching number* of G, denoted by $\operatorname{aim}(G, k)$.

Theorem 5.4.6. (Crupi–Ficarra–Lax, 2025 [33, Theorem 4.6]) *Let G be a forest. Then*

$$\operatorname{reg}(I(G)^{[k]}) = \operatorname{aim}(G, k) + k.$$

The above result solves affirmatively a conjecture due to Erey and Hibi [49, Conjecture 31].

In Chapter 6, we study binomial edge ideals and their sequentially Cohen–Macaulay property. Let G be a finite simple graph with vertex set $V(G) = 1, \ldots, n$. The *binomial edge ideal* [69, 95] of G, denoted by J_G, is the ideal of the polynomial ring $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, with K a field, generated by all binomials $x_i y_j - x_j y_i$, such that $\{i, j\}$, with $i < j$, is an edge of G.

Many algebraic properties and invariants of these ideals can be described in terms of the combinatorics of the underlying graph (see [71] for a detailed survey). A key combinatorial tool in this context is the concept of a *cutset*. For a subset $T \subset V(G)$, let

$G \setminus T$ denote the graph obtained by removing the vertices of T. The set T is called a *cutset* if the number of connected components, denoted $c(T)$, satisfies

$$c(T \setminus \{v\}) < c(T) \quad \text{for every } v \in T.$$

We write $\mathcal{C}(G)$ for the collection of all cutsets of G. For instance, the Krull dimension of the binomial edge ideal is given by

$$d = \max\{n + c(T) - |T| \;:\; T \in \mathcal{C}(G)\}. \tag{2}$$

A central theme in the theory of binomial edge ideals is the classification of Cohen–Macaulay cases. The first major contribution in this direction is due to Ene, Herzog, and Hibi [42], who classified the block graphs that are Cohen–Macaulay. Since then, substantial progress has been made on this classification problem, although a complete classification of Cohen–Macaulay binomial edge ideals remains open. A natural and well-studied extension of the Cohen–Macaulay property is sequentially Cohen–Macaulayness (see [111]).

Throughout the chapter, we study classes of graphs having sequentially Cohen–Macaulay binomial edge ideals. A natural way of classifying these graphs, follows from the introduction of a new invariant, obtained by mimicking the formula for the Krull dimension

$$\mathfrak{m}(G) = \min\{n + c(T) - |T| \;:\; T \in \mathcal{C}(G)\}.$$

In Section 6.1, we begin by studying *block star graphs* G, namely a block graph with only one cutpoint. We obtain a complete description of the modules of deficiency of J_G, leading to the prove of its sequentially Cohen–Macaulayness.

Theorem 6.1.1. (Lax–Rinaldo–Romeo, 2024 [86, Theorem 1.5]) Let G be block star graph on n vertices with $t \geq 3$ cliques attached to a single vertex v and let J_G be the binomial edge ideal of G. Then

1. $\omega^{n+1}(S/J_G)$ is a Cohen–Macaulay module of dimension $n+1$, moreover, we have $\omega^{n+1}(\omega^{n+1}(S/J_G)) \cong (J_{K_{n-1}} + (x_v, y_v))/J_{K_n}$;
2. $\omega(S/J_G) \cong \omega(S/(J_{G\setminus\{v\}} + (x_v, y_v)))$;
3. $\omega^i(S/J_G) = 0$ for all $i \neq n+1, n+t-1$.

In particular, J_G is sequentially Cohen–Macaulay.

Later, in Example 6.1.2, we show that it can be difficult to apply the same techniques of Theorem 6.1.1 in general cases. In fact, it is not always possible to get informations on the non–vanishing modules of deficiency (see for instance [121]). Based on this, we change our approach, by involving newer results [60, Proposition 16] to prove the sequentially Cohen–Macaulay property.

We are then able to characterize all graphs G with only one non–empty cutset that are sequentially Cohen–Macaulay.

Proposition 6.1.5. (Lax–Rinaldo–Romeo, 2024 [86, Proposition 1.11]) *Let G be a graph such that $\mathcal{C}(G) = \{\emptyset, T\}$, let $t = |T|$ and let $c = c(T)$ denote the number of connected components of $G \setminus T$. Then* depth $S/J_G = n - t + 2$, *and J_G is sequentially Cohen–Macaulay in the following cases:*

(i) $t = 1$;

(ii) $t \geq 2$ *and* $c = 2$;

We also prove that if $\mathfrak{m}(G) = n + 1$ and J_G is sequentially Cohen–Macaulay, then G must have at least one cutpoint, *i.e.* a cutset of cardinality 1 (Proposition 6.1.7).

In Section 6.2, we study deals with graphs that are blocks, namely graphs without cutpoints. In this setting, $\mathfrak{m}(G) \leq n$. We show that cycles (for which $\mathfrak{m}(G) = n$) and wheels (for which $\mathfrak{m}(G) = n-1$) are sequentially Cohen–Macaulay (Proposition 6.2.2, Corollary 6.2.3), by the means of Goodarzi's filtration.

In Section 6.3, we work on more general block graphs than the one studied in Theorem 6.1.1, in particular we prove that that all block graphs are sequentially Cohen–Macaulay. Although Goodarzi's filtration may yield ideals outside the class of binomial edge ideals, we demonstrate—by means of an elementary containment problem (Lemma 6.3.1)—that the special structure of block graphs allows to control the filtration effectively.

Lemma 6.3.1. (Lax–Rinaldo–Romeo, 2024 [86, Lemma 3.1]) *Let v a cutpoint of an indecomposable connected block graph G such that*

$$G \setminus \{v\} = G_1 \sqcup B_1 \sqcup \ldots \sqcup B_r$$

with $r \geq 2$, where B_i are complete graphs. Then

$$J_v^{<i>} + (J_{\overline{v}} + (x_v, y_v))^{<i>} = (J_v + (x_v, y_v))^{<i-1>}.$$

We also observe that, although Lemma 6.3.1 holds for block graphs, it is not true in general (Example 6.3.2).

Finally, in Section 6.4, we extend a construction due to Rauf and Rinaldo [98], where cones were used to generate new Cohen–Macaulay graphs and to classify bipartite Cohen–Macaulay graphs [21]. We generalize this result to the sequentially Cohen–Macaulay setting, proving that a cone is sequentially Cohen–Macaulay if and only if its components are sequentially Cohen–Macaulay.

Theorem 6.4.1. (Lax–Rinaldo–Romeo, 2024 [86, Theorem 4.1]) *Let $G = G_1 \sqcup G_2 \sqcup \ldots \sqcup G_r$ be a graph, where G_i is a connected graph for each $i = 1, \ldots, r$, with $r \geq 2$. Let v be a vertex such that $v \notin V(G)$ and $H = \mathrm{Cone}(v, G)$. Then J_H is sequentially Cohen–Macaulay if and only if $J_{G_1}, \ldots, J_{G_r}$ are sequentially Cohen–Macaulay.*

Note that in this result, the condition $r \geq 2$ is crucial. In fact, although it is true for cycles and wheels, the cone of a connected sequentially Cohen–Macaulay graph, may not be in general sequentially Cohen–Macaulay (Remark 6.4.2, Example 6.4.3).

In Chapter 7, we present a new *Macaulay2* [62] package, `SCMAlgebras` [84, 85], designed to check whether a given module or ideal satisfies the sequentially Cohen–Macaulay property. Key theoretical results are implemented, such as Theorem 1.1.42, which characterizes sequentially Cohen–Macaulay modules in terms of their modules of deficiency, and Proposition 1.1.45, involving the filter ideals. In Section 7.1 we present several algorithms, illustrating the functionalities of the package.

Chapter 1

Preliminaries

1.1 Basics on commutative algebra

In this section we collect some preliminary notions and results of commutative algebra that will be fundamental throughout this dissertation. We begin by recalling the notions of two key invariants: the Krull dimension and the depth. The relationship between these two is very important, leading to the definition of a Cohen–Macaulay module. We then proceed with a quick review on graded algebras as as a raw material for many important constructions, such as Hilbert function and minimal free resolution. The latter is crucial for the introduction of the Betti numbers, the projective dimension, the Castelnuovo–Mumford regularity. We also discuss some standard constructions, such as the Koszul complex and the local cohomology modules, which provide effective tools for calculating many invariants. Finally, we recall the sequentially Cohen–Macaulay property and its main characterizations.

Most of the theoretical tools presented in this section are classical concepts of commutative algebra. We refer to [24, 40, 68, 97] for a more detailed discussion on these topics.

1.1.1 Krull dimension and depth

Throughout the section, let R denote a commutative ring with identity, unless otherwise specified. We denote by $\mathrm{Spec}(R)$ the set of all prime ideals of R.

A *chain* of prime ideals of R is a family $\{\mathfrak{p}_j\}_{j\in J} \subset \mathrm{Spec}(R)$, such that $\mathfrak{p}_j \subset \mathfrak{p}_{j+1}$, for $j \in J$. If

$$\mathcal{C}: \quad \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n,$$

is a finite chain of prime ideals, the integer n is called the length of $\mathcal{C}$.

Definition 1.1.1. Let R be a commutative ring and let $\mathfrak{p} \in \mathrm{Spec}(R)$. The *height* of $\mathfrak{p}$, denoted by $\mathrm{ht}(\mathfrak{p})$, is the supremum of the lengths of the chains of prime ideals of R ending in $\mathfrak{p}$

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n = \mathfrak{p}.$$

If such supremum is not finite, we set $\mathrm{ht}(\mathfrak{p}) = +\infty$.

Definition 1.1.2. The *Krull dimension* of a ring R the supremum of the heights of the prime ideals of R and it is denoted by $\dim(R)$. If such supremum is not finite, we set $\dim(R) = +\infty$.

For instance, if K is a field, then $\dim(K) = 0$ since its only prime ideal is (0). One has the following,

Theorem 1.1.3. *Let R be a Noetherian ring. Then*

$$\dim(R[X]) = \dim(R) + 1.$$

As a consequence, one can prove that $\dim(K[x_1, \dots, x_n]) = n$.

More in general, one can define the Krull dimension of a R-module M. Recall that, the *annihilator* of M on R is

$$\mathrm{Ann}_R(M) = \{r \in R \ : \ rm = 0 \text{ for all } m \in M\}.$$

Definition 1.1.4. The *Krull dimension* of a R-module M is defined as the Krull dimension of the ring $R/\mathrm{Ann}(M)$, and it is denoted by $\dim_R(M)$ (or simply $\dim(M)$ if there is no chance of ambiguity).

Let us now proceed to introduce the notion of depth. Let M be a R-module. An element $x \in R$ is a *non–zero divisor* of M if $xy = 0$, for $y \in M$, implies $y = 0$.

Definition 1.1.5. Let M be a R-module. A sequence $x_1, \dots, x_n$ of elements of R is called a *regular sequence* on M if

1. $(x_1, \dots, x_n)M \neq M$;
2. x_1 is a non–zero divisor of M;
3. x_i is a non–zero divisor of $M/(x_1, \dots, x_{i-1})M$.

A regular sequence $x_1, \dots, x_n$ (contained in an ideal $I \subset R$) on M is called *maximal* (in I), if $x_1, \dots, x_n, y$ is not a regular sequence on M for any $y \in R$ ($y \in I$).

In general, for a R-module M, two maximal regular sequences contained in an ideal $I \subset R$ may have different lengths. One can show that this does not happen when R is Noetherian and if the module M is finitely generated. More specifically, one has the following result.

Theorem 1.1.6. (Rees) *Let R be a Noetherian ring, M a finitely generated R-module, $I \subset R$ an ideal such that $IM \neq M$. Then, every maximal regular sequence on M contained in I has the same length $n > 0$, such that*

1. $\mathrm{Ext}^i_R(R/I, M) = 0$, *for all $i < n$;*
2. $\mathrm{Ext}^n_R(R/I, M) \neq 0$.

The integer n, representing the common length of the maximal regular sequences contained in I, is called the *depth* of I on M and it is denoted by $\mathrm{depth}(I, M)$. By Theorem 1.1.6 one has that

$$\mathrm{depth}(I, M) = \min\{i \ : \ \mathrm{Ext}^i_R(R/I, M) \neq 0\}.$$

If $IM = M$, one sets conventionally $\mathrm{depth}(I, M) = +\infty$. If $M = R$, one simply speaks of depth of I, which is denoted by $\mathrm{depth}(I)$.

One peculiar, but common situation is the one presented in the next definition. Recall that a *local ring* $(R, \mathfrak{m})$ is a ring R with only one maximal ideal $\mathfrak{m}$. The quotient $K = R/\mathfrak{m}$ is a field, called the *residue field* of R.

Definition 1.1.7. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module. The *depth* of M is the common length of the maximal regular sequences contained in $\mathfrak{m}$, and it is denoted by $\mathrm{depth}(M)$. In other words, we define $\mathrm{depth}(M) = \mathrm{depth}(\mathfrak{m}, M)$.

Applying Theorem 1.1.6 in this special case, one has

$$\operatorname{depth}(M) = \min\{i \ : \ \operatorname{Ext}^i_R(K, M) \neq 0\},$$

where K is the residue field of R.

We recall the following result, stating some properties of the depth.

Lemma 1.1.8. (Depth Lemma) *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let*

$$0 \longrightarrow U \longrightarrow M \longrightarrow N \longrightarrow 0,$$

be an exact sequence of finitely generated R-modules. Then,

1. $\operatorname{depth}(M) \geq \min\{\operatorname{depth}(U), \operatorname{depth}(N)\}$,
2. $\operatorname{depth}(U) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(N) + 1\}$,
3. $\operatorname{depth}(N) \geq \min\{\operatorname{depth}(M), \operatorname{depth}(U) - 1\}$,

We end this section, by comparing the two invariants introduced above. Let $(R, \mathfrak{m})$ be a Noetherian local ring. For any R-module one has $\operatorname{depth}(M) \leq \dim(M)$.

Definition 1.1.9. A R-module M is *Cohen–Macaulay* if

$$\dim(M) = \operatorname{depth}(M).$$

A ring R is *Cohen–Macaulay* if it is Cohen–Macaulay as a module over itself.

Example 1.1.10. Let $S = K[x_1, \dots, x_n]$, the polynomial ring in n variables with coefficient over a field K. S is Cohen–Macaulay, in fact $\dim(S) = n$, as already observed. Moreover, $\mathbf{x} = x_1, \dots, x_n$ is a maximal regular sequence contained in the maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)$, hence $\operatorname{depth}(S) = n$. Therefore, S is Cohen–Macaulay.

An interesting property of the class of Cohen–Macaulay modules is their unmixedness. Recall that if M is a R-module, an ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is *associated* with M, if $\mathfrak{p} = \operatorname{Ann}(x) = \{r \in R \ : \ rx = 0\}$ for some $x \in M$. The set of associated primes of M is denoted by $\operatorname{Ass}_R(M)$, or by $\operatorname{Ass}(M)$ if there is no chance of confusion.

Definition 1.1.11. A R-module M is called *unmixed* if $\dim(M) = \dim(R/\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Ass}(M)$.

One has the following.

Proposition 1.1.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let M be a finitely generated R-module. Then*

$$\operatorname{depth}(M) \leq \dim(R/\mathfrak{p}) \quad \textit{for all } \mathfrak{p} \in \operatorname{Ass}(M).$$

As a consequence of Proposition 1.1.12 one has the following implication

$$\text{Cohen–Macaulay} \quad \Longrightarrow \quad \text{unmixed}$$

but the converse is not true in general.

1.1.2 Minimal graded resolutions and Betti numbers

In this paragraph we recall the fundamental facts regarding minimal free resolutions of graded modules, since they encode many information about the modules itself.

Let us start by establishing the notation that we will use for the rest of the paragraph.

Definition 1.1.13. A ring R is called *graded* if there exist a family of abelian additive subgroups $(R_d)_{d\geq 0}$, such that

1. $R = \bigoplus_{d\geq 0} R_d$ (as additive abelian groups);
2. $R_d R_\ell \subset R_{d+\ell}$, for all $d, \ell \geq 0$. In other words, if $x \in R_d$ and $y \in R_\ell$, then $xy \in R_{d+\ell}$.

Each R_d is called the dth *graded component* of R. Every element $x \in R_d$ is called *homogeneous* of *degree* d.

According to Definition 1.1.13, the zero element has arbitrary degree. The degree of an element $x \in R$ is denoted by $\deg(x)$. An element $x \in R$ has a unique presentation $x = \sum_d x_d$ as a sum of homogeneous elements $x_d \in R_d$. The elements x_d are called the *homogeneous components* of x.

Example 1.1.14. The polynomial ring $S = K[x_1, \ldots, x_n]$ in n indeterminates over a field K can be graded by assigning $\deg(x_i) = a_i$, with $a_1, \ldots, a_n$ positive integers. For instance, if $n = 3$ and $\deg(x_1) = 1, \deg(x_2) = 3, \deg(x_3) = 5$, the polynomial $x_1^2 x_2 - x_3$ is homogeneous of degree 5. If we put $\deg(x_i) = 1$, for all $i = 1, \ldots, n$, we say that S is *standard graded*.

It is also possible to define graded modules over graded rings.

Definition 1.1.15. Let R be a graded ring and let M be a R-module. M is called *graded* if there exist a family $(M_i)_{d\geq 0}$ of additive abelian subgroups of M such that

1. $M = \bigoplus_{d\geq 0} M_d$ (as additive abelian groups);
2. $R_d M_\ell \subset M_{d+\ell}$, for all $d, \ell \in \mathbb{Z}$.

Each M_d is called the dth *graded component* of M. The elements $x \in M_d$ are called *homogeneous* of *degree* d.

As per rings, the degree of an element $x \in M$ is denoted by $\deg(x)$. Moreover, an element $x \in M$ has a unique presentation $x = \sum_d x_d$ as a sum of homogeneous elements $x_d \in M_d$, called the *homogeneous components* of x.

A R-algebra A is called *graded* if there exists a family $(A_d)_{d\geq 0}$ of R-submodules of A, such that $A = \bigoplus_{d\geq 0} A_d$ (as R-modules) and for all $d, \ell \geq 0$ one has $A_d A_\ell \subset A_{d+\ell}$.

Let A be a graded R-algebra and I a two sided ideal of A. I is called *graded* (or *homogeneous*) if

$$I = \bigoplus_{d\geq 0} I_j,$$

with $I_d = I \cap A_d$, for all $d \geq 0$, where A_d is the dth graded component of A. An ideal I is graded if and only if it is generated by homogeneous elements.

Given a graded R-module M, with $M = \bigoplus_{d\geq 0} M_d$ and an integer a, one can define the R-module $M(a)$ *shifted by* a as the R-module with dth graded component

$$M(a)_d = M_{a+d} \quad \text{for all } d \geq 0.$$

In what follows, we will consider finitely generated graded K-algebras, with K a field. Although these type of algebras seem very general, they have a peculiar structure, given by the following result.

Proposition 1.1.16. *Let A be a finitely generated graded K-algebra. Then there exist a polynomial ring S over K and a graded ideal I of S such that*

$$A \cong S/I, \qquad \text{as graded } K\text{-algebras}.$$

Let K be a field and let $S = K[x_1, \dots, x_n]$ be the standard graded K-algebra of polynomials in n indeterminates over K. We report first of all the following fact, which is a graded version of Nakayama's Lemma. We then denote by $\mathfrak{m} = (x_1, \dots, x_n)$ the maximal ideal of S (which is graded).

Theorem 1.1.17. *Let $M \in \mathcal{M}$, let then $m_1, \dots, m_r$ be homogeneous elements of M. Then, the elements $m_1, \dots, m_r$ generate M if and only if $(m_1 + \mathfrak{m}M), \dots, (m_r + \mathfrak{m}M)$ generate $M/\mathfrak{m}M$.*

Definition 1.1.18. Let M, N be graded S-modules, with $M = \bigoplus_{d\geq 0} M_d$ and $N = \bigoplus_{d\geq 0} N_d$. A homomorphism $\varphi : M \longrightarrow N$ is called *homogeneous* (or *graded*) of degree ℓ, if for all $d \geq 0$ one has $\varphi(M_d) \subset N_{d+\ell}$.

If $\varphi : M \longrightarrow N$ is homogeneous of degree 0, it will be simply called a *homogeneous* (or *graded*) homomorphism.

Example 1.1.19.

1. Let $f \in S$ be a homogeneous element of degree d, then the multiplication map $S(-d) \longrightarrow S$ with $f \mapsto fg$ is a homogeneous homomorphism.

2. Let $I \subset S$ be a graded ideal, then the canonical inclusion $I \hookrightarrow S$ is a graded homomorphism.

Definition 1.1.20. Let M be a graded S-module, a S-submodule $U \subset M$ is called *graded* if the canonical inclusion $U \hookrightarrow M$ is a homogeneous homomorphism.

Let M be a graded S-module and let U be a graded submodule of M. The quotient module M/U has a natural graded structure given by

$$(M/U)_i = M_i/U_i.$$

In what follows, let $\mathcal{M}$ be the category of finitely generated graded S-modules, whose morphisms are graded homomorphisms. A module $F \in \mathcal{M}$ is called *free* if it admits a finite base $m_1, \dots m_r$ consisting of homogeneous elements. Assuming $\deg(m_i) = a_i$, for every $i = 1, \dots, r$, then one can prove that:

$$F \cong \bigoplus_{j=1}^{r} S(-a_j).$$

Let $M \in \mathcal{M}$ and let $m_1, \dots, m_r$ be the homogeneous generators of M, with $\deg(m_i) = a_i$ for every $i = 1, \dots, r$. Since M is finitely generated, there exists a surjective S-homomorphism $d_0 : F_0 \longrightarrow M$ defined as $e_i \mapsto m_i$, for all $i = 1, \dots, r$, where F_0 is the free S-module of rank r with (canonical) basis $\{e_1, \dots, e_r\}$. Assigning $\deg(e_i) = a_i$, for all $i = 1, \dots, r$, the map d_0 becomes a homogeneous map.

Thus, F_0 becomes isomorphic to the graded free module $\bigoplus_{i=1}^{r} S(-a_i)$. Moreover, set $\beta_{0j} = |\{i \ : \ a_i = j\}|$, one has

$$F_0 \cong \bigoplus_{j} S(-j)^{\beta_{0j}}.$$

Hence, setting $U_0 = \ker d_0$ we obtain the short exact sequence

$$0 \longrightarrow U_0 \xrightarrow{i_0} \bigoplus_{j} S(-j)^{\beta_{0j}} \xrightarrow{d_0} M \longrightarrow 0,$$

where i_0 is the canonical inclusion. Since U_0 is a graded submodule of F_0, thus finitely generated, we can repeat the same construction for U_0. Hence, we find a graded epimorphism $\overline{d}_1 : F_1 \cong \bigoplus_j S(-j)^{\beta_{1j}} \to U_0$. Thus, compositing such map with the inclusion $i_0 : U_0 \to \bigoplus_j S(-j)^{\beta_{0j}}$

$$\begin{tikzcd}
\bigoplus_{j} S(-j)^{\beta_{1j}} \arrow[rr, dashed, "d_1"] \arrow[dr, "\overline{d}_1"'] & & \bigoplus_{j} S(-j)^{\beta_{0j}} \arrow[r, "d_0"] & M \arrow[r] & 0, \\
& U_0 \arrow[ur, "i_0"'] & & &
\end{tikzcd}$$

yields to the exact sequence

$$\bigoplus_{j} S(-j)^{\beta_{0j}} \xrightarrow{d_1} \bigoplus_{j} S(-j)^{\beta_{0j}} \xrightarrow{d_0} M \longrightarrow 0,$$

Iterating the same process one obtains the long exact sequence of graded S-modules

$$\mathbb{F}: \ \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

with $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$, called a *graded free resolution* of M.

It is clear that such a resolution may not be unique in general. However, if in each step of the construction of the resolution, one chooses the graded representation map in a "minimal" way, then the resolution is unique up to isomorphism.

Recall that for $M \in \mathcal{M}$, a system of homogeneous generators $m_1, \ldots, m_r$ of M is called *minimal* if there is no proper part of $\{m_1, \ldots, m_r\}$ generating M. The next result will be crucial.

Lemma 1.1.21. *Let $M \in \mathcal{M}$ and let $m_1, \ldots, m_r$ be the homogeneous generators of M. Let $F_0 = S^r$ the free module of rank r, with basis $\{e_1, \ldots, e_r\}$ and let $\varepsilon : F_0 \to M$ the S-module epimorphism defined by $\varepsilon(e_i) = m_i$, for all $i = 1, \ldots, r$. The following are equivalent:*

1. *$m_1, \ldots, m_r$ is a minimal system of generators of M;*
2. $\ker \varepsilon \subset \mathfrak{m}F_0$.

Definition 1.1.22. Let $M \in \mathcal{M}$. A graded free resolution of M

$$\mathbb{F}: \ \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

is called *minimal* if $\mathrm{Im}(d_{i+1}) \subseteq \mathfrak{m}F_i$, for all $d \geq 0$.

According to Lemma 1.1.21 a minimal free resolution always exists, in fact it suffices to choose at every step the minimal generating sets of the kernels U_i, which is equivalent to the condition $\mathrm{Im}(d_{i+1}) \subseteq \mathfrak{m}F_i$.

The next result shows that the numerical informations encoded in the graded minimal free resolution of M depend only on M and not on the chosen resolution.

Proposition 1.1.23. *Let $M \in \mathcal{M}$ and let*

$$\mathbb{F}: \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

be a minimal graded free resolution of M, with $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$, for all $d \geq 0$. Then

$$\beta_{ij} = \dim_K \left(\mathrm{Tor}_i^S(K,M)_j\right) \quad \textit{for all } i,j.$$

Definition 1.1.24. Let $M \in \mathcal{M}$. The *graded Betti numbers* of M are defined as

$$\beta_{ij}(M) = \dim_K \left(\mathrm{Tor}_i^S(K,M)_j\right).$$

Moreover, one defines the *total Betti numbers* of M as

$$\beta_i(M) = \sum_j \beta_{ij}(M).$$

According to Proposition 1.1.23, the Betti numbers depend only on the module M. In fact, if $M \cong N$ are two isomorphic modules, then $\mathrm{Tor}_i^S(K,M)_j \cong \mathrm{Tor}_i^S(K,N)_j$ for all i,j, hence $\beta_{ij}(M) = \beta_{ij}(N)$, for all i,j.

Observe also that, for A module $M \in \mathcal{M}$, the ith total Betti number $\beta_i(M)$ represents the rank of the module F_i in a minimal graded free resolution, for all $d \geq 0$.

The graded Betti numbers β_{ij} of a module M are usually displayed in a table, called the *Betti table* (or *Betti diagram*) of M. Such a diagram has in the (i,j)th position the graded Betti number $\beta_{i,i+j}(M)$.

	0	1	2	$\cdots$
0	$\beta_{00}(M)$	$\beta_{11}(M)$	$\beta_{22}(M)$	$\cdots$
1	$\beta_{01}(M)$	$\beta_{12}(M)$	$\beta_{23}(M)$	$\cdots$
2	$\beta_{02}(M)$	$\beta_{13}(M)$	$\beta_{24}(M)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1.1: Betti diagram of M.

More in general, one can show that the minimal graded free resolution is unique up to isomorphism.

Theorem 1.1.25. *Let $M \in \mathcal{M}$ and let $\mathbb{F}$ and $\mathbb{G}$ be two minimal graded free resolution of M. Then, the two complexes $\mathbb{F}$ and $\mathbb{G}$ are isomorphic, that is there exist S-module graded isomorphism $\alpha_i : F_i \to G_i$ such that the following diagram:*

$$\begin{tikzcd} \cdots \arrow[r] & F_i \arrow[r] \arrow[d, "\alpha_i"] & \cdots \arrow[r] & F_1 \arrow[r] \arrow[d, "\alpha_1"] & F_0 \arrow[r] \arrow[d, "\alpha_0"] & M \arrow[r] \arrow[d, "id_M"] & 0 \\ \cdots \arrow[r] & G_i \arrow[r] & \cdots \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & M \arrow[r] & 0, \end{tikzcd}$$

is commutative.

We close this section by discussing the finiteness of the minimal free resolution, which will lead to the introduction of a new invariant.

Definition 1.1.26. Let $M \in \mathcal{M}$ and let

$$\mathbb{F}: \ \cdots \longrightarrow F_{i+1} \xrightarrow{d_{i+1}} F_i \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

be the minimal graded free resolution of M. The resolution $\mathbb{F}$ is called *finite* if there exist $k > 0$ such that $F_{k+1} = 0$ and $F_i \neq 0$, for all $i = 0, 1, \ldots, k$. In such case, k is called the *length* of $\mathbb{F}$.

Let us recall the following pivotal result due to Hilbert.

Theorem 1.1.27 (Hilbert Syzygy Theorem)**.** *Every module $M \in \mathcal{M}$ admits a finite minimal graded free resolution of length at most n.*

According to Theorem 1.1.27, M admits a minimal graded free resolution of minimum length

$$\mathbb{F}: \ 0 \longrightarrow F_p \xrightarrow{d_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \longrightarrow 0,$$

where $F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ for each $i = 0, \ldots, p$. The integer p represents the minimum length of a minimal graded free resolution of M. This number is called the *projective dimension* of M and it is denoted by $\mathrm{pd}(M)$. By Hilbert Syzygy Theorem one also has

$$\mathrm{pd}(M) \leq n.$$

The projective dimension of a module is related to its depth by the following fundamental result, which we state in our special setting.

Theorem 1.1.28 (Auslander–Buchsbaum)**.** *Let $M \in \mathcal{M}$. Then,*

$$\mathrm{pd}(M) + \mathrm{depth}(M) = n.$$

Let us now define the *Castelnuovo–Mumford regularity* of M as

$$\mathrm{reg}(M) = \max\{j \ : \ \beta_{i,i+j}(M) \neq 0 \text{ for some } d \geq 0\}.$$

We end this paragraph by recalling the definition of Hilbert function. We first recall the following result.

Proposition 1.1.29. *Let $M \in \mathcal{M}$, with $M = \bigoplus_{d\geq 0} M_d$. Then* $\dim_K M_d < +\infty$.

The above result justify the next notion.

Definition 1.1.30. Let $M \in \mathcal{M}$, with $M = \bigoplus_{d\geq 0} M_d$. The numerical function

$$\begin{aligned} H_M \ : \ \mathbb{Z}_{\geq 0} &\to \mathbb{Z}_{\geq 0} \\ d &\mapsto \dim_K M_d. \end{aligned}$$

is called the *Hilbert function* of M.

Example 1.1.31. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables. Each graded component S_d has K-basis given by the monomials of degree d. Hence,

$$H_S(d) = \binom{n+d-1}{d} = \binom{n+d-1}{n-1}.$$

One can also prove that H_S is a polynomial function of degree $n-1$.

1.1.3 Koszul complex and local cohomology modules

This paragraph is devoted to recall some homological objects that are useful in the study of a module, namely the *Koszul complex* and the *local cohomology modules*.

Let $\mathbf{f} = f_1, \dots, f_m$ be a sequence of elements of S. Let F be the free S-module of rank m with (canonical) basis $\{e_1, \dots, e_m\}$. For a subset $\mu = \{j_1 < \cdots < j_\ell\} \subset [m]$, denote by e_μ the wedge product $e_{j_1} \wedge \ldots \wedge e_{j_\ell}$. The Koszul complex $K_\bullet(\mathrm{f}; S)$ attached to the sequence $\mathbf{f}$ is the S-module complex defined as it follows:

- Define the module $K_i(\mathbf{f}; S) = \bigwedge^i F$, for all $i = 0, \dots, m$, which is a free S-module with basis consisting of all wedge products e_μ, for $\mu \subset [m]$ with $|\mu| = i$.
- Define the differential $\partial_i : K_i(\mathbf{f}; S) \to K_{i-1}(\mathbf{f}; S)$, for $i = 0, \dots, m-1$ on the generic basis element $e_\mu = e_{j_1} \wedge \ldots \wedge e_{j_i}$ as

  $$\partial_i(e_\mu) = \sum_{k=1}^{i} (-1)^{k-1} f_{j_k} e_{\mu \setminus \{j_k\}},$$

  which then extends for linearity to all $K_i(\mathbf{f}; S)$.

It is easy to observe that the composition of two consecutive differentials $\partial_i \circ \partial_{i+1} = 0$, so $K_\bullet$ is indeed a complex.

Example 1.1.32. Let $S = K[x, y]$, with K a field and let F be the free S-module of rank 2. Let us compute the Koszul complex $K_\bullet(x, y; S)$. One has the modules

$$K_0(x, y; S) = S, \qquad K_1(x, y; S) = F, \qquad K_2(x, y; S) = F \wedge F.$$

Moreover, one has the two differentials represented by the matrices

$$\partial_1 = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \partial_2 = \begin{pmatrix} x, -y \end{pmatrix}.$$

Hence, the Koszul complex $K_\bullet(x, y; S)$ is

$$0 \longrightarrow F \wedge F \xrightarrow{(x,-y)} F \xrightarrow{\binom{x}{y}} S \longrightarrow 0.$$

Let M be a S-module. Consider the complex

$$K_\bullet(\mathbf{f};M) = K_\bullet(\mathbf{f};S) \otimes_S M.$$

The module $H_i(K_\bullet(\mathbf{f};M))$ is called the ith *Koszul homology module* of $\mathbf{f}$ with respect to M and denoted by $H_i(\mathbf{f};M)$, for all $d \geq 0$. Setting $I = (\mathbf{f})$ one has

- $H_0(\mathbf{f};M) = M/IM$,
- $H_m(\mathbf{f};M) = 0 :_M I = \{x \in M \ : \ xI = 0\}$.

By [68, Theorem A.3.4.] one can characterize regular sequences in terms of Koszul homology. As a consequence one has the following result.

Theorem 1.1.33. *Let $M \in \mathcal{M}$ and let $\mathrm{f} = f_1 \dots f_k$ be a homogeneous regular sequence on M. Then,*

$$H_i(\mathbf{f};M) \cong \mathrm{Tor}_i^S(S/(\mathbf{f}), M) \quad \textit{for all } i \textit{ and } j.$$

In particular, in Theorem 1.1.33, if one sets $\mathbf{x} = x_1, \dots, x_n$, then $H_i(\mathbf{x};M) \cong \mathrm{Tor}_i^S(K, M)$. Hence, one can compute the Betti numbers as

$$\beta_{ij}(M) = \dim_K(H_i(\mathbf{x};M)_j).$$

Moreover, the depth, the projective dimension and the Castelnuovo–Mumford regularity can be computed by means of Koszul homology.

$$\begin{aligned}
\mathrm{depth}(M) &= n - \max\{i \ : \ H_i(\mathbf{x};M) \neq 0\},\\
\mathrm{pd}(M) &= \max\{i \ : \ H_i(\mathbf{x};M) \neq 0\},\\
\mathrm{reg}(M) &= \max\{j \ : \ H_i(\mathbf{x};M)_{i+j} \neq 0, \text{for some i}\}
\end{aligned}$$

Let $M \in \mathcal{M}$. Define

$$\Gamma_\mathfrak{m}(M) = \{x \in M \ : \ \mathfrak{m}^k x = 0 \text{ for some } k \geq 0\} = \bigcup_{k \geq 0} (0 :_M \mathfrak{m}^k),$$

that is the set of all elements of M annihilating some power of $\mathfrak{m}$. One can prove that $\Gamma_\mathfrak{m}(M)$ is the largest submodule of M with $\mathrm{Supp}(M) = \{\mathfrak{m}\}$. Moreover, it is easy to observe that $\Gamma_\mathfrak{m}(-)$ is a left exact additive functor. For all $d \geq 0$, one can consider the ith right derived functor $H^i_\mathfrak{m}(-)$ of $\Gamma_\mathfrak{m}(-)$, called the ith *local cohomology functor*. The module $H^i_\mathfrak{m}(M)$ is called the ith *local cohomology module* of M.

We recall the next vanishing theorem for the local cohomology modules (see [24, Theorem 3.5.7]).

Theorem 1.1.34. (Grothendieck) *Let $M \in \mathcal{M}$ and let $t = \mathrm{depth}\, M$ and $d = \dim(M)$. Then*

1. *$H^i_\mathfrak{m}(M) \neq 0$, for $i = t$ and $i = d$;*
2. *$H^i_\mathfrak{m}(M) = 0$, for all $i < t$ and $i > d$.*

As a consequence, one has the following characterization of the Cohen–Macaulay property.

Corollary 1.1.35. *A module $M \in \mathcal{M}$ is Cohen–Macaulay if and only if $H^i_\mathfrak{m}(M) = 0$ for all $i \neq \dim(M)$.*

Let $M \in \mathcal{M}$. The local cohomology modules allows to compute the invariants of M. Observe that $H^i_{\mathfrak{m}}(M)$ are naturally graded modules, one has

$$\begin{aligned} \operatorname{depth}(M) &= \min\{i \ : \ H^i_{\mathfrak{m}}(M) \neq 0\} \\ \operatorname{pd}(M) &= n - \min\{i \ : \ H^i_{\mathfrak{m}}(M) \neq 0\} \\ \operatorname{reg}(M) &= \max\{j \ : \ H^i_{\mathfrak{m}}(M)_{j-i} \neq 0, \text{for some } i\} \end{aligned}$$

1.1.4 Sequentially Cohen–Macaulay modules

The notion of sequentially Cohen–Macaulay modules was originally introduced by Stanley [111] in the graded setting, as an algebraic counterpart to the concept of nonpure shellable complexes, which had been developed by Björner and Wachs [17] in the context of combinatorics and discrete geometry. Later on, Schenzel [105] independently introduced the notion of Cohen–Macaulay filtered modules and proved the equivalence of this notion to the sequentially Cohen–Macaulay property. Since then, the theory of sequentially Cohen–Macaulay modules has gained significant attention from many authors [1, 3, 25, 26, 39, 45, 60, 64, 74, 106, 116, 121], particularly due to its deep connections with the study of algebraic properties of combinatorial objects such as simplicial complexes and graphs.

In this section we recall the notion of sequentially Cohen–Macaulay modules and its main characterizations. In the following we will work in the graded case. Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring in n variables over a field K.

Definition 1.1.36. A module $M \in \mathcal{M}$ is called a *sequentially Cohen–Macaulay* module if there exist a finite filtration of submodules of M

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{d-1} \subset M_r = M$$

such that

1. each quotient M_i/M_{i-1} is Cohen–Macaulay,
2. $\dim(M_i/M_{i-1}) < \dim(M_{i+1}/M_i)$ for all i.

A filtration satisfying the above conditions is called a *sequentially Cohen–Macaulay filtration* of M. It is easy to see that if the sequentially Cohen–Macaulay filtration exists, then it is unique.

Remark 1.1.37. Any Cohen–Macaulay module is sequentially Cohen–Macaulay, by considering the sequentially Cohen–Macaulay filtration $0 = M_0 \subset M_1 = M$. Hence, one has the implication

$$\text{Cohen–Macaulay} \implies \begin{array}{c}\text{sequentially}\\ \text{Cohen–Macaulay}\end{array}$$

but the converse is not true in general.

Proving the sequentially Cohen–Macaulay property through the construction of a sequentially Cohen–Macaulay filtration is not always an easy task. The definition itself is not operative and it is not very efficient when trying to implement it on algorithms. In order to demonstrate this condition effectively and one must rely on algebraic or combinatorial characterization.

In the ladder, two of the main characterizations of the sequentially Cohen–Macaulay property are recalled. The first one involves the notion of the modules of deficiency.

Definition 1.1.38. Let $M \in \mathcal{M}$ and let $d = \dim(M)$. For any $d \geq 0$ the *ith module of deficiency* of M is

$$\omega^i(M) = \operatorname{Ext}_S^{n-i}(M, S(-n)).$$

If $i = d$, the module $\omega^d(M) = \omega(M)$ is the *canonical module* of M.

Remark 1.1.39. The modules of deficiency were introduced and studied by Schenzel in [103] in the framework of a Noetherian local ring $(R, \mathfrak{m})$. By Local Duality [24, Section 3.5] (in the graded setting), one has that $\omega^i(M)$ is the Matlis dual of the ith local cohomology module $H^i_{\mathfrak{m}}(M)$ of M.

By Theorem 1.1.34, set $t = \operatorname{depth}(M)$ and $d = \dim(M)$, it follows that $\omega^t(M) \neq 0$ and $\omega^d(M) = \omega(M) \neq 0$, $\omega^i(M) = 0$ for $i < t$ and $i > d$, and M is a Cohen–Macaulay module if and only if $\omega^i(M) = 0$ for all $i \neq d$. In such case, the canonical module $\omega^d(M) = \omega(M)$ is also Cohen–Macaulay. Hence, the modules of deficiency of M measure how far is M from being a Cohen–Macaulay module.

Definition 1.1.40. A module $M \in \mathcal{M}$ with Cohen–Macaulay canonical module $\omega(M)$ is called a *canonically Cohen–Macaulay* module.

The following results state important properties of the modules of the deficiency that will be useful throughout the section.

Proposition 1.1.41 ([104, Lemma 1.9])**.** *Let $M \in \mathcal{M}$ and let $d = \dim(M)$. Then the following holds:*

1. $\dim\left(\omega^i(M)\right) \leq i$ *and* $\dim(\omega(M)) = d$*;*
2. *there is a natural homomorphism $M \longrightarrow \omega(\omega(M))$, which becomes a module isomorphism if and only if M satisfies the Serre condition S_2;*

The next result characterizes sequentially Cohen–Macaulay modules *via* the modules of deficiency.

Theorem 1.1.42 ([104, Theorem 5.5])**.** *Let M be a finitely generated S-module, with $d = \dim(M)$. Then, the followings are equivalent:*

1. *M is sequentially Cohen–Macaulay;*
2. *for all $0 \leq i \leq d$ the ith module of deficiency $\omega^i(M)$ is either zero or an i-dimensional Cohen–Macaulay module;*
3. *for all $0 \leq i < d$ the ith module of deficiency $\omega^i(M)$ is either zero or an i-dimensional Cohen–Macaulay module.*

The equivalence 1. $\Longleftrightarrow$ 2. in the above theorem was announced without proof in [111, Theorem 2.11], but the author mentioned a spectral sequence argument due to Peskine.

Remark 1.1.43. If M is a sequentially Cohen–Macaulay module, then by condition 2. of Theorem 1.1.42 M has a Cohen–Macaulay canonical module. Hence, it follows that a sequentially Cohen–Macaulay module is canonically Cohen–Macaulay, but the converse is not true in general.

Thus, merging the informations from Remark 1.1.37 and Remark 1.1.43, one has the following chain of implications:

$$\text{Cohen–Macaulay} \implies \begin{array}{c}\text{sequentially}\\ \text{Cohen–Macaulay}\end{array} \implies \begin{array}{c}\text{canonically}\\ \text{Cohen–Macaulay}\end{array}$$

with the reverse implications not true in general.

Although Theorem 1.1.42 remains useful in many situations (see for instance [86, Theorem 1.5],[106]), it also presents two significant challenges. Firstly, to determine which modules of deficiency vanish is often a difficult task (see [121]). Secondly, proving the Cohen–Macaulayness of the non–vanishing modules is typically non–trivial. These difficulties underscore the complexity of applying the theorem in practice, particularly when dealing with more complicated algebraic structures.

In case we work with a homogeneous ideal $I \subset S$ it is possible to characterize the sequentially Cohen–Macaulay property by using new ideals, namely the filter ideals, which are constructed from the primary decomposition of I.

Definition 1.1.44. Let $I \subset S$ be a homogeneous ideal, with $d = \dim(S/I)$, and let $I = \bigcap_{j=1}^{r} Q_j$ be the minimal primary decomposition of I. For all $1 \leq j \leq r$, let $P_j = \sqrt{Q_j}$ be the radical of Q_j. For all $-1 \leq i \leq d$, the *ith filter ideal* of I is

$$I^{<i>} = \bigcap_{\dim(S/P_j)>i} Q_j,$$

where $I^{<-1>} = I$ and $I^{<d>} = S$.

We also define the *minimum dimension* of I as the minimum integer k such that $I^{<k>} \neq I$. By definition, it is clear that minimum dimension is at least -1.

For a homogeneous ideal $I \subset S$, one has the filtration

$$I = I^{<-1>} \subseteq I^{<0>} \subseteq I^{<1>} \subseteq \ldots \subseteq I^{<d-1>} \subseteq I^{<d>} = S.$$

Moreover, rewriting the above sequence of inclusions modulo I, we obtain

$$0 \subseteq I^{<0>}/I \subseteq I^{<1>}/I \subseteq \ldots \subseteq I^{<d-1>}/I \subseteq S/I,$$

which is called the *dimension filtration* of S/I. One can show (see [60, Lemma 1]) that such filtration does not depend on the primary components, therefore it is unique. Note that, the filter ideals $I^{<i>}$ are also independent from the primary decomposition.

By the above construction, one has the following characterization of the sequentially Cohen–Macaulay property for S/I. due to .

Proposition 1.1.45. (Goodarzi, 2016 [60, Proposition 16]) *Let $I \subset S$ be a homogeneous ideal and suppose $d = \dim(S/I)$. Then, S/I is sequentially Cohen–Macaulay if and only if*

$$\operatorname{depth}\left(S/I^{<i>}\right) \geq i+1, \qquad \textit{for all } 0 \leq i < d.$$

Goodarzi's characterization provides an efficient criterion to verify the sequentially Cohen–Macaulay property for a homogeneous ideal. It has been widely used, for instance, to study the sequential Cohen–Macaulayness of the binomial edge ideals of various classes of graphs, such as closed graphs (see [45]), block graphs, wheels and cone graphs (see [86]), playing a crucial role in the proofs. In particular, for binomial edge ideals, this result also permitted in [86] the investigation of situations that were difficult to describe using Theorem 1.1.42. This breakthrough allowed for a deeper understanding of cases previously out of reach, thus highlighting the power of Goodarzi's approach.

We end this section with another useful application of filter ideals.

Definition 1.1.46. Let $I \subset S$ be a homogeneous ideal, with $d = \dim(S/I)$. For all $1 \leq i \leq d$, the *ith unmixed layer* of S/I is the S-module $I^{<i>}/I^{<i-1>}$ and it is denoted by $U_i(S/I)$.

The name unmixed layer is motivated by [105, Corollary 2.3], in fact each module $U_i(S/I)$ is unmixed, that is for all i the associated primes of $U_i(S/I)$ have the same height. The next result gives a characterization of the unmixedness of the ideal I.

Proposition 1.1.47. (Goodarzi, 2016 [60, Lemma 3]) *Let $I \subset S$ be a homogeneous ideal, with $d = \dim(S/I)$. Then S/I is unmixed if and only if $U_i(S/I) = 0$ for all $1 \leq i < d$ and $U_d(S/I) = S/I$.*

1.2 The exterior algebra

In this section, we summarize the basics on the exterior algebra. We work on monomial ideals, particularly stable and strongly stable ones. The Cartan complex, the analogue of the Koszul complex, is used to compute Betti numbers, with a key formula giving them explicitly for stable ideals. Monomial ideals with linear quotients are then introduced. Finally, we recall that ideals with linear quotients have linear resolutions when generated in one degree, hence componentwise linear, and correspond directly to squarefree monomial ideals with linear resolutions in polynomial rings.

1.2.1 Basic concepts

Let K be a field. We denote by $E = K\langle e_1, \dots, e_n\rangle$ the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$.

As a K-algebra E is standard graded with defining relations $v \wedge v = 0$, for $v \in E$ and $e_i \wedge e_j = -e_j \wedge e_i$. For any subset $\mu = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$ with $i_1 < i_2 < \cdots < i_d$ we write $e_\mu = e_{i_1} \wedge \ldots \wedge e_{i_d}$, and call e_μ a *monomial* of degree d. We set $e_\mu = 1$, if $\mu = \emptyset$. Moreover, setting $\sigma(\tau, \mu) = |\{(i, j) : i \in \tau, j \in \mu, i > j\}|$, for any $\tau, \mu \subseteq \{1, \dots, n\}$, we have $e_\tau \wedge e_\mu = (-1)^{\sigma(\tau,\mu)} e_{\tau\cup\mu}$, if $\tau \cap \mu = \emptyset$ and $e_\tau \wedge e_\mu = 0$, otherwise. The set of monomials in E forms a K-basis of E of cardinality 2^n. An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. From the previous relations, one has that $f \wedge g = (-1)^{\deg(f)\deg(g)} g \wedge f$ for any two homogeneous elements f and g in E (see, for instance, [68]). In order to simplify the notation, we put $fg = f \wedge g$ for any two elements f and g in E.

We denote by $\mathcal{G}$ the category of graded E-modules whose morphisms are the homogeneous E-module homomorphisms of degree 0 [68]. Every E-module $M \in \mathcal{G}$ has a minimal graded free resolution F over E:

$$F : \ldots \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to M \to 0,$$

where $F_i = \oplus_j E(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M) = \dim_K \mathrm{Tor}_i^E(M,K)_j$ are called the *graded Betti numbers* of M, whereas the numbers $\beta_i(M) = \sum_j \beta_{i,j}(M)$ are called the *ith total Betti numbers* of M. Moreover, the *graded Poincaré series* of M over E is defined as $\mathcal{P}_M(s,t) = \sum_{i,j\geq 0} \beta_{i,j}(M)t^i s^j$ [10].

A graded ideal I of E is a monomial ideal if I is generated by monomials.
Let $e_\mu = e_{i_1}\cdots e_{i_d} \neq 1$ be a monomial in E. We define

$$\mathrm{supp}(e_\mu) = \{i : e_i \text{ divides } e_\mu\} = \mu$$

and we write

$$\mathfrak{m}(e_\mu) = \max\{i : i \in \mathrm{supp}(e_\mu)\} = \max\{i : i \in \mu\}.$$

We set $\mathfrak{m}(e_\mu) = 0$, if $e_\mu = 1$.

Definition 1.2.1. Let I be a monomial ideal of E. I is called *stable* if for each monomial $e_\mu \in I$ and each $j < \mathfrak{m}(e_\mu)$ one has $e_j e_{\mu\setminus\{\mathfrak{m}(e_\mu)\}} \in I$. I is called *strongly stable* if for each monomial $e_\mu \in I$ and each $j \in \mu$ one has $e_i e_{\mu\setminus\{j\}} \in I$ for all $i < j$.

If I is a monomial ideal in E, we denote by $G(I)$ the unique minimal set of monomial generators of I, and by $G(I)_d$ the set of all monomials $u \in G(I)$ such that $\deg(u) = d$, $d > 0$. One can verify that the defining property of a strongly stable ideal only needs to be checked on the generators (see, for instance [4, Remark 2.2]).

We close this paragraph with some comments. As in the polynomial ring, every element of E of the type ce_μ, with $c \in K$ and e_μ a monomial, is called a *term*. In what follows, to avoid ambiguity and with the aim of simplifying the computations, we refer to any term of the form ce_μ with $c \in \{-1, 1\}$, as a monomial. In other words we generalize the classical notion of monomial recalled at the beginning of the section. This choice does not affect the classical development of the exterior algebra theory.

1.2.2 A glimpse to the Cartan Complex

The Cartan complex for the exterior algebra $E = K\langle e_1,\ldots,e_n\rangle$ plays the role of the Koszul complex for the symmetric algebra. We refer to [10, 68] for more details on this subject.

Let $\mathbb{N}$ be the set of all non negative integers and for a positive integer n, let $[n] = \{1,\ldots,n\}$. Moreover, for a n-tuple $a = (a_1,\ldots,a_n) \in \mathbb{N}^n$, let $|a| = a_1 + \cdots + a_n$.

Let $v = v_1,\ldots,v_m$ be a sequence of linear forms in E_1. The Cartan complex $C.(v;E)$ of the sequence v with values in E is defined as the complex whose i-chains $C_i(v;E)$ are the elements of degree i of the free divided power algebra $E\langle x_1,\ldots,x_m\rangle$, where $E\langle x_1,\ldots,x_m\rangle$ is the polynomial ring over E in the set of variables

$$x_i^{(j)}, \quad i = 1,\ldots,m, \quad j = 1,2,\ldots$$

modulo the relations

$$x_i^{(j)}x_i^{(k)} = \binom{j+k}{j} x_i^{(j+k)}.$$

We set $x_i^{(0)} = 1$, $x_i^{(1)} = x_i$ for $i = 1,\ldots,m$ and $x_i^{(a)} = 0$ for $a < 0$. The algebra $E\langle x_1,\ldots,x_m\rangle$ is a free E-module with basis

$$x^{(a)} = x_1^{(a_1)}x_2^{(a_2)}\cdots x_m^{(a_m)}, \quad a = (a_1,\ldots,a_m) \in \mathbb{N}^m.$$

We say that $x^{(a)}$ has degree i if $|a| = i$. If $x^{(a)} \neq 1$, we define $\mathrm{supp}(x^{(a)}) = \{i \in [m] : a_i \neq 0\}$ and set $\mathrm{supp}(x^{(a)}) = \emptyset$, for $x^{(a)} = 1$.

One has $C_i(v; E) = \oplus_{|a|=i} E x^{(a)}$. The E-linear differential ∂ on $C.(v; E)$ is defined as follows: for $x^{(a)} = x_1^{(a_1)} x_2^{(a_2)} \cdots x_m^{(a_m)}$, we set

$$\partial(x^{(a)}) = \sum_{i=1}^{m} v_i x_1^{(a_1)} \cdots x_i^{(a_i - 1)} \cdots x_m^{(a_m)}.$$

If $\mathcal{G}$ is the category of graded E-modules above defined and $M \in \mathcal{G}$, one can define the complex $C.(v; M) = M \otimes_E C.(v; E)$ and set $H_i(v; M) = H_i(C.(v; M))$. We call $H_i(v; M)$ the ith Cartan homology module of v with respect to M. One can observe that each $H_i(v; M)$ is a graded E-module. In [10, Theorem 2.2], the authors showed that the Cartan complex $C.(v; E)$ is a minimal free resolution of $E/(v)$ and, as a consequence, they proved that for any graded E-module M and each $i \geq 0$, there is the natural isomorphism $\mathrm{Tor}_i^E(M, K) \cong H_i(e_1, \ldots, e_n; M)$ of graded E-modules. Such a result allows to compute the graded Betti numbers of M.

The next important formula for determining the graded Betti numbers of a stable ideal I of E has been stated by Aramova, Herzog and Hibi via the Cartan complex [10, Corollary 3.3]:

$$\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{\mathfrak{m}(u) + i - 1}{\mathfrak{m}(u) - 1}, \quad \text{for all } i \geq 0. \tag{1.1}$$

Indeed, for every $i > 0$, a basis of the modules $H_i(e_1, \ldots, e_n; E/I)$ is given by the homology classes of the cycles

$$u_{\mathfrak{m}(u)} x^{(a)}, \;\; u \in G(I), \;\; |a| = i, \;\; \max(a) = \mathfrak{m}(u),$$

where $\max(a) = \max \mathrm{supp}(x^{(a)}) = \max\{i \in [n] : a_i \neq 0\}$ ($a \in \mathbb{N}^n$) and $u_{\mathfrak{m}(u)} = e_{\mu \setminus \{\mathfrak{m}(u)\}}$, for $u = e_\mu$.

1.2.3 A glimpse to ideals with linear quotients

Monomial ideals with linear quotients have been introduced by Herzog and Takayama [75] to study resolutions that arise as iterated mapping cones.

Definition 1.2.2. A monomial ideal I of E is said to have *linear quotients* if for some order $u_1, \ldots, u_r$ of the elements of $G(I)$ the ideals $0 : (u_1)$ and $(u_1, \ldots, u_{i-1}) : (u_i)$, for $i = 2, \ldots, r$, are generated by a subset of the set $\{e_1, \ldots, e_n\}$.

Remark 1.2.3. Differently from the definition of linear quotients over the polynomial ring, we need the condition $0 : (u_1)$ has to be generated by a subset of $\{e_1, \ldots, e_n\}$. This is always the case, since $0 : (u_1) = (\mathrm{supp}(u_1))$.

The ideal $I = (e_2, e_3e_4)$ of $E = K\langle e_1, \ldots, e_4 \rangle$ has linear quotients with respect to this order of the generators. In fact, $0 : (e_2) = (e_2)$, and $(e_2) : (e_3e_4) = (e_2, e_3, e_4)$. But I does not satisfy the conditions for linear quotients, when employing the reversed order on the generators. Indeed, for the order e_3e_4, e_2, we have $0 : (e_3e_4) = (e_3, e_4)$ and $(e_3e_4) : (e_2) = (e_2, e_3e_4)$.

Following [75], for a monomial ideal I of E with linear quotients with respect to some order of the elements $u_1, \dots, u_r$ of $G(I)$, we define

$$\mathrm{set}(u_j) = \{k \in [n] : e_k \in (u_1, \dots, u_{j-1}) : (u_j)\}, \quad \text{for } j = 2, \dots, r.$$

Moreover, we set

$$\mathrm{set}(u_1) = \{k \in [n] : e_k \in 0 : (u_1)\} = \mathrm{supp}\ u_1.$$

One can observe that in contrast to the polynomial ring context, $\mathrm{set}(u)$ contains $\mathrm{supp}(u)$ for every $u \in G(I)$.

Remark 1.2.4. A stable ideal I of E has linear quotients with respect to the reverse degree lexicographical order on the generators. Moreover, $\mathrm{set}(u) = \{i : i \le \mathfrak{m}(u)\}$, *i.e.* $\mathrm{set}(u) = [\mathfrak{m}(u)]$, for every $u \in G(I)$.

For instance, let us consider the stable ideal $I = (e_1e_2, e_1e_3, e_2e_3, e_3e_4e_5)$ of $E = K\langle e_1, \dots, e_5\rangle$. Then I has linear quotients with respect to the reverse lexicographic order $e_1e_2, e_1e_3, e_2e_3, e_3e_4e_5$. Indeed, $0 : (e_1e_2) = (e_1, e_2)$, $(e_1e_2) : (e_1e_3) = (e_1, e_2, e_3)$, $(e_1e_2, e_1e_3) : (e_2e_3) = (e_1, e_2, e_3)$ and $(e_1e_2, e_1e_3, e_2e_3) : (e_3e_4e_5) = (e_1, e_2, e_3, e_4, e_5)$. Therefore, $\mathrm{set}(e_1e_2) = \{1, 2\}$, $\mathrm{set}(e_1e_3) = \{1, 2, 3\}$, $\mathrm{set}(e_2e_3) = \{1, 2, 3\}$, $\mathrm{set}(e_3e_4e_5) = \{1, 2, 3, 4, 5\}$, and so $|\mathrm{set}(u)| = \mathfrak{m}(u)$, for all $u \in G(I)$.

Finally, we have the following hierarchy of monomial ideals in E:

$$\text{strongly stable ideals} \Longrightarrow \text{stable ideals} \Longrightarrow \begin{array}{c}\text{ideals with}\\ \text{linear quotients}\end{array}$$

Let I be a monomial ideal of E with linear quotients with respect to a minimal set of monomial generators $\{u_1, \dots, u_r\}$. We observe that not necessarily $\deg u_1 \le \cdots \le \deg u_r$. Let us consider the ideal $I = (e_1e_2, e_2e_3e_4, e_1e_3)$ of $E = K\langle e_1, \dots, e_4\rangle$. I has linear quotients for the given order of the generators. In fact, $0 : (e_1e_2) = (e_1, e_2)$, $(e_1e_2) : (e_2e_3e_4) = (e_1, e_2, e_3, e_4)$ and $(e_1e_2, e_2e_3e_4) : (e_1e_3) = (e_1, e_2, e_3)$.

Now, let $I \subset E$ be a monomial ideal with linear quotients with respect to homogeneous generators $u_1, \dots, u_r$. We say that the order $u_1, \dots, u_r$ is *degree increasing* if $\deg(u_1) \le \cdots \le \deg(u_r)$. By using exterior algebra methods [114, Lemma 4.2.1], one can prove that if $I \subset E$ is a monomial ideal with linear quotients then I has also linear quotients with respect to a degree increasing order. Such a statement can be seen as the exterior version of the "Rearrangement lemma" of Björner and Wachs [17].

Hence, from now on when we say that a monomial ideal $I \subset E$ has linear quotients with respect to some order of $G(I) = \{u_1, \dots, u_r\}$, we assume tacitly that such an order is a degree increasing order, *i.e.* $\deg(u_1) \le \cdots \le \deg(u_r)$.

We close the section with some comments on resolutions of monomial ideals with linear quotients.

It is well–known that in the polynomial ring case, monomial ideals generated in one degree d with linear quotients have d-linear resolutions (see, for instance [68, Proposition 8.2.1.]). The same property holds for monomial ideals in an exterior algebra (see, [79, Corollary 5.4.4.]).

The property of having linear quotients can be related to *componentwise linearity*. This concept has been introduced for ideals in a polynomial ring by Herzog and Hibi in [66] in order to generalize the Eagon and Reisner's result that the Stanley-Reisner ideal of a simplicial complex has a linear resolution if and only if the Alexander Dual of the complex is Cohen–Macaulay. Such a notion can be also introduced in the exterior algebra context (see, for instance, [79, Definition 5.3.1]). More precisely, a module $M \in \mathcal{M}$ is called *componentwise linear* if the submodule $M_{\langle j \rangle}$ of M, which is generated by all homogeneous elements of degree j belonging to M, has a j-linear resolution for all $j \in \mathbb{Z}$. Moreover, as in the commutative case, one can prove that if I is a monomial ideal of E with linear quotients, then I is a componentwise linear ideal [79, Theorem 5.4.5.].

Finally, it is worth mentioning a result due to Aramova, Avramov and Herzog [9, Corollary 2.2.], which states that if I is an ideal generated by squarefree monomials in a polynomial ring $S = K[x_1, \dots, x_n]$, and J denotes the corresponding monomial ideal in E, the ideal I has a linear free resolution over S if and only if the ideal J has a linear free resolution over E.

Chapter 2

Mapping cones of monomial ideals over exterior algebras

A popular topic in combinatorial commutative algebra is to detect minimal graded free resolutions of classes of monomial ideals (see, for instance, [38, 97] and the reference therein). In [75], Herzog and Takayama have shown that one can iteratively construct a free resolution of any monomial ideal in a polynomial ring by considering the mapping cone of the map between certain complexes, adding one generator at a time. Especially when the ideal has linear quotients, the complexes and the maps are well-behaved, and the procedure determines a minimal resolution [75] by means of the Koszul complex. Some meaningful examples of free resolutions which arise as iterated mapping cones are the Eliahou–Kervaire resolution of stable ideals [41] (see, also, [50]) and the Taylor resolution.

Let K be a field and let $E = K\langle e_1, \dots, e_n \rangle$ be the exterior algebra of a K-vector space V with basis $e_1, \dots, e_n$. A fundamental tool in the study of minimal graded free resolutions of graded E-modules is played by the Cartan complex [10]. It is an infinite complex which has the structure of a divided power algebra, and furthermore, as in the case of the ordinary Koszul complex over the polynomial ring, one may compute the Cartan homology of an E-module using long exact sequences attached to the partial sequences $e_1, \dots, e_i$.

The aim of this chapter is to determine a minimal graded free resolution F of a monomial ideal in E with linear quotients by iterated mapping cones and, in particular, to describe explicitly the free E-modules which appear in F and thus to get information on the graded Betti numbers of the given ideal. Our main tool is the Cartan complex. As in the polynomial case [75], we are able to provide an explicit description of the differentials in F for a special class of ideals with linear quotients which satisfy an additional condition. A significant example of monomial ideals of E with linear quotients is the class of stable ideals whose minimal graded free resolution has been completely described by Aramova, Herzog and Hibi in [10] (see, also, [9]). Recently, minimal graded free resolutions in the context of skew commutative algebras have been studied in [51, 89]. Moreover, new meaningful results related to iterated mapping cone procedure have been recently carried out in [117], where the author has replaced the polynomial ring by any commutative strongly Koszul algebra.

The chapter is organized as it follows.

In Section 2.1, we discuss the mapping cone over the exterior algebra E. We prove that if I is a monomial ideal with linear quotients, the mapping cone construction determines a minimal graded free resolution of I, and thus we describe explicitly a basis for each free module in the resolution (Theorem 2.1.1). A crucial fact is the exterior version of the “Rearrangement lemma" of Björner and Wachs [17]. Hence, if $G(I)$ is the minimal set of monomial generators of I, analyzing some special sets of positive integers that can be associated to $G(I)$, we get a formula for the graded Betti numbers of I (Corollary 2.1.3) via the notion of weak composition (Definition 2.1.2). Then, we obtain some formulas for the graded Poincaré series, the complexity and the depth of I (Corollaries 2.1.5, 2.1.6).

In Section 2.2, we introduce the notion of regular decomposition function (Definition 2.2.2) of an ideal of E with linear quotients. Our main result is Theorem 2.2.5 that gives an explicit description of the differentials of the resolutions of all ideals of E with linear quotients which admit a regular decomposition function.

Finally, in Section 2.3, firstly, we introduce the class of $\mathbf{t}$-spread strongly stable ideals in E (Definition 2.3.2), then we discuss their main properties and, as a consequence, we compute their graded Betti numbers (Corollary 2.3.5) via the results in Section 2.1. Such a formula generalizes the one stated in [10] for the ordinary strongly stable ideal in E. All examples in this chapter have been verified computationally using the software *Macaulay2* [62] and the package [4].

The result contained in this chapter appeared first in [32].

2.1 Resolutions by mapping cones

In this section, we determine a minimal graded free resolution of monomial ideals with linear quotients by *mapping cones*.

Roughly speaking, if A and B are two chain complexes of graded E-modules of the category $\mathcal{G}$ and $\psi : A \to B$ is a complex homomorphism. The mapping cone of ψ is a new chain complex denoted by $C(\psi)$ with $C(\psi)_i = A_{i-1} \oplus B_i$ for all i, and the differentials related to the ones in A and B by a special formula. We refer to [40, 120] for more details on this topic. In particular, the mapping cone of a chain map between two free resolutions gives again a free resolution which is not in general minimal.

We apply this concept in a special situation. Let I be a monomial ideal of E and assume that I has linear quotients with respect the order $u_1, \dots, u_r$ of its generators. The basic idea is to exploit the short exact sequences that arise from adding one generator of I at a time, and to iteratively construct the resolution as a mapping cone of an appropriate map between previously constructed complexes.

For this aim, set $I_j = (u_1, \dots, u_j)$ and $L_j = (u_1, \dots, u_j) : (u_{j+1})$, $j = 1, \dots, r$. Since, $I_{j+1}/I_j \simeq E/L_j$, then we get the following exact sequences of E-modules

$$0 \longrightarrow E/L_j \xrightarrow{\alpha} E/I_j \xrightarrow{\beta} E/I_{j+1} \longrightarrow 0 \ ,$$

where $\alpha : E/L_j \longrightarrow E/I_j$ is multiplication by u_{j+1}. Consider the ideal L_j. By definition $\text{set}(u_{j+1}) = \{k \in [n] : e_k \in (u_1, \dots, u_j) : (u_{j+1}) = L_j\}$, for $j = 1, \dots, r$. Thus, since I has linear quotients with respect to the order $u_1, \dots, u_r$ of its generators, we have that $L_j = (e_{k_1}, \dots, e_{k_\ell})$, with $\{k_1, \dots, k_\ell\} = \text{set}(u_{j+1})$, for $j = 1, \dots, r$. Hence, if $C^{(j)} = C.(e_{k_1}, \dots, e_{k_\ell}; E)$ is the Cartan complex for the sequence $e_{k_1}, \dots, e_{k_\ell}$, we have that $C^{(j)}$ is a minimal graded free resolution of E/L_j [10].

Let $F^{(j)}$ be a minimal graded free resolution of E/I_j. Since the modules in $C^{(j)}$ are free, thus projective, there exists a complex homomorphism $\psi^{(j)} : C^{(j)} \to F^{(j)}$, that is a sequence of maps $\psi_i^{(j)} : C_i^{(j)} \to F_i^{(j)}$ ($i \geq 0$), called the *comparison maps*, which *lifts* the map $\psi_{-1}^{(j)} = \alpha$ and makes the following diagram

$$\begin{tikzcd}
C^{(j)}: \cdots \arrow[r] & C_2^{(j)} \arrow[r, "d_2^{C^{(j)}}"] \arrow[d, "\psi_2^{(j)}"'] & C_1^{(j)} \arrow[r, "d_1^{C^{(j)}}"] \arrow[d, "\psi_1^{(j)}"'] & C_0^{(j)} \arrow[r, "d_0^{C^{(j)}}"] \arrow[d, "\psi_0^{(j)}"'] & E/L_j \arrow[r] \arrow[d, "\psi_{-1}^{(j)}=\alpha"] & 0 \\
F^{(j)}: \cdots \arrow[r] & F_2^{(j)} \arrow[r, "d_2^{F^{(j)}}"'] & F_1^{(j)} \arrow[r, "d_1^{F^{(j)}}"'] & F_0^{(j)} \arrow[r, "d_0^{F^{(j)}}"'] & E/I_j \arrow[r] & 0
\end{tikzcd}$$

commutative. The homomorphism $\psi^{(j)}$ gives rise to an acyclic complex $C(\psi^{(j)})$, called the *cone complex* of $\psi^{(j)}$, whose 0th homology module is $H_0(C(\psi^{(j)})) = \mathrm{Coker}(\alpha) = (E/I_j)/\mathrm{Im}(\alpha) \cong (E/I_j)/(E/L_j) \cong E/I_{j+1}$, that is, $C(\psi^{(j)})$ is a free resolution of E/I_{j+1} (see, for instance, [40, Appendix A3.12]). Thus by iterated mapping cones one obtains step by step a graded free resolution of E/I.

More in detail, the cone complex $C(\psi^{(j)})$ is defined as follows:

(i) let $C(\psi^{(j)})_0 = C_0^{(j)}$, and $C(\psi^{(j)})_i = C_{i-1}^{(j)} \oplus F_i^{(j)}$, for $i > 0$;

(ii) let $d_0 = \varphi \circ d_0^{F^{(j)}}$, $d_1 = (0, \psi_0 + d_1^{F^{(j)}})$, and $d_i = (-d_{i-1}^{C^{(j)}}, \psi_{i-1} + d_i^{F^{(j)}})$, for $i > 1$.

This procedure, known as the *mapping cone*, may be visualized as follows:

$$\begin{tikzcd}
C^{(j)}[-1]: \cdots \arrow[r] & C_2^{(j)} \arrow[r, "d_2^{C^{(j)}}"] \arrow[dr, "\psi_2^{(j)}"] & C_1^{(j)} \arrow[r, "d_1^{C^{(j)}}"] \arrow[dr, "\psi_1^{(j)}"] & C_0^{(j)} \arrow[r, "d_0^{C^{(j)}}"] \arrow[dr, "\psi_0^{(j)}"] & E/L_j \arrow[r] \arrow[dr] & 0 & \\
 & \oplus & \oplus & \oplus & \oplus & & \\
F^{(j)}: \cdots \arrow[r] & F_3^{(j)} \arrow[r, "d_3^{F^{(j)}}"'] & F_2^{(j)} \arrow[r, "d_2^{F^{(j)}}"'] & F_1^{(j)} \arrow[r, "d_1^{F^{(j)}}"'] & F_0^{(j)} \arrow[r, "d_0^{F^{(j)}}"'] & E/I_j \arrow[r] & 0
\end{tikzcd}$$

Here $C^{(j)}[-1]$ is the complex $C^{(j)}$ *homologically shifted* by -1.

Our aim is to prove that the free resolution $C(\psi^{(j)})$ is minimal.

The next statement follows the arguments of the corresponding result in the polynomial ring context [75, Lemma 1.5]. We elaborate on this argument and stress the needed changes due to the different context.

For a non zero n-tuple $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $\mathrm{supp}(a) = \{j \in [n] : a_j \neq 0\}$.

Theorem 2.1.1. *Let I be a monomial ideal of E with linear quotients with respect to the order $u_1, \ldots, u_r$ of the minimal generators of I and let F be the iterated mapping cone derived from the sequence $u_1, \ldots, u_r$. Then F is a minimal graded free resolution of E/I and the free E-module F_i which appears in the ith step of F has homogeneous basis*

$$\{f(a;u),\ \textit{with}\ u \in G(I),\ a \in \mathbb{N}^n,\ \mathrm{supp}(a) \subset \mathrm{set}(u),\ |a| = i-1\},$$

with $\deg f(a;u) = |a| + \deg u$, *for all* $i > 0$.

Proof. Let $F^{(j)}$, $j \geq 1$, be a minimal graded free resolution of E/I_j. We prove by induction on j, that the set $\{f(a;u)$, with $u \in G(I_j)$, $a \in \mathbb{N}^n$, $\mathrm{supp}(a) \subset \mathrm{set}(u)$, $|a| = i-1\}$ where $\deg f(a;u) = |a| + \deg u$, forms a homogeneous basis of each E-module $F_i^{(j)}$, $i \geq 1$.

For $j = 1$, the assertion is clear.

Let $j > 1$. Setting $L_j = (e_{k_1}, \ldots, e_{k_\ell})$, with $\{k_1, \ldots, k_\ell\} = \mathrm{set}(u_{j+1})$, for $j = 2, \ldots, r$, let $C^{(j)} = C.(e_{k_1}, \ldots, e_{k_\ell}; E)$ be the Cartan complex for the sequence $e_{k_1}, \ldots, e_{k_\ell}$. As we have previously pointed out, $C^{(j)}$ is a minimal graded free resolution of E/L_j.

With the same notations as in Subsection 1.2.2, in homological degree $i-1$ the Cartan complex $C^{(j)}$ has the E-basis $x^{(a)} = x_{k_1}^{(a_{k_1})} x_{k_2}^{(a_{k_2})} \cdots x_{k_\ell}^{(a_{k_\ell})}$, $a = (a_{k_1}, \ldots, a_{k_\ell}) \in \mathbb{N}^\ell$, with $|a| = \sum_{q=1}^{\ell} a_{k_q} = i-1$, $\operatorname{supp}(x^{(a)}) = \operatorname{supp}(a) \subset \operatorname{set}(u_{j+1})$.

Since $F_i^{(j+1)} = C_{i-1}^{(j)} \oplus F_i^{(j)}$, we obtain the basis we are looking for from the inductive hypothesis by identifying the elements $x^{(a)}$ with $f(a; u_{j+1})$.

In order to prove that $F^{(j+1)}$ is a minimal free resolution, it is sufficient to verify that $\operatorname{Im}(\psi^{(j)}) \subset (e_1, \ldots, e_n)F^{(j)}$.

Let $f(a; u_{j+1}) \in C_{i-1}^{(j)}$ and $\psi^{(j)}(f(a; u_{j+1})) = \sum_{i=1}^{j} \sum_b c_{b,i} f(b; u_i)$. Since we are only considering degree increasing orders, then $\deg u_{j+1} \geq \deg u_i$, for all $i = 1, \ldots, j$ and, furthermore, $|b| = |a| - 1$. Thus $\deg f(a; u_{j+1}) = |a| + \deg u_{j+1} > |b| + \deg u_i = \deg f(b; u_i)$, for all b and i. Hence, $\deg c_{b,i} > 0$ for all b and i. The assertion follows. □

As a consequence of Theorem 2.1.1, we obtain a formula for computing the graded Betti numbers of a monomial ideal with linear quotients. Next notion will be crucial for our aim.

Definition 2.1.2. A sequence $(a_1, a_2, \ldots, a_k)$ of integers fulfilling $a_i \geq 0$ for all i, and $a_1 + a_2 + \cdots + a_k = n$ is called a *weak composition* of n.

The number of weak compositions of an integer n in k parts is given by $\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$.

Corollary 2.1.3. *Let I be a monomial ideal of E with linear quotients. Then*

$$\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{i + |\operatorname{set}(u)| - 1}{|\operatorname{set}(u)| - 1},$$

for all i, j. Hence,

$$\beta_i(I) = \sum_{u \in G(I)} \binom{i + |\operatorname{set}(u)| - 1}{|\operatorname{set}(u)| - 1},$$

for all $i \geq 0$.

Proof. From Theorem 2.1.1,

$$\beta_{i,i+j}(I) = |\{a = (a_1, \ldots, a_n) \in \mathbb{N}^n : \operatorname{supp}(a) \subset \operatorname{set}(u), u \in G(I)_j \text{ and } |a| = i\}|.$$

On the other hand, one can observe that

$$|\{a = (a_1, \ldots, a_n) \in \mathbb{N}^n : \operatorname{supp}(a) \subset \operatorname{set}(u), |a| = i\}|$$

is the number of the weak compositions of the positive integer i in $|\operatorname{set}(u)|$ parts. The assertions follow. □

Example 2.1.4. Let $I = (e_1e_3, e_1e_4, e_2e_4e_6)$ be a monomial ideal of $E = K\langle e_1, \ldots, e_6\rangle$. I has linear quotients with respect to the order $e_1e_3, e_1e_4, e_2e_4e_6$. In fact, $0 : (e_1e_3) = (e_1, e_3)$, $(e_1e_3) : (e_1e_4) = (e_1, e_3, e_4)$, $(e_1e_3, e_1e_4) : (e_2e_4e_6) = (e_1, e_2, e_4, e_6)$. Therefore, setting $u_1 = e_1e_3$, $u_2 = e_1e_4$, $u_3 = e_2e_4e_6$, one has

$$\operatorname{set}(u_1) = \{1, 3\}, \quad \operatorname{set}(u_2) = \{1, 3, 4\}, \quad \operatorname{set}(u_3) = \{1, 2, 4, 6\}.$$

Clearly $\beta_0(I) = 3$. In fact, $\binom{0+|\operatorname{set}(u)|-1}{|\operatorname{set}(u)|-1} = 1$, for all $u \in G(I)$. Moreover,

$$\begin{aligned}\beta_1(I) &= |\{a = (a_1, \dots, a_6) \in \mathbb{N}^6 : \operatorname{supp}(a) \subset \operatorname{set}(u), u \in G(I) \text{ and } |a| = 1\}| \\ &= \sum_{i=1}^{3} \binom{1 + |\operatorname{set}(u_i)| - 1}{|\operatorname{set}(u_i)| - 1} = \binom{2}{1} + \binom{3}{2} + \binom{4}{3} = 9.\end{aligned}$$

Let us compute $\beta_2(I)$. It is

$$\begin{aligned}\beta_2(I) &= |\{a = (a_1, \dots, a_6) \in \mathbb{N}^6 : \operatorname{supp}(a) \subset \operatorname{set}(u), u \in G(I) \text{ and } |a| = 2\}| \\ &= \sum_{i=1}^{3} \binom{2 + |\operatorname{set}(u_i)| - 1}{|\operatorname{set}(u_i)| - 1} = \binom{3}{1} + \binom{4}{2} + \binom{5}{3} = 19.\end{aligned}$$

And so on. The Betti table of I displayed by *Macaulay2* [62] is the following one:

	0	1	2	3	4	5	6	$\cdots$
total:	3	9	19	34	55	83	119	$\cdots$
2:	2	5	9	14	20	27	35	$\cdots$
3:	1	4	10	20	35	56	84	$\cdots$

TABLE 2.1: Betti table of the ideal $I = (e_1e_3, e_1e_4, e_2e_4e_6)$.

Corollary 2.1.3 yields the next results.

Corollary 2.1.5. *Let I be a monomial ideal of E with linear quotients. Then the graded Poincaré series of I over E is*

$$\mathcal{P}_I(s,t) = \sum_{i\geq 0}\sum_{j\geq 0}\sum_{u\in G(I)_j} (1+t)^{i+|\operatorname{set}(u)|-1} s^j.$$

Proof. Since $\mathcal{P}_I(s,t) = \sum_{i,j\geq 0} \beta_{i,j}(I)t^i s^j$, the desired formula follows from Corollary 2.1.3. □

Recall that if $\mathcal{M}$ is the category of finitely generated $\mathbb{Z}$-graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$ and $m \in M$ the *complexity* of M, which measures the growth rate of the Betti numbers of M, is defined as follows [9] (see also [79]):

$$\operatorname{cx}_E(M) = \inf\left\{c \in \mathbb{Z} \;:\; \beta_i(M) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\right\}.$$

In [9], the authors introduced an important invariant of an E-module, the depth, in a similar way as the depth of a module over a polynomial ring. More in detail, for $M \in \mathcal{M}$, a linear form $v \in E_1$ is called M-regular if $0 :_M (v) = vM$. A sequence of linear forms $v_1, \dots, v_s$ is called an M-regular sequence if v_1 is M-regular, v_i is $M/(v_1, \dots, v_{i-1})M$-regular for $i = 2, \dots, s$ and $M/(v_1, \dots, v_s)M \neq 0$ [9, 90]. It is shown in [9] that all maximal M-regular sequences have the same length. This length is called the *depth* of M over E and is denoted by $\operatorname{depth}_E(M)$. Such an invariant is closely related to the complexity at least if the ground field is infinite. Indeed, in such case, Aramova, Avramov and Herzog [9, Theorem 3.2] proved that, for $M \in \mathcal{M}$

$$n = \operatorname{cx}_E(M) + \operatorname{depth}_E(M). \tag{2.1}$$

Corollary 2.1.6. *Let I be a monomial ideal of E with linear quotients. Then,*

$$\operatorname{cx}_E(E/I) \;=\; \max\left\{|\operatorname{set}(u)| : u \in G(I)\right\}.$$

Moreover, if K is an infinite field, then

$$\operatorname{depth}_E(E/I) \;=\; \min\left\{n - |\operatorname{set}(u)| : u \in G(I)\right\}.$$

Proof. We have $\operatorname{cx}_E(E/I) = \operatorname{cx}_E(I)$. By Corollary 2.1.3 we have

$$\begin{aligned}
\beta_i(I) &= \sum_{u \in G(I)} \binom{i + |\operatorname{set}(u)| - 1}{|\operatorname{set}(u)| - 1} \\
&= \sum_{k=1}^{n} \Big[\sum_{u \in G(I) : |\operatorname{set}(u)| = k} \binom{i+k-1}{k-1} \Big] \\
&= \sum_{k=1}^{n} \left|\left\{u \in G(I) : |\operatorname{set}(u)| = k\right\}\right| \binom{i+k-1}{i}.
\end{aligned}$$

The kth binomial coefficient in the previous sum is a polynomial in i of degree $k-1$ and the number $\max\left\{|\operatorname{set}(u)| : u \in G(I)\right\}$ is the maximal k which appears in the sum. It follows that $\operatorname{cx}_E(E/I) = \max\left\{|\operatorname{set}(u)| : u \in G(I)\right\}$. If K is an infinite field, then by (2.1), $\operatorname{depth}_E(E/I) = n - \operatorname{cx}_E(E/I)$ and the assertion follows. □

Remark 2.1.7. If I is a stable ideal of E, then the formula for the graded Betti numbers stated in Corollary 2.1.3 becomes formula (1.1), whereas the formula for the complexity in Corollary 2.1.6 becomes the formula in [79, Lemma 3.1.4]. In fact, I has linear quotients with respect to the reverse lexicographic order and, moreover, in such a case $|\operatorname{set}(u)| = \mathfrak{m}(u)$, for all $u \in G(I)$ (Remark 1.2.4).

2.2 Regular decomposition functions

In this section we describe an explicit resolution of all ideals in E with linear quotients which satisfy an extra condition. For our aim we introduce the notion of regular decomposition function.

Let I be a monomial ideal with linear quotients with respect to the sequence of generators $u_1, \ldots, u_r$, and set as before $I_j = (u_1, \ldots, u_j)$ for $j = 1, \ldots, r$. Let $M(I)$ be the set of all monomials in I. Define the map $g : M(I) \to G(I)$ as follows: $g(u) = u_j$, if j is the smallest number such that $u \in I_j$.

The next statement holds.

Lemma 2.2.1. (a) *For all $u \in M(I)$ one has $u = g(u)c(u)$ with some complementary factor $c(u)$, and $\operatorname{set}(g(u)) \cap \operatorname{supp}(c(u)) = \emptyset$.*

(b) *Let $u \in M(I)$, $u = vw$ with $v \in G(I)$ and $\operatorname{set}(v) \cap \operatorname{supp}(w) = \emptyset$. Then $v = g(u)$.*

(c) *Let $u, v \in M(I)$ such that $\operatorname{supp}(u) \cap \operatorname{supp}(v) = \emptyset$. Then $g(uv) = g(u)$ if and only if $\operatorname{set}(g(u)) \cap \operatorname{supp}(v) = \emptyset$.*

Proof. The proof of Herzog and Takayama for ideals over polynomial ring [68, Lemma 1.8] carries over. □

One can observe that any function $M(I) \to G(I)$ satisfying Lemma 2.2.1(a) is uniquely determined because of Lemma 2.2.1(b). We call g the *decomposition function* of I.

Definition 2.2.2. The decomposition function $g : M(I) \to G(I)$ is *regular*, if $\mathrm{set}(g(e_s u)) \subseteq \mathrm{set}(u)$ for all $s \in \mathrm{set}(u)$ such that $s \notin \mathrm{supp}(u)$ and $u \in G(I)$.

If I is a stable ideal, then its decomposition function is regular with respect to the reverse lexicographic order. In general, the decomposition function for an ideal with linear quotients is not regular as next example shows.

Example 2.2.3. Let $E = K\langle e_1, \dots, e_4\rangle$ and $I = (e_2e_4, e_1e_2, e_1e_3)$. One can verify that I has linear quotients. In particular, $0 : (e_2e_4) = (e_2, e_4)$, $(e_2e_4) : (e_1e_2) = (e_1, e_2, e_4)$ and $(e_2e_4, e_1e_2) : (e_1e_3) = (e_1, e_2, e_3)$. On the other hand, $g(e_2(e_1e_3)) = e_1e_2$ but $\mathrm{set}(e_1e_2) = \{1, 2, 4\} \nsubseteq \mathrm{set}(e_1e_3) = \{1, 2, 3\}$ and g is not regular.

Let us consider a new order e_1e_2, e_2e_4, e_1e_3 of the generators of I. In such a case $0 : (e_1e_2) = (e_1, e_2)$, $(e_1e_2) : (e_2e_4) = (e_1, e_2, e_4)$ and $(e_1e_2, e_2e_4) : (e_1e_3) = (e_1, e_2, e_3)$. Thus I has linear quotients also with respect the new order of the generators. On the other hand, we can note that $\mathrm{set}(e_1e_2) = \{1, 2\} \subset \mathrm{set}(u)$, for all $u \in G(I) \setminus \{e_1e_2\}$. As a consequence, since $g(e_s u) = e_1e_2$, for all $s \in \mathrm{set}(u) \setminus \mathrm{supp}(u)$ and $u \in G(I) \setminus \{e_1e_2\}$, it follows that g is regular. Indeed, we have that $\mathrm{set}(g(e_s u)) = \{1, 2\} \subset \mathrm{set}(u)$, for all $s \in \mathrm{set}(u) \setminus \mathrm{supp}(u)$ and $u \in G(I)$.

The next statement can be proved using the same techniques as the corresponding result in the polynomial case [75, Lemma 1.11].

Lemma 2.2.4. *If the decomposition function $g : M(I) \to G(I)$ is regular, then*

$$g(e_s(g(e_t u))) = g(e_t(g(e_s u))),$$

for all $u \in M(I)$ and all $s, t \in \mathrm{set}(u)$ such that $s, t \notin \mathrm{supp}(u)$.

Let $\varepsilon_1, \dots, \varepsilon_n$ denote the canonical basis of $\mathbb{N}^n$. From Subsection 1.2.2, one knows that the differential of the Cartan complex $C(e_1, \dots, e_n; E/I)$ is given by the formula:

$$\partial(cx^{(a)}) = \sum_{i=1}^{n} ce_i x_1^{(a_1)} \cdots x_i^{(a_i-1)} \cdots x_n^{(a_n)} = \sum_{j \in \mathrm{supp}(a)} ce_j x^{(a-\varepsilon_j)}, \quad c \in E/I, \quad a \in \mathbb{N}^n.$$

For convenience, we extend the definition introduced in Theorem 2.1.1 setting $f(a; u) = 0$ if $\mathrm{supp}(a) \nsubseteq \mathrm{set}(u)$, for $u \in M(I)$.

Suppose u and v are monomials of E with $\mathrm{supp}(v) \subseteq \mathrm{supp}(u)$. Then, there exists a unique monomial $u' \in E$ such that $u'v = u$. We set

$$u' = \frac{u}{v}.$$

With this convention, the following identities hold:

$$u\frac{w}{v} = \frac{uw}{v} \quad \text{and} \quad \frac{u}{v}\frac{v}{z} = \frac{u}{z}.$$

In the following theorem, for our convenience, we use a modified version of the Cartan complex $C.(v; E)$, where v is a sequence of linear forms of E. Let u be a monomial of E. We substitute the differential ∂ with $(-1)^{\deg(u)}\partial$ and denote the new differential again by ∂. We denote this modified Cartan complex by $(-1)^{\deg(u)}C.(v; E)$. It is clear that this complex provides again a minimal free resolution of $E/(v)$.

The next theorem is a direct analog of [75, Theorem 1.12].

Theorem 2.2.5. *Let I be a monomial ideal of E with linear quotients, and F the graded minimal free resolution of E/I. Suppose that the decomposition function $g : M(I) \to G(I)$ is regular. Then the maps in the resolution F are given by $d(f(0;u)) = (-1)^{\deg(u)}u$, for $u \in G(I)$, and for $a \in \mathbb{N}^n$, $a \neq 0$, $u \in G(I)$ and $\operatorname{supp}(a) \subset \operatorname{set}(u)$,*

$$
\begin{aligned}
d(f(a;u)) &= -\sum_{t \in \operatorname{supp}(a)} (-1)^{\deg(u)} e_t f(a - \varepsilon_t; u) \\
&\quad + \sum_{t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u)} (-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} f(a - \varepsilon_t; g(e_t u)).
\end{aligned}
$$

Proof. Let I have linear quotients with respect to the sequence $u_1, \ldots, u_r$. We proceed by induction on $r = |G(I)|$. For $r = 1$, $I = (u_1) = (u)$ is a principal ideal and the Cartan complex $(-1)^{\deg(u)} C.(e_{k_1}, \ldots, e_{k_\ell}; E)$ with $\operatorname{set}(u) = \operatorname{supp}(u) = \{k_1, \ldots, k_\ell\}$ provides a minimal free resolution of E/I. In this case, the maps are as required.

Now, suppose that $r > 1$ and that our statement holds for $r - 1$. As before, we set $I_{r-1} = (u_1, \ldots, u_{r-1})$ and $L_{r-1} = (u_1, \ldots, u_{r-1}) : (u_r)$, then we get the following exact sequences of E-modules

$$
0 \longrightarrow E/L_{r-1} \longrightarrow E/I_{r-1} \longrightarrow E/I \longrightarrow 0 \ ,
$$

where $E/L_{r-1} \longrightarrow E/I_{r-1}$ is multiplication by u_r. Let $F^{(r-1)}$ be the graded minimal free resolution of E/I_{r-1}, $C^{(r-1)} = (-1)^{\deg(u_r)} C.(e_{k_1}, \ldots, e_{k_\ell}; E)$ the Cartan complex for the sequence $e_{k_1}, \ldots, e_{k_\ell}$ with $\operatorname{set}(u_r) = \{k_1, \ldots, k_\ell\}$.

Let F be the mapping cone of $\psi^{(r-1)} : C^{(r-1)} \to F^{(r-1)}$ and for simplicity set $\psi^{(r-1)} = \psi$ and $u_r = u$. Since $F^{(r-1)}$ is a subcomplex of F, it follows from our inductive hypothesis that the formula for the map given in the statement holds for all $f(a; u_j)$ with $j < r$. Hence, it remains to check it for $f(a; u_r) = f(a; u)$.

By the definition of the mapping cone of ψ, $d(f(a;u)) = -\partial(f(a;u)) + \psi(f(a;u))$. Thus, to prove the asserted formula it is enough to show that we can define ψ as

$$
\psi(f(a;u)) = \sum_{t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u)} (-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} f(a - \varepsilon_t; g(e_t u))
$$

if $a \neq 0$ and $\psi(f(0;u)) = (-1)^{\deg(u)} u$, otherwise. That is, the following diagram is commutative

$$
\begin{array}{ccccccc}
C^{(r-1)} : \cdots & \longrightarrow & C_i^{(r-1)} & \xrightarrow{\partial_i} & C_{i-1}^{(r-1)} & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \psi_i} & & \Big\downarrow{\scriptstyle \psi_{i-1}} & & \\
F^{(r-1)} : \cdots & \longrightarrow & F_i^{(r-1)} & \xrightarrow[d_i]{} & F_{i-1}^{(r-1)} & \longrightarrow & \cdots
\end{array}
$$

Here, with abuse of notation, we also let d denote the chain map of $F^{(r-1)}$.

Let $t \in \operatorname{set}(u)$. Then

$$
(\psi \circ \partial)(f(\varepsilon_t; u)) = \psi((-1)^{\deg(u)} e_t f(0;u)) = (-1)^{\deg(u)} (-1)^{\deg(u)} e_t u = e_t u.
$$

On the other hand, $d(f(0; g(e_t u))) = (-1)^{\deg(g(e_t u))} g(e_t u)$. Hence, if $t \notin \operatorname{supp}(u)$, we have that

$$
\begin{aligned}
(d \circ \psi)(f(\varepsilon_t; u)) &= (-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} d(f(0; g(e_t u))) \\
&= \frac{e_t u}{g(e_t u)} g(e_t u) = e_t u.
\end{aligned}
$$

Otherwise, if $t \in \operatorname{supp}(u)$, then $\psi(f(\varepsilon_t; u)) = 0$. In this case, $(d \circ \psi)(f(\varepsilon_t; u)) = 0$ but also $(\psi \circ \partial)(f(\varepsilon_t; u)) = e_t u = 0$. So the first square in the diagram commutes.

For the other squares, let $f(a; u)$ with $\operatorname{supp}(a) \subseteq \operatorname{set}(u)$ and $|a| > 1$. Then

$$
\begin{aligned}
(\psi \circ \partial)(f(a; u)) &= \sum_{t \in \operatorname{supp}(a)} (-1)^{\deg(u)} e_t \psi(f(a - \varepsilon_t; u)) \\
&= \sum_{\substack{t \in \operatorname{supp}(a) \\ t \notin \operatorname{supp}(u)}} \sum_{\substack{s \in \operatorname{supp}(a) \setminus \{t\} \\ s \notin \operatorname{supp}(u)}} (-1)^{\deg(u) + \deg(g(e_s u))} \frac{e_t e_s u}{g(e_s u)} f(a - \varepsilon_t - \varepsilon_s; g(e_s u)) \\
&= \sum_{\substack{t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u) \\ s \in \operatorname{supp}(a) \setminus (\operatorname{supp}(u) \cup \{t\}) \\ s > t}} (-1)^{\deg(u) + \deg(g(e_s u))} \frac{e_t e_s u}{g(e_s u)} f(a - \varepsilon_t - \varepsilon_s; g(e_s u)) \\
&- \sum_{\substack{t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u) \\ s \in \operatorname{supp}(a) \setminus (\operatorname{supp}(u) \cup \{t\}) \\ s < t}} (-1)^{\deg(u) + \deg(g(e_s u))} \frac{e_s e_t u}{g(e_s u)} f(a - \varepsilon_t - \varepsilon_s; g(e_s u)).
\end{aligned}
$$

Here, to get the second equality, we have noted that if $s = t$ or $t \in \operatorname{supp}(u)$ or $s \in \operatorname{supp}(u)$, the corresponding term in the sum is zero. Exchanging the role of t and s in the second sum, we obtain

$$
\begin{aligned}
(\psi \circ \partial)(f(a; u)) = \sum_{\substack{t, s \in \operatorname{supp}(a) \\ t, s \notin \operatorname{supp}(u) \\ s > t}} (-1)^{\deg(u) + \deg(g(e_s u))} \frac{e_t e_s u}{g(e_s u)} f(a - \varepsilon_t - \varepsilon_s; g(e_s u)) \\
- \sum_{\substack{t, s \in \operatorname{supp}(a) \\ t, s \notin \operatorname{supp}(u) \\ s > t}} (-1)^{\deg(u) + \deg(g(e_t u))} \frac{e_t e_s u}{g(e_t u)} f(a - \varepsilon_t - \varepsilon_s; g(e_t u)).
\end{aligned}
\tag{2.2}
$$

As for the composition counterclockwise, we have

$$
(d \circ \psi)(f(a; u)) = \sum_{t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u)} (-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} d(f(a - \varepsilon_t; g(e_t u))).
$$

Let $t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u)$ and assume for a moment that $\operatorname{supp}(a - \varepsilon_t) \subseteq \operatorname{set}(g(e_t u))$. Then,

$$
\begin{aligned}
d(f(a - \varepsilon_t; g(e_t u))) = - \sum_{s \in \operatorname{supp}(a - \varepsilon_t)} (-1)^{\deg(g(e_t u))} e_s f(a - \varepsilon_t - \varepsilon_s; g(e_t u)) \\
+ \sum_{s \in \operatorname{supp}(a - \varepsilon_t) \setminus \operatorname{supp}(g(e_t u))} (-1)^{\deg(g(e_s g(e_t u)))} \frac{e_s g(e_t u)}{g(e_s g(e_t u))} f(a - \varepsilon_t - \varepsilon_s; g(e_s g(e_t u))).
\end{aligned}
\tag{2.3}
$$

We claim that this expression is also true if $\operatorname{supp}(a - \varepsilon_t) \nsubseteq \operatorname{set}(g(e_t u))$. In this case, the symbol $f(a-\varepsilon_t; g(e_t u))$ is zero. Hence, the right-hand side of (2.3) should be zero, too.

Indeed, pick $s \in \operatorname{supp}(a-\varepsilon_t)$. If $\operatorname{supp}(a-\varepsilon_t-\varepsilon_s) \not\subseteq \operatorname{set}(g(e_t u))$, then by the regularity of g we have $\operatorname{set}(g(e_s g(e_t u))) \subseteq \operatorname{set}(g(e_t u))$. Therefore $\operatorname{supp}(a-\varepsilon_t-\varepsilon_s) \not\subseteq \operatorname{set}(g(e_s g(e_t u)))$ too, making the right hand side of (2.3) zero.

Otherwise, $\operatorname{supp}(a-\varepsilon_t-\varepsilon_s) \subseteq \operatorname{set}(g(e_t u))$. In this case $s \notin \operatorname{set}(g(e_t u))$, otherwise we would have $\operatorname{supp}(a-\varepsilon_t) \subseteq \operatorname{set}(g(e_t u))$, against our assumption. Hence, by Lemma 2.2.1(c) we have that $g(e_s g(e_t u)) = g(e_t u)$. Furthermore, in such a case $s \in \operatorname{supp}(a-\varepsilon_t) \setminus \operatorname{supp}(g(e_t u))$ and so

$$(-1)^{\deg(g(e_s g(e_t u)))} \frac{e_s g(e_t u)}{g(e_s g(e_t u))} f(a-\varepsilon_t-\varepsilon_s; g(e_s g(e_t u))) = (-1)^{\deg(g(e_t u))} e_s f(a-\varepsilon_t-\varepsilon_s; g(e_t u)).$$

Therefore, we see that the right hand side of (2.3) is zero, as all summands are either zero or cancel against each other.

Now, observe that

$$\begin{aligned} -(-1)^{\deg(g(e_t u))}(-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} e_s &= -\frac{e_t u}{g(e_t u)} e_s \\ &= -(-1)^{\deg(e_t u)-\deg(g(e_t u))} e_s \frac{e_t u}{g(e_t u)} \\ &= (-1)^{\deg(u)+\deg(g(e_t u))} \frac{e_s e_t u}{g(e_t u)} \end{aligned}$$

and

$$\begin{aligned} &(-1)^{\deg(g(e_t u))+\deg(g(e_s g(e_t u)))} \frac{e_t u}{g(e_t u)} \frac{e_s g(e_t u)}{g(e_s g(e_t u))} \\ &= (-1)^{\deg(g(e_t u))+\deg(g(e_s g(e_t u)))} (-1)^{\deg(g(e_t u))} \frac{e_t u}{g(e_t u)} \frac{g(e_t u) e_s}{g(e_s g(e_t u))} \\ &= (-1)^{\deg(g(e_s g(e_t u)))} \frac{e_t u e_s}{g(e_s g(e_t u))} = (-1)^{\deg(g(e_s g(e_t u)))+\deg(u)} \frac{e_t e_s u}{g(e_s g(e_t u))}. \end{aligned}$$

Consequently, by (2.3) and the previous calculations, we have

$$\begin{aligned} (d\circ\psi)(f(a;u)) = &\sum_{\substack{t \in \operatorname{supp}(a) \\ s \in \operatorname{supp}(a)\setminus(\operatorname{supp}(u)\cup\{t\})}} (-1)^{\deg(u)+\deg(g(e_t u))} \frac{e_s e_t u}{g(e_t u)} f(a-\varepsilon_t-\varepsilon_s; g(e_t u)) \\ + &\sum_{\substack{t \in \operatorname{supp}(a) \\ s \in \operatorname{supp}(a-\varepsilon_t)\setminus \operatorname{supp}(g(e_t u))}} (-1)^{\deg(u)+\deg(g(e_s g(e_t u)))} \frac{e_t e_s u}{g(e_s g(e_t u))} f(a-\varepsilon_t-\varepsilon_s; g(e_s g(e_t u))). \end{aligned}$$

Note that in the first sum we can write $t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u)$, otherwise the corresponding term in the sum is zero. Likewise, in the second sum, we can write $t \in \operatorname{supp}(a) \setminus \operatorname{supp}(u)$ and moreover $s \in \operatorname{supp}(a) \setminus (\operatorname{supp}(u) \cup \{t\})$. Indeed, if $t \in \operatorname{set}(u) \setminus \operatorname{supp}(u)$, it is clear that $\operatorname{supp}(g(e_t u)) \subset \operatorname{supp}(u) \cup \{t\}$. Therefore,

$$\operatorname{supp}(a) \setminus (\operatorname{supp}(u) \cup \{t\}) \subseteq \operatorname{supp}(a-\varepsilon_t) \setminus \operatorname{supp}(g(e_t u)).$$

If s belongs to the second set and does not belong to the first one, then $s \in \operatorname{supp}(u)$ and the corresponding summand is zero. Hence, we can rewrite the previous equation as

$$(d\circ\psi)(f(a;u)) = \sum_{\substack{t\in \mathrm{supp}(a)\setminus\mathrm{supp}(u)\\ s\in\mathrm{supp}(a)\setminus(\mathrm{supp}(u)\cup\{t\})}} (-1)^{\deg(u)+\deg(g(e_tu))}\frac{e_se_tu}{g(e_tu)}f(a-\varepsilon_t-\varepsilon_s;g(e_tu))$$
$$+\sum_{\substack{t\in \mathrm{supp}(a)\setminus\mathrm{supp}(u)\\ s\in\mathrm{supp}(a)\setminus(\mathrm{supp}(u)\cup\{t\})}} (-1)^{\deg(u)+\deg(g(e_sg(e_tu)))}\frac{e_te_su}{g(e_sg(e_tu))}f(a-\varepsilon_t-\varepsilon_s;g(e_sg(e_tu))).$$

Consequently, we get

$$\begin{aligned}(d\circ\psi)(f(a;u)) = &\sum_{\substack{t,s\in \mathrm{supp}(a)\\ t,s\notin\mathrm{supp}(u)\\ s<t}} (-1)^{\deg(u)+\deg(g(e_tu))}\frac{e_se_tu}{g(e_tu)}f(a-\varepsilon_t-\varepsilon_s;g(e_tu))\\
&-\sum_{\substack{t,s\in \mathrm{supp}(a)\\ t,s\notin\mathrm{supp}(u)\\ s>t}} (-1)^{\deg(u)+\deg(g(e_tu))}\frac{e_te_su}{g(e_tu)}f(a-\varepsilon_t-\varepsilon_s;g(e_tu))\\
&+\sum_{\substack{t,s\in \mathrm{supp}(a)\\ t,s\notin\mathrm{supp}(u)\\ s>t}} (-1)^{\deg(u)+\deg(g(e_sg(e_tu)))}\frac{e_te_su}{g(e_sg(e_tu))}f(a-\varepsilon_t-\varepsilon_s;g(e_sg(e_tu)))\\
&-\sum_{\substack{t,s\in \mathrm{supp}(a)\\ t,s\notin\mathrm{supp}(u)\\ s<t}} (-1)^{\deg(u)+\deg(g(e_sg(e_tu)))}\frac{e_se_tu}{g(e_sg(e_tu))}f(a-\varepsilon_t-\varepsilon_s;g(e_sg(e_tu))).\end{aligned}$$

By Lemma 2.2.4, the last two sums cancel out by exchanging the roles of s and t in one of them. Moreover, we get an expression identical to (2.2) by exchanging the roles of s and t in the first sum. Thus, we see that $(d\circ\psi)(f(a;u)) = (\psi\circ\partial)(f(a;u))$. The conclusion follows. □

2.3 An application: vector–spread monomial ideals

In this section we introduce the notion of strongly stable vector–spread ideal in the exterior algebra $E = K\langle e_1,\dots,e_n\rangle$. This notion has been recently introduced in [55] in the polynomial ring as a generalization of the concept of t-spread strongly stable ideal given by Ene, Herzog and Qureshi in [43]. We will address the polynomial ring case more in depth in the next two chapters.

Definition 2.3.1. Given $n\ge 1, \mathbf{t} = (t_1,\dots,t_{d-1})\in\mathbb{N}^{d-1}, d\ge 2$ and $e_\mu = e_{i_1}\cdots e_{i_\ell}\neq 1$ a monomial of E, with $1\le i_1<i_2<\cdots<i_\ell\le n$, $\ell\le d$ we say that e_μ is $\mathbf{t}$-*spread* if $i_{j+1}-i_j\ge t_j$, for all $j=1,\dots,\ell-1$. We say that a monomial ideal I of E is $\mathbf{t}$-*spread* if it is generated by $\mathbf{t}$-spread monomials.

For instance, the monomial ideal $I = (e_1e_3, e_2e_6, e_2e_4e_8)$ of the exterior algebra $E = K\langle e_1,\dots,e_8\rangle$ is a $(2,2)$-spread monomial ideal, but not a $(3,2)$-spread monomial ideal.

Note that any monomial ideal of E is $\mathbf{1}$-spread, where $\mathbf{1} = (1,\dots,1)\in\mathbb{N}^n$. The set of all monomials of E will be denoted by $\mathrm{Mon}(E)$, whereas the set of all $\mathbf{t}$-spread monomials of E will be denote by $\mathrm{Mon}_{\mathbf{t}}(E)$. For $\mathbf{t}=\mathbf{1}$, $\mathrm{Mon}_{\mathbf{1}}(E) = \mathrm{Mon}(E)$. One can quickly observe that if ℓ is a positive integer, then there does exist a $\mathbf{t}$-spread monomial of degree ℓ in E if and only if $n\ge 1+\sum_{j=1}^{\ell-1}t_j$.

Definition 2.3.2. A $\mathbf{t}$-spread monomial ideal I is called $\mathbf{t}$-*spread strongly stable* if for all $\mathbf{t}$-spread monomial $e_\mu \in I$, all $j \in \mathrm{supp}(e_\mu)$ and all $i < j$, such that $e_i e_{\mu\setminus\{j\}}$ is $\mathbf{t}$-spread, it follows that $e_i e_{\mu\setminus\{j\}} \in I$.

For instance, the ideal $I = (e_1e_3, e_1e_4, e_2e_4e_6)$ is a $(2,2)$-spread strongly stable monomial ideal of $E = K\langle e_1, \dots, e_6\rangle$.

Note that a $\mathbf{1}$-spread strongly stable ideal is the ordinary strongly stable ideal introduced in [10].

The defining property of a $\mathbf{t}$-spread strongly stable ideal needs to be checked only for the set of $\mathbf{t}$-spread monomial generators.

Proposition 2.3.3. *Let I be a $\mathbf{t}$-spread ideal and suppose that for all $e_\mu \in G(I)$, and for all integers $1 \le i < j \le n$ such that $j \in \mu$ and $e_i e_{\mu\setminus\{j\}}$ is $\mathbf{t}$-spread, one has $e_i e_{\mu\setminus\{j\}} \in I$. Then I is $\mathbf{t}$-spread strongly stable.*

Proof. Let $e_\nu \in I$ be a $\mathbf{t}$-spread monomial and $1 \le i < j \le n$ integers such that $j \in \nu$ and $e_i e_{\nu\setminus\{j\}}$ is $\mathbf{t}$-spread. There exists $e_\mu \in G(I)$ and a monomial $e_\rho \in E$ such that $e_\nu = e_\mu e_\rho$ in E. We distinguish two cases: $j \in \mu$, $j \in \rho$.

If $j \in \mu$, then $e_i e_{\mu\setminus\{j\}} \in I$ by assumption, and so $e_i e_{\nu\setminus\{j\}} = e_i e_{\mu\setminus\{j\}} e_\rho \in I$.

If $j \in \rho$, then $e_i e_{\nu\setminus\{j\}} = e_i e_\mu e_{\rho\setminus\{j\}} \in I$. □

In what follows, let $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{N}^{d-1}$, $d \ge 2$.

The proof of the next statement is verbatim the same as the corresponding result in the polynomial ring, see [28, Theorem 2.2] and [55, Proposition 5.1]. Thus we omit it.

Theorem 2.3.4. *Let I be a $\mathbf{t}$-spread strongly stable ideal. Then I has linear quotients with $G(I)$ ordered with respect to the lexicographic order. In particular, I is componentwise linear.*

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring on n variables. By abuse of notation, and in order to simplify the notation, if u is a monomial in E, we denote again by u the corresponding squarefree monomial in S. For instance, if $u = e_1e_2e_3 \in \mathrm{Mon}(E)$ then we denote by u the squarefree monomial $x_1x_2x_3$ of S, too. Moreover, if I is a monomial ideal in E, we will denote again by I the corresponding squarefree ideal in S.

Let I be a $\mathbf{t}$-spread strongly stable ideal of E with $G(I) = \{u_1, \dots, u_r\}$ ordered with respect to the pure lexicographic order [68]. Using the notation discussed above, I can be considered as a $\mathbf{t}$-spread strongly stable ideal of S with $G(I) = \{u_1, \dots, u_r\}$ ordered with respect to the pure lexicographic order. Hence, we set

$$\mathrm{set}_S(u_j) = \{i : x_i \in (u_1, \dots, u_{j-1}) :_S (u_j)\}, \quad \mathrm{set}_E(u_j) = \{i : e_i \in (u_1, \dots, u_{j-1}) :_E (u_j)\},$$

for $j = 1, \dots, r$. It is clear that

$$\mathrm{set}_E(u_j) = \mathrm{set}_S(u_j) \cup \mathrm{supp}(u_j), \quad j = 1, \dots, r.$$

Now, by [28, Corollary 2.3], if $u_k = e_{j_1} \cdots e_{j_\ell}$, then

$$\mathrm{set}_S(u_k) = [\mathfrak{m}(u_k) - 1] \setminus \bigcup_{h=1}^{\ell-1} \big[j_h, j_h + (t_h - 1)\big],$$

where $[a, b] = \{c \in \mathbb{N} : a \le c \le b\}$ and $a, b \in \mathbb{N}$.

Therefore,

$$\mathrm{set}_E(u_k) = \mathrm{set}_S(u_k) \cup \mathrm{supp}(u_k) = [\mathfrak{m}(u_k)] \setminus \bigcup_{h=1}^{\ell-1} \big[j_h + 1, j_h + (t_h - 1)\big] \tag{2.4}$$

and consequentially $|\,\mathrm{set}_E(u_k)| = \mathfrak{m}(u_k) - \sum_{h=1}^{\ell-1}(t_h - 1)$.

Theorem 2.3.4 yields the following corollary which allow us to state a formula for computing the graded Betti numbers of a $\mathbf{t}$-spread strongly stable ideal.

Corollary 2.3.5. *Let I be a $\mathbf{t}$-spread strongly stable ideal. Then*

$$\beta_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{\mathfrak{m}(u) - \sum_{h=1}^{j-1}(t_h - 1) + i - 1}{i}.$$

Proof. From Theorem 2.3.4, I has linear quotients. Hence, the assertion follows from Corollary 2.1.3 and (2.4). ☐

Remark 2.3.6. For $\mathbf{t} = \mathbf{1}$, the formula in Corollary 2.3.5 becomes the well–known formula (1.1).

Notes

Recently in [52] Ferraro and Utkina have extended the Herzog–Takayama construction to the setting of skew polynomial rings, *i.e.* rings $R = K[x_1, \dots, x_n]$ where the multiplication is defined by the relations $x_i x_j = q_{ij} x_j x_i$, with $q_{ij} \in K^*$ such that $q_{ij}^{-1} = q_{ji}$.

The authors, for a monomial ideal with linear quotients $I = (u_1, \dots, u_m)$ of R, construct a new complex [52, Construction 3.6], called the *skew Herzog–Takayama complex* $\mathbb{HT}_\bullet(u_1, \dots, u_m)$, arising as iterated mapping cone, following a similar argument to the one we followed in Section 2.1. In this construction, the role played by the Koszul complex in the polynomial ring case [75] and by the Cartan complex in the exterior algebra (Section 2.1) is played by the *skew Taylor resolution*. The authors also point out that, while in the commutative case, the Taylor resolution coincide with the Koszul complex, in the skew–commutative framework they do not coincide in general, but they are isomorphic [52, Proposition 3.3]. They are able to prove that this construction yields a minimal free resolution of R/I. More in depth, they prove the following.

Theorem. (Ferraro–Utkina, 2024 [52, Theorem 3.6]) *Let $I = (u_1, \dots, u_m) \subset R$ be an ideal with linear quotients (for the given order on the generators) and regular decomposition function. Then, the complex $\mathbb{HT}_\bullet(u_1, \dots, u_m)$ is a minimal free resolution of R/I.*

For such a resolution they also provide an explicit description for the modules and the differentials.

The above result allows also the author to provide similar results to corollaries 2.1.3 and 2.1.5, to express explicitly the bigraded Poincaré series ([52, Corollary 3.15]) and Betti numbers ([52, Corollary 3.16]).

Chapter 3

Macaulay's theorem for vector–spread algebras

One of the main well-studied and important numerical invariant of a graded ideal in a standard graded polynomial ring is its Hilbert function which gives the sizes of the graded components of the ideal. There is an extensive literature on this topic (see, for instance, [68] and the references therein). Usually, Hilbert functions are described using the well–known Macaulay's expansion with binomials. This fact often implies the use of combinatorial tools and furthermore the arguments consist of very clever computations with binomials. The crucial idea of Macaulay is that there exist special monomial ideals, the so called *lex ideals*, that attain all possible Hilbert functions. The pivotal property is that a lex ideal grows as slowly as possible. The "squarefree" analogue of Macaulay's theorem is known as the Kruskal–Katona theorem. Indeed, if Macaulay's theorem describes the possible Hilbert functions of the graded ideals in polynomial rings, the possible f-vectors of a simplicial complex are characterized in the theorem of Kruskal–Katona [68, 79, 82]. In fact, the Hilbert function of the Stanley-Reisner ring of a simplicial complex Δ is determined by the f-vector of Δ, and vice versa. Kruskal–Katona's theorem is a fundamental result in topological combinatorics and discrete geometry which quickly have aroused much interest in face enumeration questions for various classes of simplicial complexes, polytopes, and manifolds. Furthermore, it may be also interpreted as a theorem on Hilbert functions of quotients of exterior algebras [10]. The lex ideals as well as squarefree lex ideals play a key role in the study of the minimal free resolutions of monomial ideals. Indeed, if one considers the stable and squarefree stable ideals and the formulas for computing their graded Betti numbers [68] one can deduce the Bigatti-Hulett theorem [15, 76] which says that lex ideals have the largest graded Betti numbers among all graded ideals with the same Hilbert function (see also [77, 96]).

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring, with K a field, and let $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$, $d \geq 2$, be a $(d-1)$-tuple whose entries are non negative integers. The class of $\mathbf{t}$-*spread strongly stable ideals* has been introduced by Ficarra in [55]. It is a special class of monomial ideals which generalizes the class of t-spread strongly stable ideals in [43] (see also, [6, 36] and the references therein), t non negative integer. More in detail, a monomial $u = x_{j_1}x_{j_2}\cdots x_{j_\ell}$ $(1 \leq j_1 \leq j_2 \leq \cdots \leq j_\ell \leq n)$ of degree $\ell \leq d$ of S is called a *vector–spread monomial of type* $\mathbf{t}$ or simply a $\mathbf{t}$-*spread monomial* if $j_{i+1} - j_i \geq t_i$, for $i = 1, \dots, \ell - 1$ and a $\mathbf{t}$-*spread monomial ideal* is a monomial ideal of S generated by $\mathbf{t}$-spread monomials. A $\mathbf{t}$-*spread strongly stable ideal* is a $\mathbf{t}$-*spread monomial ideal* with an additional combinatorial property (Definition 3.1.5). For $\mathbf{t} = (0, \dots, 0), (1, \dots, 1)$, one obtains the classical notions of strongly stable ideal and squarefree strongly stable ideal, respectively [68]. In this chapter we present a generalization of Macaulay's theorem for the class of $\mathbf{t}$-spread strongly stable ideals.

The crucial role is played by the class of **t**-*spread lex ideal* (Definition 3.1.5). Since, $\mathbf{t} \in \mathbb{Z}_{\geq 0}^{d-1}$, *i.e.* the entries of **t** can be also zero, in order to unify the theory about the classification of the Hilbert functions of graded ideals of S, we put our attention on the classification of the possible $f_{\mathbf{t}}$-vector (Definition 3.1.2) of **t**-spread strongly stable ideals. More in detail, we answer to the following question:

Question. *Under which conditions a given sequence of positive integers*

$$f = (f_{-1}, f_0, \dots, f_{d-1}),$$

is the $f_{\mathbf{t}}$*-vector of a* **t***-spread strongly stable ideal?*

The chapter is structured as it follows. We introduce the notion of **t**-*spread shadow* of a set of monomials of S and the notions of **t**-*spread strongly stable set (ideal)* and **t**-*spread lex set (ideal)* (Definitions 3.1.4, 3.1.5). The combinatorics of such sets is deeply analyzed in Section 3.2. The key result in the section is Theorem 3.2.4 which allows us to prove that to every **t**-spread strongly stable ideal I one can associate a unique **t**-spread lex ideal which has the same $f_{\mathbf{t}}$-vector as I (Corollary 3.2.6). Moreover, we point out why this is not possible for an arbitrary **t**-spread monomial ideal. Section 3.3 contains the main result in the chapter which gives the classification of all suitable $f_{\mathbf{t}}$-vectors of a **t**-spread strongly stable ideal (Theorem 3.3.8). The classification is obtained by introducing a new operator (Definition 3.3.2) which, for suitable values of **t**, is analog either to the operator $a \longrightarrow a^{\langle d\rangle}$ or to the operator $a \longrightarrow a^{(d)}$ which are involved in Macaulay's theorem and in Kruskal–Katona 's theorem, respectively [68]. Finally, in Section 3.4, as an application of the results in the previous sections, we state an upper bound for the graded Betti numbers of the class of all **t**-spread strongly stable ideals with a given $f_{\mathbf{t}}$-vector (Theorem 3.4.1). We prove that the **t**-spread lex ideals give the maximal Betti numbers among all **t**-spread strongly stable ideals with a given $f_{\mathbf{t}}$-vector. Such a statement generalizes the well–known result proved independently by Bigatti [15] and Hulett [76] for graded ideals in a polynomial ring over a field of characteristic zero and afterwards generalized by Pardue [96] to any characteristic. The chapter contains some examples illustrating the main results developed using *Macaulay2* [62].

The results contained in this chapter first appeared in [31].

3.1 Vector-spread monomial ideals

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring, with K a field, and let $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$, $d \geq 2$, be a $(d-1)$-tuple whose entries are non negative integers. A monomial $u = x_{j_1}x_{j_2}\cdots x_{j_\ell}$ $(1 \leq j_1 \leq j_2 \leq \cdots \leq j_\ell \leq n)$ of degree $\ell \leq d$ of S is called a *vector–spread monomial of type* **t** or simply a **t**-*spread monomial* if $j_{i+1} - j_i \geq t_i$, for $i = 1, \dots, \ell - 1$.

If I is a graded ideal of S we denote by I_j the j-graded component of I and by $\text{indeg}(I)$ the initial degree of I, *i.e.* the smallest j such that $I_j \neq 0$. Moreover, for a monomial ideal $I \subset S$, we denote by $G(I)$ the unique minimal set of monomial generators. Furthermore, if $j \geq 0$ is an integer, we set $G(I)_j = \{u \in G(I) : \deg(u) = j\}$.

A **t**-*spread monomial ideal* is a monomial ideal of S generated by **t**-spread monomials. For instance, $I = (x_1x_4^2x_5, x_1x_4^2x_6, x_1x_5x_7)$ is a $(3,0,1)$-spread monomial ideal of the polynomial ring $S = K[x_1, \dots, x_7]$, but it is not $(3,0,2)$-spread as $x_1x_4^2x_5 \in G(I)$ is not a $(3,0,2)$-spread monomial.

Note that any monomial (ideal) is **0**-spread, where $\mathbf{0} = (0,0,\dots,0)$. If $t_i \geq 1$, for all i, a **t**-spread monomial (ideal) is a *squarefree* monomial (ideal).

We denote by $M_{n,\ell,\mathbf{t}}$ the set of all $\mathbf{t}$-spread monomials of degree ℓ in S. If $\ell \le d$, by [55, Corollary 2.4],

$$|M_{n,\ell,\mathbf{t}}| = \binom{n+(\ell-1)-\sum_{j=1}^{\ell-1} t_j}{\ell}. \tag{3.1}$$

Definition 3.1.1. For a set L of monomials of S, one defines the *vector–spread shadow* or simply the $\mathbf{t}$-*spread shadow* of L to be the set

$$\mathrm{Shad}_{\mathbf{t}}(L) \;=\; \big\{wx_j \;:\; w\in L, wx_j \text{ is } \mathbf{t}\text{-spread}, j=1,\dots,n\big\}.$$

Note that $\mathrm{Shad}_{\mathbf{t}}(M_{n,\ell,\mathbf{t}}) = M_{n,\ell+1,\mathbf{t}}$ for all $\ell \ge 0$ and that $\mathrm{Shad}_{\mathbf{t}}(L) = \emptyset$ whenever all monomials in L have degrees $\ge d$. Moreover, one can quickly observe that if L is a set of monomials of S, then the definition of $\mathrm{Shad}_{\mathbf{0}}(L)$ coincides with the classical notion of shadow of L [68, Chapter 6].

Definition 3.1.2. If I is a monomial ideal of S, we denote by $[I_j]_{\mathbf{t}}$ the set of all $\mathbf{t}$-spread monomials in I_j. Furthermore, we set

$$f_{\mathbf{t},\ell-1}(I) = |M_{n,\ell,\mathbf{t}}| - |[I_\ell]_{\mathbf{t}}|, \qquad 0 \le \ell \le d.$$

and define the vector

$$f_{\mathbf{t}}(I) = (f_{\mathbf{t},-1}(I), f_{\mathbf{t},0}(I), \dots, f_{\mathbf{t},d-1}(I)).$$

Such a vector is called the $f_{\mathbf{t}}$-*vector* of I. Note that $f_{\mathbf{t},-1}(I) = 1$.

Remark 3.1.3. For $\mathbf{t} = (t_1,\dots,t_{d-1})$ with $t_i \ge 1$, for all i, then I is the Stanley-Reisner ideal I_Δ of a uniquely determined simplicial complex Δ on vertex set $\{1,\dots,n\}$ with $f_{\mathbf{1}}(I)$ as f-vector, where $\mathbf{1} = (1,\dots,1)$.

Definition 3.1.4. Let $L \subseteq M_{n,\ell,\mathbf{t}}$. L is called a $\mathbf{t}$-*spread strongly stable set* if for all $u \in L$, $j < i$ such that x_i divides u and $x_j(u/x_i)$ is $\mathbf{t}$-spread, then $x_j(u/x_i) \in L$. L is called a $\mathbf{t}$-*spread lex set*, if for all $u \in L$, $v \in M_{n,\ell,\mathbf{t}}$ such that $v \ge_{\mathrm{lex}} u$, then $v \in L$.

Here $\ge_{\mathrm{lex}}$ stands for the lex order induced by $x_1 > \cdots > x_n$ [68].

For our convenience, throughout the chapter, we assume the empty set to be both a $\mathbf{t}$-spread strongly stable set and a lex set.

Definition 3.1.5. Let I be a $\mathbf{t}$-*spread ideal*. I is said to be a $\mathbf{t}$-*spread strongly stable ideal* if $[I_\ell]_{\mathbf{t}}$ is a $\mathbf{t}$-*spread strongly stable set*, for all ℓ.

I is said to be a $\mathbf{t}$-*spread lex ideal* if $[I_\ell]_{\mathbf{t}}$ is a $\mathbf{t}$-*spread lex set*, for all ℓ.

One can observe that any $\mathbf{t}$-spread lex set (ideal) is a $\mathbf{t}$-spread strongly stable set (ideal). Moreover, for $\mathbf{t} = \mathbf{0}$ ($\mathbf{t} = \mathbf{1} = (1,\dots,1)$) one obtains the classical notions of (squarefree) strongly stable ideal and (squarefree) lex ideal [68].

3.2 Vector-spread shadow and Bayer's theorem

In this section, if $\mathbf{t} = (t_1,\dots,t_{d-1}) \in \mathbb{Z}_{\ge 0}^{d-1}$, $d \ge 2$, we deal with the combinatorics of the $\mathbf{t}$-spread shadows of $\mathbf{t}$-spread strongly stable sets and $\mathbf{t}$-spread lex sets. As a consequence, given a $\mathbf{t}$-spread strongly stable ideal I of the polynomial ring S, we prove the existence of a unique $\mathbf{t}$-spread lex ideal of S with the same $f_{\mathbf{t}}$-vector of I.

For i, j integers, we set $[i, j] = \{k \in \mathbb{Z} : i \leq k \leq j\}$. Note that $[i, j] \neq \emptyset$ if and only if $i \leq j$.

Let u be a monomial. We set $\max(u) = \max\{i \ : \ x_i \text{ divides } u\}$.

Lemma 3.2.1. *Let $L \subseteq M_{n,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread strongly stable set. Then*

$$\mathrm{Shad}_{\mathbf{t}}(L) = \big\{wx_j \ : \ w \in L, j \geq \max(w), wx_j \text{ is } \mathbf{t}\text{-spread}\big\}.$$

Proof. Let $u \in \mathrm{Shad}_{\mathbf{t}}(L)$. Then, $u = wx_j$ for some $w \in L$. If $\max(w) \leq j$ there is nothing to prove. Suppose $j < \max(w)$. We can write $u = x_{\max(u)}w'$, with $w' = x_j(u/x_{\max(u)})$ and $\max(w') \leq \max(u)$. The proof is complete if we show that $w' \in L$. Let $u = x_{j_1} \cdots x_{j_{\ell+1}}$ with $j_1 \leq \cdots \leq j_{\ell+1}$. Then, $j = j_p$ for some $p < \ell + 1$ and $w' = x_{j_1} \cdots x_{j_\ell} = x_{j_p}(u/x_{\max(u)})$ is a $\mathbf{t}$-spread monomial because w is $\mathbf{t}$-spread. Moreover, $w' \in L$ since $j_p < \max(u) = j_{\ell+1}$ and L is a $\mathbf{t}$-spread strongly stable set. □

Proposition 3.2.2. *Let $L \subseteq M_{n,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread set.*

(a) *If L is a $\mathbf{t}$-spread strongly stable set, then $\mathrm{Shad}_{\mathbf{t}}(L) \subseteq M_{n,\ell+1,\mathbf{t}}$ is a $\mathbf{t}$-spread strongly stable set.*

(b) *If L is a $\mathbf{t}$-spread lex set, then $\mathrm{Shad}_{\mathbf{t}}(L) \subseteq M_{n,\ell+1,\mathbf{t}}$ is a $\mathbf{t}$-spread lex set.*

Proof. Let $u = wx_j \in \mathrm{Shad}_{\mathbf{t}}(L)$. For the proofs of both (a) and (b), by Lemma 3.2.1, we can assume that $\max(w) \leq j$.

(a) Let $i < k$ such that x_k divides u and $u' = (u/x_k)x_i$ is $\mathbf{t}$-spread. We prove that $u' \in \mathrm{Shad}_{\mathbf{t}}(L)$. If $k = j$, then $u' = wx_i \in \mathrm{Shad}_{\mathbf{t}}(L)$ by definition. Suppose $k \neq j$. Since $j = \max(u)$, then $i < k < j$ and consequently x_k divides w. Therefore, $u' = x_i(w/x_k)x_j \in \mathrm{Shad}_{\mathbf{t}}(L)$ because $x_i(w/x_k) \in L$ as $x_i(w/x_k)$ is a $\mathbf{t}$-spread monomial, $i < k$ and L is a $\mathbf{t}$-spread strongly stable set.

(b) Let $v \in M_{n,\ell+1,\mathbf{t}}$ with $v >_{\mathrm{lex}} u$. We prove that $v \in \mathrm{Shad}_{\mathbf{t}}(L)$. By definition of the lex order, it follows that $v/x_{\max(v)} \geq_{\mathrm{lex}} u/x_{\max(u)}$. The hypothesis on L guarantees that $v/x_{\max(v)} \in L$. Hence, $v = (v/x_{\max(v)})x_{\max(v)} \in \mathrm{Shad}_{\mathbf{t}}(L)$. □

Let $L \subseteq M_{n,\ell,\mathbf{t}}$ be a set of monomials, where $\ell \leq d$. For every $i \in [1, n]$ we denote by $m_i(L)$ the number of monomials $u \in L$ such that $\max(u) = i$ and then we set $m_{\leq j}(L) = \sum_{i=1}^{j} m_i(L)$. Note that $m_i(L) = 0$ if $i \leq \sum_{j=1}^{\ell-1} t_j$.

Lemma 3.2.3. *Let $L \subseteq M_{n,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread strongly stable set with $\ell < d$. Then*

(a) *$m_i(\mathrm{Shad}_{\mathbf{t}}(L)) = m_{\leq i-t_\ell}(L)$ for all i;*

(b) *$|\mathrm{Shad}_{\mathbf{t}}(L)| = \sum_{k=1+\sum_{j=1}^{\ell-1} t_j}^{n-t_\ell} m_{\leq k}(L)$.*

Proof. (a) If $i \leq \sum_{j=1}^{\ell} t_j$ the proof is trivial. Let $i \geq 1 + \sum_{j=1}^{\ell} t_j$. Consider the map

$$\varphi : \{u \in \mathrm{Shad}_{\mathbf{t}}(L) \ : \ \max(u) = i\} \to \{w \in L \ : \ \max(w) \leq i - t_\ell\},$$

defined as follows. Let $u \in \mathrm{Shad}_{\mathbf{t}}(L)$ with $\max(u) = i$. By Lemma 3.2.1, $u = wx_i$ where $w \in L$ is the unique monomial such that $\max(u) = i$. Thus, we set $\varphi(u) = w$. The map φ is well defined by the uniqueness of w. To prove (a), it is enough to show that φ is a bijection. φ is clearly injective. To prove that φ is surjective, let $w \in L$ with $\max(w) \leq i - t_\ell$. Then, $u = wx_i$ is $\mathbf{t}$-spread because $\max(w) \leq i - t_\ell$. Since $\max(u) = i$, then u belongs to the domain of φ and $\varphi(u) = w$, as desired.

(b) Since

$$\operatorname{Shad}_{\mathbf{t}}(L)=\bigcup_{i=1+\sum_{j=1}^{\ell}t_j}^{n}\big\{u\in\operatorname{Shad}_{\mathbf{t}}(L)\ :\ \max(u)=i\big\},$$

where the union is disjoint, by (a), $|\{u\in\operatorname{Shad}_{\mathbf{t}}(L):\max(u)=i\}|=m_i(\operatorname{Shad}_{\mathbf{t}}(L))=m_{\le i-t_\ell}(L)$, and so

$$\begin{aligned}|\operatorname{Shad}_{\mathbf{t}}(L)|&=\sum_{i=1+\sum_{j=1}^{\ell}t_j}^{n}m_i(\operatorname{Shad}_{\mathbf{t}}(L))\\&=\sum_{i=1+\sum_{j=1}^{\ell}t_j}^{n}m_{\le i-t_\ell}(L)\\&=\sum_{k=1+\sum_{j=1}^{\ell-1}t_j}^{n-t_\ell}m_{\le k}(L).\end{aligned}$$

□

The following result is a vector–spread analogue of a well–known theorem due to Bayer, see [68, Theorem 6.3.3].

Theorem 3.2.4. *Let $L\subset M_{n,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread lex set and let $N\subset M_{n,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread strongly stable set. Suppose $|L|\le|N|$. Then, $m_{\le i}(L)\le m_{\le i}(N)$.*

Proof. We first observe that $N=N_0\cup N_1x_n$, where N_0 and N_1 are the unique $\mathbf{t}$-spread strongly stable sets such that

$$N_0=\{u\in N\ :\ \max(u)<n\},\qquad N_1=\{u/x_n\ :\ u\in N,\ \max(u)=n\}.$$

Similarly, we can write $L=L_0\cup L_1x_n$, where L_0 and L_1 are $\mathbf{t}$-spread lex sets defined as above.

We proceed by induction on $n\ge 1$, with the base case being trivial. Let $n>1$. Firstly, observe that $m_{\le n}(L)=|L|$ and $|N|=m_{\le n}(N)$. Hence, the assertion holds for $i=n$. Note that $m_{\le n-1}(L)=|L_0|$ and $m_{\le n-1}(N)=|N_0|$. Thus, to say that $m_{\le n-1}(L)\le m_{\le n-1}(N)$ is equivalent to prove that

$$|L_0|\le|N_0|.\tag{3.2}$$

Assume for a moment that inequality (3.2) holds. Then, applying our inductive hypothesis to the sets $L_0,N_0\subset M_{n-1,\ell,\mathbf{t}}$ we obtain

$$m_{\le i}(L)=m_{\le i}(L_0)\le m_{\le i}(N_0)=m_{\le i}(N)\quad\text{for}\quad i=1,\dots,n-1,$$

as desired. Thus, it remains to prove the inequality (3.2).

Let $N_0^*\subset M_{n-1,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread lex set with $|N_0^*|=|N_0|$ and $N_1^*\subset M_{n-t_{\ell-1},\ell-1,\mathbf{t}}$ be a $\mathbf{t}$-spread lex set with $|N_1^*|=|N_1|$. Let $N^*=N_0^*\cup N_1^*x_n$. We claim that N^* is a $\mathbf{t}$-spread strongly stable set. Let $u\in N^*$. We shall prove that for every $j<i$ such that x_i divides u and $x_j(u/x_i)$ is $\mathbf{t}$-spread, then $x_j(u/x_i)\in N^*$. If $u\in N_0^*$ there is nothing to prove since N_0^* is a $\mathbf{t}$-spread lex set. Suppose $u\in N_1^*x_n$, then we can write $u=wx_n$, where $w\in N_1^*$. If $i<n$, then $w'=x_j(w/x_i)$ belongs to N_1^* and

$x_j(u/x_i) = w'x_n \in N_1^*x_n$. If $i = n$, then $x_j(u/x_i) = wx_j$. Now, if x_n divides wx_j, then again $x_jw \in N_1^*x_n$. Otherwise, if x_n does not divide wx_j, then $wx_j \in N^*$ if and only if $wx_j \in N_0^*$. Thus, we must show that $\mathrm{Shad}_{\mathbf{t}}(N_1^*) \subset N_0^*$. For this aim, it is sufficient to prove that $|\mathrm{Shad}_{\mathbf{t}}(N_1^*)| \le |N_0^*|$, as both sets are $\mathbf{t}$-spread lex sets (Proposition 3.2.2(b)). By Lemma 3.2.3 and the induction hypothesis we obtain

$$\begin{aligned}|\mathrm{Shad}_{\mathbf{t}}(N_1^*)| &= \sum_{i=1+\sum_{j=1}^{\ell-2}t_j}^{n-t_{\ell-1}} m_{\le i}(N_1^*) \le \sum_{i=1+\sum_{j=1}^{\ell-2}t_j}^{n-t_{\ell-1}} m_{\le i}(N_1)\\ &= |\mathrm{Shad}_{\mathbf{t}}(N_1)| \le |N_0| = |N_0^*|.\end{aligned}$$

Finally, N^* is a $\mathbf{t}$-spread strongly stable set.

Since $|N| = |N^*|$, we may replace N by N^* and assume that N_0 is a $\mathbf{t}$-spread lex set. We suppose $n > 1 + \sum_{j=1}^{\ell-1} t_j$, otherwise $M_{n,\ell,\mathbf{t}} = \{x_1x_{1+t_1}\cdots x_{1+\sum_{j=1}^{\ell-1}t_j}\}$ and the assertion is trivial.

Let $m = x_{j_1}\cdots x_{j_\ell}$ be a $\mathbf{t}$-spread monomial and $\alpha : M_{n,\ell,\mathbf{t}} \to M_{n,\ell,\mathbf{t}}$ be the map defined as follows:

(a) if $j_\ell \ne n$, then $\alpha(m) = m$;

(b) if $j_\ell = n$ and $m \ne \min_{>_{\mathrm{lex}}} M_{n,\ell,\mathbf{t}} = x_{n-\sum_{j=1}^{\ell-1}t_j}x_{n-\sum_{j=2}^{\ell-1}t_j}\cdots x_{n-t_{\ell-1}}x_n$, then there exists $r \in [2,\ell]$ such that $j_r > j_{r-1} + t_{r-1}$. Hence, if r is the largest integer with this property, we define

$$\alpha(m) = x_{j_1}\cdots x_{j_{r-1}}x_{j_r-1}\cdots x_{j_{\ell-1}-1}x_{n-1};$$

(c) if $j_\ell = n$ and $m = \min_{>_{\mathrm{lex}}} M_{n,\ell,\mathbf{t}} = x_{n-\sum_{j=1}^{\ell-1}t_j}x_{n-\sum_{j=2}^{\ell-1}t_j}\cdots x_{n-t_{\ell-1}}x_n$, then

$$\alpha(m) = x_{n-1-\sum_{j=1}^{\ell-1}t_j}x_{n-1-\sum_{j=2}^{\ell-1}t_j}\cdots x_{n-1-t_{\ell-1}}x_{n-1}.$$

Such map α is well defined and is easily seen to be a lexicographic order preserving map, *i.e.* if $m_1, m_2 \in M_{n,\ell,\mathbf{t}}$ and $m_1 <_{\mathrm{lex}} m_2$, then $\alpha(m_1) <_{\mathrm{lex}} \alpha(m_2)$, too.

To prove (3.2), since both L_0 and N_0 are $\mathbf{t}$-spread lex sets, it is enough to show that $\min_{>_{\mathrm{lex}}} L_0 \ge_{\mathrm{lex}} \min_{>_{\mathrm{lex}}} N_0$.

Let $u = \min_{>_{\mathrm{lex}}} L = x_{i_1}\cdots x_{i_\ell}$ and $v = \min_{>_{\mathrm{lex}}} N = x_{j_1}\cdots x_{j_\ell}$. We claim that $\alpha(u) = \min_{>_{\mathrm{lex}}} L_0$ and $\alpha(v) = \min_{>_{\mathrm{lex}}} N_0$. Indeed,

(a) if $v \in N_0$, then $\alpha(v) = v \in N_0$

(b) if $v \in N_1x_n$ and $\alpha(v) = x_{j_1}\cdots x_{j_{r-1}}x_{j_r-1}\cdots x_{j_{\ell-1}-1}x_{n-1}$, where $r \in [2,\ell]$ is the largest integer such that $j_r > j_{r-1} + t_{r-1}$, then

  (i) if $r = \ell$, then $\alpha(v) = x_{j_1}\cdots x_{j_{\ell-1}}x_{n-1} = (v/x_n)x_{n-1} \in N_0$, because N is a $\mathbf{t}$-spread strongly stable set.

  (ii) if $r < \ell$, since N is a $\mathbf{t}$-spread strongly stable set, we have

$$\begin{aligned}v_1 &= x_{j_k-1}(v/x_{j_k}) \in N,\\ v_2 &= x_{j_{k+1}-1}(v_1/x_{j_{k+1}}) \in N,\\ &\ \ \vdots\\ v_{\ell-r} &= x_{j_{\ell-1}-1}(v_{\ell-k-1}/x_{j_{\ell-1}}) \in N,\end{aligned}$$

then $\alpha(v) = x_{n-1}(v_{\ell-k}/x_n) \in N_0$.

(c) if $v \in N_1 x_n$ and $\alpha(v) = x_{j_1-1}\cdots x_{j_{\ell-1}-1}x_{n-1}$, we have

$$\begin{aligned} v_1 &= x_{j_1-1}(v/x_{j_1}) \in N, \\ v_2 &= x_{j_2-1}(v_1/x_{j_2}) \in N, \\ &\vdots \\ v_{\ell-1} &= x_{j_\ell-1}(v_{\ell-2}/x_{j_\ell-1}) \in N, \end{aligned}$$

and $\alpha(v) = x_{n-1}(v_{\ell-1}/x_n) \in N_0$, because N is a **t**-spread strongly stable set.

Hence, in all possible cases $\alpha(v) \in N_0$. Thus, we have $\min_{>_{\text{lex}}} N_0 \leq_{\text{lex}} \alpha(v)$, and $\min_{>_{\text{lex}}} N_0 \geq_{\text{lex}} v = \min_{>_{\text{lex}}} N$. Since $\max(\min_{>_{\text{lex}}} N_0) < n$, we have

$$\min_{>_{\text{lex}}} N_0 = \alpha(\min_{>_{\text{lex}}} N_0) \geq_{\text{lex}} \alpha(v) \geq_{\text{lex}} \min_{>_{\text{lex}}} N_0,$$

and so $\min_{>_{\text{lex}}} N_0 = \alpha(v)$. Similarly one can prove that $\min_{>_{\text{lex}}} L_0 = \alpha(u)$.

Finally, since L is a **t**-spread lex set and $|L| \leq |N|$, we have $u \geq_{\text{lex}} v$. Consequently, $\min_{>_{\text{lex}}} L_0 = \alpha(u) \geq_{\text{lex}} \alpha(v) = \min_{>_{\text{lex}}} N_0$ and the proof is completed. □

As a striking consequence of the previous result, we prove that to every **t**-spread strongly stable ideal I one can associate a unique **t**-spread lex ideal which shares the same $f_{\mathbf{t}}$-vector of I. It is necessary to highlight that if I is an arbitrary **t**-spread ideal of S, then such a **t**-spread lex ideal does not always exist as pointed out in the next example.

Example 3.2.5. ([8, Remark 2]) Let $I = (x_2x_8, x_2x_6, x_2x_4) \subset K[x_1, \dots, x_8]$, which is a **t**-spread ideal, with $\mathbf{t} = (2)$. Such I is not strongly stable, and we have the impossibility to construct a **t**-spread lex ideal with the same $f_{\mathbf{t}}$-vector as I.

Nevertheless, there can exist a **t**-spread ideal I of S which is not **t**-spread strongly stable but for which there exists a **t**-spread lex ideal with the same $f_{\mathbf{t}}$-vector of I (see, for instance, [6, Remark 4.10] and [8, Remark 2]).

Proposition 3.2.6. *Let $I \subset S$ be a **t**-spread strongly stable ideal. Then, there exists a unique **t**-spread lex ideal $I^{\mathbf{t},\text{lex}} \subset S$ such that $f_{\mathbf{t}}(I) = f_{\mathbf{t}}(I^{\mathbf{t},\text{lex}})$.*

Proof. We construct a **t**-spread lex ideal J verifying $f_{\mathbf{t}}(J) = f_{\mathbf{t}}(I)$ as follows. For all $\ell \in [0, d]$, let L_ℓ be the unique **t**-spread lex set of $M_{n,\ell,\mathbf{t}}$ with $|L_\ell| = |[I_\ell]_{\mathbf{t}}|$. Whereas, for $\ell > d$ we set $L_\ell = \emptyset$. For all $\ell \geq 0$, we denote by J_ℓ the K-vector space spanned by the monomials in the set

$$L_\ell \cup \text{Shad}_{\mathbf{0}}(B_{\ell-1}),$$

where $B_{-1} = \emptyset$ and for $\ell \geq 1$, $B_{\ell-1}$ is the set of monomials in $J_{\ell-1}$.

Then, we set $J = \bigoplus_{\ell \geq 0} J_\ell$. We claim that J satisfies our statement. Firstly, we must show that J is a **t**-spread lex ideal. For this purpose, it is enough to observe that $\text{Shad}_{\mathbf{0}}(B_{\ell-1}) \subseteq B_\ell$ for all $\ell \geq 1$ by construction

It remains to prove that $f_{\mathbf{t}}(J) = f_{\mathbf{t}}(I)$, *i.e.* $|[J_\ell]_{\mathbf{t}}| = |[I_\ell]_{\mathbf{t}}|$, for all $\ell \in [0, d]$. Let $\delta = \text{indeg}(I) = \text{indeg}(J)$. Then, $\delta \leq d$ and $|[J_\ell]_{\mathbf{t}}| = |[I_\ell]_{\mathbf{t}}| = 0$ for all $\ell \in [0, \delta - 1]$. Now, let $\ell \geq \delta$. Since $|L_\ell| = |[I_\ell]_{\mathbf{t}}|$, then

$$|[J_\ell]_{\mathbf{t}}| = |L_\ell \cup \text{Shad}_{\mathbf{t}}(B_{\ell-1})| = |[I_\ell]_{\mathbf{t}}|$$

if and only if $\text{Shad}_{\mathbf{t}}(B_{\ell-1}) \subseteq L_\ell$. We proceed by finite induction on $\ell \in [\delta, d]$. For the base case $\ell = \delta$, just note that $B_{\delta-1} = \emptyset$. Now, let $\ell > \delta$, then $\text{Shad}_{\mathbf{t}}(B_{\ell-2}) \subseteq L_{\ell-1}$ by the inductive hypothesis. Hence,

$$\begin{aligned}\text{Shad}_{\mathbf{t}}(B_{\ell-1}) &= \text{Shad}_{\mathbf{t}}(L_{\ell-1} \cup \text{Shad}_{\mathbf{0}}(B_{\ell-2}))\\ &= \text{Shad}_{\mathbf{t}}(L_{\ell-1} \cup \text{Shad}_{\mathbf{t}}(B_{\ell-2}))\\ &= \text{Shad}_{\mathbf{t}}(L_{\ell-1}).\end{aligned}$$

Indeed,

$$\text{Shad}_{\mathbf{t}}(\text{Shad}_{\mathbf{0}}(B_{\ell-2})) = \text{Shad}_{\mathbf{t}}(\text{Shad}_{\mathbf{t}}(B_{\ell-2})).$$

It is clear that the second set is included in the first one. For the other inclusion, let $u \in \text{Shad}_{\mathbf{t}}(\text{Shad}_{\mathbf{0}}(B_{\ell-2}))$. Then, by Lemma 3.2.1 $u = vx_ix_j$ with $i \in [\max(v), j]$, $v \in B_{\ell-2}$ and $\deg(v) = \ell - 2$. Since u is $\mathbf{t}$-spread and $\max(u) = j$, then vx_i is $\mathbf{t}$-spread as well. Hence, $vx_i \in \text{Shad}_{\mathbf{t}}(B_{\ell-2})$ and so $u \in \text{Shad}_{\mathbf{t}}(\text{Shad}_{\mathbf{t}}(B_{\ell-2}))$.

Thus, it remains to prove that $\text{Shad}_{\mathbf{t}}(L_{\ell-1}) \subseteq L_\ell$. Both sets are $\mathbf{t}$-spread lex sets. Therefore, the previous inclusion holds if and only if $|\text{Shad}_{\mathbf{t}}(L_{\ell-1})| \leq |L_\ell|$. By Lemma 3.2.3(b) and Theorem 3.2.4 applied to the sets $L_{\ell-1}$ and $[I_{\ell-1}]_{\mathbf{t}}$ satisfying $|L_{\ell-1}| = |[I_{\ell-1}]_{\mathbf{t}}|$, we have,

$$\begin{aligned}|\text{Shad}_{\mathbf{t}}(L_{\ell-1})| &= \sum_{i=1+\sum_{j=1}^{\ell-2} t_j}^{n-t_{\ell-1}} m_{\leq i}(L_{\ell-1}) \leq \sum_{i=1+\sum_{j=1}^{\ell-2} t_j}^{n-t_{\ell-1}} m_{\leq i}([I_{\ell-1}]_{\mathbf{t}})\\ &= |\text{Shad}_{\mathbf{t}}([I_{\ell-1}]_{\mathbf{t}})| \leq |[I_\ell]_{\mathbf{t}}| = |L_\ell|.\end{aligned}$$

The inductive proof is complete.

We denote J by $I^{\mathbf{t},\text{lex}}$. It is clear that $I^{\mathbf{t},\text{lex}}$ is the unique ideal meeting the requirements of the statement. □

Example 3.2.7. Let $\mathbf{t} = (1, 0, 2)$ and $n = 6$. Consider the following $\mathbf{t}$-spread strongly stable ideal of $S = K[x_1, \dots, x_6]$:

$$I = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4^2, x_3x_4^2x_6).$$

Then,

$$\begin{aligned}[I_\ell]_{\mathbf{t}} &= \emptyset, \text{ for } \ell = 0, 1,\\ [I_2]_{\mathbf{t}} &= \{x_1x_2, x_1x_3, x_1x_4, x_2x_3\},\\ [I_3]_{\mathbf{t}} &= \{x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_3x_6,\\ &\quad x_1x_4^2, x_1x_4x_5, x_1x_4x_6, x_2x_3^2, x_2x_3x_4, x_2x_3x_5, x_2x_3x_6, x_2x_4^2\},\\ [I_4]_{\mathbf{t}} &= \{x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2^2x_6, x_1x_2x_3x_5, x_1x_2x_3x_6, x_1x_2x_4x_6, x_1x_3^2x_5, x_1x_3^2x_6,\\ &\quad x_1x_3x_4x_6, x_1x_4^2x_6, x_2x_3^2x_5, x_2x_3^2x_6, x_2x_3x_4x_6, x_2x_4^2x_6, x_3x_4^2x_6\},\\ [I_\ell]_{\mathbf{t}} &= \emptyset, \text{ for all } \ell \geq 5.\end{aligned}$$

Therefore,

$$\begin{aligned}f_{\mathbf{t}}(I) &= (f_{\mathbf{t},-1}(I), f_{\mathbf{t},0}(I), f_{\mathbf{t},1}(I), f_{\mathbf{t},2}(I), f_{\mathbf{t},3}(I))\\ &= (1, 6, 11, 18, 0).\end{aligned}$$

Note that the value of $f_{\mathbf{t},3}(I)$ depends on the fact that $[I_4]_{\mathbf{t}} = M_{6,4,\mathbf{t}}$.

Moreover, $L_\ell = \emptyset$ for $\ell = 0, 1$ and for $\ell \geq 5$. Whereas, for $\ell = 2, 3, 4$, we have

$$\begin{aligned}
L_2 &= \{x_1x_2, x_1x_3, x_1x_4, x_1x_5\},\\
L_3 &= \{x_1x_2^2, x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_1x_2x_6, x_1x_3^2, x_1x_3x_4, x_1x_3x_5, x_1x_3x_6,\\
&\quad x_1x_4^2, x_1x_4x_5, x_1x_4x_6, x_1x_5^2, x_1x_5x_6, x_1x_6^2, x_2x_3^2, x_2x_3x_4\},\\
L_4 &= \{x_1x_2^2x_4, x_1x_2^2x_5, x_1x_2^2x_6, x_1x_2x_3x_5, x_1x_2x_3x_6, x_1x_2x_4x_6, x_1x_3^2x_5, x_1x_3^2x_6,\\
&\quad x_1x_3x_4x_6, x_1x_4^2x_6, x_2x_3^2x_5, x_2x_3^2x_6, x_2x_3x_4x_6, x_2x_4^2x_6, x_3x_4^2x_6\}.
\end{aligned}$$

Hence,

$$I^{\mathbf{t},\mathrm{lex}} = (x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_1x_6^2, x_2x_3^2, x_2x_3x_4, x_2x_4^2x_6, x_3x_4^2x_6).$$

3.3 The vector–spread Macaulay's theorem

The purpose of this section is to give a classification of all possible $f_{\mathbf{t}}$-vectors of a $\mathbf{t}$-spread strongly stable ideal. We follow the steps of the classical Macaulay's theorem, see [68, Theorem 6.3.8].

We quote the following result from [68, Lemma 6.3.4].

Lemma 3.3.1. *Let ℓ be a positive integer. Then, each positive integer a has a unique expansion*

$$a = \binom{a_\ell}{\ell} + \binom{a_{\ell-1}}{\ell-1} + \cdots + \binom{a_p}{p},$$

with $a_\ell > a_{\ell-1} > \cdots > a_p \geq p \geq 1$.

The previous expansion is called the *binomial expansion* (or *Macaulay expansion*) of a with respect to ℓ. Hereafter, suppose we can write a positive integer a as

$$a = \binom{a_\ell}{\ell} + \cdots + \binom{a_p}{p} + \cdots + \binom{a_1}{1}, \tag{3.3}$$

where $a_\ell > a_{\ell-1} > \cdots > a_p \geq p$ and $a_j < j$ for $j = 1, \dots, p-1$. Then, by Lemma 3.3.1, $a = \sum_{j=p}^{\ell} \binom{a_j}{j}$ is the (unique) binomial expansion of a with respect to ℓ. However, for our convenience, we refer to (3.3) also as a binomial expansion of a.

Definition 3.3.2. Let n, ℓ be positive integers, $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$, $d \geq 2$ such that $n > \sum_{j=1}^{d-1} t_j$ and $\ell < d$. For all $\ell \in [1, d-1]$, we define a $\mathbf{t}$-*spread operator* as follows: for any positive integer $a \leq |M_{n,\ell,\mathbf{t}}|$, let $a = \sum_{j=p}^{\ell} \binom{a_j}{j}$ be the binomial expansion of a with respect to ℓ. We define

$$a^{(\ell,\mathbf{t})} = \sum_{j=p+1}^{\ell+1} \binom{a_{j-1} + 1 - t_\ell}{j}.$$

Let $u \in M_{n,\ell,\mathbf{t}}$. We define the *initial* $\mathbf{t}$-*spread lexsegment set determined by* u to be the set

$$\mathcal{L}_{\mathbf{t}}^i(u) = \{v \in M_{n,\ell,\mathbf{t}} : v \geq_{\mathrm{lex}} u\}.$$

Note that any $\mathbf{t}$-spread lex set $L \subset M_{n,\ell,\mathbf{t}}$ is an initial $\mathbf{t}$-spread lexsegment set.

Definition 3.3.2 is justified by the next result.

Theorem 3.3.3. *Let $u \in M_{n,\ell,\mathbf{t}}$ with $\ell < d$ and $a = |M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u)|$. Then,*

$$\left|M_{n,\ell+1,\mathbf{t}} \setminus \mathrm{Shad}_{\mathbf{t}}(\mathcal{L}_{\mathbf{t}}^i(u))\right| \;=\; a^{(\ell,\mathbf{t})}.$$

In order to prove the theorem, we need some preliminary lemmata.

Given $\emptyset \neq A \subseteq [1,n]$, we set

$$M_{A,\ell,\mathbf{t}} = M_{n,\ell,\mathbf{t}} \cap K[x_a : a \in A].$$

Moreover, if $\mathbf{t} = (t_1,\dots,t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$, we set $\mathbf{t}_{\geq k} = (t_k,\dots,t_{d-1})$.

Lemma 3.3.4. *Let $u = x_{i_1}\cdots x_{i_\ell} \in M_{n,\ell,\mathbf{t}}$. Then*

$$M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u) = \bigcup_{k=1}^{\ell} x_{i_1}\cdots x_{i_{k-1}} M_{[i_k+1,n],\ell-(k-1),\,\mathbf{t}_{\geq k}}. \tag{3.4}$$

This union is disjoint, and the binomial expansion of $\left|M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u)\right|$ is

$$\left|M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u)\right| = \sum_{j=1}^{\ell}\binom{a_j}{j}, \tag{3.5}$$

where $a_j = n - i_{\ell-(j-1)} + j - 1 - \sum_{h=\ell-(j-1)}^{\ell-1} t_h$, for all $j \in [1,\ell]$.

Proof. Since $\geq_{\mathrm{lex}}$ is a total order, we have $M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u) = \{v \in M_{n,\ell,\mathbf{t}} : v <_{\mathrm{lex}} u\}$. Let $v = x_{j_1}\cdots x_{j_\ell} \in M_{n,\ell,\mathbf{t}}$, with $v <_{\mathrm{lex}} u$. Then, $i_1 = j_1,\dots,i_{k-1} = j_{k-1}$ and $i_k < j_k$, for some $k \in [1,\ell]$. Hence, $v = x_{i_1}\cdots x_{i_{k-1}}w$, where $w \in M_{[i_k+1,n],\ell-(k-1),\mathbf{t}_{\geq k}}$ and (3.4) follows.

To prove (3.5) one can apply (3.1), observing that the union in (3.4) is disjoint and that $|x_{i_1}\cdots x_{i_{k-1}}M_{[i_k+1,n],\ell-(k-1),\mathbf{t}_{\geq k}}| = |M_{n-i_k,\ell-(k-1),\mathbf{t}_{\geq k}}|$. In fact,

$$\begin{aligned}
\left|M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u)\right| &= \left|\bigcup_{k=1}^{\ell} x_{i_1}\cdots x_{i_{k-1}} M_{[i_k+1,n],\ell-(k-1),\,\mathbf{t}_{\geq k}}\right| \\
&= \sum_{k=1}^{\ell}\left|M_{n-i_k,\ell-(k-1),\mathbf{t}_{\geq k}}\right| \\
&= \sum_{k=1}^{\ell}\binom{n-i_k+(\ell-(k-1)-1)-\sum_{h=k}^{\ell-1}t_h}{\ell-(k-1)} \\
&= \sum_{j=1}^{\ell}\binom{n-i_{\ell-(j-1)}+j-1-\sum_{h=\ell-(j-1)}^{\ell-1}t_h}{j},
\end{aligned}$$

where in the last equality we set $j = \ell-(k-1)$.

It remains to prove that (3.5) is the binomial expansion of $\left|M_{n,\ell,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^i(u)\right|$. Let $p = \min\{j : a_j \geq j\}$. By Lemma 3.3.1, it is enough to show the following facts:

(i) $a_\ell > a_{\ell-1} > \cdots > a_p \geq p$, and

(ii) $a_j < j$, for $j = 1,\dots,p-1$.

Statement (ii) follows from the definition of p. For the proof of (i), let $\ell > j \geq p$. Then, we have

$$a_{j+1} - a_j = i_{\ell-(j-1)} - i_{\ell-j} + 1 - t_{\ell-j} \geq t_{\ell-j} + 1 - t_{\ell-j} = 1,$$

since $i_{\ell-(j-1)} - i_{\ell-j} \geq t_{\ell-j}$. Thus,

$$a_{j+1} \geq a_j + 1,$$

and so $a_\ell > a_{\ell-1} > \cdots > a_p \geq p$, as desired. □

Let $L \subseteq M_{n,\ell,\mathbf{t}}$ be a $\mathbf{t}$-spread lex set, with $\ell < d$. By Proposition 3.2.2(b), $\mathrm{Shad}_{\mathbf{t}}(L) \subseteq M_{n,\ell+1,\mathbf{t}}$ is again a $\mathbf{t}$-spread lex set. Let

$$u = \min_{>_{\mathrm{lex}}} L = x_{i_1}x_{i_2}\cdots x_{i_\ell}.$$

Then, $L = \mathcal{L}_{\mathbf{t}}^i(u)$. Hence, if we set $\widetilde{L} = \mathrm{Shad}_{\mathbf{t}}(L)$ and $\widetilde{u} = \min_{>_{\mathrm{lex}}} \mathrm{Shad}_{\mathbf{t}}(L)$, then $\widetilde{L} = \mathcal{L}_{\mathbf{t}}^i(\widetilde{u})$. Therefore, to determine the $\mathbf{t}$-spread shadow $\widetilde{L}$ of L it is enough to determine the monomial $\widetilde{u}$. This is accomplished in the next lemma.

Lemma 3.3.5. *With the notation and assumptions as above, we have*

$$\widetilde{u} = \Big(\prod_{m=1}^{\ell-r} x_{i_m}\Big)\Big(\prod_{j=1}^{r+1} x_{n-\sum_{p=\ell-(r-j)}^{\ell} t_p}\Big), \tag{3.6}$$

where we set $i_0 = t_0 = 0$ and

$$r = \min\Big\{s \in [0,\ell] \;:\; n - i_\ell + \sum_{h=1}^{s}(i_{\ell-(h-1)} - i_{\ell-h} - t_{\ell-h}) \geq t_\ell\Big\}. \tag{3.7}$$

Proof. Let us prove that $\widetilde{u}$ belongs to $\widetilde{L}$. For this aim, it is enough to show that $v = \widetilde{u}/x_n \in L$. Note that

$$n - i_\ell + \sum_{h=1}^{s}(i_{\ell-(h-1)} - i_{\ell-h} - t_{\ell-h}) = n - i_{\ell-s} - \sum_{h=1}^{s} t_{\ell-h},$$

for all $s \in [0,\ell]$. Thus, $r = \min\big\{s \in [0,\ell] : n - i_{\ell-s} - \sum_{h=1}^{s} t_{\ell-h} \geq t_\ell\big\}$. Hence,

$$\Big(n - \sum_{h=1}^{r} t_{\ell-h}\Big) - i_{\ell-r} \geq t_\ell.$$

By definition of $>_{\mathrm{lex}}$ we have $v \geq_{\mathrm{lex}} u$. Since v is $\mathbf{t}$-spread, it follows that $v \in L$.

To prove that $\widetilde{u} = \min_{>_{\mathrm{lex}}} \widetilde{L}$, suppose by contradiction that there exists $w \in \widetilde{L}$ such that $w <_{\mathrm{lex}} \widetilde{u}$. Write $w = x_{j_1}\cdots x_{j_{\ell+1}}$, $\widetilde{u} = x_{k_1}\cdots x_{k_{\ell+1}}$. Then, $j_1 = k_1, \dots, j_{q-1} = k_{q-1}$ and $j_q > k_q$, for some $q \in [1,\ell+1]$.

If $q \geq \ell - r + 1$, then $j_q > k_q = n - \sum_{p=q}^{\ell} t_p$. This is absurd, because all monomials $x_{s_1}\cdots x_{s_{\ell+1}} \in M_{n,\ell+1,\mathbf{t}}$ satisfy the inequalities $s_q \leq n - \sum_{h=q}^{\ell} t_h$, $q \in [1,\ell+1]$.

If $1 \leq q \leq \ell - r$, then $j_q > k_q = i_q$. By Lemma 3.2.1, $w/x_{j_{\ell+1}} = w' \in L$. Hence, $\min_{>_{\mathrm{lex}}} L = u >_{\mathrm{lex}} w'$, a contradiction. Finally, $\widetilde{u} = \min_{>_{\mathrm{lex}}} \widetilde{L}$. □

The next example illustrates the previous lemma.

Example 3.3.6. Let $\mathbf{t} = (2,1,2)$, $S = K[x_1,\dots,x_8]$, $L = \mathcal{L}^i(u)$ for some $u \in M_{8,3,\mathbf{t}}$. Set $\text{Shad}_{\mathbf{t}}(L) = \widetilde{L}$ and $\widetilde{u} = \min_{>_{\text{lex}}} \widetilde{L}$. Let r the integer defined in (3.7).

Let $u = x_2x_4x_6$. Since $n - \max(u) = 2 = t_3$, then $r = 0$ and $\widetilde{u} = ux_n = x_2x_4x_6x_8$.

Let $u = x_2x_6x_7$. Then, $r = 2$ and $\widetilde{u} = x_2x_5x_6x_8$.

Let $u = x_4x_6x_7$. Then, $r = 3$ and $\widetilde{u} = x_3x_5x_6x_8$. In such a case $\text{Shad}_{\mathbf{t}}(L) = M_{8,4,\mathbf{t}}$.

Proof of Theorem 3.3.3. As before, let $L = \mathcal{L}^i_{\mathbf{t}}(u)$, $\widetilde{L} = \text{Shad}_{\mathbf{t}}(L)$ and $\widetilde{u} = \min_{>_{\text{lex}}} \widetilde{L}$. Write $\widetilde{u} = x_{k_1}x_{k_2}\cdots x_{k_{\ell+1}}$, where the indices k_j are determined in (3.6) and

$$r = \min\Big\{s \in [0,\ell] \ : \ n - i_\ell + \sum_{h=1}^{s}(i_{\ell-(h-1)} - i_{\ell-h} - t_{\ell-h}) \geq t_\ell\Big\}.$$

Then, by Lemma 3.3.4, we have the binomial expansions

$$|M_{n,\ell,\mathbf{t}} \setminus L| \ = \ \sum_{j=1}^{\ell}\binom{a_j}{j}, \qquad |M_{n,\ell+1,\mathbf{t}} \setminus \widetilde{L}| \ = \ \sum_{j=1}^{\ell+1}\binom{\widetilde{a}_j}{j},$$

where

(a) $a_j = n - i_{\ell-(j-1)} + j - 1 - \sum_{h=\ell-(j-1)}^{\ell-1} t_h$, for all $j \in [1,\ell]$, and

(b) $\widetilde{a}_j = n - k_{\ell+1-(j-1)} + j - 1 - \sum_{h=\ell+1-(j-1)}^{\ell} t_h$, for all $j \in [1,\ell+1]$.

It remains to prove that $|M_{n,\ell+1,\mathbf{t}} \setminus \widetilde{L}| = a^{(\ell,\mathbf{t})}$. Firstly, we establish how the coefficients a_j and $\widetilde{a}_j$ are related. Note that, for $j \in [1, r+1]$, we have

$$\begin{aligned}
\widetilde{a}_j &= n - k_{\ell+1-(j-1)} + j - 1 - \sum_{h=\ell+1-(j-1)}^{\ell} t_h \\
&= n - \Big(n - \sum_{p=\ell+1-(j-1)}^{\ell} t_p\Big) + j - 1 - \sum_{h=\ell+1-(j-1)}^{\ell} t_h \\
&= j - 1.
\end{aligned}$$

Since $\binom{j-1}{j} = 0$, we may write as well

$$|M_{n,\ell+1,\mathbf{t}} \setminus \widetilde{L}| = \sum_{j=r+2}^{\ell+1}\binom{\widetilde{a}_j}{j}.$$

Instead, since $k_{\ell+1-(j-1)} = i_{\ell-(j-1)}$, for $j \in [r+2, \ell+1]$, we have

$$\widetilde{a}_j = a_{j-1} + 1 - t_\ell.$$

Therefore,

$$\big|M_{n,\ell+1,\mathbf{t}} \setminus \widetilde{L}\big| = \sum_{j=r+2}^{\ell+1}\binom{a_{j-1} + 1 - t_\ell}{j}.$$

Let $p = \min\{j \ : \ a_j \geq j\}$. The theorem is proved if we show that

$$\left|M_{n,\ell+1,\mathbf{t}} \setminus \widetilde{L}\right| = \left|M_{n,\ell,\mathbf{t}} \setminus L\right|^{(\ell,\mathbf{t})} = \sum_{j=p+1}^{\ell+1} \binom{a_{j-1}+1-t_\ell}{j}.$$

If $p+1 = r+2$ this is clear. Suppose $p+1 > r+2$. Then, it is enough to show that

$$\binom{a_{j-1}+1-t_\ell}{j} = 0, \quad \text{for all } j \in [r+2, p].$$

If $j \leq p$, then $a_{j-1} < j-1$. Hence, $a_{j-1}+1-t_\ell \leq a_{j-1}+1 < j$ and $\binom{a_{j-1}+1-t_\ell}{j} = 0$, as desired. Now let $r+2 > p+1$. We must prove that

$$\binom{a_{j-1}+1-t_\ell}{j} = 0,$$

for all $j \in [p+1, r+1]$. Set $a_{\ell+1} = n - \sum_{j=1}^{\ell-1} t_j$. Then

$$r = \min\{s \in [0,\ell] \ : \ a_{s+1} - s \geq t_\ell\}.$$

If $j \leq r+1$, then $j-2 \leq r-1$. Hence, $a_{(j-2)+1} - (j-2) = a_{j-1} - (j-2) < t_\ell$. It follows that $a_{j-1}+1-t_\ell < a_{j-1}+2-t_\ell < j$ and $\binom{a_{j-1}+1-t_\ell}{j} = 0$, as desired. $\square$

Example 3.3.7. Let $n = 31$, $\mathbf{t} = (0,1,3,1)$, $a = 2023$ and $\ell = 3$. Then

$$a = \sum_{j=1}^{\ell} \binom{a_j}{j} = \binom{23}{3} + \binom{22}{2} + \binom{21}{1}$$

is the binomial expansion of a with respect to ℓ. Therefore, since $r = 0$, we have

$$\begin{aligned} a^{(\ell,\mathbf{t})} = 2023^{(3,(0,1,3,1))} &= \sum_{j=r+2}^{\ell+1} \binom{a_{j-1}+1-t_\ell}{j} \\ &= \binom{19}{2} + \binom{20}{3} + \binom{21}{4} = 7296. \end{aligned}$$

Now, we can state and prove the main result in the chapter.

Theorem 3.3.8. *Let $f = (f_{-1}, f_0, \dots, f_{d-1})$ be a sequence of non-negative integers. The following conditions are equivalent:*

1. *there exists a $\mathbf{t}$-spread strongly stable ideal $I \subset S = K[x_1,\dots,x_n]$ such that*

$$f_{\mathbf{t}}(I) = f;$$

2. *$f_{-1} = 1$ and $f_{\ell+1} \leq f_\ell^{(\ell+1,\mathbf{t})}$, for all $\ell = -1,\dots,d-2$.*

Proof. 1. $\Longrightarrow$ 2. Assume that $f_{\mathbf{t}}(I) = f$. By Proposition 3.2.6, we may replace I by $I^{\mathbf{t},\mathrm{lex}}$ without changing the $f_{\mathbf{t}}$-vector. Thus, we may assume as well that I is a $\mathbf{t}$-spread lex ideal. Then, $f_{-1} = f_{\mathbf{t},-1}(I) = 1$ and for all $\ell \in [-1, d-2]$ we have

$\text{Shad}_{\mathbf{t}}([I_{\ell+1}]_{\mathbf{t}}) \subseteq [I_{\ell+2}]_{\mathbf{t}}$. Hence,

$$\begin{aligned} f_{\ell+1} = f_{\mathbf{t},\ell+1}(I) = |M_{n,\ell+2,\mathbf{t}}| - |[I_{\ell+2}]_{\mathbf{t}}| &\leq |M_{n,\ell+2,\mathbf{t}}| - |\text{Shad}_{\mathbf{t}}([I_{\ell+1}]_{\mathbf{t}})| \\ &= |M_{n,\ell+2,\mathbf{t}} \setminus \text{Shad}_{\mathbf{t}}([I_{\ell+1}]_{\mathbf{t}})| \\ &= f_{\mathbf{t},\ell}(I)^{(\ell+1,\mathbf{t})} = f_{\ell}^{(\ell+1,\mathbf{t})}, \end{aligned}$$

where the last equality follows from Theorem 3.3.3. Statement 2. is proved.

2. $\Longrightarrow$ 1. First we prove that

$$f_{\ell+1} \leq |M_{n,\ell+2,\mathbf{t}}|, \quad \text{for all} \quad \ell = -2, \dots, d-2.$$

For $\ell = -2$, $f_{-1} = 1 = |M_{n,0,\mathbf{t}}|$ because there is only one $\mathbf{t}$-spread monomial of degree 0, namely $u = 1$. Now we proceed by induction. Let $\ell \geq -1$. By the hypothesis (ii), we have $f_{\ell+1} \leq f_{\ell}^{(\ell+1,\mathbf{t})}$ and, by induction, $f_{\ell} \leq |M_{n,\ell+1,\mathbf{t}}|$. Thus, there exists a unique monomial $u \in M_{n,\ell+1,\mathbf{t}}$ such that $|M_{n,\ell+1,\mathbf{t}} \setminus \mathcal{L}_{\mathbf{t}}^{i}(u)| = f_{\ell}$. By Theorem 3.3.3, we have $f_{\ell}^{(\ell+1,\mathbf{t})} = |M_{n,\ell+2,\mathbf{t}} \setminus \text{Shad}_{\mathbf{t}}(\mathcal{L}_{\mathbf{t}}^{i}(u))|$. This shows that $f_{\ell}^{(\ell+1,\mathbf{t})} \leq |M_{n,\ell+2,\mathbf{t}}|$ and consequently we have $f_{\ell+1} \leq |M_{n,\ell+2,\mathbf{t}}|$, as desired.

For all $\ell \in [0,d]$, let L_{ℓ} be the unique $\mathbf{t}$-spread lex set of $M_{n,\ell,\mathbf{t}}$ such that $|L_{\ell}| = |M_{n,\ell,\mathbf{t}}| - f_{\ell-1}$. For $\ell > d$ we set $L_{\ell} = \emptyset$. As in Proposition 3.2.6, we construct the ideal $I = \bigoplus_{\ell \geq 0} I_{\ell}$ where I_{ℓ} is the K-vector space spanned by the set

$$L_{\ell} \cup \text{Shad}_{\mathbf{0}}(B_{\ell-1}),$$

where $B_{-1} = \emptyset$ and for $\ell \geq 1$, $B_{\ell-1}$ is the set of monomials generating $I_{\ell-1}$. As in Proposition 3.2.6, one shows that I is a $\mathbf{t}$-spread lex ideal. Hence, it remains to prove that $f_{\mathbf{t}}(I) = f$. As in the proof of Proposition 3.2.6, this boils down to proving that $\text{Shad}_{\mathbf{t}}(L_{\ell+1}) \subseteq L_{\ell+2}$, for all $\ell \in [-1, d-2]$. Since $f_{\ell+1} \leq f_{\ell}^{(\ell+1,\mathbf{t})}$ we have

$$|M_{n,\ell+2,\mathbf{t}} \setminus L_{\ell+2}| \leq |M_{n,\ell+1,\mathbf{t}} \setminus L_{\ell+1}|^{(\ell+1,\mathbf{t})} = |M_{n,\ell+2,\mathbf{t}} \setminus \text{Shad}_{\mathbf{t}}(L_{\ell+1})|,$$

where the last equality follows from Theorem 3.3.3. Thus, $|\text{Shad}_{\mathbf{t}}(L_{\ell+1})| \leq |L_{\ell+2}|$. Hence, $\text{Shad}_{\mathbf{t}}(L_{\ell+1}) \subseteq L_{\ell+2}$, because both are $\mathbf{t}$-spread lex sets. The proof is complete. □

Example 3.3.9. Let $\mathbf{t} = (1,0,2)$, $d = 4$ and $n = 6$. Consider the following vector

$$f = (f_{-1}, f_0, f_1, f_2, f_3) = (1, 6, 11, 18, 0).$$

Then, $f_{-1} = 1$ and $f_{\ell+1} \leq f_{\ell}^{(\ell+1,\mathbf{t})}$, for all $\ell = -1, \dots, 2$. Therefore, from Theorem 3.3.8 there exists a $\mathbf{t}$-spread strongly stable ideal of $S = K[x_1, \dots, x_6]$ that has f as a $f_{\mathbf{t}}$-vector. The ideal I of Example 3.2.7 is such an ideal.

3.4 A formula for Betti numbers

In this final section, as an application we recover the vector–spread version of the well–known result proved by Bigatti [15] and Hulett [76], independently (see, also, [77, 96]). More precisely, we prove that in the class of all $\mathbf{t}$-spread strongly stable ideals with a given $f_{\mathbf{t}}$-vector, the $\mathbf{t}$-spread lex ideals have the largest graded Betti numbers.

Theorem 3.4.1. *Let $I \subset S = K[x_1,\dots,x_n]$ be a* $\mathbf{t}$*-spread strongly stable ideal. Then,*

$$\beta_{i,j}(I) \ \le\ \beta_{i,j}(I^{\mathbf{t},\mathrm{lex}}), \quad \textit{for all } i \textit{ and } j.$$

Proof. By [55, Corollary 5.2], we have

$$\beta_{i,i+j}(I) \ =\ \sum_{u\in G(I)_j} \binom{\max(u)-1-\sum_{h=1}^{j-1} t_h}{i}. \tag{3.8}$$

We are going to write (3.8) in a more suitable way. We observe that I is a $\mathbf{t}$-spread ideal and thus

$$G(I)_j \ =\ [I_j]_{\mathbf{t}} \setminus \mathrm{Shad}_{\mathbf{t}}([I_{j-1}]_{\mathbf{t}}).$$

Hence, we can write the Betti number in (3.8) as a difference $A-B$, where

$$\begin{aligned}
A \ &=\ \sum_{u\in G([I_j]_{\mathbf{t}})} \binom{\max(u)-1-\sum_{h=1}^{j-1} t_h}{i} = \sum_{k=1}^{n} m_k([I_j]_{\mathbf{t}})\binom{k-1-\sum_{h=1}^{j-1} t_h}{i}\\
&= \sum_{k=1}^{n}\Big(m_{\le k}([I_j]_{\mathbf{t}})-m_{\le k-1}([I_j]_{\mathbf{t}})\Big)\binom{k-1-\sum_{h=1}^{j-1} t_h}{i}\\
&= \sum_{k=1}^{n} m_{\le k}([I_j]_{\mathbf{t}})\binom{k-1-\sum_{h=1}^{j-1} t_h}{i} - \sum_{k=1}^{n-1} m_{\le k}([I_j]_{\mathbf{t}})\binom{k-\sum_{h=1}^{j-1} t_h}{i}
\end{aligned}$$

and

$$\begin{aligned}
B \ &=\ \sum_{u\in \mathrm{Shad}_{\mathbf{t}}([I_{j-1}]_{\mathbf{t}})} \binom{\max(u)-1-\sum_{h=1}^{j-1} t_h}{i}\\
&=\ \sum_{k=1+\sum_{h=1}^{j-1} t_h}^{n} m_k(\mathrm{Shad}_{\mathbf{t}}([I_{j-1}]_{\mathbf{t}}))\binom{k-1-\sum_{h=1}^{j-1} t_h}{i}\\
&=\ \sum_{k=1+\sum_{h=1}^{j-1} t_h}^{n} m_{\le k-t_{j-1}}([I_{j-1}]_{\mathbf{t}})\binom{k-1-\sum_{h=1}^{j-1} t_h}{i},
\end{aligned}$$

where the last equality follows from Lemma 3.2.3(a).

Furthermore, we can write $A = A_1 - A_2$ with

$$\begin{aligned}
A_1 \ &= m_{\le n}([I_j]_{\mathbf{t}})\binom{n-1-\sum_{h=1}^{j-1} t_h}{i},\\
A_2 \ &= \sum_{k=1}^{n-1} m_{\le k}([I_j]_{\mathbf{t}})\left[\binom{k-\sum_{h=1}^{j-1} t_h}{i} - \binom{k-1-\sum_{h=1}^{j-1} t_h}{i}\right]\\
&= \sum_{k=1+\sum_{h=1}^{j-1} t_h}^{n-1} m_{\le k}([I_j]_{\mathbf{t}})\binom{k-1-\sum_{h=1}^{j-1} t_h}{i-1}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\beta_{i,i+j}(J) &= m_{\leq n}([I_j]_{\mathbf{t}})\binom{n-1-\sum_{h=1}^{j-1}t_h}{i}\\ &-\sum_{k=1+\sum_{h=1}^{j-1}t_h}^{n-1} m_{\leq k}([I_j]_{\mathbf{t}})\binom{k-1-\sum_{h=1}^{j-1}t_h}{i-1}\\ &-\sum_{k=1+\sum_{h=1}^{j-1}t_h}^{n} m_{\leq k-t_{j-1}}([I_{j-1}]_{\mathbf{t}})\binom{k-1-\sum_{h=1}^{j-1}t_h}{i}.\end{aligned}\tag{3.9}$$

Now, we compute the graded Betti numbers $\beta_{i,i+j}(I^{\mathbf{t},\mathrm{lex}})$. Recall that I and $I^{\mathbf{t},\mathrm{lex}}$ share the same $f_{\mathbf{t}}$-vector. Therefore, $|[I_j^{\mathbf{t},\mathrm{lex}}]_{\mathbf{t}}| = |[I_j]_{\mathbf{t}}|$, for all j. Applying Theorem 3.2.4, we have $m_{\leq k}([I_j^{\mathbf{t},\mathrm{lex}}]_{\mathbf{t}}) \leq m_{\leq k}([I_j]_{\mathbf{t}})$ for all $k \in [n]$. Moreover,

$$m_{\leq n}([I_j]_{\mathbf{t}}) = |[I_j]_{\mathbf{t}}| = |[I_j^{\mathbf{t},\mathrm{lex}}]_{\mathbf{t}}| = m_{\leq n}([I_j^{\mathbf{t},\mathrm{lex}}]_{\mathbf{t}}).$$

Therefore, replacing in (3.9), for all k and j, every occurrence of $m_{\leq k}([I_j]_{\mathbf{t}})$ with $m_{\leq k}([I_j^{\mathbf{t},\mathrm{lex}}]_{\mathbf{t}})$, we get the Betti number $\beta_{i,i+j}(I^{\mathbf{t},\mathrm{lex}})$. Finally, $\beta_{i,i+j}(I) \leq \beta_{i,i+j}(I^{\mathbf{t},\mathrm{lex}})$, for all $i, j \geq 0$. □

Remark 3.4.2. Note that in the previous result, we allow K to be an arbitrary field.

Example 3.4.3. Consider again the $\mathbf{t}$-spread strongly stable ideal I of Example 3.2.7. Then, the Betti tables of I and $I^{\mathbf{t},\mathrm{lex}}$ are, respectively,

	0	1	2
2:	4	4	1
3:	1	2	1
4:	1	2	1

	0	1	2	3	4
2:	4	6	4	1	–
3:	3	7	7	4	1
4:	2	4	2	–	–

TABLE 3.1: Comparison of the Betti tables of the ideals I and $I^{\mathbf{t},\mathrm{lex}}$.

From these tables we see that $\beta_{i,i+j}(I) \leq \beta_{i,i+j}(I^{\mathbf{t},\mathrm{lex}})$ for all i and j.

Chapter 4

Vector-spread principal Borel ideals

Let $S = K[x_1, \ldots, x_n]$ be the standard graded polynomial ring over a field K. Throughout the chapter, monomials $u \in S$ of degree ℓ will be written as $u = x_{j_1}x_{j_2}\cdots x_{j_\ell}$ with increasing indices $1 \leq j_1 \leq \cdots \leq j_\ell \leq n$.

Following [43, 55], we consider the notion of $\mathbf{t}$-*spread monomials* for a fixed vector $\mathbf{t} = (t_1, \ldots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$. A monomial $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in S$, with $\ell \leq d$, is said to be $\mathbf{t}$-*spread* if it satisfies the inequality $j_{k+1} - j_k \geq t_k$ for all $k = 1, \ldots, \ell - 1$.

For a monomial ideal $I \subset S$, let $G(I)$ denote its unique minimal set of monomial generators. We call I a $\mathbf{t}$-*spread monomial ideal* if every generator $u \in G(I)$ is $\mathbf{t}$-spread. Furthermore, we define a $\mathbf{t}$-*spread strongly stable ideal* to be a $\mathbf{t}$-spread monomial ideal $I \subset S$ such that for every $\mathbf{t}$-spread monomial $u \in I$, and any indices $j < i$ such that x_i divides u and $x_j(u/x_i)$ is again $\mathbf{t}$-spread, it follows that $x_j(u/x_i) \in I$. This generalizes the classical notions: for $\mathbf{t} = \mathbf{0} = (0, \ldots, 0)$, one recovers the standard notion of a strongly stable ideal, while the case $\mathbf{t} = \mathbf{1} = (1, \ldots, 1)$ corresponds to squarefree strongly stable ideals (see, for instance, [68]).

Given $\mathbf{t}$-spread monomials $u_1, \ldots, u_m \in S$, we denote by $B_\mathbf{t}(u_1, \ldots, u_m)$ the smallest $\mathbf{t}$-spread strongly stable ideal of S containing all u_i, with respect to the inclusion order. The monomials $u_1, \ldots, u_m$ are referred to as the $\mathbf{t}$-*spread Borel generators* of $B_\mathbf{t}(u_1, \ldots, u_m)$. In the case where $m = 1$, i.e., $I = B_\mathbf{t}(u)$, the ideal I is called a *principal* $\mathbf{t}$-*spread Borel ideal*, with $\mathbf{t}$-spread Borel generator u.

The theory of strongly stable ideals (corresponding to the case $\mathbf{t} = \mathbf{0}$) is by now classical and well-developed; see [68] for a comprehensive treatment. In characteristic zero, it is known that the generic initial ideal of any homogeneous ideal of S is strongly stable [68, Proposition 4.6.2]. Moreover, the structure and minimal graded free resolutions of such ideals have been described explicitly by Eliahou and Kervaire in their seminal work [41], a cornerstone for numerous developments in Combinatorial Commutative Algebra.

Likewise, the theory of vector–spread strongly stable ideals has undergone significant advances in recent years (see, for instance, [5, 7, 28, 36, 93] and the references therein). Furthermore, vector–spread ideals naturally arise in various contexts. They occur as initial ideals of the defining ideals of fiber cones of monomial ideals in two variables [72]. They serve as fundamental tools in the study of restricted classes of toric algebras of Veronese type [36] and, moreover, they emerge in the classification of edge ideals whose matching powers are all bi-Cohen–Macaulay [30].

When the number m of the $\mathbf{0}$-spread Borel generators of a strongly stable ideal $I \subset S$ becomes too large, certain algebraic properties of I tend to deteriorate. For instance, in the case $m = 1$, *i.e.* when I is a principal Borel ideal, it is known that the Rees algebra of I is Koszul [35]. Addressing a question posed by Conca, it was shown in [37] that the Koszul property of the Rees algebra persists when $m = 2$. However, explicit examples demonstrate that this property generally fails for $m \geq 3$. These observations indicate that the algebraic behavior of a $\mathbf{t}$-spread principal Borel ideal is often better behaved than that of an arbitrary $\mathbf{t}$-spread strongly stable ideal.

The present work aims to investigate the structure of principal vector–spread Borel ideals through the lens of Alexander duality. To this end, we restrict our attention to the squarefree case, and henceforth assume that $t_i \geq 1$ for all $i = 1, \dots, d-1$.

In Section 4.1, we describe explicitly the minimal primary decomposition of a squarefree principal vector–spread Borel ideal (Theorem 4.1.2). The notion of $\mathbf{t}$-*spread support* of a $\mathbf{t}$-spread monomial plays a key role in the formulation of this result.

In Section 4.2, we show that squarefree principal vector–spread Borel ideals are sequentially Cohen–Macaulay (Theorem 4.2.1). To this end, we prove that their Alexander duals are vertex splittable in the sense of Moradi and Khosh-Ahang [92]. Whether this result holds in the non–squarefree setting is an open question.

Finally, in Section 4.3, we classify the squarefree principal vector–spread Borel ideals whose ordinary and symbolic powers coincide (Theorem 4.3.1). Our approach mainly follows the one outlined in [93]. We point out, however, that the proof of the implication (b) $\Rightarrow$ (c) of Theorem 4.3.1 differs from that given in [93, Theorem 5.13]. One can see that the proof in [93] remains valid only when $t_1 \geq \cdots \geq t_{d-1}$.

The computational packages [53, 84] have been employed to verify several examples supporting our theoretical results.

4.1 Primary decomposition

The following notation is fixed throughout the chapter.

Setup 4.1.1. Let $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 1}^{d-1}$, let $u = x_{j_1} \cdots x_{j_d} \in S$ be a $\mathbf{t}$-spread monomial with $1 \leq j_1 < \cdots < j_{d-1} < j_d = n$ and $d \geq 2$ and let $B_{\mathbf{t}}(u)$ be the squarefree vector–spread principal Borel ideals with $\mathbf{t}$-spread Borel generator u.

The conditions $j_d = n$ and $d \geq 2$ are not restrictive. Indeed if $j_d < n$, then $B_{\mathbf{t}}(u)$ can be regarded as a monomial ideal of the polynomial ring $K[x_1, \dots, x_{j_d}]$ with fewer variables than S. Moreover, if $d = 1$ then $B_{\mathbf{t}}(u)$ is generated by variables.

For an integer $n \geq 1$, we set $[n] = \{1, \dots, n\}$. Given a non–empty subset $A \subseteq [n]$, let $P_A = (x_i : i \in A)$. For integers $j, k \geq 1$, we set $[j, k] = \{h \in \mathbb{N} : j \leq h \leq k\}$. The $\mathbf{t}$-*spread support* of a $\mathbf{t}$-spread monomial $v = x_{i_1} x_{i_2} \cdots x_{i_\ell} \in S$ is defined as

$$\operatorname{supp}_{\mathbf{t}}(v) = \bigcup_{s=1}^{\ell-1} [i_s, i_s + (t_s - 1)].$$

Moreover, we set $\operatorname{supp}(v) = \{i_1, \dots, i_\ell\} = \{j : x_j \mid v\}$.

Theorem 4.1.2. *The minimal primary decomposition of $B_{\mathbf{t}}(u)$ is*

$$B_{\mathbf{t}}(u) = \Big(\bigcap_{G \in \mathcal{G}} P_{[n] \setminus G} \Big) \cap \Big(\bigcap_{v \in G(B_{\mathbf{t}}(u))} P_{[n] \setminus \operatorname{supp}_{\mathbf{t}}(v)} \Big),$$

where $\mathcal{G}$ is the family of all subsets of $[n]$ of the form

$$\operatorname{supp}_{\mathbf{t}}(x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s}) \cup [j_s + 1, n],$$

such that $x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s} \in G(B_{\mathbf{t}}(x_{j_1} x_{j_2} \cdots x_{j_{s-1}} x_{j_s}))$, for some $s = 1, \dots, d$.

Proof. Set $I = B_{\mathbf{t}}(u)$. Since I is squarefree, there exists a unique simplicial complex Δ on the vertex set $[n]$ such that $I = I_\Delta$ is the Stanley-Reisner ideal of Δ. Therefore, by [68, Lemma 1.5.4], the minimal primary decomposition of I is

$$I = \bigcap_{F \in \mathcal{F}(\Delta)} P_{[n]\setminus F},$$

where $\mathcal{F}(\Delta)$ is the set of facets of Δ. Hence, we must prove that

$$\mathcal{F}(\Delta) = \mathcal{G} \cup \{\mathrm{supp}_{\mathbf{t}}(v) \ : \ v \in G(B_{\mathbf{t}}(u))\}. \tag{4.1}$$

Observe that $F \in \mathcal{F}(\Delta)$ if and only if $\mathbf{x}_F \notin I$ and $\mathbf{x}_{F\cup\{i\}} \in I$, for all $i \in [n] \setminus F$.

We show that each set $F = \mathrm{supp}_{\mathbf{t}}(v)$, with $v \in B_{\mathbf{t}}(u)$ is a facet of Δ. Let $v = x_{\ell_1}x_{\ell_2}\cdots x_{\ell_d} \in G(B_{\mathbf{t}}(u))$. We have

$$F \ = \ \{\ell_1, \ell_1+1, \dots, \ell_1+(t_1-1), \ \dots, \ \ell_{d-1}, \ell_{d-1}+1, \dots, \ell_{d-1}+(t_{d-1}-1)\},$$

with $\ell_1 \le j_1, \ \ell_2 \le j_2, \ \dots, \ \ell_{d-1} \le j_{d-1}$. Clearly, $\mathbf{x}_F \notin I$, because, by construction, $\mathbf{x}_F$ is not a multiple of any $\mathbf{t}$-spread monomial of degree d of S. Hence, $F \in \Delta$.

To prove that F is a facet, it suffices to show that $\mathbf{x}_{F\cup\{i\}} \in I$, for all $i \in [n] \setminus F$, that is, $F \cup \{i\} \notin \Delta$, for all $i \in [n] \setminus F$. Let $i \in [n] \setminus F$. We have that

- if $i < \ell_1$, $H = \{i, \ell_1+(t_1-1), \dots, \ell_{d-1}+(t_{d-1}-1)\} \not\subseteq F$, and $\mathbf{x}_H \in I = B_{\mathbf{t}}(u)$, that is, $H \notin \Delta$. Indeed, $i < \ell_1 \le j_1$, $\ell_1+(t_1-1) < \ell_1+t_1 \le j_1+t_1 \le j_2, \dots,$ $\ell_{d-1}+(t_{d-1}-1) < \ell_{d-1}+t_{d-1} \le j_{d-1}+t_{d-1} \le j_d$;

- if $\ell_k+(t_k-1) < i < \ell_{k+1}$, for some $1 \le k \le d-2$, then

$$H = \{\ell_1, \dots, \ell_k, i, \ell_{k+1}+(t_{k+1}-1), \dots, \ell_{d-1}+(t_{d-1}-1)\} \not\subseteq F,$$

  and $\mathbf{x}_H \in I = B_{\mathbf{t}}(u)$, arguing as before;

- if $\ell_{d-1}+(t_{d-1}-1) < i$, then $H = \{\ell_1, \dots, \ell_{d-1}, i\} \not\subseteq F$, and $\mathbf{x}_H \in I = B_{\mathbf{t}}(u)$, in fact $i \le j_d = n$.

In each case, $\mathbf{x}_{F\cup\{i\}}$ is a multiple of a $\mathbf{t}$-spread monomial of degree d, belonging to $B_{\mathbf{t}}(u)$. Therefore $\mathbf{x}_{F\cup\{i\}} \in I$ and F is a facet.

Analogously, one proves that any $F \in \mathcal{G}$ is a facet of Δ. Indeed, let $F \in \mathcal{G}$. Then, there exists $s \in \{1, \dots, d-1\}$ such that

$$F = \mathrm{supp}_{\mathbf{t}}(x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{s-1}}x_{j_s}) \cup [j_s+1, n],$$

with $x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{s-1}}x_{j_s} \in G(B_{\mathbf{t}}(x_{j_1}x_{j_2}\cdots x_{j_{s-1}}x_{j_s}))$. If $s = 1$, then $F = [j_1+1, n]$, hence $F \in \mathcal{F}(\Delta)$. Suppose $2 \le s \le d$, then $\mathbf{x}_F \notin I$. To prove that F is a facet, we must show that $\mathbf{x}_{F\cup\{i\}} \in I$, for all $i \in [n] \setminus F$. Let $i \in [n] \setminus F$, we have

- if $i < \ell_1$, then

$$H = \{i, \ell_1+(t_1-1), \dots, \ell_{s-1}+(t_{s-1}-1), j_s+(t_s-1), \dots, j_s+(t_s+\ldots+t_{d-1}-1)\} \not\subseteq F$$

  and $\mathbf{x}_H \in I = B_{\mathbf{t}}(u)$, since $i < \ell_1 \le j_1$, $\ell_k+(t_k-1) < \ell_k+t_k \le j_k+t_k \le j_{k+1}$ for $2 \le k \le s-1$, and $j_s+(t_s+\cdots+t_{s+k}-1) < j_s+t_s+t_{s+1}\cdots+t_{s+k} \le$ $j_{s+1}+t_{s+1}+\cdots+t_{s+k} \le \cdots \le j_{s+k}+t_{s+k} \le j_{s+k+1}$ for $0 \le k \le d-s-1$;

– if $\ell_k + (t_k - 1) < i < \ell_{k+1}$, for some $1 \leq k \leq s-1$, then

$$H = \{\ell_1, \dots, \ell_k, i, \ell_{k+1} + (t_{k+1} - 1), \dots, \ell_{d-1} + (t_{d-1} - 1)\} \nsubseteq F,$$

and $\mathbf{x}_H \in I = B_{\mathbf{t}}(u)$, arguing as before.

Conversely, we show that the only facets of Δ are the ones determined before. This is equivalent to prove that for every face G of Δ, there exists $F \in \mathcal{F}(\Delta)$ of the forms described in (4.1) which contains G.

For this aim, let $G \in \Delta$. We define the following integers

$$\begin{aligned} \ell_1 &= \min(G), \\ \ell_i &= \min\{r \in G : r \geq \ell_{i-1} + t_{i-1}\}, \quad \text{for } 2 \leq i \leq d. \end{aligned}$$

Suppose $\ell_i \leq j_i$, for all i. In such case, the sequence $\ell_1 < \ell_2 < \dots < \ell_k$ can have at most $d-1$ elements. Indeed, if $k \geq d$, then $G \supseteq \{\ell_1, \ell_2, \dots, \ell_d\}$ with $\ell_i \geq \ell_{i-1} + t_{i-1}$ for $2 \leq i \leq d$. This implies that $\{\ell_1, \ell_2, \dots, \ell_d\}$ should be a face of Δ, an absurd. In fact, since $\ell_i \leq j_i$, for all i, then the $\mathbf{t}$-spread monomial $x_{\ell_1} \cdots x_{\ell_d}$ must belong to I. Therefore $k \leq d-1$, and G has the following form

$$G = \{\ell_1, \ell_1 + 1, \dots, \ell_1 + r_1, \dots, \ \ell_k, \ell_k + 1, \dots, \ell_k + r_k\}$$

with $0 \leq r_i \leq t_i - 1$, for $i = 1, \dots, k$ and $\ell_i \geq \ell_{i-1} + t_{i-1}$, for $i = 2, \dots, k$. Moreover,

$$\begin{aligned} G \ \subseteq \ F = \{&\ell_1, \ell_1 + 1, \dots, \ell_1 + (t_1 - 1), \ \dots, \ \ell_k, \ell_k + 1, \dots, \ell_k + (t_k - 1), \\ &j_{k+1}, j_{k+1} + 1, \dots, j_{k+1} + (t_{k+1} - 1), \ \dots, \ j_{d-1}, \dots, j_{d-1} + (t_{d-1} - 1)\}, \end{aligned}$$

and F is a facet as observed before.

Suppose now $\ell_i > j_i$ for some i. Let $s = \min\{i : \ \ell_i > j_i\}$, then G has the following form

$$G = \{\ell_1, \ell_1 + 1, \dots, \ell_1 + r_1, \dots, \ \ell_{s-1}, \ell_{s-1} + 1, \dots, \ell_{s-1} + r_{s-1}\}$$

with $0 \leq r_i \leq t_i - 1$, for $i = 1, \dots, s-1$ and $\ell_i \geq \ell_{i-1} + t_{i-1}$, for $i = 2, \dots, s-1$. On the other hand,

$$\begin{aligned} G \ \subseteq \ F = \{&\ell_1, \ell_1 + 1, \dots, \ell_1 + (t_1 - 1), \ \dots, \ \ell_{s-1}, \ell_{s-1} + 1, \dots, \ell_{s-1} + (t_{s-1} - 1), \\ &j_s + 1, j_s + 2, \dots, n\} = \operatorname{supp}_{\mathbf{t}}(x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s}) \cup [j_s + 1, n] \in \mathcal{G}. \end{aligned}$$

This proves the result. □

Theorem 4.1.2 allows to recover [28, Corollary 2.4] for the family of squarefree vector–spread principal Borel ideals. For a monomial $u \in S$, set $\min(u) = \min\{j : x_j \mid u\}$.

Corollary 4.1.3. *We have* $\operatorname{ht}(B_{\mathbf{t}}(u)) = \min(u)$.

Proof. Set $I = B_{\mathbf{t}}(u)$. By Theorem 4.1.2, if Δ is the simplicial complex whose Stanley-Reisner ideal I_Δ is I, that is $I = I_\Delta$, we have that $\mathcal{F}(\Delta)$ consists of sets of one of the following forms:

– $F = \{\ell_1, \ell_1 + 1, \dots, \ell_1 + (t_1 - 1), \dots, \ell_{d-1}, \ell_{d-1} + 1, \dots, \ell_{d-1} + (t_{d-1} - 1)\}$, with $\ell_k \leq j_k$ for $1 \leq k \leq d-1$ and $\ell_k - \ell_{k-1} \geq t_{k-1}$ for $2 \leq k \leq d-1$;

– $F' = \{\ell_1, \ell_1+1, \dots, \ell_1+(t_1-1), \dots, \ell_{s-1}, \ell_{s-1}+1, \dots, \ell_{s-1}+(t_{s-1}-1),$
$j_s+1, j_s+2, \dots, n\}$,
with $1 \le s \le d-1$, $\ell_k \le j_k$ for $1 \le k \le s-1$ and $\ell_k - \ell_{k-1} \ge t_{k-1}$ for $2 \le k \le s$.

Note that for $s = 1$, $F' = [j_1+1, n]$. Hence,

$$\mathrm{ht}(I) = \min\left\{n - \sum_{i=1}^{d-1} t_i,\ j_1,\ n - \sum_{i=2}^{s-1} t_i\right\} = j_1 = \min(u).$$

Indeed, since $s \le d-1$ and $u = x_{j_1}x_{j_2}\cdots x_{j_d}$ is a $\mathbf{t}$-spread monomial, then $j_1 \le n - \sum_{i=1}^{d-1} t_i \le n - \sum_{i=2}^{s-1} t_i$. □

4.2 Vertex splittings and sequential Cohen–Macaulayness

In this section we analyze the sequential Cohen–Macaulayness of the principal $\mathbf{t}$-spread Borel ideal $B_{\mathbf{t}}(u)$ via the notion of *vertex splitting*.

The goal of this section is to prove the following theorem.

Theorem 4.2.1. *$B_{\mathbf{t}}(u)$ is sequentially Cohen–Macaulay.*

The proof of this result requires some preparation.

For a graded ideal $I \subset S$, we denote by $I_{\langle i\rangle}$ the ideal generated by the elements of I having degree i. We say that I is componentwise linear if $I_{\langle i\rangle}$ has linear resolution for all i. By [68, Theorem 8.2.15], monomial ideals with linear quotients are componentwise linear.

Let $I \subset S$ be a squarefree ideal and let Δ be the unique simplicial complex on the vertex set $[n]$ such that $I_\Delta = I$. The Alexander dual of I is the ideal

$$I^\vee = \bigcap_{u \in G(I)} P_{\mathrm{supp}(u)}.$$

We need the following result [68, Theorem 8.2.20].

Theorem 4.2.2. *Let $I \subset S$ be a squarefree monomial ideal. Then I is sequentially Cohen–Macaulay if and only if $I^\vee$ is componentwise linear.*

Following [92], we say that a monomial ideal $I \subset S$ is *vertex splittable* if it can be obtained by the following recursive procedure:

(a) if u is a monomial and $I = (u)$, $I = (0)$ or $I = S$, then I is vertex splittable;

(b) if there exists a variable x_i and vertex splittable ideals $I_1 \subset S$ and $I_2 \subset K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ such that $I = x_iI_1 + I_2$, $I_2 \subseteq I_1$ and $G(I)$ is the disjoint union of $G(x_iI_1)$ and $G(I_2)$, then I is vertex splittable.

In the case (ii), the decomposition $I = x_iI_1 + I_2$ is called a *vertex splitting* of I and x_i is called a *splitting vertex* of I. Vertex splittable ideals have linear quotients, and so they are componentwise linear [92]. Recall that a monomial ideal $I \subset S$ has *linear quotients* if the set $G(I)$ can be ordered as $u_1, \dots, u_m$ such that for each $i = 2, \dots, m$, the ideal $(u_1, \dots, u_{i-1}) : (u_i)$ is generated by variables.

The next lemma is crucial.

Lemma 4.2.3. *The ideal $B_{\mathbf{t}}(u)^\vee$ is vertex splittable. In particular one has*

$$B_{\mathbf{t}}(u)^\vee = x_1 B'_{\mathbf{t}}(u)^\vee + B'_{\mathbf{t}'}(u/x_{j_1})^\vee, \tag{4.2}$$

where $B'_{\mathbf{t}}(u) = B_{\mathbf{t}}(u) \cap K[x_2, \dots, x_n]$ and $\mathbf{t}' = (t_2, \dots, t_{d-1})$.

Proof. Let $I = B_{\mathbf{t}}(u) = I_\Delta$. It follows from [92, Lemma 2.2] that

$$I^\vee = I_\Delta^\vee = x_1 I_{\Delta_1}^\vee + I_{\Delta_2}^\vee$$

with $I_{\Delta_2}^\vee \subseteq I_{\Delta_1}^\vee$, and where Δ_1 and Δ_2 are defined as follows

$$\Delta_1 = \{F \in \Delta : 1 \notin F\}, \quad \Delta_2 = \{F \in \Delta : 1 \notin F \text{ and } F \cup \{1\} \in \Delta\}.$$

We are going to verify that $I_{\Delta_1} = B'_{\mathbf{t}}(u)$ and $I_{\Delta_2} = B'_{\mathbf{t}'}(u/x_{j_1})$.

Firstly we prove that $I_{\Delta_1} = B'_{\mathbf{t}}(u)$. The inclusion $B'_{\mathbf{t}}(u) \subseteq I_{\Delta_1}$ is trivial since the facets of the simplicial complex associated to the squarefree ideal $B'_{\mathbf{t}}(u)$ are exactly the facets of Δ not containing 1. To prove the other inclusion, let $F \in \Delta$ be a facet and suppose $1 \notin \Delta$. By Theorem 4.1.2, we have that F can be of the following two forms:

- $F = \text{supp}_{\mathbf{t}}(v)$, where $v \in G(I)$, or
- $F = \text{supp}_{\mathbf{t}}(x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s}) \cup [j_s + 1, n] \in \mathcal{G}$, where $s \in \{1, \dots, d-1\}$ and $x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s} \in G(B_{\mathbf{t}}(x_{j_1} \cdots x_{j_s}))$.

Firstly, let $F = \text{supp}_{\mathbf{t}}(v)$ for some $v \in G(I)$. Then $v \in B'_{\mathbf{t}}(u)$ since $1 \notin F$. Hence $\mathbf{x}_{[n]\setminus F} \in B'_{\mathbf{t}}(u)^\vee$.

Now, let $F \in \mathcal{G}$ as above. Then $x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s} \in B_{\mathbf{t}}(x_{j_1} \cdots x_{j_s}) \cap K[x_2, \dots, x_n]$ since $1 \notin F$. It follows that $\mathbf{x}_{[n]\setminus F} \in (B_{\mathbf{t}}(x_{j_1} \cdots x_{j_s}) \cap K[x_2, \dots, x_n])^\vee$. Note that, since $B_{\mathbf{t}}(u) \subset B_{\mathbf{t}}(x_{j_1} \cdots x_{j_s})$, one has $(B_{\mathbf{t}}(x_{j_1} \cdots x_{j_s}) \cap K[x_2, \dots, x_n])^\vee \subset B'_{\mathbf{t}}(u)^\vee$. Hence, $\mathbf{x}_{[n]\setminus F} \in B'_{\mathbf{t}}(u)^\vee$ once again. This proves that $I_{\Delta_1} \subseteq B'_{\mathbf{t}}(u)$.

With an analogous argument, one can verify that $I_{\Delta_2} = B'_{\mathbf{t}'}(u/j_1)$.

Finally, we prove that $I^\vee$ is vertex splittable. We proceed by double induction on the initial degree d of I and on the number of variables.

If $d = 1$, then I is a monomial prime ideal, $I^\vee$ is a monomial principal ideal and the statement follows for any number of variables.

Now, let $d > 1$. Notice that $n \geq 1 + t_1 + \cdots + t_{d-1}$, otherwise $I = (0)$. We proceed by induction on $n \geq 1 + t_1 + \cdots + t_{d-1}$. If $n = 1 + t_1 + \cdots + t_{d-1}$ then $I = (x_1 x_{1+t_1} \cdots x_{1+t_1+\cdots+t_{d-1}})$ and so $I^\vee = (x_1, x_{1+t_1}, \dots, x_{1+t_1+\cdots+t_{d-1}})$ is a monomial prime ideal which is vertex splittable. Now let $n > 1 + t_1 + \cdots + t_{d-1}$. By the first part of the proof we have

$$I = x_1 B'_{\mathbf{t}}(u) + B'_{\mathbf{t}'}(u/x_{j_1}). \tag{4.3}$$

Notice that $B'_{\mathbf{t}}(u)$ can be regarded as an ideal of $K[x_2, \dots, x_n]$ and so by induction on n is vertex splittable. Moreover, since $\deg(u/x_{j_1}) = d - 1 < d$, by induction on d the ideal $B'_{\mathbf{t}'}(u/x_{j_1})$ is also vertex splittable. Since $B'_{\mathbf{t}'}(u/x_{j_1}) = I_{\Delta_2} \subseteq I_{\Delta_1} = B'_{\mathbf{t}}(u)$, it follows that I is also vertex splittable, and this ends the inductive proof. □

Now, we are in the position to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $I = B_{\mathbf{t}}(u)$. By Lemma 4.2.3, $I^\vee$ is vertex splittable, hence by [92, Theorem 2.4] it has linear quotients. Therefore, by [68, Corollary 8.2.22], $I^\vee$ is componentwise linear. It follows, by [68, Theorem 8.2.20] that I is sequentially Cohen–Macaulay as stated. □

Lemma 4.2.3 combined with [92, Theorem 2.3] imply immediately

Corollary 4.2.4. *Let $B_{\mathbf{t}}(u) = I_\Delta$. Then Δ is vertex decomposable.*

The next example illustrates the previous results.

Example 4.2.5. Let $\mathbf{t} = (2, 1)$ and $I = B_{\mathbf{t}}(x_2x_5x_8) \subset K[x_1, \dots, x_8]$. Following Lemma 4.2.3, we can write $I = x_1I_2 + I_1$ where

$$\begin{aligned} I_1 &= (x_3x_4,\ x_3x_5,\ x_3x_6,\ x_3x_7,\ x_3x_8,\ x_4x_5,\ x_4x_6,\ x_4x_7,\ x_4x_8,\ x_5x_6,\ x_5x_7,\ x_5x_8),\\ I_2 &= (x_2x_4x_5,\ x_2x_4x_6,\ x_2x_4x_7,\ x_2x_4x_8,\ x_2x_5x_6,\ x_2x_5x_7,\ x_2x_5x_8). \end{aligned}$$

Let $\mathbf{t}' = (1)$. Then $I^\vee = x_1I_1^\vee + I_2^\vee$ is vertex splittable, with

$$\begin{aligned} I_1^\vee &= B'_{\mathbf{t}}(x_2x_5x_8)^\vee = (x_2,\ x_4x_5,\ x_4x_6x_7x_8,\ x_5x_6x_7x_8),\\ I_2^\vee &= B'_{\mathbf{t}'}(x_5x_8)^\vee = (x_3x_4x_5,\ x_3x_4x_6x_7x_8,\ x_3x_5x_6x_7x_8,\ x_4x_5x_6x_7x_8). \end{aligned}$$

Using the *Macaulay2* [62] package `SCMAlgebras` [84, 85] one can verify that I is sequentially Cohen–Macaulay. Alternatively, using the package `HomologicalShiftIdeals` [53] we can check that $I^\vee$ has linear quotients, thus it is componentwise linear.

4.3 Symbolic powers and normal torsionfreeness

Let $I = P_{F_1} \cap \cdots \cap P_{F_m}$ be the minimal primary decomposition of a squarefree monomial ideal $I \subset S$. We say that I is *normally torsionfree* if $\mathrm{Ass}(I^k) \subseteq \mathrm{Ass}(I)$ for all $k \geq 1$. The kth *symbolic power* of I is the monomial ideal defined as

$$I^{(k)} = P_{F_1}^k \cap \cdots \cap P_{F_m}^k.$$

We have $I^k \subseteq I^{(k)}$ for all $k \geq 1$. By [68, Theorem 1.4.6], the equality $I^{(k)} = I^k$ holds for all $k \geq 1$ if and only if I is normally torsionfree.

The goal of this last section is to prove the following characterization.

Theorem 4.3.1. *Let $I = B_{\mathbf{t}}(u)$. The following statements are equivalent.*

(a) *I is normally torsionfree.*

(b) *$I^{(k)} = I^k$, for all integers $k \geq 1$.*

(c) *$j_i \leq \sum_{s=1}^{i} t_s$ for all $i = 1, \dots, d-1$.*

The proof of this theorem requires some preparation. Let $I \subset S$ be a monomial ideal and let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$. The $\mathbf{a}$-*restriction* of I is defined as the monomial ideal $I^{\leq \mathbf{a}} \subset S$ with minimal generating set

$$G(I^{\leq \mathbf{a}}) = \{x_1^{b_1} \cdots x_n^{b_n} \in G(I) :\ b_i \leq a_i \text{ for all } i\}.$$

We recall the following result proved by Sayedsadeghi, Nasernejad and Qureshi in [101] (see also [102]). Let $\mathbf{e}_i \in \mathbb{Z}^n_{\geq 0}$ be the vector with ith component equal to 1 and with the other components equal to zero, and set $\mathbf{1} = \mathbf{e}_1 + \cdots + \mathbf{e}_n = (1, \dots, 1) \in \mathbb{Z}^n_{\geq 0}$.

Theorem 4.3.2. (Sayedsadeghi–Nasernejad–Qureshi, 2022 [101, Theorem 3.7],[102]) *Let $I \subset S$ be a squarefree monomial ideal. Suppose there exists a squarefree monomial $v \in S$ such that $v \in P \setminus P^2$ for all $P \in \mathrm{Ass}(I)$, and $(I^{\leq \mathbf{1}-\mathbf{e}_i})^{(k)} = (I^{\leq \mathbf{1}-\mathbf{e}_i})^k$ for all $k \geq 1$ and all $i \in \mathrm{supp}(v)$. Then $I^{(k)} = I^k$ for all $k \geq 1$.*

The following lemma will be needed later.

Lemma 4.3.3. *Let $J \subset S$ be a monomial ideal, let $u \in S$ be a monomial and set $I = uJ$. Then, $I^{(k)} = I^k$ for all $k \geq 1$, if and only if, $J^{(k)} = J^k$ for all $k \geq 1$*

Proof. It is shown in [57, Proposition 6.3] that $I^{(k)} = u^k J^{(k)}$ for all $k \geq 1$. The assertion now follows at once. □

Let $I \subset S$ be a graded ideal and let $\mathcal{R}(I) = \bigoplus_{k\geq 0} I^k t^k$ be its Rees algebra. The *analytic spread* of I, denoted by $\ell(I)$, is defined as the Krull dimension of the ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) = \bigoplus_{k\geq 0}(I^k/\mathfrak{m}I^k)t^k$, where $\mathfrak{m} = (x_1, \dots, x_n)$.

Proposition 4.3.4. *Let $I \subset S$ be a graded ideal such that $\mathcal{R}(I)$ is Cohen–Macaulay. Suppose that $\ell(I) < n$. Then $\mathfrak{m} = (x_1, \dots, x_n) \notin \mathrm{Ass}(I^k)$ for all $k \gg 0$.*

Proof. By [68, Proposition 10.3.2], $\lim_{k\to\infty} \operatorname{depth} S/I^k = n - \ell(I)$. Our assumption implies that $\operatorname{depth} S/I^k > 0$ for all $k \gg 0$. Hence $\mathfrak{m} \notin \mathrm{Ass}(I^k)$ for all $k \gg 0$. □

Corollary 4.3.5. *Let $I = B_{\mathbf{t}}(u) \subset S$ be a squarefree principal vector–spread Borel ideal. If $\ell(I) < n$ then $\mathfrak{m} \notin \mathrm{Ass}(I^k)$ for all $k \geq 1$.*

Proof. By [29, Proposition 1], I is vertex splittable. By [91, Theorem 3.1(a)-(b)] it follows that $\mathcal{R}(I)$ is Cohen–Macaulay and $\mathrm{Ass}(I) \subseteq \mathrm{Ass}(I^2) \subseteq \mathrm{Ass}(I^3) \subseteq \cdots$. Proposition 4.3.4 and our assumption imply that $\mathfrak{m} \notin \mathrm{Ass}(I^k)$ for all $k \gg 0$. The previous ascending chain of inclusions yields that $\mathfrak{m} \notin \mathrm{Ass}(I^k)$ for all $k \geq 1$. □

Let $I \subset S$ be a monomial ideal with $G(I) = \{u_1, \dots, u_m\}$. The *linear relation graph* Γ of I is defined as the graph with the edge set

$$E(\Gamma) = \{\{i, j\} : \text{there exist } u_k, u_\ell \in G(I) \text{ such that } x_i u_k = x_j u_\ell\},$$

and with the vertex set $V(\Gamma) = \bigcup_{\{i,j\}\in E(\Gamma)}\{i, j\}$.

If $I \subset S$ is a monomial ideal generated in degree d having linear relations (e.g. I has linear resolution) then [36, Lemma 5.2] guarantees that

$$\ell(I) = r - s + 1 \tag{4.4}$$

where r is the number of vertices and s is the number of connected components of the linear relation graph of I.

Let $I = B_{\mathbf{t}}(u)$ as in the Setup 4.1.1, and define the sets

$$B_r = \Big[\sum_{s=1}^{r-1} t_s + 1,\, j_r\Big], \quad r = 1, \dots, d.$$

Notice that $v = x_{i_1} \cdots x_{i_d} \in G(I)$ if and and only if $i_r \in B_r$, for all $r = 1, \dots, d$, and $i_r - i_{r-1} \geq t_r$ for all $r = 2, \dots, d$.

Proposition 4.3.6. *Let $I = B_{\mathbf{t}}(u)$ and $B_i = [\sum_{s=1}^{i-1} t_s + 1, j_i]$ for $i = 1, \dots, d$. Let Γ be the linear relation graph of I. The following statements hold.*

(a) *We have $|B_i| \geq 2$ if and only if $B_i \subseteq V(\Gamma)$.*

(b) *Suppose that $j_i \leq \sum_{s=1}^{i} t_s$ for all $i = 1, \dots, d-1$. Then $B_p \cap B_q = \emptyset$ for all $1 \leq p < q \leq d$ and $\mathfrak{m} \notin \mathrm{Ass}(I^k)$ for all $k \geq 1$.*

Proof. (a) Suppose that $B_i \subseteq V(\Gamma)$ and assume for a contradiction that $|B_i| = 1$, for some i. Then $j_i = (\sum_{s=1}^{i-1} t_s) + 1$. It follows that each generator of $B_{\mathbf{t}}(u)$ is a multiple of x_{j_i} and this implies that $B_i = \{j_i\} \nsubseteq V(\Gamma)$, a contradiction.

Conversely, suppose that $|B_i| \geq 2$. We claim that if $|B_i| \geq 2$, then the induced subgraph of Γ on B_i is a complete graph. This implies that $B_i \subseteq V(\Gamma)$.

Let $h_1, h_2 \in B_i$ with $h_1 < h_2 \leq j_i$. If $i = 1$, then $x_{h_1}(u/x_{j_1}), x_{h_2}(u/x_{j_1}) \in G(I)$ since I is $\mathbf{t}$-spread. Hence $\{h_1, h_2\} \in E(\Gamma)$. If $i = d$, then $vx_{h_1}, vx_{h_2} \in G(I)$ where $v = x_1 x_{1+t_1} \cdots x_{1+t_1+\dots+t_{d-2}}$. Hence $\{h_1, h_2\} \in E(\Gamma)$. Finally, let $1 < i < d$. Then $\sum_{s=1}^{i-1} t_s + 1 \leq h_1 < h_2 \leq j_i$. We have $wx_{h_1}, wx_{h_2} \in G(I)$ where $w = x_{j_d}(v/x_{1+t_1+\dots+t_{i-1}})$. Hence $\{h_1, h_2\} \in E(\Gamma)$ as desired.

(b) Suppose for a contradiction that $B_p \cap B_q \neq \emptyset$ for some $1 \leq p < q \leq d$. Then $j_p \geq \sum_{s=1}^{q-1} t_s + 1 > \sum_{s=1}^{p} t_s$. Since by assumption we have $j_p \leq \sum_{s=1}^{p} t_s$, we reach the desired contradiction.

Next, we claim that Γ does not contain any edge with one endpoint in B_p and the other endpoint in B_q for some $1 \leq p < q \leq d$. If $|B_p| = 1$ or $|B_q| = 1$, then (a) implies the assertion. Now, let $|B_p| \geq 2$ and $|B_q| \geq 2$. Recall that $B_p \cap B_q = \emptyset$ for all $1 \leq p < q \leq d$. Take $h_1 \in B_p$ and $h_2 \in B_q$ for some $1 \leq p, q \leq d$ with $p \neq q$. Then for any $v \in G(I)$ with x_{h_1} dividing v, we have $x_{h_2}(v/x_{h_1}) \notin G(I)$. Indeed, v is not $\mathbf{t}$-spread because the support of this monomial contains two distinct elements from B_q. This yields the assertion.

Now, if $I = (x_1 x_{1+t_1} \cdots x_{1+t_1+\dots+t_{d-1}})$ is a principal ideal, there is nothing to prove. Suppose that I is not principal. By Corollary 4.3.5 it is enough to show that $\ell(I) < n$. By [55, Corollary 5.3(c)], I has linear resolution. Hence, formula (4.4) holds. So it is enough to prove that the linear relation graph Γ of I has more than one connected component. Since I is not principal, we have $j_i > (\sum_{s=1}^{i-1} t_s) + 1$ for some i. Hence $|B_i| \geq 2$ for some i. By part (a) and the previous claim, it follows that Γ is the disjoint union of the complete graphs on the vertex sets B_i for which $|B_i| \geq 2$. Hence Γ has only one connected component if and only if $|B_d| \geq 2$ and $|B_i| = 1$, for all $i = 1, \dots, d-1$. In this case, $\ell(I) = |B_d| < n$. Otherwise, Γ has at least two connected components and we have again $\ell(I) < n$, as desired. □

Let $I \subset S$ be a monomial ideal and $P \subset S$ be a monomial prime ideal. The *monomial localization* of I with respect to P is the monomial ideal of the polynomial ring $R = K[x_j : x_j \in P]$ defined as $I(P) = \varphi(I)$ where $\varphi : S \to R$ is the R-algebra homomorphism defined by setting $\varphi(x_j) = 1$ for all $x_j \notin P$.

We are now ready to prove Theorem 4.3.1.

Proof. The equivalence between (a) and (b) follows from [68, Theorem 1.4.6].

(b) $\Rightarrow$ (c) Let $I = B_{\mathbf{t}}(u)$ and assume that $I^{(k)} = I^k$ for all $k \geq 1$. If $d = 2$ then the statement follows from [93, Theorem 5.13]. Hence, we assume that $d \geq 3$.

Suppose by contradiction that (c) does not hold. Then, there is at least one index $1 \leq \ell \leq d-1$ such that $j_\ell > \sum_{s=1}^{\ell} t_s$. Let ℓ be the smallest such integer. Let $v = x_{h_1} \cdots x_{h_d} \in G(I)$. Notice that for any $r = 1, \dots, \ell - 1$ we have

$$h_r \in B_r \subseteq \Big[\sum_{s=1}^{r-1} t_s + 1, \sum_{s=1}^{r} t_s\Big], \tag{4.5}$$

and for $r = \ell, \dots, d$ we have $h_r \geq (\sum_{s=1}^{\ell-1} t_s) + 1$.

Let $Q = P_{[(\sum_{s=1}^{\ell-1} t_s)+1, n]}$ and $R = K[x_j : x_j \in Q]$. The inclusion (4.5) implies that $I(Q) = B_{\mathbf{s}}(v) \cap R$ where $\mathbf{s} = (t_\ell, \dots, t_{d-1})$ and $v = u/(x_{j_1} \cdots x_{j_{\ell-1}})$. Hence $I(Q)$ is a

$\mathbf{s}$-spread monomial ideal in the polynomial ring R with $\min(v) = j_\ell \geq 1 + \sum_{s=1}^{\ell} t_s$. Moreover, $I(Q)^{(k)} = I(Q)^k$ for all $k \geq 1$. Therefore, this reduction shows that we may assume from the very beginning that $\ell = 1$ and that $\min(u) = j_1 \geq t_1 + 1$.

To finish the proof, we will show that $I^{(2)} \neq I^2$, reaching the desired contradiction. To this end, we claim that

$$w = (x_1 \cdots x_{t_1} x_{t_1+1})(x_{j_2} \cdots x_{j_d}) \in I^{(2)} \setminus I^2.$$

Firstly, let us show that $w \in I^{(2)}$. Since $I^{(2)} = \bigcap_{P \in \mathrm{Ass}(I)} P^2$, we must prove that $w \in P^2$ for all $P \in \mathrm{Ass}(I)$.

Let $P \in \mathrm{Ass}(I)$. By Theorem 4.1.2, either $P = P_{[n]\setminus \mathrm{supp}_{\mathbf{t}}(v)}$ for some $v \in G(I)$ or $P = P_{[n]\setminus G}$ for some $G \in \mathcal{G}$, where $\mathcal{G}$ is defined in the statement of Theorem 4.1.2.

Suppose $P = P_{[n]\setminus \mathrm{supp}_{\mathbf{t}}(v)}$. If $\min(v) \geq 3$ then $x_1x_2 \in P^2$ and so $w \in P^2$. If $\min(v) = 2$, then $[2, t_1 + 1] \subseteq \mathrm{supp}_{\mathbf{t}}(v)$. Hence $x_1 \in P$ but $x_{1+t_1} \notin P$. Since $x_{1+t_1} x_{j_2} \cdots x_{j_d} \in I \subset P$, $x_{1+t_1} \notin P$ and P is a prime ideal, it follows that $x_{j_s} \in P$ for some $s = 2, \dots, d$. Hence $x_1 x_{j_s} \in P^2$ and so $w \in P^2$. Finally, if $\min(v) = 1$, then $[1, t_1] \subseteq \mathrm{supp}_{\mathbf{t}}(v)$. Hence $x_{1+t_1} \in P$ but $x_1 \notin P$. Since $x_1 x_{j_2} \cdots x_{j_d} \in I \subset P$ it follows that $x_{j_s} \in P$ for some $s = 2, \dots, d$. Hence $x_{1+t_1} x_{j_s} \in P^2$ and so $w \in P^2$.

Suppose now $P = P_{[n]\setminus G}$ with $G \in \mathcal{G}$. Then $G = \mathrm{supp}_{\mathbf{t}}(x_{\ell_1} \cdots x_{\ell_{s-1}} x_{j_s}) \cup [j_s + 1, n]$ where $x_{\ell_1} x_{\ell_2} \cdots x_{\ell_{s-1}} x_{j_s} \in G(B_{\mathbf{t}}(x_{j_1} x_{j_2} \cdots x_{j_{s-1}} x_{j_s}))$, for some $s = 1, \dots, d$. If $s = 1$, then $x_1 x_2 \in P^2$ and so $w \in P^2$ too. Suppose $s > 1$. Then $x_{j_s} \in P$. Moreover, applying the same reasoning as before on the integer ℓ_1, it follows that $x_1x_2 \in P^2$ or $x_1 \in P$ or $x_{t_1+1} \in P$. In each possible case, we have $w \in P^2$, as desired.

Hence $w \in I^{(2)}$. Suppose for a contradiction that $w \in I^2$. Then, there would exist squarefree monomials $v_1, v_2 \in G(I)$, $g \in S$ such that

$$w \;=\; (x_1 \cdots x_{t_1} x_{t_1+1})(x_{j_2} \cdots x_{j_d}) \;=\; v_1 v_2 g. \tag{4.6}$$

Write $v_1 = x_{p_1} \cdots x_{p_d}$. Then $p_1 \in [1, t_1 + 1] \cup \{j_2, \dots, j_d\}$. We claim that $p_1 \in [1, t_1]$. Suppose, otherwise, that $p_1 \in \{j_2, \dots, j_d\}$. Since $p_d > \cdots > p_2 > p_1$ it would follow from (4.6) that v_1 divides $x_{j_2} \cdots x_{j_d}$. This is impossible because v_1 has degree d. Hence $p_1 \in [1, t_1 + 1]$. Since $p_2 \geq p_1 + t_1$ it follows that $p_2 \in \{j_2, \dots, j_d\}$. Therefore $x_{p_2} \cdots x_{p_d} = x_{j_2} \cdots x_{j_d}$. By equation (4.6), it then follows that

$$x_1 \cdots x_{p_1-1} x_{p_1+1} \cdots x_{t_1+1} \;=\; v_2 g \in I.$$

This is impossible, because v_2 is a $\mathbf{t}$-spread monomial of degree $d \geq 3$.

(c) $\Rightarrow$ (a) By assumption we have $j_i \leq \sum_{s=1}^{i} t_s$ for all $i = 1, \dots, d-1$. We prove by finite induction on $\sum_{i \in \mathrm{supp}(u)} i$ that $I = B_{\mathbf{t}}(u)$ is normally torsionfree.

When $\sum_{i \in \mathrm{supp}(u)} i$ is minimal, then $u = x_1 x_{1+t_1} \cdots x_{1+t_1+\cdots+t_{d-1}}$, I is a principal ideal and so I is normally torsionfree.

Now suppose that $\sum_{i \in \mathrm{supp}(u)} i$ is not minimal.

Notice that $j_i \geq \sum_{r=1}^{i-1} t_r + 1$ for all $r = 1, \dots, d$. We may assume that $j_i > \sum_{r=1}^{i-1} t_r + 1$ for all $i = 1, \dots, d$. Indeed suppose that this is not the case and let r be the biggest integer with this property. Then $B_{\mathbf{t}}(u) = (x_{j_1} \cdots x_{j_r}) B_{\mathbf{s}}(v)$ where $\mathbf{s} = (t_{r+1}, \dots, t_{d-1})$ and $v = u/(x_{j_1} \cdots x_{j_r})$, and where $B_{\mathbf{s}}(v)$ is a $\mathbf{s}$-spread ideal in the polynomial ring $K[x_{j_r+t_r}, \dots, x_n]$ with fewer variables. Then, by Lemma 4.3.3, $B_{\mathbf{t}}(u)$ is normally torsionfree if and only if $B_{\mathbf{s}}(v)$ is such.

Hence we can assume that $j_i > \sum_{r=1}^{i-1} t_r + 1$ for all $i = 1, \dots, d$. To prove (a) we apply Theorem 4.3.2.

Let $I = B_{\mathbf{t}}(u)$ and let $u = x_{j_1}\cdots x_{j_d}$ be the vector–spread Borel generator of I. Firstly, let us verify that $u \in P \setminus P^2$ for all $P \in \text{Ass}(I)$. Since $I \subset P$ for all $P \in \text{Ass}(I)$, we only need to verify that $u \notin P^2$ for all $P \in \text{Ass}(I)$. This is equivalent to show that $|G(P) \cap \{x_{j_1}, \dots, x_{j_d}\}| \le 1$ for all $P \in \text{Ass}(I)$.

Let $P \in \text{Ass}(I)$. By Theorem 4.1.2, either $P = P_{[n]\setminus \text{supp}_{\mathbf{t}}(v)}$ for some $v \in G(I)$, or else $P = P_{[n]\setminus G}$ for some $G \in \mathcal{G}$. We distinguish the two cases.

Suppose $P = P_{[n]\setminus \text{supp}_{\mathbf{t}}(v)}$. We claim that $G(P) \cap \{x_{j_1}, \dots, x_{j_d}\} = \{x_{j_d}\} = \{x_n\}$ (Setup 4.1.1) which implies the desired assertion. From the definition of $\mathbf{t}$-spread support it follows immediately that $x_{j_d} \notin \text{supp}_{\mathbf{t}}(v)$ and so $x_{j_d} = x_n \in G(P)$. Now we show that $x_{j_s} \notin G(P)$ for $s = 1, \dots, d-1$. Equivalently, we show that $j_s \in \text{supp}_{\mathbf{t}}(v)$ for $s = 1, \dots, d-1$. Let $v = x_{i_1}\cdots x_{i_d}$. Then $i_s \in B_s = [\sum_{r=1}^{s-1} t_r + 1, j_s]$, for all $s = 1, \dots, d-1$. It follows that

$$i_s + t_s - 1 \ge \sum_{r=1}^{s-1} t_r + 1 + t_s - 1 = \sum_{r=1}^{s} t_r \ge j_s \ge i_s.$$

Hence $j_s \in [i_s, i_s + t_s - 1] \subseteq \text{supp}_{\mathbf{t}}(v)$, as desired.

Suppose $P = P_{[n]\setminus G}$ with $G \in \mathcal{G}$. Then $G = \text{supp}_{\mathbf{t}}(x_{\ell_1}\cdots x_{\ell_{s-1}}x_{j_s}) \cup [j_s + 1, n]$ for some $x_{\ell_1}x_{\ell_2}\cdots x_{\ell_{s-1}}x_{j_s} \in G(B_{\mathbf{t}}(x_{j_1}x_{j_2}\cdots x_{j_{s-1}}x_{j_s}))$ and some $s = 1, \dots, d$. We claim that $G(P) \cap \{x_{j_1}, \dots, x_{j_d}\} = \{x_{j_s}\}$. Notice that $[n] \setminus G = [j_s] \setminus \text{supp}_{\mathbf{t}}(x_{\ell_1}\cdots x_{\ell_{s-1}}x_{j_s})$. Then, the same argument as before shows that $G(P) \cap \{x_{j_1}, \dots, x_{j_d}\} = \{x_{j_s}\}$.

Now to apply Theorem 4.3.2 it remains to prove that $(I^{\le \mathbf{1}-\mathbf{e}_{j_i}})^{(k)} = (I^{\le \mathbf{1}-\mathbf{e}_{j_i}})^k$ for all $k \ge 1$ and all $i = 1, \dots, d$.

Let $1 \le i \le d$ and $B_i = [\sum_{s=1}^{i-1} t_s + 1, j_i]$, for $i = 1, \dots, d$. Our assumption and Proposition 4.3.6 imply that $|B_i| \ge 2$ for all i and that the B_i's are pairwise disjoint. Set $w = x_{j_i-1}(u/x_{j_i})$. If $j_i - j_{i-1} > t_i$, then it is clear that $I^{\le \mathbf{1}-\mathbf{e}_{j_i}} = B_{\mathbf{t}}(w)$. Since $j_r \le \sum_{s=1}^{r} t_s$, for all $r = 1, \dots, d$ with $r \ne i$, $j_i - 1 < j_i \le \sum_{s=1}^{i} t_s$ and $\sum_{i \in \text{supp}(w)} i < \sum_{i \in \text{supp}(u)} i$, it follows by induction that $(I^{\le \mathbf{1}-\mathbf{e}_{j_i}})^{(k)} = (I^{\le \mathbf{1}-\mathbf{e}_{j_i}})^k$ for all $k \ge 1$, as wanted. Hence, Theorem 4.3.2 implies that $I^{(k)} = I^k$ for all $k \ge 1$.

If otherwise $j_i - j_{i-1} = t_i$, then we define the integer

$$\ell \;=\; \max\{m :\ j_p - j_{p-1} = t_{p-1} \text{ for all } p = m, \dots, i\},$$

and we set

$$v \;=\; \begin{cases} \Big(\prod_{r=1}^{\ell-2} x_{j_r}\Big)\Big(\prod_{r=\ell-1}^{i} x_{j_r-1}\Big)\Big(\prod_{r=i+1}^{d} x_{j_r}\Big) & \text{if } j_i - j_{i-1} = t_{i-1}, \\ x_{j_i-1}(u/x_{j_i}) & \text{otherwise.} \end{cases}$$

We claim that $I^{\le \mathbf{1}-\mathbf{e}_{j_i}} = B_{\mathbf{t}}(v)$. If $v = x_{j_i-1}(u/x_{j_i})$ this is clear. Now, assume we are in the first case and write $v = x_{h_1}\cdots x_{h_d}$. Notice that $h_r \ge (\sum_{s=1}^{r-1} t_s) + 1$ for all $r = 1, \dots, d$, because $j_r > \sum_{s=1}^{r-1} t_s + 1$ for all $r = 1, \dots, d$. Moreover, the definition of ℓ implies immediately that v is $\mathbf{t}$-spread.

Let $w = x_{q_1}\cdots x_{q_d} \in G(B_{\mathbf{t}}(v))$. Then w is $\mathbf{t}$-spread, $\sum_{s=1}^{r-1} t_s + 1 \le q_r \le h_r \le j_r$ for all $r \in [1, \ell-2] \cup [i+1, d]$, and $\sum_{s=1}^{r-1} t_s + 1 \le q_r \le h_r \le j_r - 1$, for all $r \in [\ell-1, i]$. In particular, $\sum_{s=1}^{i-1} t_s + 1 \le q_i \le h_i \le j_i - 1$. This implies that x_{j_i} does not divide w and so $w \in G(I^{\le \mathbf{1}-\mathbf{e}_{j_i}})$.

Conversely, take $w = x_{q_1}\cdots x_{q_d} \in G(I^{\le \mathbf{1}-\mathbf{e}_{j_i}})$. Then w is $\mathbf{t}$-spread with $q_r \in B_r$ for all r and $j_i \notin \text{supp}(w)$. We claim that $q_r \le h_r$ for all $r = 1, \dots, d$. For $r \in [1, \ell-2] \cup [i+1, d]$

this is clear. For $r = i$ we have $q_i \in B_i$ and $q_i \neq j_i$. Hence $q_i \leq j_i - 1 = h_i$. Now suppose for a contradiction that q_{i-1} is not less or equal to $j_{i-1} - 1 = h_{i-1}$. Then, since $q_{i-1} \in B_{i-1}$ we have $q_{i-1} = j_{i-1}$. Since w is $\mathbf{t}$-spread we have $q_i - q_{i-1} - t_{i-1} \geq 0$. However, since $q_i \leq j_i - 1$ and $q_{i-1} = j_{i-1}$ we then have

$$q_i - q_{i-1} - t_{i-1} = j_i - 1 - j_{i-1} - t_{i-1} = -1$$

by the meaning of ℓ, and this is absurd. Hence $q_{i-1} \leq h_{i-1}$. Proceeding in a similar way we obtain that $q_r \leq h_r = j_r - 1$ for all $r = \ell - 1, \dots, i$. Hence $I^{\leq \mathbf{1}-\mathbf{e}_{j_i}} \subseteq B_{\mathbf{t}}(v)$.

Therefore $I^{\leq \mathbf{1}-\mathbf{e}_{j_i}} = B_{\mathbf{t}}(v)$. Since $h_r \leq j_r \leq \sum_{s=1}^{r} t_s$ for all r and $\sum_{i \in \operatorname{supp}(v)} i < \sum_{i \in \operatorname{supp}(u)} i$, it follows by induction that $(I^{\leq \mathbf{1}-\mathbf{e}_{j_i}})^{(k)} = (I^{\leq \mathbf{1}-\mathbf{e}_{j_i}})^k$ for all $k \geq 1$, as we wanted to show. Finally, Theorem 4.3.2 implies that $I^{(k)} = I^k$ for all $k \geq 1$. $\square$

Chapter 5

Squarefree powers of edge ideals via Betti splittings

All the graphs we consider in this chapter are finite simple graphs. Let G be a graph with the vertex set $V(G) = \{1, \dots, n\}$ and the edge set $E(G)$. A *k-matching* of G is a subset M of $E(G)$ of size k such that $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$. We denote by $V(M)$ the vertex set of M, that is, the set $\{i \in V(G) \ : \ i \in e \text{ for } e \in M\}$. We say that a graph H is a *subgraph of* G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is said an *induced subgraph* if for any two vertices i, j in H, $\{i, j\} \in E(H)$ if and only if $\{i, j\} \in E(G)$. If A is a subset of $V(G)$, the *induced subgraph* on A, denoted by G_A, is the graph with vertex set A and the edge set $\{\{i, j\} : i, j \in A \text{ and } \{i, j\} \in E(G)\}$. A matching M is called an *induced matching* if $E(G_{V(M)}) = M$. The *matching number* of G, denoted by $\nu(G)$, is the maximum size of a matching of G. Whereas, the *induced matching number* of G, denoted by $\text{indm}(G)$, is the maximum size of an induced matching of G. It is clear that $\text{indm}(G) \le \nu(G)$ for any graph G.

Matchings play a pivotal role in graph theory [65]. Hereafter, we set $[n] = \{1, \dots, n\}$ if $n \ge 1$ is an integer, and $[0] = \emptyset$. Let G be a graph on the vertex set $[n]$, and let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring over a field K. For a non–empty subset A of $[n]$, we set $\mathbf{x}_A = \prod_{i \in A} x_i$. Let $1 \le k \le \nu(G)$. We denote by $I(G)^{[k]}$ the squarefree monomial ideal generated by $\mathbf{x}_{V(M)}$ for all k-matchings M of G. If $k = 1$, then $I(G)^{[1]}$ is the well–known ideal, called the *edge ideal* of G [118, 119], and we denote it simply by $I(G)$. More recently, matching theory has been related to Commutative Algebra *via* the notion of *squarefree powers* [16, 47].

Let us explain the connection with the concept of squarefree power (see, for instance, [16]). Let $I \subset S$ be a squarefree monomial ideal and $G(I)$ be its unique minimal set of monomial generators. The *kth squarefree power* of I, denoted by $I^{[k]}$, is the ideal generated by the squarefree monomials of I^k. Thus, $u_1 u_2 \cdots u_k$, $u_i \in G(I)$, $i \in [k]$, belongs to $G(I^{[k]})$ if and only if $u_1, u_2, \dots, u_k$ is a regular sequence. Let $\nu(I)$ be the *monomial grade* of I (*i.e.* the maximum among the lengths of a monomial regular sequence contained in I). Then $I^{[k]} \neq (0)$ if and only if $k \le \nu(I)$. Hence, the ideal $I(G)^{[k]}$ is the kth squarefree power of $I(G)$ and $\nu(I(G)) = \nu(G)$.

The study of ordinary powers of ideals is a classical subject in Commutative Algebra. The fascination with this topic is due to the fact that many algebraic invariants behave asymptotically well, that is, stabilize or show a regular behaviour for sufficiently high powers. The first result in this area was obtained by Brodmann [23] who showed that the depth of the powers of an ideal I of a Noetherian ring R is eventually constant: there exists $k_0 > 0$ such that $\text{depth}(R/I^k) = \text{depth}(R/I^{k_0})$ for all $k \ge k_0$. The study of the initial behaviour of the depth function of powers of graded ideals $I \subset S$ was initiated by Herzog and Hibi [67]. They made the boldest guess possible: any bounded convergent function $\varphi : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$ is the depth function of some suitable graded ideal in a polynomial ring. Recently, this conjecture has been settled in affirmative by Hà et al. [63, Theorem 4.1].

Let $I \subset S$ be a squarefree monomial ideal and $k \in [\nu(I)]$. Let $d_k = \text{indeg}(I^{[k]})$ be the *initial degree* of $I^{[k]}$, that is, the minimum degree of a monomial of $G(I^{[k]})$. In [48, Proposition 1.1], Erey et al. established that

$$\text{depth}(S/I^{[k]}) \geq d_k - 1$$

for all $k \in [\nu(I)]$. Based on this, they introduced the *normalized depth function* of I, as follows

$$g_I(k) = \text{depth}(S/I^{[k]}) - (d_k - 1)$$

where $k \in [\nu(I)]$. In contrast to the behaviour of the depth function of ordinary powers, it was predicted in [48] as follows.

Conjecture: *For any $I \subset S$, the function $g_I(k)$ is a non–increasing function.*

Squarefree powers were introduced in [16] and are currently studied by many researchers [46, 49, 47, 48, 54, 56, 108, 109].

Forests are among the simplest graphs. In this chapter, we deeply investigate the squarefree powers of edge ideals of forests. Our result are based on the technique of *Betti splitting*. In their groundbreaking article, Eliahou and Kervaire considered an earlier version of Betti splitting [41, Proposition 3.1]. Later on, Francisco et al., inspired by the results in [41], introduced the notion of Betti splitting [58, 115]. Roughly speaking, if $I \subset S$ is a monomial ideal and I_1 and I_2 are sub–ideals of I such that $G(I)$ is the disjoint union of $G(I_1)$ and $G(I_2)$, then $I = I_1 + I_2$ is said to be a Betti splitting if the minimal free resolution of I can be recovered from the minimal free resolutions of I_1, I_2 and $I_1 \cap I_2$.

The chapter is structured as follows.

In Section 5.1, we consider Betti splittings of squarefree powers. Theorem 5.1.2 is a technical but powerful criterion that can be used to establish whether a certain decomposition $I = I_1 + I_2$ is a Betti splitting. This fact is used in Lemma 5.1.5 to reprove a result in [56] under weaker assumptions (Proposition 5.1.6).

Section 5.2 deeply investigates some Betti splittings of squarefree powers of edge ideals of forests. These Betti splittings will serve as important tools in the proofs in Sections 5.3, and 5.4. For a vertex v in G, we say w is a *neighbor* of v if $\{v, w\} \in E(G)$. We denote the set of all neighbors of v by $N_G(v)$. $N_G(v)$ is called the *neighborhood* of v. The *degree* of v, $\text{deg}_G(v)$, is the number $|N_G(v)|$. If $\text{deg}_G(v) = 1$, v is called a *leaf*. Any forest has at least two leaves. A leaf $v \in V(G)$, with unique neighbor w, is called a *distant leaf* if at most one of the neighbors of w is not a leaf. The existence of a distant leaf in any forest G (Proposition 5.2.1) provides a natural Betti splitting for $I(G)^{[k]}$ (Proposition 5.2.5).

In Section 5.3, using induction, we prove that $g_{I(G)}$ is non–increasing, as stated in Theorem 5.3.4. We also compute the normalized depth function of the edge ideal of a path and use this result to obtain a general upper bound for the normalized depth function of the edge ideal of any graph.

In the last section, we compute the Castelnuovo–Mumford regularity $\text{reg}(I(G)^{[k]})$ in terms of the combinatorics of the forest G (Theorem 5.4.6), solving affirmatively a conjecture due to Erey and Hibi [49, Conjecture 31].

Finally, we would like to remark that not all squarefree monomial ideals admit a Betti splitting, as pointed out by Bolognini [18, Example 4.6]; see also [58, Example 4.2]. On the other hand, Francisco, Hà and Van Tuyl showed that the edge ideal of any graph always admits a Betti splitting [58, Corollary 3.1]. It is an open question whether all the squarefree powers of an edge ideal admit a Betti splitting.

The results contained in this chapter first appeared in [33].

5.1 Betti splittings for squarefree powers

Let $S = K[x_1, \dots, x_n]$ be the standard graded polynomial ring over a field K. Let $I \subset S$ be a monomial ideal. By $G(I)$, we denote the unique minimal set of monomial generators of I. Let $I_1, I_2 \subset S$ monomial ideals such that $G(I)$ is the disjoint union of $G(I_1)$ and $G(I_2)$. We say that $I = I_1 + I_2$ is a *Betti splitting* [58, Definition 1.1] if

$$\beta_{i,j}(I) = \beta_{i,j}(I_1) + \beta_{i,j}(I_2) + \beta_{i-1,j}(I_1 \cap I_2), \quad \text{for all } i, j \geq 0. \tag{5.1}$$

When $I = I_1 + I_2$ is a Betti splitting, important homological invariants of I can be deduced from the corresponding invariants of the smaller ideals. More in detail, the next result holds true.

Proposition 5.1.1. (Francisco–Hà–Van Tuyl, 2009 [58, Corollary 2.2]) *Let $I = I_1 + I_2$ be a Betti splitting. Then*

$$\begin{aligned} \operatorname{reg}(I) &= \max\{\operatorname{reg}(I_1), \operatorname{reg}(I_2), \operatorname{reg}(I_1 \cap I_2) - 1\}, \\ \operatorname{pd}(I) &= \max\{\operatorname{pd}(I_1), \operatorname{pd}(I_2), \operatorname{pd}(I_1 \cap I_2) + 1\}, \end{aligned}$$

where $\operatorname{reg}(-)$ *is the regularity, and* $\operatorname{pd}(-)$ *is the projective dimension.*

Consider the natural short exact sequence

$$0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to I \to 0.$$

Then (5.1) holds if and only if the following induced maps in Tor of the above sequence

$$\operatorname{Tor}_i^S(K, I_1 \cap I_2) \to \operatorname{Tor}_i^S(K, I_1) \oplus \operatorname{Tor}_i^S(K, I_2)$$

are zero for all $i \geq 0$ [58, Proposition 2.1]. In this case, we say that the inclusion map $I_1 \cap I_2 \to I_1 \oplus I_2$ is Tor-*vanishing* (see [2, 94] for more details on this subject).

The next criterion gives a useful method to prove that a map $J \to L$ is Tor-vanishing, where $(0) \neq J \subset L$ are monomial ideals of S. It appeared implicitly for the first time in [41] (see also the proof of [94, Lemma 4.2]).

Theorem 5.1.2. (Eliahou–Kervaire, 1990 [41, Proposition 3.1]) *Let $J, L \subset S$ be non–zero monomial ideals with $J \subset L$. Suppose there exists a map $\varphi : G(J) \to G(L)$ such that for any $\emptyset \neq \Omega \subseteq G(J)$ we have*

$$\operatorname{lcm}(u \ : \ u \in \Omega) \in \mathfrak{m}(\operatorname{lcm}(\varphi(u) : u \in \Omega)),$$

where $\mathfrak{m} = (x_1, x_2, \dots, x_n)$. Then the inclusion map $J \to L$ is Tor-*vanishing.*

Let I be a monomial ideal; following [94], we denote by $\partial^* I$ the ideal generated by the elements of the form f/x_i, where f is a minimal monomial generator of I and x_i is a variable dividing f.

Lemma 5.1.3. (Nguyen–Vu, 2019 [94, Proposition 4.4]) *Let $J, L \subset S$ be non–zero monomial ideals. Suppose that $\partial^* J \subseteq L$. Then $J \subseteq \mathfrak{m} L$ and there exists a map $\varphi : G(J) \to G(L)$ satisfying the assumption of Theorem 5.1.2. In particular, the inclusion map $J \to L$ is* Tor-*vanishing.*

Note that for all $1 \leq \ell < k \leq \nu(I)$, we have $\partial^* I^{[k]} \subset I^{[k-1]} \subseteq I^{[\ell]}$. Hence, the previous lemma implies immediately the following result.

Corollary 5.1.4. *Let $I \subset S$ be a squarefree monomial ideal. Then the map $I^{[k]} \longrightarrow I^{[\ell]}$ is* Tor*-vanishing for all $1 \leq \ell < k \leq \nu(I)$.*

Let $I = I_1 + I_2$ with $G(I_2) = \{u \in G(I) \,:\, x \text{ divides } u\}$ and $G(I_1) = G(I) \setminus G(I_2)$, for some variable x. We say that $I = I_1 + I_2$ is an *x-partition*. In addition, if it is a Betti splitting, we say that $I = I_1 + I_2$ is an *x-splitting* [58, Definition 2.6].

Lemma 5.1.5. *Let $I \subset K[x_1, \dots, x_{n-1}]$ be a squarefree monomial ideal. Then for all $1 < k \leq \nu(I)$*

$$(I, x_n)^{[k]} = I^{[k]} + x_n I^{[k-1]} \subset S \tag{5.2}$$

is a Betti splitting.

Proof. Note that $(I, x_n)^{[k]} = I^{[k]} + x_n I^{[k-1]}$ is a x_n-partition. Moreover,

$$I^{[k]} \cap x_n I^{[k-1]} = x_n[I^{[k]} \cap I^{[k-1]}] = x_n I^{[k]}.$$

Thus, to prove that (5.2) is a Betti splitting, it remains to be shown that the inclusion map $x_n I^{[k]} \longrightarrow I^{[k]} \oplus x_n I^{[k-1]}$ is Tor-vanishing, that is, for all i and all $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, the map

$$\operatorname{Tor}_i^S(K, x_n I^{[k]})_{\mathbf{a}} \longrightarrow \operatorname{Tor}_i^S(K, I^{[k]})_{\mathbf{a}} \oplus \operatorname{Tor}_i^S(K, x_n I^{[k-1]})_{\mathbf{a}}$$

is zero. But if $\operatorname{Tor}_i^S(K, x_n I^{[k]})_{\mathbf{a}} \neq 0$, then $a_n > 0$ since x_n divides all minimal generators of $x_n I^{[k]}$. On the other hand, if $a_n > 0$, then $\operatorname{Tor}_i^S(K, I^{[k]})_{\mathbf{a}} = 0$ because x_n does not divide any generator of $I^{[k]}$. Therefore, the map $x_n I^{[k]} \longrightarrow I^{[k]} \oplus x_n I^{[k-1]}$ is Tor-vanishing if and only if for all i and all $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $a_n > 0$, the map $\operatorname{Tor}_i^S(K, x_n I^{[k]})_{\mathbf{a}} \longrightarrow \operatorname{Tor}_i^S(K, x_n I^{[k-1]})_{\mathbf{a}}$ is zero. This is equivalent to saying that $x_n I^{[k]} \longrightarrow x_n I^{[k-1]}$ is Tor-vanishing. This latter condition holds if and only if the map $I^{[k]} \longrightarrow I^{[k-1]}$ is such. But this is obviously the case by Corollary 5.1.4. □

The next result was proved in [56, Proposition 2.4] under the additional assumption that all squarefree powers $I^{[k]}$ are componentwise linear. This assumption was made to ensure that $I^{[k]} + x_n I^{[k-1]}$ is a Betti splitting [18, Theorem 3.3]. Thanks to Lemma 5.1.5, this hypothesis can be removed. Therefore, we have the following.

Proposition 5.1.6. *Let $I \subset K[x_1, \dots, x_{n-1}]$ be a squarefree monomial ideal. Set $J = (I, x_n)$ and $d_k = \operatorname{indeg}(I^{[k]})$ for $1 \leq k \leq \nu(I)$, $g_I(0) = g_I(\nu(I) + 1) = +\infty$ and $d_0 = 0$. Then $\nu(J) = \nu(I) + 1$ and for all $1 \leq k \leq \nu(J)$,*

$$g_J(k) \;=\; \min\{g_I(k) + d_k - d_{k-1} - 1,\; g_I(k-1)\}.$$

5.2 Some Betti splittings of squarefree powers of edge ideals of forests

In this section we prove some results useful for the goal of this chapter, that is, computing explicitly the regularity function $\operatorname{reg}(I(G)^{[k]})$ and proving that the normalized depth function $g_{I(G)}(k)$ is non–increasing when G is a forest.

A *cycle* of a graph G of length q is a subgraph C of G such that

$$E(C) = \{\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{q-1}, i_q\}, \{i_q, i_1\}\},$$

where $i_1, i_2, \dots, i_q$ are distinct vertices of G.

A *forest* is a graph G with no cycle. A connected forest is called a *tree*. A *leaf* v of a graph G is a vertex incident to only one edge. Any tree possesses at least two leaves. Let $v \in V(G)$ be a leaf and w be the unique neighbor of v. Following [49], we say that v is a *distant leaf* if at most one of the neighbors of w is not a leaf. In this case, we say that $\{w, v\}$ is a *distant edge*. The following picture displays this situation.

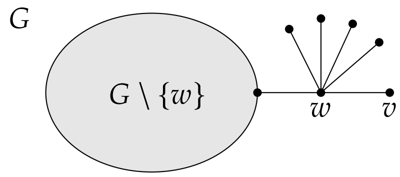


FIGURE 5.1: A graph G with distant edge $\{w, v\}$.

The gray area represents the graph $G \setminus \{w\}$. Next picture clarifies Figure 5.1. It describes a forest G with $\{w, v\}$ as a distant edge.

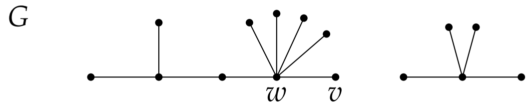


FIGURE 5.2: A forest G with distant edge $\{w, v\}$.

The next result will be crucial for the development of the chapter. Indeed, both the results obtained in this section and in the next one, are based upon the following proposition. It already appeared in [49] and [78]. But for the sake of completeness we include a different shorter proof.

Proposition 5.2.1. ([49, Lemma 20], [78, Proposition 9.1.1]) *Let G be a forest. Then G has a distant leaf.*

Proof. It is enough to assume that G is a tree. Let $v_1, v_2, \dots, v_r$ be an induced path of G of maximal length. We claim that v_r is a distant leaf. Indeed, let $N_G(v_{r-1}) = \{w_1, \dots, w_s\}$. Up to a relabeling, we may assume $w_1 = v_{r-2}$ and $w_2 = v_r$. Since $\deg_G(w_2) = 1$, it is enough to show that $\deg_G(w_i) = 1$ for $i = 3, \dots, s$. Suppose for a contradiction that this is not the case. In other words, we have that $\deg_G(w_i) > 1$ for some $i \in \{3, \dots, s\}$. Let u be a neighbor of w_i different from v_{r-1}. Then $v_1, v_2, \dots, v_{r-1}, w_i, u$ is an induced path of G of length $r + 1$, a contradiction. The assertion follows. $\square$

We fix the following setup. We will consider forests G satisfying $\nu(G) \geq 3$. The cases $\nu(G) = 1, 2$ will be addressed directly in the proof of Theorem 5.3.4.

Setup 5.2.2. Let G be a forest with the vertex set $[n]$ and $\nu(G) \geq 3$. Let $n \in V(G)$ be a distant leaf with distant edge $\{n-1, n\} \in E(G)$ such that $N_G(n-1) = \{i_1, \dots, i_t, n-2, n\}$ with $t \geq 0$, $\deg_G(i_j) = 1$ for $j = 1, \dots, t$, $\deg_G(n) = 1$ and $\deg_G(n-2) \geq 1$. Let G_1, G_2 and G_3 be the induced subgraphs of G on the vertex sets $[n-1]$, $[n-2]$ and $[n-3]$, respectively. Note that $|V(G_1)|, |V(G_2)|, |V(G_3)| < n$, and moreover $I(G_3) \neq 0$ because $\nu(G) \geq 3$. Indeed, if $I(G_3) = 0$, then either G_3 contains no vertices, or all vertices of

G_3 are adjacent to $n-2$. Then $\{n-2, n-1\} \in E(G)$ and the other edges $\{i, j\}$ of G have $i = n-2$ or $i = n-1$. Then it is clear that $\nu(G) \le 2$, against our assumption.

Lemma 5.2.3. *Assume Setup 5.2.2. Then for all $1 \le k \le \nu(G)$,*

$$I(G_1)^{[k]} = x_{n-1}x_{n-2}I(G_3)^{[k-1]} + \sum_{j=1}^{t} x_{n-1}x_{i_j}I(G_2)^{[k-1]} + I(G_2)^{[k]}. \tag{5.3}$$

Proof. Let $u = e_1 \cdots e_k \in G(I(G_1)^{[k]})$ with $e_j = x_{p_j}x_{q_j}$ for $j = 1, \dots, k$. This means that $M = \{\{p_j, q_j\} : j = 1, \dots, k\}$ is a k-matching of G_1. If $n-1 \notin V(M)$, none of the vertices from $\{i_1, \dots, i_t\}$ belongs to $V(M)$. Thus $u \in G(I(G_2)^{[k]})$. Otherwise, if $n-1 \in V(M)$, then either $\{n-1, n-2\} \in M$ or $\{n-1, i_j\} \in M$ for some $j \in [t]$. In the first case, $M' = M \setminus \{\{n-1, n-2\}\}$ is a $(k-1)$-matching of G_3. Then $u \in G(x_{n-1}x_{n-2}I(G_3)^{[k-1]})$. In the second case, $M' = M \setminus \{\{n-1, i_j\}\}$ is a $(k-1)$-matching of G_2 and $u \in G(x_{n-1}x_{i_j}I(G_2)^{[k-1]})$. These cases show that equation (5.3) holds. ☐

Lemma 5.2.4. *Assume Setup 5.2.2. Set $J = I(G_1)^{[k]} \cap x_nx_{n-1}I(G_2)^{[k-1]}$. Then for all $1 \le k \le \nu(G)$,*

$$J = x_nx_{n-1}\big[I(G_3)^{[k]} + x_{n-2}I(G_3)^{[k-1]} + \sum_{j=1}^{t} x_{i_j}I(G_2)^{[k-1]}\big]. \tag{5.4}$$

Proof. To get the assertion, we apply Lemma 5.2.3 showing the followings:

(a) $x_{n-1}x_{n-2}I(G_3)^{[k-1]} \cap x_{n-1}x_nI(G_2)^{[k-1]} = x_nx_{n-1}x_{n-2}I(G_3)^{[k-1]}$;

(b) $x_{n-1}x_{i_j}I(G_2)^{[k-1]} \cap x_{n-1}x_nI(G_2)^{[k-1]} = x_nx_{n-1}x_{i_j}I(G_2)^{[k-1]}$, for $j \in [t]$;

(c) $I(G_2)^{[k]} \cap x_{n-1}x_nI(G_2)^{[k-1]} = x_nx_{n-1}I(G_2)^{[k]}$.

Let us prove (a). From Setup 5.2.2, G_2 and G_3 are induced subgraphs of G on the vertex sets $[n-2]$ and $[n-3]$, respectively. Hence, $I(G_3) \subset I(G_2)$ and consequently $x_{n-2}I(G_3)^{[k-1]} \subset I(G_3)^{[k-1]} \subset I(G_2)^{[k-1]}$ and $x_{n-1}x_{n-2}I(G_3)^{[k-1]} \subset x_{n-1}I(G_2)^{[k-1]}$. On the other hand, setting $L = x_{n-1}x_{n-2}I(G_3)^{[k-1]} \cap x_{n-1}x_nI(G_2)^{[k-1]}$ we have

$$L = (\mathrm{lcm}(u, v) \ : \ u \in G(x_{n-1}x_{n-2}I(G_3)^{[k-1]}), v \in G(x_{n-1}x_nI(G_2)^{[k-1]}).$$

Hence observing that x_n does not divide any of the monomial generators of $I(G_3)$ and $I(G_2)$, equation (a) follows.

Equation (b) is justified similarly, given that x_n, and x_{i_j} do not divide any generator of $I(G_2)$. Lastly, equation (c) is derived from the containment $I(G_2)^{[k]} \subset I(G_2)^{[k-1]}$ and the fact that none of the generators of $I(G_2)$ are divisible by x_{n-1} and x_n.
In light of equations (a), (b) and (c), we deduce the equation (5.3) in Lemma 5.2.3 by establishing the following:

$$I(G_2)^{[k]} + x_{n-2}I(G_3)^{[k-1]} = I(G_3)^{[k]} + x_{n-2}I(G_3)^{[k-1]} \tag{5.5}$$

The inclusion "$\supseteq$" is clear. For the other inclusion, consider $u = e_1 \cdots e_k \in G(I(G_2)^{[k]})$ such that $e_i \in G(I(G_2))$ for each i. If x_{n-2} divides u, then x_{n-2} divides u_i for some i. Without loss of generality, we may assume $i = 1$. This implies that $x_{n-2}u_2 \cdots u_k$

belongs to $G(x_{n-2}I(G_3)^{[k-1]})$. On the other hand, if x_{n-2} does not divide u, then u is an element of $G(I(G_3)^{[k]})$, as required. Hence, formula (5.5) is established. (a), (b) and (c) combined with (5.3) and (5.5) imply the desired formula (5.4). $\square$

Proposition 5.2.5. *Assume Setup 5.2.2. Then for all $1 \leq k \leq \nu(G)$,*

$$I(G)^{[k]} = I(G_1)^{[k]} + x_n x_{n-1} I(G_2)^{[k-1]} \tag{5.6}$$

is a Betti splitting.

Proof. Firstly, we prove that (5.6) holds. The sum of the right-hand side is evidently a subset of $I(G)^{[k]}$. Hence, we only need to establish the reverse inclusion. Indeed, let $u = e_1 \cdots e_k \in G(I(G)^{[k]})$ where $e_j = x_{p_j}x_{q_j}$ for $j = 1, \dots, k$. Then $M = \{\{p_j, q_j\} : j = 1, \dots, k\}$ is a k-matching of G. If $n \in V(M)$, then $\{p_j, q_j\} = \{n-1, n\}$ for some j, because n is a leaf. Then $M' = M \setminus \{\{n-1, n\}\}$ is a $(k-1)$-matching of G_2. Hence $u \in G(x_n x_{n-1} I(G_2)^{[k-1]})$ in this case. Otherwise, assume that $n \notin V(M)$, then M is a k-matching of G_1 and $u \in G(I(G_1)^{[k]})$. These two cases show that (5.6) holds.

Note that (5.6) is an x_n-partition because $G(I(G)^{[k]})$ is the disjoint union of $G(I(G_1)^{[k]})$ and $G(x_n x_{n-1} I(G_2)^{[k-1]})$.

To conclude that (5.6) is indeed a Betti splitting, we must show that the inclusion map $J \to I(G_1)^{[k]} \oplus x_n x_{n-1} I(G_2)^{[k-1]}$ is Tor-vanishing. Observe that, while x_n does not divide any generator of $I(G_1)^{[k]}$, it divides all generators of J according to (5.4). Therefore, arguing as in the proof of Lemma 5.1.5, it is enough to show that the inclusion map $J \to x_n x_{n-1} I(G_2)^{[k-1]}$ is Tor-vanishing. For this purpose, we define the map

$$\varphi : G(J) \to G\big(x_n x_{n-1} I(G_2)^{[k-1]}\big)$$

considering the following three cases:

(i) Let $x_n x_{n-1} u \in G(x_n x_{n-1} I(G_3)^{[k]})$. By Lemma 5.1.3, there exists a map $\widetilde{\varphi} : G(I(G_3)^{[k]}) \to G(I(G_3)^{[k-1]})$ verifying Theorem 5.1.2. Then we have $\widetilde{\varphi}(u) \in G(I(G_3)^{[k-1]}) \subset G(I(G_2)^{[k-1]})$ and we set $\varphi(x_n x_{n-1} u) = x_n x_{n-1} \widetilde{\varphi}(u)$.

(ii) Let $x_n x_{n-1} x_{n-2} u \in G(x_n x_{n-1} x_{n-2} I(G_3)^{[k-1]})$. Then $u \in G(I(G_2)^{[k-1]})$ and we set $\varphi(x_n x_{n-1} x_{n-2} u) = x_n x_{n-1} u$.

(iii) Let $x_n x_{n-1} x_{i_j} u \in G(x_n x_{n-1} x_{i_j} I(G_2)^{[k-1]})$, for some $j \in [t]$. Then we set $\varphi(x_n x_{n-1} x_{i_j} u) = x_n x_{n-1} u$.

The map φ is well-defined by equation (5.4). By Theorem 5.1.2, it suffices to show that for any subset $\Omega \subseteq G(J)$,

$$\operatorname{lcm}(u \ : \ u \in \Omega) \in \mathfrak{m}(\operatorname{lcm}(\varphi(u) \ : \ u \in \Omega)), \tag{5.7}$$

where $\mathfrak{m} = (x_1, \dots, x_n)$ is the maximal ideal of S. Let $\Omega \subseteq G(J)$. Then

$$\Omega = \Big(\bigcup_{\ell}\{x_n x_{n-1} u_\ell\}\Big) \cup \Big(\bigcup_{r}\{x_n x_{n-1} x_{n-2} w_r\}\Big) \cup \Big(\bigcup_{j=1}^{t}\bigcup_{s}\{x_n x_{n-1} x_{i_j} z_{j,s}\}\Big).$$

where $u_\ell \in G(I(G_3)^{[k]})$, $w_r \in G(I(G_3)^{[k-1]})$ and $z_{j,s} \in G(I(G_2)^{[k-1]})$.

It follows that

$$\operatorname{lcm}(u \ : \ u \in \Omega) = x_n x_{n-1} \operatorname{lcm}(u_\ell, x_{n-2}w_r, x_{i_j}z_{j,s} \ : \ \ell, r, s, \ j \in [t])$$

and

$$\operatorname{lcm}(\varphi(u) \ : \ u \in \Omega) = x_n x_{n-1} \operatorname{lcm}(\widetilde{\varphi}(u_\ell), w_r, z_{j,s} \ : \ \ell, r, s, \ j \in [t]).$$

From these expressions, we deduce that $\operatorname{lcm}(u : u \in \Omega) \in \mathfrak{m}(\operatorname{lcm}(\varphi(u) : u \in \Omega))$. Specifically, if at least one w_r or at least one $z_{j,s}$ appears in Ω, then either x_{n-2} divides $\operatorname{lcm}(u : u \in \Omega)$ but does not divide $\operatorname{lcm}(\varphi(u) : u \in \Omega)$, or x_{i_j} divides $\operatorname{lcm}(u : u \in \Omega)$ but does not divide $\operatorname{lcm}(\varphi(u) : u \in \Omega)$. Conversely, if neither w_r nor $z_{j,s}$ appears in Ω, the monomial $\operatorname{lcm}(u : u \in \Omega)$ is strictly divided by the monomial $\operatorname{lcm}(\varphi(u) : u \in \Omega)$, due to the conditions met by $\widetilde{\varphi}$ as described in Theorem 5.1.2. This concludes our proof. □

Now consider (5.4) in Lemma 5.2.4. Set

$$J_1 = x_n x_{n-1}[I(G_3)^{[k]} + x_{n-2}I(G_3)^{[k-1]}],$$
$$J_2 = x_n x_{n-1} \sum_{j=1}^{t} x_{i_j} I(G_2)^{[k-1]} = x_n x_{n-1}(x_{i_1}, \dots, x_{i_t}) I(G_2)^{[k-1]}.$$

With this notation, $J = J_1 + J_2$. Note further that $J_2 \neq (0)$ if and only if $t > 0$.

Proposition 5.2.6. *Assume Setup 5.2.2 and suppose $t > 0$. With the notation above, we have*

$$J_1 \cap J_2 = (x_{i_1}, \dots, x_{i_t}) J_1 \tag{5.8}$$

and $J = J_1 + J_2$ is a Betti splitting.

Proof. We first show that (5.8) holds. This equality follows from the observation that none of the generators of J_1 are divisible by x_{i_j} for any $j \in [t]$ and the following chain of containments:

$$I(G_3)^{[k]} + x_{n-2}I(G_3)^{[k-1]} \subset I(G_2)^{[k]} + x_{n-2}I(G_2)^{[k-1]} \subset I(G_2)^{[k-1]}.$$

It remains to prove that the map $J_1 \cap J_2 \to J_1 \oplus J_2$ is Tor-vanishing. Note that each generator of $J_1 \cap J_2$ is divisible by x_{i_j} for some $j \in [t]$, while none of the generators of J_1 are divisible by any of the x_{i_j}. Hence, arguing as in Lemma 5.1.5, it is enough to show that the inclusion $J_1 \cap J_2 \to J_2$ is Tor-vanishing. For this purpose, we apply Theorem 5.1.2 again. Let

$$\varphi : G((x_{i_1}, \dots, x_{i_t})J_1) \to G(J_2)$$

be defined considering the following cases:

(i) Let $x_n x_{n-1} x_{i_j} u \in G((x_{i_1}, \dots, x_{i_t})J_1)$ with $u \in G(I(G_3)^{[k]})$ and $j \in [t]$. By Lemma 5.1.3, there exists $\widetilde{\varphi} : G(I(G_3)^{[k]}) \to G(I(G_3)^{[k-1]})$ satisfying Theorem 5.1.2. Then we set $\varphi(x_n x_{n-1} x_{i_j} u) = x_n x_{n-1} x_{i_j} \widetilde{\varphi}(u)$.

(ii) Let $x_n x_{n-1} x_{n-2} x_{i_j} u \in G((x_{i_1}, \dots, x_{i_t})J_1)$ with $u \in G(I(G_3)^{[k-1]})$, $j \in [t]$. Then $u \in G(I(G_2)^{[k-1]})$ and we set $\varphi(x_n x_{n-1} x_{n-2} x_{i_j} u) = x_n x_{n-1} x_{i_j} u$.

The map φ is well defined by (5.8). Arguing as in the proof of Proposition 5.2.5, we see that φ verifies the condition in Theorem 5.1.2, as desired. □

5.3 Behaviour of the normalized depth function of forests

In this section, we prove that the normalized depth function $g_{I(G)}(k)$ of $I(G)$ is a non–increasing function when G is a forest using the results in the previous section and an auxiliary lemma.

For its proof, we recall two general facts that we will use repeatedly in what follows. Let $u \in S$ be a monomial and $L \subset S$ be a monomial ideal, then $\operatorname{depth}(S/uL) = \operatorname{depth}(S/L)$. Moreover, if $\{x_{p_1}, \dots, x_{p_s}\}$ is a collection of variables not dividing any generator of L, then by [73, Corollary 3.2]

$$\operatorname{depth}(S/(x_{p_1}, \dots, x_{p_s})L) = \operatorname{depth}(S/L) - (s-1).$$

The next remark will be crucial for the sequel.

Remark 5.3.1. Let $I \subset S$ be a squarefree monomial ideal, where $S = K[x_1, \dots, x_n]$ and consider the ring extension $S' = S[y_1, \dots, y_m] = K[x_1, \dots, x_n, y_1, \dots, y_m]$. Then $\nu(I) = \nu(IS')$. With abuse of notation, we denote again by I the ideal IS', that is, the extension of I in S'. Note that $\operatorname{depth}(S'/I) = \operatorname{depth}(S/I) + m$. Let g_I, h_I be the normalized depth functions of $I \subset S$, and $I \subset S'$, respectively. Then $g_I(k) = h_I(k) - m$ for all $1 \le k \le \nu(I)$. Thus g_I is non–increasing if and only if h_I is non–increasing.

To avoid unnecessary distinctions, we will regard $I(G), I(G_1), I(G_2)$ and $I(G_3)$ as ideals of $S = K[x_1, \dots, x_n]$. Therefore, $g_{I(G_i)}(k) = \operatorname{depth}(S/I(G_i)^{[k]}) - (2k-1)$ for all i and k.

Lemma 5.3.2. *Assume Setup 5.2.2.*

(a) *If $t = 0$, then*

$$\begin{aligned} g_{I(G)}(k) = \min\{&g_{I(G_1)}(k),\ g_{I(G_2)}(k-1) - 2, \\ &g_{I(G_3)}(k-1) - 3,\ g_{I(G_3)}(k) - 2\} \end{aligned} \tag{5.9}$$

for all $1 \le k \le \nu(G)$.

(b) *If $t > 0$, then*

$$\begin{aligned} g_{I(G)}(k) = \min\{&g_{I(G_1)}(k),\ g_{I(G_2)}(k-1) - 2 - t, \\ &g_{I(G_3)}(k-1) - 2 - t,\ g_{I(G_3)}(k) - 1 - t\} \end{aligned} \tag{5.10}$$

for all $1 \le k \le \nu(G)$.

Proof. First, we compute $\operatorname{depth}(S/I(G)^{[k]})$. By Proposition 5.2.5, the decomposition (5.6) is a Betti splitting. Hence, from Proposition 5.1.1 and the Auslander–Buchsbaum formula (see for instance [68, Corollary A.4.3]),

$$\operatorname{depth}(S/I(G)^{[k]}) = \min\{\operatorname{depth}(S/I(G_1)^{[k]}),\ \operatorname{depth}(S/I(G_2)^{[k-1]}),\ \operatorname{depth}(S/J) - 1\},$$

where J is given in (5.4). Now we distinguish two cases.

Case 1. Let $t = 0$. Then $J = J_1 = x_n x_{n-1}[I(G_3)^{[k]} + x_{n-2}I(G_3)^{[k-1]}]$ is a Betti splitting by Lemma 5.1.5. Since $I(G_3)^{[k]} \cap x_{n-2}I(G_3)^{[k-1]} = x_{n-2}I(G_3)^{[k]}$, we have

$$\begin{aligned} \operatorname{depth}(S/J) &= \min\{\operatorname{depth}(S/I(G_3)^{[k]}),\ \operatorname{depth}(S/I(G_3)^{[k-1]}),\ \operatorname{depth}(S/I(G_3)^{[k]}) - 1\} \\ &= \min\{\operatorname{depth}(S/I(G_3)^{[k-1]}),\ \operatorname{depth}(S/I(G_3)^{[k]}) - 1\}. \end{aligned}$$

We have

$$\begin{aligned}
\operatorname{depth}(S/I(G_1)^{[k]}) - (2k-1) &= g_{I(G_1)}(k),\\
\operatorname{depth}(S/I(G_2)^{[k-1]}) - (2k-1) &= \operatorname{depth}(S/I(G_2)^{[k-1]}) - (2(k-1)-1) - 2\\
&= g_{I(G_2)}(k-1) - 2.
\end{aligned}$$

Similar computations yield $\operatorname{depth}(S/I(G_3)^{[k-1]}) - (2k-1) = g_{I(G_3)}(k-1) - 2$ and $\operatorname{depth}(S/I(G_3)^{[k]}) - (2k-1) - 1 = g_{I(G_3)}(k) - 1$. Hence, (5.9) holds in this case.

Case 2. Let $t > 0$. Now we compute $\operatorname{depth}(S/J)$. By Proposition 5.2.6, $J = J_1 + J_2$ is a Betti splitting and $J_1 \cap J_2 = (x_{i_1}, \dots, x_{i_t})J_1$. Thus,

$$\begin{aligned}
\operatorname{depth}(S/J) - 1 &= \min\{\operatorname{depth}(S/J_1) - 1, \operatorname{depth}(S/J_2) - 1, \operatorname{depth}(S/J_1) - (t-1) - 1\}\\
&= \min\{\operatorname{depth}(S/I(G_2)^{[k-1]}) - t, \operatorname{depth}(S/J_1) - t\}. \qquad (5.11)
\end{aligned}$$

Moreover, from Case 1

$$\operatorname{depth}(S/J_1) - t = \min\{\operatorname{depth}(S/I(G_3)^{[k-1]}) - t, \operatorname{depth}(S/I(G_3)^{[k]}) - t - 1\}.$$

Thus, from (5.11), $\operatorname{depth}(S/I(G)^{[k]})$ is equal to

$$\begin{aligned}
&\min\{\operatorname{depth}(S/I(G_1)^{[k]}), \operatorname{depth}(S/I(G_2)^{[k-1]}), \operatorname{depth}(S/J) - 1\}\\
=\ &\min\{\operatorname{depth}(S/I(G_1)^{[k]}), \operatorname{depth}(S/I(G_2)^{[k-1]}), \operatorname{depth}(S/I(G_2)^{[k-1]}) - t,\\
&\quad \operatorname{depth}(S/I(G_3)^{[k-1]}) - t, \operatorname{depth}(S/I(G_3)^{[k]}) - t - 1\}\\
=\ &\min\{\operatorname{depth}(S/I(G_1)^{[k]}), \operatorname{depth}(S/I(G_2)^{[k-1]}) - t, \operatorname{depth}(S/I(G_3)^{[k-1]}) - t,\\
&\quad \operatorname{depth}(S/I(G_3)^{[k]}) - t - 1\}.
\end{aligned}$$

We have $\operatorname{depth}(S/I(G_1)^{[k]}) - (2k-1) = g_{I(G_1)}(k)$. Moreover,

$$\operatorname{depth}(S/I(G_2)^{[k-1]}) - (2k-1) - t = g_{I(G_2)}(k-1) - 2 - t.$$

Similarly,

$$\begin{aligned}
\operatorname{depth}(S/I(G_3)^{[k-1]}) - (2k-1) - t &= g_{I(G_3)}(k-1) - 2 - t,\\
\operatorname{depth}(S/I(G_3)^{[k]}) - (2k-1) - t - 1 &= g_{I(G_3)}(k) - 1 - t.
\end{aligned}$$

Finally, we see that (5.10) holds. □

The next example clarifies Lemma 5.3.2

Example 5.3.3. Consider the graph G on eleven vertices depicted below.

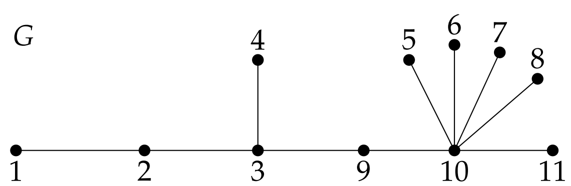
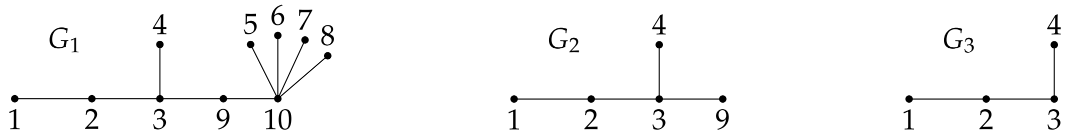


Figure 5.3: A forest G on eleven vertices.

Notice that $\nu(G) = 3$ and $\{9, 10\}$ is a distance edge of G. Using the notation in Setup 5.2.2, with $n = 11$, we have $N_G(10) = \{i_1, \dots, i_t, n-2, n\} = \{5, 6, 7, 8, 9, 11\}$, with $t = 4 > 0$. The graphs G_1, G_2, G_3 are depicted below:



In the previous picture we have not drawn the isolated vertices in G_2 and G_3.
Let $k = 2$. According to Lemma 5.3.2(b), since $t = 4$, we have

$$\begin{aligned} g_{I(G)}(2) &= \min\{g_{I(G_1)}(2),\ g_{I(G_2)}(1) - 2 - t,\ g_{I(G_3)}(1) - 2 - t,\ g_{I(G_3)}(2) - 1 - t\} \\ &= \min\{g_{I(G_1)}(2),\ g_{I(G_2)}(1) - 6,\ g_{I(G_3)}(1) - 6,\ g_{I(G_3)}(2) - 5\}. \end{aligned}$$

Next, by using *Macaulay2* [54, 62] $g_{I(G_1)}(2) = 3$, $g_{I(G_2)}(2) = 7$, $g_{I(G_3)}(1) = 8$ and $g_{I(G_3)}(2) = 7$. Thus,

$$g_{I(G)}(2) = \min\{3,\ 1,\ 2,\ 2\} = 1.$$

Now we are ready to prove the main result in the section.

Theorem 5.3.4. *Let G be a forest. Then $g_{I(G)}$ is non–increasing.*

Proof. Let $G_1, \dots, G_c$ be the connected components of G. Each G_i is a tree with at least two vertices. Let $m = \max_i |V(G_i)|$. If $m = 2$, then up to a relabeling, $I(G) = (x_1x_2, x_3x_4, \dots, x_{n-1}x_n)$ is a complete intersection. Setting $y_i = x_{2i-1}x_{2i}$, $i = 1, \dots, \frac{n}{2}$, and $L = (y_1, \dots, y_{\frac{n}{2}})$, we have $\nu(G) = \nu(L) = \frac{n}{2}$ and $\mathrm{pd}(I(G)^{[k]}) = \mathrm{pd}(L^{[k]})$ for all k. Since $L^{[k]}$ is a squarefree Veronese ideal generated in degree k, we have $\mathrm{pd}(L^{[k]}) = \frac{n}{2} - k$ for $k = 1, \dots, \nu(L)$. Thus, for all $k = 1, \dots, \frac{n}{2}$, from the Auslander–Buchsbaum formula, we have

$$g_{I(G)}(k) = \mathrm{depth}(S/I(G)^{[k]}) - (2k-1) = n - 1 - \mathrm{pd}(I^{[k]}) - (2k-1) = \frac{n}{2} - k.$$

Hence, $g_{I(G)}(k)$ is non–increasing in this case.

Now suppose $m > 2$. Then G has a distant leaf. We proceed by induction on $n = |V(G)|$. If $\nu(G) = 1$ there is nothing to prove. Similarly if $\nu(G) = 2$, then $g_{I(G)}(2) = 0$ [48, Corollary 3.5] and $g_{I(G)}(1) \geq 0$. Thus we may suppose that $\nu(G) \geq 3$. Assume Setup 5.2.2. Let $k < \nu(G)$. Suppose $t = 0$. Then, Lemma 5.3.2 gives

$$\begin{aligned} g_{I(G)}(k+1) = \min\big\{&g_{I(G_1)}(k+1),\ g_{I(G_2)}(k) - 2, \\ &g_{I(G_3)}(k) - 3,\ g_{I(G_3)}(k+1) - 2\big\}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} g_{I(G)}(k) = \min\big\{&g_{I(G_1)}(k),\ g_{I(G_2)}(k-1) - 2, \\ &g_{I(G_3)}(k-1) - 3,\ g_{I(G_3)}(k) - 2\big\}. \end{aligned} \tag{5.13}$$

It remains to prove that $g_{I(G)}(k+1) - g_{I(G)}(k) \leq 0$. If $k + 1 = \nu(G)$, then $g_{I(G)}(k+1) = 0$ by [48, Corollary 3.5] and there is nothing to prove.

Suppose now $k + 1 < \nu(G)$. Then we have $k \leq \nu(G) - 2$. It follows directly from the definition of matching numbers that $\nu(G_3) \leq \nu(G_2) \leq \nu(G_1) \leq \nu(G)$ and $\nu(G) = \nu(G_3) + 1$. Thus $k \leq \nu(G_3) - 1$, and this inequality guarantees that all terms $g_{I(G_i)}(\ell)$ appearing in the minimum taken in (5.12) and in the minimum taken in (5.13) have $\ell \leq \nu(G_i)$ for $i = 1, 2, 3$.

Now, to end the proof, it is enough to distinguish the four cases arising from (5.13). Suppose $g_{I(G)}(k) = g_{I(G_1)}(k)$. Then, from (5.12), we have $g_{I(G)}(k+1) \le g_{I(G_1)}(k+1)$ and

$$g_{I(G)}(k+1) - g_{I(G)}(k) \ \le\ g_{I(G_1)}(k+1) - g_{I(G_1)}(k) \le 0,$$

where the last inequality follows by the inductive hypothesis, since $|V(G_1)| < n$. For the other three cases, one can argue similarly. The case $t > 0$ is analogous. $\square$

Proposition 5.2.5 together with Proposition 5.2.6 and a simple inductive argument imply the following interesting consequence.

Corollary 5.3.5. *Let G be a forest. Then the graded Betti numbers of the squarefree powers $I(G)^{[k]}$ do not depend upon the characteristic of the field K.*

We end the section with a general upper bound for the normalized depth function of the edge ideal of any graph G in terms of the longest induced path of G.

We denote by P_n the *path on n vertices*, that is the graph with $V(P_n) = [n]$ and $E(P_n) = \{\{1,2\},\{2,3\},\dots,\{n-1,n\}\}$. It is well known that $\nu(P_n) = \lfloor \frac{n}{2} \rfloor$.

Theorem 5.3.6. *We have*

$$g_{I(P_n)}(k) \ = \ \begin{cases} \left\lceil \dfrac{n}{3} \right\rceil - k & \text{if } k = 1,\dots,\lceil \frac{n}{3} \rceil, \\ 0 & otherwise. \end{cases}$$

Proof. We proceed by induction on $n \ge 3$. For $n = 3,4,5$ one can easily verify the above formula. Let $n \ge 6$, then $\nu(G) \ge 3$ and n is a distant leaf with distant edge $\{n-1,n\}$. Using the notation in Setup 5.2.2, we have $t = 0$, $G_1 = P_{n-1}$, $G_2 = P_{n-2}$ and $G_3 = P_{n-3}$. Hence, by Lemma 5.3.2,

$$g_{I(P_n)}(k) = \min\{g_{I(P_{n-1})}(k),\ g_{I(P_{n-2})}(k-1)-2,\ g_{I(P_{n-3})}(k-1)-3,\ g_{I(P_{n-3})}(k)-2\}.$$

If $k \ge \lceil \frac{n}{3} \rceil$, then $k-1 \ge \lceil \frac{n-3}{3} \rceil$. By the inductive hypothesis $g_{I(P_{n-3})}(k-1) - 3 = 0$, and by the above equation $g_{I(P_n)}(k) = 0$, too. In this case, the assertion follows.

Suppose now $1 \le k < \lceil \frac{n}{3} \rceil$, then all terms appearing in the minimum taken above are greater than zero, by the inductive hypothesis. Hence,

$$\begin{aligned} g_{I(P_n)}(k) &= \min\left\{\left\lceil \frac{n-1}{3} \right\rceil - k + 1,\ \left\lceil \frac{n-2}{3} \right\rceil - (k-1),\ \left\lceil \frac{n-3}{3} \right\rceil - (k-1), \right. \\ &\qquad \left. \left\lceil \frac{n-3}{3} \right\rceil - k + 1\right\} \\ &= \min\left\{\left\lceil \frac{n-1}{3} \right\rceil - k + 1,\ \left\lceil \frac{n-2}{3} \right\rceil - k + 1,\ \left\lceil \frac{n}{3} \right\rceil - k\right\} \\ &= \left\lceil \frac{n}{3} \right\rceil - k, \end{aligned}$$

as desired. The inductive proof is complete. $\square$

Denote by $\ell(G)$ the number of vertices of the longest induced path of G.

Corollary 5.3.7. *Let G be a connected graph with n vertices. Then $\nu(G) \ge \lfloor \frac{\ell(G)}{2} \rfloor$ and*

$$g_{I(G)}(k) \le \begin{cases} \lceil \frac{3n-2\ell(G)}{3} \rceil - k & \text{for } k = 1,\dots,\lceil \frac{\ell(G)}{3} \rceil, \\ n - \ell(G) & \text{for } k = \lceil \frac{\ell(G)}{3} \rceil + 1,\dots,\lfloor \frac{\ell(G)}{2} \rfloor. \end{cases}$$

Proof. Let P be an induced path of G with $|V(P)| = \ell(G)$, and set $S_P = K[x_v : v \in V(P)]$. Then $\nu(G) \geq \nu(P) = \lfloor\frac{\ell(G)}{2}\rfloor$. Let $k \leq \lfloor\frac{\ell(G)}{2}\rfloor$. By [47, Corollary 1.3], we have $\operatorname{depth}(S/I(G)^{[k]}) \leq \operatorname{depth}(S/I(P)^{[k]}) = \operatorname{depth}(S_P/I(P)^{[k]}) + n - \ell(G)$. Hence, the assertion follows from Theorem 5.3.6. □

5.4 The Erey–Hibi Conjecture

In this last section, we turn to the regularity of the squarefree powers of $I(G)$, when G is a forest, and we prove a conjecture due to Erey and Hibi [49].

In [49], Erey and Hibi introduced the concept of *k-admissible matching* of a graph. Let n and k be positive integers. A sequence $(a_1, \dots, a_n)$ of positive integers is called a *k-admissible sequence* if $a_1 + \cdots + a_n \leq n + k - 1$.

Definition 5.4.1. Let G be a graph, M be a matching of G and $k \in [\nu(G)]$. We say that M is a *k-admissible matching* if there exists a sequence $M_1, \dots, M_r$ of non–empty subsets of M satisfying the following conditions:

(a) $M = M_1 \cup \cdots \cup M_r$;

(b) $M_i \cap M_j = \emptyset$ for all $i \neq j$;

(c) for all $i \neq j$, if $e_i \in M_i$ and $e_j \in M_j$, then $\{e_i, e_j\}$ is a *gap* in G, that is, $\{e_i, e_j\}$ is an induced matching of size 2;

(d) the sequence $(|M_1|, \dots, |M_r|)$ is k-admissible;

(e) the induced subgraph of G on $V(M_i)$ is a forest, for all $i \in [r]$.

In such a case, we say that $M = M_1 \cup \cdots \cup M_r$ is a *k-admissible partition* of G. The *k-admissible matching number* of G, denoted by $\operatorname{aim}(G, k)$, is the number

$$\operatorname{aim}(G, k) = \max\{|M| \ : \ M \text{ is a } k\text{-admissible matching of } G\}$$

for $k \in [\nu(G)]$. We set $\operatorname{aim}(G, k) = 0$, if G has no k-admissible matching.

If G is a forest, condition (e) is vacuously verified.

Remark 5.4.2. (Erey–Hibi, 2021 [49, Remark 14]) Note that $\operatorname{aim}(G, k) \geq k$ for all $k \in [\nu(G)]$. If H is an induced subgraph of G, then $\operatorname{aim}(H, k) \leq \operatorname{aim}(G, k)$. Moreover,

$$\operatorname{indm}(G) = \operatorname{aim}(G, 1) \leq \operatorname{aim}(G, 2) \leq \cdots \leq \operatorname{aim}(G, \nu(G)) = \nu(G).$$

For the proof of the next corollary, we recall some facts. Let $u \in S$ be a monomial and $L \subset S$ be a monomial ideal, then $\operatorname{reg}(uL) = \operatorname{reg}(L) + \deg(u)$. Moreover, if $\{x_{p_1}, \dots, x_{p_s}\}$ is a collection of variables not dividing any generator of L, then by [73, Corollary 3.2], $\operatorname{reg}((x_{p_1}, \dots, x_{p_s})L) = \operatorname{reg}(L) + 1$. Furthermore, if H is an induced subgraph of G, [47, Corollary 1.3] gives

$$\operatorname{reg}\left(I(H)^{[k]}\right) \leq \operatorname{reg}\left(I(G)^{[k]}\right).$$

Corollary 5.4.3. *With the notation and assumptions of Proposition 5.2.5, we have*

$$\operatorname{reg}(I(G)^{[k]}) = \max\left\{\operatorname{reg}\left(I(G_1)^{[k]}\right), \operatorname{reg}\left(I(G_2)^{[k-1]}\right) + 2, \operatorname{reg}\left(I(G_3)^{[k]}\right) + 1\right\}$$

for $k = 1, \dots, \nu(G)$.

Proof. By Proposition 5.2.5, since $I(G)^{[k]}$ is a Betti splitting, we have (Proposition 5.1.1)

$$\begin{aligned}\operatorname{reg}(I(G)^{[k]}) &= \max\{\operatorname{reg}(I(G_1)^{[k]}),\ \operatorname{reg}(x_nx_{n-1}I(G_2)^{[k-1]}),\ \operatorname{reg}(J)-1\}\\ &= \max\{\operatorname{reg}(I(G_1)^{[k]}),\ \operatorname{reg}(I(G_2)^{[k-1]})+2,\ \operatorname{reg}(J)-1\},\end{aligned}\tag{5.14}$$

where J is given in (5.4). Since $\operatorname{reg}((x_{i_1},\dots,x_{i_t})J_1)-1=\operatorname{reg}(J_1)$, Proposition 5.2.6 gives

$$\begin{aligned}\operatorname{reg}(J) &= \max\{\operatorname{reg}(J_1),\ \operatorname{reg}(J_2),\ \operatorname{reg}((x_{i_1},\dots,x_{i_t})J_1)-1\}\\ &= \max\{\operatorname{reg}(J_1),\ \operatorname{reg}(J_2)\},\end{aligned}$$

Since $J_1=x_nx_{n-1}[I(G_3)^{[k]}+x_{n-2}I(G_3)^{[k-1]}]$ is a Betti splitting, by Lemma 5.1.5 we have

$$\begin{aligned}\operatorname{reg}(J_1) = \max\{&\operatorname{reg}(x_nx_{n-1}I(G_3)^{[k]}),\ \operatorname{reg}(x_nx_{n-1}x_{n-2}I(G_3)^{[k-1]}),\\ &\operatorname{reg}(x_nx_{n-1}x_{n-2}I(G_3)^{[k]})-1\}\\ = \max\{&\operatorname{reg}(I(G_3)^{[k]})+2,\ \operatorname{reg}(I(G_3)^{[k-1]})+3\}.\end{aligned}\tag{5.15}$$

Now we distinguish two cases.

Case 1. Suppose $t>0$, then $J_2=x_nx_{n-1}(x_{i_1},\dots,x_{i_t})I(G_2)^{[k-1]}\neq(0)$, and we have $\operatorname{reg}(J_2)=\operatorname{reg}(I(G_2)^{[k-1]})+3$. Since $\operatorname{reg}(I(G_3)^{[k-1]})\le\operatorname{reg}(I(G_2)^{[k-1]})$,

$$\begin{aligned}\operatorname{reg}(J) &= \max\{\operatorname{reg}(I(G_3)^{[k]})+2,\ \operatorname{reg}(I(G_3)^{[k-1]})+3,\ \operatorname{reg}(I(G_2)^{[k-1]})+3\}\\ &= \max\{\operatorname{reg}(I(G_3)^{[k]})+2,\ \operatorname{reg}(I(G_2)^{[k-1]})+3\}.\end{aligned}\tag{5.16}$$

Hence, combining (5.14) with (5.16), the assertion follows.

Case 2. Suppose $t=0$, then $J_2=(0)$ and $J=J_1$. Combining (5.15) with (5.14), the assertion follows because $\operatorname{reg}(I(G_3)^{[k-1]})\le\operatorname{reg}(I(G_2)^{[k-1]})$. □

We need the following lemmata proved in [49].

Lemma 5.4.4. (Erey–Hibi, 2021 [49, Lemma 15]) *Let G be a forest and let $2\le k\le\nu(G)$. Then*

$$\operatorname{aim}(G,k)\le\operatorname{aim}(G,k-1)+1.$$

Lemma 5.4.5. (Erey–Hibi, 2021 [49, Lemma 23]) *Let G be a forest and let $\{w,v\}$ be a distant edge of G. Then for all $2\le k\le\nu(G)$,*

$$\operatorname{aim}(G\setminus\{w,v\},k-1)+1\le\operatorname{aim}(G,k).$$

We are in the position to prove Erey–Hibi conjecture [49, Conjecture 31].

Theorem 5.4.6. *Let G be a forest. Then*

$$\operatorname{reg}(I(G)^{[k]})=\operatorname{aim}(G,k)+k,\quad \textit{for } k=1,\dots,\nu(G).$$

Proof. As in the proof of Theorem 5.3.4, let $G_1,\dots,G_c$ be the connected components of G, and let $m=\max_i|V(G_i)|$.

If $m=2$, up to a relabeling, $I(G)=(x_1x_2,x_3x_4,\dots,x_{n-1}x_n)$. Since $M=\{\{1,2\},\{3,4\},\dots,\{n-1,n\}\}$ is a k-admissible matching for all $k=1,\dots,\nu(G)$, we have $\operatorname{aim}(G,k)=\frac{n}{2}$ for all $k=1,\dots,\nu(G)$, and $\operatorname{indm}(G)=\nu(G)=\frac{n}{2}$. In this case, the equality $\operatorname{reg}(I(G)^{[k]})=\operatorname{aim}(G,k)+k$ follows from [49, Proposition 30].

Suppose $m > 2$. We proceed by induction on $|V(G)| \geq 3$. We assume $\nu(G) \geq 3$. Indeed, $\operatorname{reg}(I(G)^{[1]}) = \operatorname{aim}(G,1)+1$ and $\operatorname{reg}(I(G)^{[\nu(G)]}) = \operatorname{aim}(G,\nu(G))+\nu(G) = 2\nu(G)$ have already been proved in [14, Theorem 4.7] and [16, Theorem 5.1] (see also [49]). So, if $\nu(G) \leq 2$, there is nothing to prove. Thus, we may assume Setup 5.2.2. Then Corollary 5.4.3 together with the inductive hypothesis yield

$$\begin{aligned}\operatorname{reg}(I(G)^{[k]}) &= \max\{\operatorname{reg}(I(G_1)^{[k]}),\ \operatorname{reg}(I(G_2)^{[k-1]})+2,\ \operatorname{reg}(I(G_3)^{[k]})+1\}\\ &= \max\{\operatorname{aim}(G_1,k)+k,\ \operatorname{aim}(G_2,k-1)+k+1,\ \operatorname{aim}(G_3,k)+k+1\}.\end{aligned}$$

To prove the assertion, it is enough to show that

$$\operatorname{aim}(G,k) = \max\{\operatorname{aim}(G_1,k),\ \operatorname{aim}(G_2,k-1)+1,\ \operatorname{aim}(G_3,k)+1\}. \tag{5.17}$$

Firstly, we prove that

$$\operatorname{aim}(G,k) \geq \max\{\operatorname{aim}(G_1,k),\ \operatorname{aim}(G_2,k-1)+1,\ \operatorname{aim}(G_3,k)+1\}. \tag{5.18}$$

Indeed, $\operatorname{aim}(G,k) \geq \operatorname{aim}(G_1,k)$ by Remark 5.4.2. Moreover, by Lemma 5.4.5 it follows that $\operatorname{aim}(G,k) \geq \operatorname{aim}(G_2,k-1)+1$, because $G_2 = G \setminus \{n-1,n\}$ and $\{n-1,n\}$ is a distant edge. Finally, we prove $\operatorname{aim}(G,k) \geq \operatorname{aim}(G_3,k)+1$. Indeed, let M be a k-admissible matching of G_3 with $|M| = \operatorname{aim}(G_3,k)$. Then $M = M_1 \cup \cdots \cup M_r$ with $|M_1|+\cdots+|M_r| \leq r+k-1$ as in Definition 5.4.1. Note that $\{n-1,n\}$ forms a gap with all $e \in E(G_3)$. Thus, setting $M' = M \cup \{\{n-1,n\}\}$ we have that M' is a k-admissible matching of G, because M' admits the k-admissible partition $M_1\cup\cdots\cup M_r\cup\{\{n-1,n\}\}$ since

$$|M_1|+\cdots+|M_r|+|\{\{n-1,n\}\}| \leq (r+1)+k-1.$$

This shows that $\operatorname{aim}(G,k) \geq \operatorname{aim}(G_3,k)+1$.

Now we show that (5.17) holds. If $\operatorname{aim}(G,k) = \operatorname{aim}(G_2,k-1)+1$, then equality follows by (5.18). Suppose now that $\operatorname{aim}(G,k) > \operatorname{aim}(G_2,k-1)+1$. By Lemma 5.4.4, $\operatorname{aim}(G_2,k-1)+1 \geq \operatorname{aim}(G_2,k)$. Hence $\operatorname{aim}(G,k) > \operatorname{aim}(G_2,k)$. So we can find a k-admissible matching M of G with $|M| = \operatorname{aim}(G,k) > \operatorname{aim}(G_2,k)$ such that $M = M_1 \cup \cdots \cup M_r$ is a k-admissible partition. Thus $n-1 \in V(M)$, otherwise M is also a k-admissible matching of G_2, which is impossible. One of the following possibilities occurs: $\{n-1,n-2\} \in M$, $\{n-1,n\} \in M$ or $\{n-1,i_j\} \in M$, $j \in [t]$.

Suppose $\{n-1,n-2\} \in M$, up to relabeling we may assume $\{n-1,n-2\} \in M_1$. Set $M' = M \setminus \{\{n-1,n-2\}\}$. We claim that M_1 is a singleton. If not, then

$$|M_1 \setminus \{\{n-1,n-2\}\}|+|M_2|+\cdots+|M_r| = |M|-1 \leq r+(k-1)-1.$$

By the definition of admissible $(k-1)$-matching, the above inequality implies that $\operatorname{aim}(G_3,k-1) \geq |M|-1$. Hence, $\operatorname{aim}(G_2,k-1)+1 \geq \operatorname{aim}(G_3,k-1)+1 \geq |M| = \operatorname{aim}(G,k)$, against our assumption.

Hence, $M_1 = \{\{n-1,n-2\}\}$ and M' is a k-admissible matching of G_3, because $|M_2|+\cdots+|M_r| \leq (r-1)+k-1$. Thus $\operatorname{aim}(G_3,k)+1 \geq \operatorname{aim}(G,k)$. By (5.18), $\operatorname{aim}(G,k) \geq \operatorname{aim}(G_3,k)+1$. Hence $\operatorname{aim}(G,k) = \operatorname{aim}(G_3,k)+1$ and (5.17) holds.

The cases $\{n-1,n\} \in M$, $\{n-1,i_j\} \in M$, $j \in [t]$, can be treated similarly. Indeed, $N_G(n-1) = \{i_1,\dots,i_t,n-2,n\}$ with $t \geq 0$, as stated in Setup 5.2.2. □

Notes

We remark that the formula we established in Theorem 5.4.6 has recently been generalized by Chau, Das, Roy, and Saha in [27]. The authors extended the result to the larger class of block graphs. They proved the following.

Theorem. (Chau–Das–Roy–Saha, 2025 [27, Theorem 4.12]) *Let G be a block graph. Then for all* $1 \leq k \leq \nu(G)$*,*

$$\operatorname{reg}(I(G)^{[k]}) = \operatorname{aim}(G,k) + k.$$

Since the family of block graphs includes forests, as we will see in the next chapter, the above result extends the regularity formula of squarefree powers of edge ideals of forests we proved in Section 5.4.

This development suggests that the formula could extend to broader families of graphs. Motivated by this perspective, in [27] they also conjectured the following.

Conjecture. (Chau–Das–Roy–Saha, 2025 [27, Conjecture 6.2]) *Let G be a chordal graph. Then, for all* $1 \leq k \leq \nu(G)$*,*

$$\operatorname{reg}(I(G)^{[k]}) = \operatorname{aim}(G,k) + k.$$

The authors were also able to prove the conjecture for the second squarefree power of edge ideals of Cohen–Macaulay chordal graphs (see [70]).

Theorem. (Chau–Das–Roy–Saha, 2025 [27, Theorem 5.8]) *Let G be a Cohen–Macaulay chordal graph, with* $\nu(G) \geq 2$*. Then*

$$\operatorname{reg}(I(G)^{[2]}) = \operatorname{aim}(G,2) + 2.$$

These advances highlight how the formula, conjectured in [49] and proved in Theorem 5.4.6, has initiated a promising line of research, with evidence pointing toward its validity for the entire class of chordal graphs.

Chapter 6

Sequentially Cohen–Macaulay binomial edge ideals

Let G be a simple graph with the vertex set $[n]$ and the edge set $E(G)$. The notion of binomial edge ideals was introduced by Herzog et al. in [69], and independently by Ohtani in [95]. Many algebraic properties and invariants of such ideals were described *via* the combinatorics of the underlying graph, see [71] for a nice survey on this topic. The most important one is the so-called cutset property. Let $G \setminus T$ denote the graph obtained by removing the vertices of $T \subset [n]$. T is called a *cutset* if the number of connected components of $G \setminus T$, denoted by $c(T)$, satisfies $c(T) > c(T \setminus \{v\})$ for any $v \in T$. We denote by $\mathcal{C}(G)$ the set of all cutsets for G. For example, the Krull dimension of the ring induced by the binomial edge ideal is

$$d = \max\{n + c(T) - |T| \ : \ T \in \mathcal{C}(G)\}. \tag{6.1}$$

One of the main topics of the study of binomial edge ideals is to classify the Cohen–Macaulay ones. As an example, after their introduction, the first paper is [42] where the main result is to classify the block graphs that are Cohen–Macaulay. After that, many authors focused on the problem (see [11],[21],[42],[98], [99]). Recently, Gorenstein binomial edge ideals ([59]) and licci binomial edge ideals ([44]) have been characterized. A full classification of graphs whose binomial edge ideals are Cohen–Macaulay is not yet known. However, significant progress in this direction has been recently made in [19],[20],[22],[80],[81],[87],[100]. A nice and deeply studied generalization of the Cohen–Macaulay ring is the sequentially Cohen–Macaulay one (see [111]).

As in the case of the Cohen–Macaulay property, for studying sequentially Cohen–Macaulay binomial edge ideals, one can reduce to connected indecomposable graphs as proved in [45]. In particular, in [45], the authors classify the binomial edge ideals with quadratic Gröbner bases that are sequentially Cohen–Macaulay, that is those ideals which are associated with closed graphs (see [42]). Two are the main ingredients to obtain such result: 1) a characterization of Goodarzi [60] on sequentially Cohen–Macaulay homogeneous ideals; 2) the initial ideal of closed graphs. In [106], the authors classified the complete bipartite graphs that are sequentially Cohen–Macaulay studying the deficiency modules (see [104]).

In Section 6.1, we focus on the graphs with one cutset. In particular we use the Goodarzi's filtration and the deficiency modules. An interesting observation of this section is obtained by the following invariant

$$\mathfrak{m}(G) = \min\{n + c(T) - |T| \ : \ T \in \mathcal{C}(G)\}.$$

If $\mathfrak{m}(G) = n + 1$, namely $\mathfrak{m}(G) = \dim S/P_\emptyset$, and J_G is sequentially Cohen–Macaulay, then G has at least a cutpoint, namely a cutset of cardinality 1 (see Proposition 6.1.7).

In Section 6.2, we focus on the family of graphs that are blocks, namely graphs without cutpoints. Hence, in this case $\mathfrak{m}(G) \leq n$ and we prove that cycles ($\mathfrak{m}(G) = n$) and wheels ($\mathfrak{m}(G) = n - 1$) are sequentially Cohen–Macaulay by using the Goodarzi's filtration (see also [122]).

In Section 6.3, we prove that block graphs are sequentially Cohen–Macaulay. We underline that in general the filtration of Goodarzi gives us new ideals, induced by the primary decomposition of J_G, that are not anymore binomial edge ideals. Nevertheless, using elementary containment problem (see Lemma 3.1) by good properties of the block graphs, it is possible to control the filtration using the Mayer–Vietoris sequence (see Theorem 3.2).

In [98], the authors provide a useful tool to generate new Cohen–Macaulay graphs using cones. As an application, this has been used to completely classify all bipartite Cohen–Macaulay graphs [21]. In Section 6.4, we extend this result to sequentially Cohen–Macaulay property, obtaining that a cone is sequentially Cohen–Macaulay if and only if its components are sequentially Cohen–Macaulay (see Theorem 6.4.1).

The results presented in this chapter first appeared in [86].

6.1 Graphs with only one non–empty cutset

Let G be a graph with the vertex set $V(G) = [n] = \{1, \dots, n\}$ and the edge set $E(G)$. A subset $C \subset V(G)$ is called a *clique* of G if for all $i, j \in C$ with $i \neq j$ one has $\{i, j\} \in E(G)$. A *free vertex* of G is a vertex that belongs to exactly one maximal clique of G, with respect to inclusion. A *block* of G is a connected subgraph of G that has no cutpoints, which is maximal with respect to this property. A *block graph* is a graph in which every block is a complete graph. A connected graph G is *decomposable* if there exist two subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$ and $V(G_1) \cap V(G_2) = \{v\}$, where v is a free vertex of G_1 and G_2.

Let K be a field and $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$ be the standard graded polynomial K-algebra in $2n$ indeterminates. Let also $\mathfrak{m} = (x_1, \dots, x_n, y_1, \dots, y_n)$ the maximal graded ideal of S. The *binomial edge ideal* of G is the ideal J_G of S generated by all binomials $f_{ij} = x_i y_j - x_j y_i$, such that $i < j$ and $\{i, j\} \in E(G)$.

From now on, throughout the chapter we will use the following exact sequence

$$0 \longrightarrow S/J_G \longrightarrow S/J_{G_v} \oplus S/(J_{G\setminus\{v\}} + (x_v, y_v)) \longrightarrow S/(J_{G_v\setminus\{v\}} + (x_v, y_v)) \longrightarrow 0, \qquad (6.2)$$

where $G \setminus \{v\}$ denotes the induced subgraph of G on the vertex set $V(G) \setminus \{v\}$, G_v is the graph on the vertex set $V(G)$ and edge set $E(G) \cup \{\{u, v\} : u, v \in N_G(v)\}$, where $N_G(v)$ is the neighborhood of v in G.

We begin our investigation by focusing on a *block star graphs* G, namely a block graph with only one cutpoint (see Figure 6.1). Assume G has t cliques attached to the cutpoint v. Such a graph has $\mathcal{C}(G) = \{\emptyset, \{v\}\}$, thus

$$J_{G_v} = P_\emptyset = J_{K_n}, \qquad J_{G\setminus\{v\}} + (x_v, y_v) = P_{\{v\}}.$$

Moreover, $J_{G_v\setminus\{v\}} + (x_v, y_v) = J_{G_v} + (J_{G\setminus\{v\}} + (x_v, y_v)) = J_{K_{n-1}} + (x_v, y_v)$.

In the following we also assume that $t \geq 3$. In fact, if $t = 2$, then J_G is Cohen–Macaulay since G is a decomposable graph (see [45]). With this assumption, by (6.1) we have

$$\dim S/J_G = n + t - 1.$$

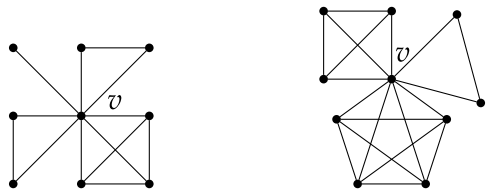


FIGURE 6.1: Examples of block star graphs

Theorem 6.1.1. *Let G be block star graph on n vertices with $t \geq 3$ cliques attached to a single vertex v and let J_G be the binomial edge ideal of G. Then*

1. *$\omega^{n+1}(S/J_G)$ is a Cohen–Macaulay module of dimension $n+1$, moreover, we have $\omega^{n+1}(\omega^{n+1}(S/J_G)) \cong (J_{K_{n-1}} + (x_v, y_v))/J_{K_n}$;*
2. *$\omega(S/J_G) \cong \omega(S/(J_{G\setminus\{v\}} + (x_v, y_v)))$;*
3. *$\omega^i(S/J_G) = 0$ for all $i \neq n+1, n+t-1$.*

In particular, J_G is sequentially Cohen–Macaulay.

Proof. We will show the statement by applying Theorem 1.1.42. In the following we will proceed similarly to the proof of [106, Theorem 4.1].

Consider the Mayer–Vietoris sequence

$$0 \longrightarrow S/J_G \longrightarrow S/J_{K_n} \oplus S/(J_{G\setminus\{v\}} + (x_v, y_v)) \longrightarrow S/(J_{K_{n-1}} + (x_v, y_v)) \longrightarrow 0. \qquad (6.3)$$

We now apply the local cohomology functors to the sequence (6.3). Since S/J_{K_n} is a Cohen–Macaulay module of dimension $n+1$, we have that $H^i_{\mathfrak{m}}(S/J_{K_n}) = 0$ for all $i \neq n+1$ and, since $S/(J_{G\setminus\{v\}} + (x_v, y_v))$, $S/(J_{K_{n-1}} + (x_v, y_v))$ are Cohen–Macaulay modules of dimension $n+t-1$ and n, respectively, we have that $H^i_{\mathfrak{m}}(S/(J_{G\setminus\{v\}} + (x_v, y_v))) = 0$ for all $i \neq n+t-1$ and $H^i_{\mathfrak{m}}(S/(J_{K_{n-1}} + (x_v, y_v))) = 0$ for all $i \neq n$. Moreover, since depth $S/J_G = n+1$ (see [42]) and dim $S/J_G = n+t-1$, we have that $H^i_{\mathfrak{m}}(S/J) = 0$ for all $i < n+1$ or $i > n+t-1$. Merging all this informations gives us the short exact sequence

$$0 \longrightarrow H^n_{\mathfrak{m}}(S/(J_{K_{n-1}} + (x_v, y_v))) \longrightarrow H^{n+1}_{\mathfrak{m}}(S/J_G) \longrightarrow H^{n+1}_{\mathfrak{m}}(S/J_{K_n}) \longrightarrow 0 \qquad (6.4)$$

and the isomorphism

$$H^{n+t-1}_{\mathfrak{m}}(S/J_G) \cong H^{n+t-1}_{\mathfrak{m}}(S/J_{G\setminus\{v\}} + (x_v, y_v)). \qquad (6.5)$$

Moreover, $H^i(S/J_G) = 0$, for all $i \notin \{n+1, n+t-1\}$. This implies condition (3) of the Theorem; in fact, by 1.1.39 we have

$$\omega^i(S/J_G) = 0, \qquad \text{for all } i \notin \{n+1, n+t-1\},$$

as stated. By applying Local Duality to sequence (6.4), we obtain the short exact sequence on the modules of deficiency,

$$0 \longrightarrow \omega^{n+1}(S/J_{K_n}) \longrightarrow \omega^{n+1}(S/J_G) \longrightarrow \omega^n(S/(J_{K_{n-1}} + (x_v, y_v)) \longrightarrow 0,$$

which, by applying again the local cohomology functor and dualizing, leads to the short exact sequence

$$0 \longrightarrow \omega^{n+1}(\omega^{n+1}(S/J_G)) \longrightarrow \omega^{n+1}(\omega^{n+1}(S/J_{K_n})) \longrightarrow$$
$$\longrightarrow \omega^n(\omega^n(J_{K_{n-1}} + (x_v, y_v))) \longrightarrow \omega^n(\omega^{n+1}(S/J_G)) \longrightarrow 0.$$

Note that we took into account that $\omega^{n+1}(S/J_{K_n})$ is a Cohen–Macaulay module of dimension $n+1$ and $\omega^n(S/(J_{K_{n-1}} + (x_v, y_v))$ is a Cohen–Macaulay module of dimension n. Moreover, by Depth Lemma, we obtain $\operatorname{depth} \omega^{n+1}(S/J_G) \geq n$. Consider now the map $\omega^{n+1}(\omega^{n+1}(S/J_{K_n})) \longrightarrow \omega^n(\omega^n(S/(J_{K_{n-1}} + (x_v, y_v))))$, then there exist an epimorphism f which lifts the map and makes the following diagram

$$\begin{tikzcd}
S/J_{K_n} \arrow[r, "f"] \arrow[d] & S/(J_{K_{n-1}} + (x_v, y_v)) \arrow[d] \\
\omega(\omega(S/J_{K_n})) \arrow[r] & \omega(\omega(S/(J_{K_{n-1}} + (x_v, y_v)))
\end{tikzcd}$$

commutative. Observe that we wrote $\omega(\omega(S/J_{K_n}))$ instead of $\omega^{n+1}(\omega^{n+1}(S/J_{K_n}))$ and $\omega(\omega(S/(J_{K_{n-1}} + (x_v, y_v))))$ instead of $\omega^n(\omega^n(S/(J_{K_{n-1}} + (x_v, y_v))))$ since we are dealing with canonical modules. Moreover, by Proposition 1.1.41, the vertical maps are isomorphisms. Then we obtain the short exact sequence,

$$\begin{aligned} 0 \longrightarrow \omega^{n+1}(\omega^{n+1}(S/J_G)) \longrightarrow S/J_{K_n} &\xrightarrow{f} S/(J_{K_{n-1}} + (x_v, y_v)) \\ &\longrightarrow \omega^n(\omega^{n+1}(S/J_G)) \longrightarrow 0. \end{aligned} \tag{6.6}$$

Since f is a surjective map, then $\omega^n(\omega^{n+1}(S/J_G)) = 0$. Hence, $\operatorname{depth} \omega^{n+1}(S/J_G) = \dim \omega^{n+1}(S/J_G) = n+1$. It follows that $\omega^{n+1}(S/J_G)$ is a Cohen–Macaulay module of dimension $n+1$. Moreover, the sequence (6.6) induces the isomorphism $\omega^{n+1}(\omega^{n+1}(S/J_G)) \cong (J_{K_{n-1}} + (x_v, y_v))/J_{K_n}$ and this completes the proof of point 1. of the statement.

Finally, the isomorphism (6.5) leads to $\omega(S/J_G) \cong \omega(S/J_{G \setminus \{v\}} + (x_v, y_v))$, hence $\omega(S/J_G)$ is a Cohen–Macaulay module of dimension $n+t-1$ and point 2. of the statement is also proved.

Lastly, by conditions 1. and 2. and Theorem 1.1.42, it easily follows that S/J_G is sequentially Cohen–Macaulay. □

In the previous result, we have a complete description of the modules of deficiency of S/J_G, when G is a block star graph. Nevertheless, it can be difficult to apply the same techniques of Theorem 6.1.1 in general cases. In fact, by sequence (6.2), it is not always possible to retrieve informations on the non–vanishing local cohomology modules, hence the modules of deficiency (see for instance [121]).

Example 6.1.2. Let G be the tree on six vertices as shown in the picture below.

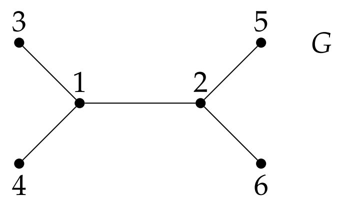


FIGURE 6.2: A tree G on six vertices.

Such a graph has the following cutsets

$$\emptyset, \{1\}, \{2\}, \{1,2\},$$

hence sequence (6.2), by choosing $v = 1$, becomes

$$0 \to S/J_G \to S/J_{G_1} \oplus S/(J_{G\setminus\{1\}} + (x_1, y_1)) \to S/(J_{G_1\setminus\{1\}} + (x_1, y_1)) \to 0.$$

One has $\operatorname{depth} S/J_G = 7$ and $\dim S/J_G = 8$. Moreover, $\dim S/J_{G_1} = \dim S/(J_{G\setminus\{1\}} + (x_1, y_1)) = 8$ and $\dim S/(J_{G_1\setminus\{1\}} + (x_1, y_1)) = 7$ By applying local cohomology functors, we obtain the sequence

$$\begin{aligned} 0 \to H^6_{\mathfrak{m}}(S/(J_{G_1\setminus\{1\}} + (x_1, y_1))) \to H^7_{\mathfrak{m}}(S/J_G) \to H^7_{\mathfrak{m}}(S/J_{G_1}) \to \\ \to H^7_{\mathfrak{m}}(S/(J_{G_1\setminus\{1\}} + (x_1, y_1))) \to H^8_{\mathfrak{m}}(S/J_G) \to \\ \to H^8_{\mathfrak{m}}(S/J_{G_1}) \oplus H^8_{\mathfrak{m}}(S/(J_{G\setminus\{1\}} + (x_1, y_1))) \to 0, \end{aligned} \tag{6.7}$$

where we also kept in mind that $\operatorname{depth} S/J_{G_1} = 7$, $\operatorname{depth} S/(J_{G\setminus\{1\}} + (x_1, y_1)) = 8$ and $\operatorname{depth} S/(J_{G_1\setminus\{1\}} + (x_1, y_1)) = 6$.

The local cohomology modules appearing in sequence (6.7) are all non–zero, hence it is difficult to retrieve informations about $H^7_{\mathfrak{m}}(S/J_G)$ and $H^8_{\mathfrak{m}}(S/J_G)$.

For this reason, we will use Proposition 1.1.45 ([60, Proposition 16]) in what follows, in order to give a deeper classification of those graphs with sequentially Cohen–Macaulay binomial edge ideal.

Observe also that Proposition 1.1.45 can be reformulated, using the notation illustrated in Section 1.1.4, in terms of

$$\mathcal{D}(I) = \left\{ \dim S/P_j \;:\; P_j = \sqrt{Q_j} \right\} = \{d_1, \ldots, d_\ell\}.$$

Lemma 6.1.3. *Let $I \subset S$ be a homogeneous ideal and $\mathcal{D}(I) = \{d_1, \ldots, d_\ell\}$, with $d_1 < \ldots < d_\ell = d$. Then S/I is sequentially Cohen–Macaulay if and only if*

$$\operatorname{depth} S/I^{<d_j-1>} = d_j, \qquad \text{for all } j = 1, \ldots, \ell.$$

Proof. Suppose S/I is sequentially Cohen–Macaulay, then by Proposition 1.1.45, we have that

$$\operatorname{depth} S/I^{<i>} \geq i + 1, \qquad \text{for all } 0 \leq i < d,$$

in particular

$$\operatorname{depth} S/I^{<d_j-1>} \geq d_j, \qquad \text{for all } j = 1, \ldots, \ell.$$

Moreover, for all $j = 1, \ldots, \ell$, we have $\operatorname{depth} S/I^{<d_j-1>} \leq \dim S/I^{<d_j-1>}$ and, by the definition of $I^{<d_j-1>}$, by [24, Proposition 1.2.13] that $\operatorname{depth} S/I^{<d_j-1>} \leq d_j$. Therefore, for all $j = 1, \ldots, \ell$, we have

$$\operatorname{depth} S/I^{<d_j-1>} = d_j.$$

Conversely, suppose

$$\operatorname{depth} S/I^{<d_j-1>} = d_j, \qquad \text{for all } j = 1, \ldots, \ell$$

and show that S/I is sequentially Cohen–Macaulay. Let $0 \leq i < d$, we distinguish two cases.

Case 1. Suppose $i = d_j - 1$ for some j, then $\operatorname{depth} S/I^{<i>} = d_j = i + 1$.

Case 2. Suppose $d_j \leq i < d_{j+1}$ for some j, then $\operatorname{depth} S/I^{<i>} = d_{j+1} \geq i + 1$.

Therefore, for all $0 \leq i < d$,

$$\operatorname{depth} S/I^{<i>} \geq i + 1,$$

and S/I is sequentially Cohen–Macaulay. ◻

We end this section by providing some sufficient conditions and some necessary conditions for a graph S/J_G to be sequentially Cohen–Macaulay. For this aim the next notion will be crucial.

A graph G on n vertices is *ℓ-vertex-connected* (or simply ℓ-connected) if $\ell < n$ and for every subset $T \subset V(G)$ of vertices such that $|T| < \ell$, the graph $G \setminus T$ is connected. The *vertex-connectivity* (or simply *connectivity*) of G, denoted by $\kappa(G)$, is the maximum integer ℓ such that G is ℓ-vertex-connected.

In [11], the authors proved the following result.

Theorem 6.1.4. (Banerjee–Núñez-Betancourt, 2017 [11, Theorem 3.19]) *Let G be a non-complete, connected graph on n vertices and J_G the corresponding binomial edge ideal. Then*

$$\operatorname{depth} S/J_G \leq n - \kappa(G) + 2,$$

where $\kappa(G)$ is the connectivity of the graph G.

The next Proposition characterizes all sequentially Cohen–Macaulay graphs with only one non–empty cutset.

Proposition 6.1.5. *Let G be a graph such that $\mathcal{C}(G) = \{\emptyset, T\}$, let $t = |T|$ and let $c = c(T)$ denote the number of connected components of $G \setminus T$. Then* $\operatorname{depth} S/J_G = n - t + 2$, *and J_G is sequentially Cohen–Macaulay in the following cases:*

(i) $t = 1$;

(ii) $t \geq 2$ *and* $c = 2$;

Proof. From $\mathcal{C}(G) = \{\emptyset, T\}$ we have

$$J_G = P_\emptyset \cap P_T.$$

We recall that $P_\emptyset = J_{K_n}$ and we observe that since $(\{x_v, y_v\}_{v\in T}) \subset P_T$, then $P_\emptyset + P_T = J_{K_{n-t}} + (\{x_v, y_v\}_{v\in T})$. Let $X_T = \{x_1, \dots, x_n, y_1 \dots y_n\} \setminus \{\{x_v, y_v\} : v \in T\}$. Hence, we have the Mayer–Vietoris sequence

$$0 \to S/J_G \to S/J_{K_n} \oplus S/P_T \to K[X_T]/J_{K_{n-t}} \to 0.$$

In particular, from Depth Lemma

$$\begin{aligned}\operatorname{depth} S/J_G &\geq \min\{n+1, n-t+c, (n-t+1)+1\} \\ &= n - t + 2,\end{aligned}$$

because $t \geq 1$ and $c \geq 2$. Since G has connectivity t, from Theorem 6.1.4 we have

$$\operatorname{depth} S/J_G \leq n - t + 2,$$

thus $\operatorname{depth} S/J_G = n - t + 2$. Therefore, J_G is sequentially Cohen–Macaulay if for $i \geq n-t+2$ we have $J_G^{<i>} \neq J_G$, namely

$$J_G^{<i>} \in \{J_{P_T}, J_{K_n}\}.$$

To achieve the above condition, it is sufficient to require that $J_G^{<n-t+2>} \in \{J_{P_T}, J_{K_n}\}$. To obtain

$$J_G^{<n-t+2>} = J_{P_T},$$

we should have $\dim S/J_{K_n} = n+1 \leq n-t+2$. Similarly, to obtain

$$J_G^{<n-t+2>} = J_{K_n},$$

we should have $n-t+c \leq n-t+2$. To sum up, we have the conditions

(i) $n-t+2 \geq n+1 \Rightarrow t \leq 1$,

(ii) $n-t+2 \geq n-t+c \Rightarrow c \leq 2$,

and the assertion follows. □

The next Lemma illustrates a necessary condition for the binomial edge ideal J_G to be sequentially Cohen–Macaulay.

Lemma 6.1.6. *Let G be a non-complete, connected graph on n vertices. Let $\overline{T} \in \mathcal{C}(G)$ such that* $\dim S/P_{\overline{T}} = \mathfrak{m}(G)$. *If J_G is sequentially Cohen–Macaulay, then*

$$\kappa(G) - |\overline{T}| + c(\overline{T}) \leq 2,$$

where $\kappa(G)$ is the connectivity of the graph G.

Proof. We verify the statement by contradiction. Suppose

$$\kappa(G) - |\overline{T}| + c(\overline{T}) \geq 3,$$

we must prove that J_G is not sequentially Cohen–Macaulay. We proceed in the following way:

$$\begin{aligned}\kappa(G) - |\overline{T}| + c(\overline{T}) \geq 3 \implies & \; n - |\overline{T}| + c(\overline{T}) \geq n - \kappa(G) + 3 \\ \implies & \; n - |\overline{T}| + c(\overline{T}) \geq (n - \kappa(G) + 2) + 1\end{aligned}$$

Since $\dim S/P_{\overline{T}} = n - |\overline{T}| + c(\overline{T})$, it is clear that

$$J_G^{<n-\kappa(G)+2>} = J_G.$$

Hence, we obtain

$$\begin{aligned}\operatorname{depth} S/J_G^{<n-\kappa(G)+2>} &= \operatorname{depth} J_G \\ &\leq n - \kappa(G) + 2,\end{aligned}$$

which, by Proposition 1.1.45, contradicts the fact that J_G is sequentially Cohen–Macaulay. □

As a consequence of Lemma 6.1.6, we are able to prove the following:

Proposition 6.1.7. *Let G be a non-complete, connected graph on n vertices. Suppose* $\mathfrak{m}(G) = \dim S/P_\emptyset = n+1$. *If J_G is sequentially Cohen–Macaulay, then G has at least one cutpoint.*

Proof. By Lemma 6.1.6, we know that

$$\kappa(G) - |\overline{T}| + c(\overline{T}) \leq 2,$$

where $\overline{T} = \emptyset$ and $c(\overline{T}) = 1$. Hence,

$$\kappa(G) + 1 \leq 2 \implies \kappa(G) \leq 1.$$

Since G is a connected graph, necessarily $\kappa(G) = 1$ and the assertion follows. □

Inspired by Proposition 6.1.7, in Section 6.3, we will study block graphs. In fact, they satisfy the hypothesis of the Proposition 6.1.7 recursively, by removing cutpoints.

Observe that being sequentially Cohen–Macaulay and with at least a cutpoint does not imply that $m(G) = n + 1$ (e.g. Example 6.4.4).

6.2 Cycles and wheels

In this section we focus on some families of graphs with no cutpoints: we prove that cycles and wheels have sequentially Cohen–Macaulay binomial edge ideal. We recall that a module M is called *almost Cohen–Macaulay* if $\operatorname{depth} M \geq \dim M - 1$, and M is said *approximately Cohen–Macaulay* (introduced in [61]) if and only if M is almost Cohen–Macaulay and sequentially Cohen–Macaulay (see [105, Proposition 4.5]). Let C_n be a cycle with n vertices. We first remark that cycle graphs are not Cohen–Macaulay in general if $n \geq 4$. It has been proved by Zafar in [122] that $\operatorname{depth} S/J_{C_n} = n$ (see [122, Theorem 4.5]). Since, $\dim S/J_{C_n} = n + 1$, then S/J_{C_n} is almost Cohen-Macaulay. In the paper [122], Zafar also proves that cycles are approximately Cohen-Macaulay, and hence sequentially Cohen–Macaulay. Nevertheless, we provide a shorter proof for the sequential Cohen–Macaulayness, by using an elementary property of cycles. Without loss of generality, from now on, we will assume $n \geq 4$.

Lemma 6.2.1. *Let C_n be a cycle with n vertices. Then the nonempty cutsets of C_n have cardinalities at least 2. Moreover, if $T \neq \emptyset$ is a cutset then $|T|$ is equal to the number of connected components of $G \setminus T$.*

Proof. Let $V(C_n) = \{1, \dots, n\}$, and $E(C_n) = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\} \cup \{\{1, n\}\}$. Since C_n is biconnected we need at least 2 vertices to break the cycle. Moreover, without loss of generality we may assume that a nonempty cutset contains the vertex 1. We claim that $T = \{i_1 = 1, \dots, i_r\}$ with $i_k > i_{k-1} + 1$ for all $2 \leq k \leq r$, and $i_r < n$. That is T is a cutset if and only if it is a subset of $\{1, \dots, n\}$ of cardinality at least 2 and such that there are no adjacent vertices in T. Hence, $G \setminus T$ is a collection of $|T|$ paths and isolated vertices as stated. □

Proposition 6.2.2. *Let C_n be a cycle with n vertices. Then J_{C_n} is sequentially Cohen–Macaulay.*

Proof. We first observe that, by Lemma 6.2.1, it easily follows that

$$\dim S/J_{C_n} = n + 1.$$

In fact, given a cutset T of C_n, if $T = \emptyset$, then $\dim S/P_{\emptyset} = n + 1$. Otherwise, if $T \neq \emptyset$, by Lemma 6.2.1 we have that $c(T) = |T|$, thus $\dim S/P_T = n - |T| + c(T) = n$. Therefore,

$$\dim S/J_{C_n} = \max\{n, n+1\} = n + 1,$$

as stated.

Our aim is to apply Lemma 6.1.3 to prove that S/J_{C_n} is sequentially Cohen–Macaulay. We observe that $\mathcal{D}(J_G) = \{n, n+1\}$, and S/J_{C_n} is sequentially Cohen–Macaulay if

$$\operatorname{depth} S/{J_{C_n}}^{<n-1>} = n \text{ and } \operatorname{depth} S/{J_{C_n}}^{<n>} = n+1.$$

From ${J_{C_n}}^{<n-1>} = J_{C_n}$ and $S/{J_{C_n}}^{<n>} = P_\emptyset$, the assertion follows. □

In order to prove a consequence of the previous result, we recall that if G is a graph on the vertex set $V(G) = [n]$ and $v \notin V(G)$, the *cone* of v on G, denoted by $\mathrm{Cone}(v, G)$, is the graph on the vertex set $V(G) \cup \{v\}$ and edge set $E(G) \cup \{\{v, w\} : w \in V(G)\}$. The *wheel graph* is $W_n = \mathrm{Cone}(n+1, C_n)$. Moreover, any nonempty cutset T of W_n is $T' \cup \{n+1\}$ where T' is a nonempty cutset of C_n. Hence, $c(T) = |T| - 1$. It is known that (see [83, Corollary 3.11])

$$\operatorname{depth} S/J_{W_n} = n.$$

Corollary 6.2.3. *Let W_n be a wheel graph. Then J_{W_n} is sequentially Cohen–Macaulay.*

Proof. With a similar arguing to that of Proposition 6.2.2, we obtain

$$\dim S/J_{W_n} = \max\{n, n+2\} = n+2.$$

As done in Proposition 6.2.2, by applying Lemma 6.1.3, the assertion follows. □

From the proofs of Proposition 6.2.2 and Corollary 6.2.3, it arises that $\mathfrak{m}(C_n) = n$ and $\mathfrak{m}(W_n) = (n+1) - 1$.

The class of blocks that are (non-)sequentially Cohen–Macaulay is a big one, as the following and easy example shows.

Example 6.2.4. Let G be a graph with a unique non–empty cutset of cardinality greater than 1, then G is a block. By Proposition 6.1.5 and its notation, if $c = 2$ we construct an infinite family of graphs that are sequentially Cohen–Macaulay. If $c > 2$ we have an infinite family of graphs that are non-sequentially Cohen–Macaulay.

6.3 Block graphs

In this Section we prove that all block graphs have sequentially Cohen–Macaulay binomial edge ideals. As observed in [45, Remark 1.7], we reduce to the study of indecomposable connected block graphs. Moreover, for the sake of simplicity we set $J_v := J_{G_v}$ and $J_{\overline{v}} := J_{G \setminus \{v\}}$ in the Mayer–Vietoris sequence (6.2).

Lemma 6.3.1. *Let v a cutpoint of an indecomposable connected block graph G such that*

$$G \setminus \{v\} = G_1 \sqcup B_1 \sqcup \ldots \sqcup B_r$$

with $r \geq 2$, where B_i are complete graphs. Then

$$J_v^{<i>} + (J_{\overline{v}} + (x_v, y_v))^{<i>} = (J_v + (x_v, y_v))^{<i-1>}.$$

Proof. We claim that if G is an indecomposable block graph, then $\mathcal{C}(G_v) = \mathcal{C}(G \setminus \{v\})$. We first observe that the inclusion $\mathcal{C}(G \setminus \{v\}) \subseteq \mathcal{C}(G_v)$ easily holds. Thus, it suffices to show that $\mathcal{C}(G_v) \subseteq \mathcal{C}(G \setminus \{v\})$. Moreover, if T is a cutset of G that does not intersect $N_{G_1}(v)$, then T is obviously a cutset of both graphs. Let T be a cutset of G_v and assume T is not cutset of $G \setminus \{v\}$. From the above observation, $T \cap N_{G_1}(v) \neq \emptyset$ and hence there exists $u \in T \cap N_{G_1}(v)$ such that

$c_{G_1}(T \setminus \{u\}) = c_{G_1}(T)$. That is, u is not a cutpoint, namely u is free in the graph G_1. This implies $N_G(u) = N_{G_1}(u) \cup \{v\}$ and G is decomposable in u, a contradiction.

Observe now that the block containing v in G_v, namely B_v, is a block of G_v itself, such that $|V(B_v)| \geq 4$ and having at least 3 free vertices. Thanks to this we have that

$$J_v^{<i>} + (x_v, y_v) = (J_v + (x_v, y_v))^{<i-1>}.$$

In fact,

$$J_v^{<i>} = \bigcap_{\dim S/P_T > i} P_T = (\bigcap P'_T) + J_{F_v},$$

where F_v is the complete graph containing the free vertices of B_v, and

$$J_v^{<i>} + (x_v, y_v) = (\bigcap P'_T) + J_{F_v \setminus \{v\}} = J_v^{<i-1>}.$$

Hence, it is sufficient to prove that for all i

$$J_v^{<i>} + (x_v, y_v) \supseteq (J_{\overline{v}} + (x_v, y_v))^{<i>}.$$

For any $T \in \mathcal{C}(G_v) = \mathcal{C}(G \setminus \{v\})$ let P_T^v and $P_T^{\overline{v}}$ be primes of J_v and $J_{\overline{v}}$, respectively. Then

$$\dim P_T^v = n - |T| + c_{G_v}(T) \text{ and } \dim P_T^{\overline{v}} = (n-1) - |T| + c_{G \setminus \{v\}}(T).$$

Observe that $c_{G\setminus\{v\}}(T) = c_{G_v}(T) + r$. Since $r \geq 2$, then

$$\dim P_T^{\overline{v}} = (n-1) - |T| + c_{G_v}(T) + r = \dim P_T^v + r - 1.$$

Let $\mathcal{D}(J_{G_v}) = \{d_0, \ldots, d_l\}$ and $\mathcal{D}(J_{G\setminus\{v\}}) = \{\overline{d}_0, \ldots, \overline{d}_l\}$ where $d_0 = \dim P_\emptyset = n+1$. We have proved that $d_k = \overline{d}_k - r + 1$. we show that for any k

$$J_v^{<d_k - 1>} + (x_v, y_v) \supseteq (J_{\overline{v}} + (x_v, y_v))^{<d_k - 1>}.$$

If $k = 0$, and $d_0 - 1 < \overline{d}_0 - 1$, then

$$J_v^{<d_0-1>} + (x_v, y_v) = J_v + (x_v, y_v) \supseteq (J_{\overline{v}} + (x_v, y_v))^{<d_0-1>} = J_{\overline{v}} + (x_v, y_v)$$

because there is no dimension shift, *i.e.* the filtered ideals $(J_{\overline{v}} + (x_v, y_v))^{<i>}$, for $i = 0, 1, \ldots, d_0 - 1$, coincide with $J_{\overline{v}} + (x_v, y_v)$ itself.

Let $0 < k \leq l$, we observe that since $G \setminus v$ is not connected then

$$(J_{\overline{v}} + (x_v, y_v))^{<\overline{d}_k - 1>} = \left(\bigcap_{\dim S/P'_T \geq \overline{d}_k - r} P'_T \right) + \sum_{i=1}^{r} J_{B_i} + (x_v, y_v).$$

Since J_{F_v} is the binomial edge ideal induced by the free vertices of the block containing v in G_v, then

$$J_v^{<d_k - 1>} = \bigcap_{\dim S/P_T > d_k} P_T = \left(\bigcap_{\dim S/P'_T \geq d_k - 1} P'_T \right) + J_{F_v}.$$

Since $\sum_{i=1}^{r} J_{B_i} \subset J_{F_v}$ and $\overline{d}_k - r = d_k - 1$, we have

$$J_v^{<d_k-1>} + (x_v, y_v) \supseteq (J_{\overline{v}} + (x_v, y_v))^{<\overline{d}_k - 1>} \supseteq (J_{\overline{v}} + (x_v, y_v))^{<d_k - 1>}$$

and the assertion follows. □

Whilst Lemma 6.3.1 holds for block graphs, it is not true in general, as showed in the next example.

Example 6.3.2. Let G be the graph on 7 vertices obtained by attaching a vertex to the complete bipartite graph $K_{2,4}$ as shown in the figure below.

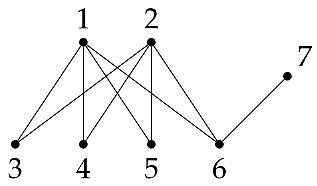


FIGURE 6.3: The graph $K_{2,4}$ with a whisker

Such a graph has the cutpoint 6. By a direct computation, using *Macaulay2* [62], one can show that

$$J_{G_6}^{<7>} + (J_{G\setminus\{6\}} + (x_6, y_6))^{<7>} \neq (J_{G_6\setminus\{6\}} + (x_6, y_6))^{<6>}.$$

We are now in position to prove the main result.

Theorem 6.3.3. *Let G be an indecomposable, connected block graph. Then J_G is sequentially Cohen–Macaulay.*

Proof. Our aim is to apply Proposition 1.1.45 to S/J_G, proving that for all $0 \leq i < d = \dim S/J_G$,

$$\operatorname{depth}(S/J_G^{<i>}) \geq i + 1.$$

We proceed by induction on the number r of the cutpoints of G. If $r = 1$, then G is a block star graph. Let v be the unique cutpoint of G and let $c_v = c(G \setminus \{v\})$. Hence,

$$J_G = J_v \cap (J_{\overline{v}} + (x_v, y_v)),$$

and $d = \dim S/J_G = \dim S/J_{\overline{v}} = n + c_v - 1$. Moreover, it is clear that

$$J_G^{<i>} = \begin{cases} J_G & \text{if } 0 \leq i \leq n, \\ (J_{\overline{v}} + (x_v, y_v)) & \text{if } n < i < d, \\ S & \text{if } i = d. \end{cases}$$

Therefore, if $0 \leq i \leq n$, we have

$$\operatorname{depth}(S/J_G^{<i>}) = n + 1 \geq i + 1.$$

Otherwise, if $n < i < d$

$$\begin{aligned} \operatorname{depth}(S/J_G^{<i>}) &= \operatorname{depth}(S/(J_{\overline{v}} + (x_v, y_v))) \\ &= \dim(S/(J_{\overline{v}} + (x_v, y_v))) \\ &= d \\ &\geq i + 1, \end{aligned}$$

and this proves the base case.

Suppose now $r > 1$ and assume the statement true for $r-1$. Consider the Mayer–Vietoris sequence

$$0 \to S/J_G \to S/J_v \oplus S/J_{\overline{v}} \to S/(J_v + (J_{\overline{v}} + (x_v, y_v))) \to 0.$$

Fix now i such that $0 \le i < d$, the above sequence induces the following sequence on the filtered ideals:

$$0 \to S/J_G^{<i>} \to S/J_v^{<i>} \oplus S/(J_{\overline{v}} + (x_v, y_v))^{<i>} \to S/(J_v^{<i>} + (J_{\overline{v}} + (x_v, y_v))^{<i>}). \quad (6.8)$$

In order to apply the Depth Lemma to sequence (6.8), we express the ideal $J_v^{<i>} + (J_{\overline{v}} + (x_v, y_v))^{<i>}$ in a more suitable form. By Lemma 6.3.1, we know that

$$J_v^{<i>} + (J_{\overline{v}} + (x_v, y_v))^{<i>} = (J_v + (x_v, y_v))^{<i-1>},$$

hence sequence (6.8) becomes

$$0 \to S/J_G^{<i>} \to S/J_v^{<i>} \oplus S/(J_{\overline{v}} + (x_v, y_v))^{<i>} \to (J_v + (x_v, y_v))^{<i-1>}, \quad (6.9)$$

and, by applying the Depth Lemma to the above sequence, we obtain

$$\begin{aligned}\operatorname{depth}(S/J_G^{<i>}) \ge \min\big\{ & \operatorname{depth}(S/J_v^{<i>} \oplus S/(J_{\overline{v}} + (x_v, y_v))^{<i>}),\\ & \operatorname{depth}(S/(J_v + (x_v, y_v))^{<i-1>}) + 1\big\}.\end{aligned}$$

Observe that $\operatorname{depth} S/J_v^{<i>} \le \operatorname{depth} S/(J_{\overline{v}} + (x_v, y_v))^{<i>}$, since G_v is a block graph and $G \setminus \{v\}$ is tensor product of block graphs. Therefore,

$$\operatorname{depth}(S/J_v^{<i>} \oplus S/(J_{\overline{v}} + (x_v, y_v))^{<i>}) = \operatorname{depth} S/J_v^{<i>}.$$

Then

$$\operatorname{depth}(S/J_G^{<i>}) \ge \min\big\{ \operatorname{depth} S/J_v^{<i>}, \operatorname{depth}(S/(J_v + (x_v, y_v))^{<i-1>}) + 1\big\}.$$

By inductive hypothesis, we have that

$$\operatorname{depth} S/J_v^{<i>} \ge i+1, \qquad \operatorname{depth}(S/(J_v + (x_v, y_v))^{<i-1>}) \ge i,$$

and the assertion follows. □

6.4 Cone graphs

Cone graphs have been studied by many authors (e.g. [21],[83],[98]), to control the behavior of the depth and the Cohen–Macaulay property.

In this section we study the binomial edge ideals of cone graphs and we specify under which conditions they are sequentially Cohen–Macaulay.

Theorem 6.4.1. *Let $G = G_1 \sqcup G_2 \sqcup \ldots \sqcup G_r$ be a graph, where G_i is a connected graph for each $i = 1, \ldots, r$, with $r \ge 2$. Let v be a vertex such that $v \notin V(G)$ and $H = \operatorname{Cone}(v, G)$. Then J_H is sequentially Cohen–Macaulay if and only if $J_{G_1}, \ldots, J_{G_r}$ are sequentially Cohen–Macaulay.*

Proof. We first observe that, since H is a cone graph, then

$$H_v = K_{n+1}, \qquad H \setminus \{v\} = G.$$

Hence, we write
$$J_H = J_{K_{n+1}} \cap (J_G + (x_v, y_v)).$$
Moreover,
$$S/(J_G + (x_v, y_v)) \cong S_1/J_{G_1} \otimes \cdots \otimes S_r/J_{G_r},$$
where $S_i = K[x_u, y_u \ : \ u \in V(G_i)]$. Thus, S_i/J_{G_i} are sequentially Cohen–Macaulay if and only if $S/(J_G + (x_v, y_v))$ is sequentially Cohen–Macaulay.

Since $J_H^{<i>} = J_{K_{n+1}}^{<i>} \cap (J_G + (x_v, y_v))^{<i>}$ and $\dim S/J_{K_{n+1}} = n+2$, for $i \geq n+2$
$$J_H^{<i>} = (J_G + (x_v, y_v))^{<i>}.$$
Therefore, if $i \geq n+2$, we have $\operatorname{depth}(S/J_H^{<i>}) \geq i+1$ if and only if $\operatorname{depth}(S/(J_G + (x_v, y_v))^{<i>}) \geq i+1$.

Thus, we now deal with the case $i \leq n+1$. We observe that
$$\dim S/P_{\{v\}} = n + r \geq n+2,$$
thus $P_{\{v\}}$ appears in the primary decomposition of any $(J_G + (x_v, y_v))^{<i>}$ for any $i \leq n+1$. Then we write
$$\begin{aligned} J_H^{<i>} &= J_H^{<i>} \cap P_{\{v\}} \\ &= P_\emptyset \cap (J_G + (x_v, y_v))^{<i>} \cap P_{\{v\}} \\ &= (P_\emptyset \cap P_{\{v\}}) \cap (J_G + (x_v, y_v))^{<i>}. \end{aligned}$$
Note that $P_\emptyset \cap P_{\{v\}} = J_{B_v}$, where B_v is a block star graph with cutpoint v. Now, we consider the Mayer–Vietoris sequence
$$0 \to S/J_H^{<i>} \to S/J_{B_v} \oplus S/(J_G + (x_v, y_v))^{<i>} \to S/J_{B_v} + (J_G + (x_v, y_v))^{<i>} \to 0. \quad (6.10)$$
Since
$$(J_G + (x_v, y_v))^{<i>} \subset P_{\{v\}},$$
and any cutset of H contains v, that is
$$(x_v, y_v) \subset (J_G + (x_v, y_v))^{<i>}.$$
Therefore,
$$P_{\{v\}} = J_{B_v} + (x_v, y_v) \subseteq J_{B_v} + (J_G + (x_v, y_v))^{<i>} \subseteq J_{B_v} + P_{\{v\}} = P_{\{v\}},$$
yielding $J_{B_v} + (J_G + (x_v, y_v))^{<i>} = P_{\{v\}}$. Hence, by the Depth Lemma applied to (6.10), we have
$$\begin{aligned} \operatorname{depth}(S/J_H^{<i>}) &\geq \min\left\{ \operatorname{depth}(S/J_{B_v}), \operatorname{depth}(S/(J_G + (x_v, y_v))^{<i>}), \operatorname{depth}(S/P_{\{v\}}) + 1 \right\} \\ &= \min\left\{ n+2, \operatorname{depth}(S/(J_G + (x_v, y_v))^{<i>}), n + r + 1 \right\} \end{aligned}$$
and
$$\begin{aligned} \min\left\{ \operatorname{depth}(S/(J_G + (x_v, y_v))^{<i>}), \operatorname{depth}(S/J_{B_v}) \right\} &\geq \min\left\{ \operatorname{depth}(S/J_H^{<i>}), \operatorname{depth}(S/P_{\{v\}}) \right\} \\ &= \min\left\{ \operatorname{depth}(S/J_H^{<i>}), n + r \right\} \end{aligned}$$
Since $i + 1 \leq n+2$, we have that
$$\operatorname{depth} S/J_H^{<i>} \geq i+1 \Longleftrightarrow \operatorname{depth} S/(J_G + (x_v, y_v))^{<i>} \geq i+1,$$

and the assertion follows. □

Remark 6.4.2. The hypothesis $r \geq 2$ of Theorem 6.4.1 is fundamental. Indeed, even if the wheel W_n is a cone on a cycle C_n, and they are both sequentially Cohen–Macaulay, there are examples of sequentially Cohen–Macaulay graphs G such that the cone from a vertex v to G is not sequentially Cohen–Macaulay.

Example 6.4.3. Let $G = K_{1,3}$ be the claw graph on vertices $\{1, 2, 3, 4\}$, then $H = \text{Cone}(5, G)$ is such that $\mathcal{C}(G) = \{\emptyset, T = \{1, 5\}\}$ and $c(T) = 3$. From Theorem 6.1.1 G is sequentially Cohen–Macaulay, while from Proposition 6.1.5, H is not sequentially Cohen–Macaulay.

FIGURE 6.4: A cone H on a connected graph G

Theorem 6.4.1 is also useful to construct a sequentially Cohen–Macaulay graph G with a cutpoint such that $\mathfrak{m}(G) \neq |V(G)| + 1$.

Example 6.4.4. Let $G = \text{Cone}(9, C_4 \sqcup C_4)$ be the cone on the disjoint union of two cycles C_4 (see Figure 6.5). Such a graph is sequentially Cohen–Macaulay by Theorem 6.4.1 and has cutpoint 9. Nevertheless, $\mathfrak{m}(G) = 8 < 10 = |V(G)| + 1$.

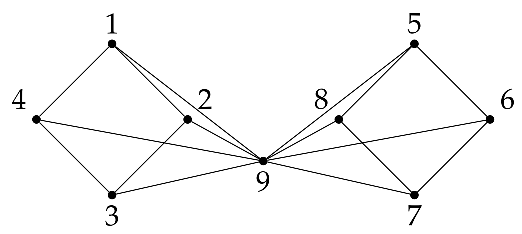


FIGURE 6.5: A cone graph on two graphs C_4

Chapter 7

A Macaulay2 package for sequential Cohen–Macaulayness

This chapter is devoted to present a new *Macaulay2* [62] package, `SCMAlgebras` [84, 85], designed to check whether a given module or ideal satisfies the sequentially Cohen–Macaulay property. This package implements key theoretical results, such as Theorem 1.1.42, which characterizes sequentially Cohen–Macaulay modules in terms of their modules of deficiency, and Proposition 1.1.45 due to Goodarzi [60], which provides an efficient criterion for verifying the sequentially Cohen–Macaulay property for homogeneous ideal. To support these results, the package introduces new computational tools, such as functions for calculating the modules of deficiency (see Definition 1.1.38) and generating filter ideals (see Definition 1.1.44), both of which are crucial for testing the sequentially Cohen–Macaulay property of the input objects. Moreover, the filter ideals serve as useful tools for checking the unmixedness of an ideal (Proposition 1.1.47), *via* the notion of unmixed layer.

These tools extend the functionality of the software, playing as a starting point for future development in the manipulation of sequentially Cohen–Macaulay property in both theoretical and applied contexts.

7.1 Example usage of the package

In this section we describe the main functions of the package `SCMAlgebras` by showing some examples of usage and algorithms.

Example 7.1.1. Let $S = K[x_1, x_2, x_3, x_4, x_5]$ and let M be the S-module defined as the Coker of the map represented by the matrix $\begin{pmatrix} x_1x_2 & x_3x_4 & 0 & 0 \\ 0 & x_1x_5 & x_2x_4 & 0 \end{pmatrix}$. Such a module has depth $M = 3$ and $\dim M = 4$. We use the `deficiencyModule` function to compute the modules of deficiency of M.

```
i1 :  loadPackage "SCMAlgebras";
i2 :  S=QQ[x_1..x_5];
i3 :  M=coker matrix{{x_1*x_2,x_3*x_4,0,0},{0,x_1*x_5,x_2*x_4,0}}
o3 =  cokernel | x x   x x   0     0 |
                  1 2   3 4
               | 0     x x   x x   0 |
                        1 5   2 4
                                 2
o3 :  S-module, quotient of S

i4 :  deficiencyModule(M,1)
o4 =  0
o4 :  S-module
```

```
i5 :  deficiencyModule(M,3)
                            2      2
o5 =  cokernel | x x    x x    x x x  |
                  1 5    3 4    1 2 4
                                    1
o5 :  S-module, quotient of S
```

The function `deficiencyModule(M,i)` uses Definition 1.1.38 and computes the module $\mathrm{Ext}_S^{n-i}(M, S(-n))$. According to Remark 1.1.39, in our case, the function returns the zero module whenever $i < 3$ or $i > 4$, and the only non vanishing modules of deficiency are $\omega^3(M)$ and $\omega^4(M) = \omega(M)$, *i.e.* the canonical module of M. Moreover, the function `canonicalModule(M)` allows to compute the canonical module $\omega(M)$ directly, as the module of deficiency corresponding to dimension.

```
i6 :  canonicalModule(M) == deficiencyModule(M,4)
o6 =  true
```

Hence, one can check if M is sequentially Cohen–Macaulay using the `isSCM` function, involving the modules of deficiency, as seen in Theorem 1.1.42.

```
i7 :  isSCM M
o7 =  false
```

Moreover, using the `isCCM` function, one can check that M is canonically Cohen–Macaulay, by checking the Cohen–Macaulayness of its canonical module.

```
i8 :  isCCM M
o8 =  true
```

Remark 7.1.2. Note that, while in Example 7.1.1, the functions `deficiencyModule` and `canonicalModule` are applied to a module, they can also be applied to an ideal.

Example 7.1.3. Consider the graph G on ten vertices depicted below.

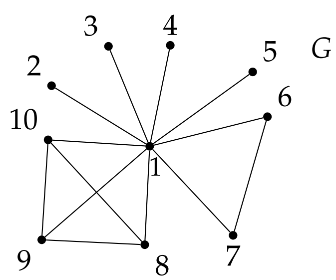


FIGURE 7.1: A block star graph G on ten vertices.

Consider the ideal J_G in the polynomial ring $S = K[x_1, \ldots, x_{10}, y_1, \ldots, y_{10}]$ generated by all the binomials $f_{ij} = x_i y_j - x_j y_i$, such that $\{i, j\}$ with $i < j$ is an edge of G, namely the *binomial edge ideal* (see [69, 95]) of G.

```
i1 :  loadPackage "SCMAlgebras";
i2 :  S = QQ[x_1..x_10,y_1..y_10];
i3 :  E = {{1,2},{1,3},{1,4},{1,5},{1,6},{1,7},{1,8},{1,9},{1,10},
      {6,7},{8,9},{8,10},{9,10}};
i4 :  J=ideal(for e in E list x_(e#0)*y_(e#1)-x_(e#1)*y_(e#0));
```

The function `filterIdeals` computes the filter ideals of J_G, according to Definition 1.1.44. Moreover, the function `minimumDimension` returns the smallest integer such that the filter ideals differ from J_G.

```
i5 :  filterIdeal(J,13)
o5 =  ideal (y , x , x  y  - x y  , x  y  - x y  , x y  - x y ,
              1   1   10 9    9 10   10 8    8 10   9 8    8 9
      ------------------------------------------------------------
      x y  - x y )
       7 6    6 7
o5 :  Ideal of S
i6 :  minimumDimension J
o6 =  11
i7 :  filterIdeal(J,7) == J
o7 =  true
i8 :  filterIdeal(J,11) == J
o8 =  false
```

Using the function `unmixedLayer`, one can compute the unmixed layers of J_G, according to Definition 1.1.46.

```
i9   :  unmixedLayer(J,13)
o9   =  subquotient (| y   x   x  y  - x y    x  y  - x y
                        1   1   10 9    9 10   10 8    8 10
        x y  - x y   x y  - x y  |, | y   x   x  y  - x y
         9 8    8 9   7 6    6 7       1   1   10 9    9 10
        x  y  - x y    x y  - x y   x y  - x y  |)
         10 8    8 10   9 8    8 9   7 6    6 7
                                   1
o9   :  S-module, quotient of S
```

To check the unmixedness of J_G, the function `isUnmixed` can be applied to the ideal `J`. In this case Proposition 1.1.47 is used.

```
i10 :  isUnmixed J
o10 =  false
```

Finally, one can check that S/J_G is sequentially Cohen–Macaulay (as proved in [86, Theorem 1.5]) using the `isSCM` function. This time we apply the function to the ideal `J`, in such case Proposition 1.1.45 is used.

```
i11 :  isSCM J
o11 :  true
```

Remark 7.1.4. At the end of Example 7.1.1 we applied the `isSCM` function to `J` and Goodarzi's characterization (Proposition 1.1.45) was used to check the sequentially Cohen–Macaulay property for S/J_G. Note that, to make the check involving Theorem 1.1.42, one can apply the function to `S^1/J` instead of `J`,

```
i10 :  isSCM(S^1/J)
o10 :  true
```

getting, as expected, the same result.

A full list of the functions provided by the package `SCMAlgebras` is summarized in table 7.1.

Function	Description
`deficiencyModule(M,i)`	Computes the ith module of deficiency $\omega^i(M)$ of M
`canonicalModule(M)`	Computes the canonical module $\omega(M)$ of M
`minimumDimension(I)`	Computes the minimum dimension of I
`filterIdeal(I,i)`	Computes the ith filter ideal $I^{<i>}$ of I
`unmixedLayer(I,i)`	Computes the ith unmixed layer $U_i(I)$ of I
`isUnmixed(I)`	Checks if I is unmixed
`isSCM(M)`	Checks if M is sequentially Cohen–Macaulay
`isCCM(M)`	Checks if M is canonically Cohen–Macaulay

TABLE 7.1: Available functions in `SCMAlgebras v1.1`

7.2 Further developments

The examples and the algorithms presented in this chapter are part of the *Macaulay2* package `SCMAlgebras v1.1` [84, 85].

The tests were performed with version `1.25.05` of *Macaulay2* [62].

We are confident that the methods and functionalities provided herein may be valuable for further applications. Moreover, the structure of the package allows for future developments, specifically by implementing additional key results and techniques. This task is currently under study by the author.

Conclusions

This thesis has focused on minimal resolutions, algebraic properties and invariants of graded ideals, studied in both the exterior algebra and the polynomial ring frameworks. Throughout the dissertation, we primarily followed a combinatorial approach; we also used various computational tools to verify the results through examples. We obtained many results in this direction; nevertheless, there are several open problems and further investigations that can be dealt with.

In Chapter 5, we analyzed the following conjecture.

Conjecture. (Erey–Herzog–Hibi–Saeedi Madani, 2023 [48]) *For any squarefree ideal $I \subset S$, the function $g_I(k)$ is a non–increasing function.*

Although counterexamples show that such conjecture does not hold in general for every squarefree monomial ideal [107], there is still hope that it is true for every edge ideal. Hence, it would be interesting to study the normalized depth function for other classes of graphs (see for instance [48, 56]).

Moreover, we provided a formula for the normalized depth function of a path graph (Theorem 5.3.6), thus proving its non–increasingness. This also yields the proof of an upper bound for the normalized depth function of any connected graphs in terms of induced paths (Corollary 5.3.7). Let C_n be the cycle graph on n vertices. Our experiments using *Macaulay2* [62] suggest that $g_{I(P_n)}(k) = g_{I(C_n)}(k)$, for all $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$. Hence, the following question arises naturally.

Question. *Let C_n denote the cycle on n vertices. That is, $V(C_n) = [n]$ and $E(C_n) = E(P_n) \cup \{\{1, n\}\}$. Then $\nu(C_n) = \nu(P_n)$. It is well known that* $\operatorname{depth}(S/I(P_n)) = \lceil \frac{n}{3} \rceil$ *and* $\operatorname{depth}(S/I(C_n)) = \lceil \frac{n-1}{3} \rceil$*, thus $g_{I(P_n)}(1)$ and $g_{I(C_n)}(1)$ differ at most by one. Is it true that*

$$g_{I(C_n)}(k) = g_{I(P_n)}(k)$$

for all $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$?

Note that if this question is answered in the affirmative, then an analogue of Corollary 5.3.7 in terms of cycles in a graph may be deduced.

In Chapter 6, we studied the sequentially Cohen–Macaulay property of various classes of graphs: cycles, wheels, block graphs, cones. As for the Cohen–Macaulay property, a full classification of graphs with sequentially Cohen–Macaulay binomial edge ideal is not yet known. Moreover, following Theorem 6.1.1, it would be interesting to find classes of graph for which a full description of the modules of deficiency can be achieved (see [106]).

Our results show that all trees, as well as all (even) cycles have sequentially Cohen–Macaulay binomial edge ideals. Moreover, the only complete bipartite graphs are C_4 [106] and trees (Section 6.3). A complete characterization of bipartite Cohen–Macaulay graphs has been done in [21]. Hence, the following question arises.

Question. *What are bipartite graphs with sequentially Cohen–Macaulay binomial edge ideals?*

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