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On Degenerate Nonlocal Carrier's Problems

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Declaration of Authorship

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"...però quanto a te, quanto a quel che non puoi fare che tu, per te qualcosa da poter fare dovrebbe esserci, ma non ti costringerà nessuno. Pensa da te stesso, decidi da te stesso, che cosa adesso tu stesso debba fare. Beh, che tu non abbia rammarichi..."

Ryōji Kaji - Neon Genesis Evangelion

UNIVERSITY OF MESSINA

Abstract

Department of Mathematical and Computer Sciences,
Physical Sciences and Earth Sciences

Doctor of Philosophy

On Degenerate Nonlocal Carrier's Problems

by GIUSEPPE FAILLA

The aim of this thesis is to present some original results of existence, nonexistence and multiplicity of solutions for a class of nonlocal degenerate Carrier's problems

$$\begin{cases} -a\left(\int_\Omega u^q dx\right)\Delta_p u = f(u) & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, a and f are continuous functions fulfilling some additional conditions, and $q \geq 1$. The main issue of this class of problems lies in the loss of variational structure. This feature forces us to gain our results thought new techniques. In particular, we provide some general results for existence and multiplicity of positive solutions for the nonlocal elliptic problem (1) where the degenerate weight function could be sign-changing with a known number of positive bumps. Our results will be extended to problems of type (1) driven by non-homogeneous operators as $p(x)$-Laplacian and Double Phase operator and for problems with x-dependence on the reaction term. A careful analysis will show that similar results can be provided also for Neumann problems. Moreover, some nonexistence results permit us to enrich the scenario.

The main approach is based on the following steps:

- first we freeze the nonlocal term in the positive bumps of the weight degenerate function a and we obtain a variational elliptic problem;
- then, we provide, through sub-super solutions methods and variational tools, the existence of at least one positive solution of the frozen problem;
- next, when it is the case, we provide the uniqueness of the solution for the frozen problem or in any case, the existence of the unique smallest solution in a suitable interval;
- finally, we build a one-dimensional fixed point problem that allows us to provide the existence of at least two fixed points. In particular, each fixed point of this map will be a positive solution of our original problem (1).

We emphasize that, in the first part of this work, we obtain our results by imposing a monotonicity assumption involving the reaction term, i.e., we assume that the map

$$t \mapsto \frac{f(t)}{t^{p-1}}$$

is strictly decreasing in a suitable interval. This hypothesis, exploiting a Díaz-Saá-type argument, permits us to obtain the uniqueness of solution of our problem with frozen nonlocal term. In the second part, we are able to remove this monotonicity constrain. This new challenge require a deeper analysis and the introduction of a suitable set-valued map, in order to provide the existence of the unique smallest solution in an appropriate interval. In both cases a sophisticated combination of sub-super solutions methods, variational tools and fixed point theory allows us to come back to the solutions of our original problem.

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Special thanks to Prof. Scivoli. If I had not met her fifteen years ago, I would not be here today (trying to) do mathematics.

List of Symbols

a.e.	almost everywhere
a.a.	almost any
w.r.t.	with respect to
l.s.c.	lower semi-continuous
u.s.c.	upper semi-continuous
$\mathbb{R}$	real line
$\mathbb{R}^N$	N-dimensional Euclidean space
$\mathbb{R}_+$	the set of positive real number, i.e., $(0, +\infty)$
$\lvert \cdot \rvert$	Lebesgue measure
$\lvert \cdot \rvert_{\mathcal{H}^{N-1}}$	$(N-1)$-dimensional Hausdorff measure
$\to$	strong convergence
$\rightharpoonup$	weak convergence
$\hookrightarrow$	embedding
$(P, \leq)$	partially ordered set
Ω	bounded domain in $\mathbb{R}^N$
$\partial\Omega$	boundary of Ω
$\overline{\Omega}$	closure of Ω, i.e., $\Omega \cup \partial\Omega$
$B(x_0, R)$	open ball or radius R centered at x_0
$\frac{\partial u}{\partial n}$	outward normal derivative of u
u^-	negative part of the function u, i.e., $u^- = \max\{0, -u\}$
u^+	positive part of the function u, i.e., $u^+ = \max\{0, u\}$
$C(\Omega)$	space of continuous functions in Ω
$C^{1,\nu}(\Omega)$	space of Hölder-continuous functions in Ω
$C^{0,1}(\Omega)$	space of Lipschitz-continuous functions in Ω
$C_0^\infty(\Omega)$	space of C^∞-functions with compact support
C_+	positive cone of C^1-functions
X^*	dual space of X
$L^0(\Omega)$	space of measurable functions $u : \Omega \to \mathbb{R}$
$L^p(\Omega)$	Lebesgue space
$W^{1,p}(\Omega)$	Sobolev space
$L^{p(x)}(\Omega)$	Lebesgue space with variable exponent
$W^{1,p(x)}(\Omega)$	Sobolev space with variable exponent
$L^{\mathcal{H}}(\Omega)$	Musielak-Orlicz space
$W^{1,\mathcal{H}}(\Omega)$	Sobolev-Musielak-Orlicz space
$\lVert \cdot \rVert_X$	norm in X
$\rho_{p(x)}$	modular function for variable exponent framework
$\rho_{\mathcal{H}}$	modular function for Musielak-Orlicz framework
p'	Hölder conjugate exponent of p
p^*	Sobolev critical exponent of p
∇u	gradient of u
div A	divergence of A
Δ_p	p-Laplacian $\Delta_p u = \operatorname{div}\left(\lvert\nabla u\rvert^{p-2}\nabla u\right)$

$\lambda_{1,p}$	first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$
$\Delta_{p(x)}$	$p(x)$-Laplacian $\Delta_{p(x)}u = \text{div}\left(\lvert\nabla u\rvert^{p(x)-2}\nabla u\right)$
$\lambda_{1,p(x)}$	first eigenvalue of $\left(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$
Δ_p^μ	weighted p-Laplacian $\Delta_p^\mu u = \text{div}\left(\mu(x)\lvert\nabla u\rvert^{p-2}\nabla u\right)$
$\mathcal{G} = \Delta_p^\mu + \Delta_q$	Double Phase operator
φ_1	first positive eigenfunction of the related operator
e_1	first eigenfunction of the related operator normalized in L^∞-norm
φ_t	solution of the torsion problem
e_t	solution of the torsion problem normalized in L^∞-norm
$\mathscr{S}$	set-valued map
$\mathscr{S}^+(A)$	strong inverse of A under the set-valued map $\mathscr{S}$
$\mathscr{S}^-(A)$	weak inverse of A under the set-valued map $\mathscr{S}$
2^Y	all subsets of Y, i.e., $\{Z : Z \subseteq Y\}$

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To Myself,
to My past self for believing,
to My future self for never stopping believing.

Chapter 1

Introduction

Over the past eighty years, in the wide sea of problems that have appeared in mathematical analysis, a special place is reserved for nonlocal problems. However, even in this last class of problems, many lines of research have opened up. One of the most fascinating certainly concerns the problems introduced by Carrier [19, 20] between 1945 and 1949; here, the author, studying the deflection of beams, proposed a mathematical model

$$\left[1+\frac{\alpha^{-2}}{2\pi}\int_0^{\pi}\varphi^2(\xi,\eta)d\xi\right]\frac{\partial^2\varphi}{\partial\xi^2}=\frac{\partial^2\varphi}{\partial\eta^2},$$

that would be destined to arouse widespread interest in the subsequent years. We refer the interest reader to the works of Dickley [46, 47], Greenberg-Hu [75], Rowland [114] and the references therein for an overview on the Carrier's model applied on string vibration theory.

Although these models have been originated in the context of mechanical theory, in the last years of the previous century and in the early part of the new millennium, the biology field has sparked renewed interest in such models. In particular, we refer to the papers of Carillo-Chipot [17], Chang-Chipot [22], Chipot-Corrêa [25], Chipot-Lovat [26, 27], Chipot-Molinet [28], Chipot-Rodrigues [29], Chipot-Siegwart [30], Corrêa [33], Corrêa-Menezes-Ferreira [34], and the related references. In these works, the authors provide a deeper analysis of the following equation

$$-\mathrm{div}\left[a\left(\int_\Omega u dx\right)\nabla u\right]=f-\lambda u \ \text{ in }\Omega$$

and for parabolic counterpart. What is inferred is the possibility of using such models to describe the following feature: let us consider u as the density of bacteria in a container Ω. Let f be the inflow of external resources and $\lambda>0$ be the mortality rate. The above equation permits to describe a growth model where we may assume that the velocity of dispersion of the bacteria's population depends by the gradient of density, i.e.,

$$v=-a\left(\int_\Omega u dx\right)\nabla u.$$

A brief but significant digression is certainly in order on Kirchhoff-type problems, see the seminal book of Kirchhoff [88]. Similarly with respect to the Carrier-type one, the problem presents a nonlocal term

$$-a\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u=f$$

but, on the contrary, in this case, the dependence of the weight function a on the gradient gives a fully variational structure to Kirchhoff's models. The lack of variational structure on Carrier-type models has led to the study of such problems with different techniques. In particular, let us consider the problem

$$\begin{cases} -a\left(\int_\Omega |u|^q dx\right)\Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

for some $q \geq 1$. We refer the interested reader to the works of Bueno-Ercole-Ferreira-Zumpano [12], where the authors obtained an existence result by Schauder's fixed point Theorem, Corrêa-de Morais Filho [35] in which the Galerkin method produces solutions for Carrier problems, Dai [43] for iterative methods for Kirchhoff-Carrier problems, Figueiredo Sousa-Morales Rodrigo-Suárez [61] for linear and sublinear cases with bifurcation argument, Fijałkowski-Przeradzki [62] for Krasnosel'skii fixed point theorem use, Infante [84] and Yan-Wang [125] for fixed point index approach, Jiang-Zhai [85] for applications of monotone and mixed monotone operators theory, Jin-Yan [86] where the use of Leray-Schauder's degree gives a multiplicity result with sign information on the solutions, do Ó-Lorca-Sánchez-Ubilla [108] for result exploiting a fixed point theorem of cone expansion-compression type, Santos-Santos [116] for a combination of sub-super solutions methods with comparison principle in singular framework, Santos-Santos-Mishra [117] for the singular case treated with bifurcation and ε-perturbation methods or Yan-Ma [123] and Yan-Ren [124] for existence and multiplicity of solutions gained through sub-super solutions and Rabinowitz's bifurcation techniques. In addition, we address the curious reader to Furter-Grinfeld [65], for a clear panoramic on the differences between local and nonlocal models, and Lehoucq-Silling [91, Section 6.5] and Stańczy [119] for a wide overview on nonlocal theory in all the open research lines, which are not the main focus of this thesis.

A crucial point in the study of these nonlocal problems lies in the sign of the weight function. In all the aforementioned papers, the weight a is strictly positive and separated from zero, i.e., there exists some positive constant a_0 such that

$$\inf_{t\in[0,+\infty)} a(t) \geq a_0 > 0.$$

First steps in new directions was taken, in 2017, by Ambrosetti-Arcoya [3], where the authors consider a weight function in a Kirchhoff problem, that could approach zero at zero or infinity. Based on this seminal idea, in 2018, Santos Júnior and Siciliano [118] provided an existence and multiplicity result for a class of Kirchhoff problems such that the weight function could be zero in multiple points. Both the above results exploited the variational structure of the Kirchhoff equation.

To obtain results for the nonvariational Carrier-type problem we have to wait until 2019-2020, when, in two papers devoted to problem (1.1) driven by the Laplace operator (i.e., $p = 2$), Gasiński-Santos Júnior [70, 71] obtained results for existence, nonexistence and multiplicity of positive solutions and Gasiński-Santos Júnior- Siciliano [72] for Laplacian problem with possible singularities. Finally, in 2020, Candito-Gasiński-Livrea-Santos Júnior [15] improved the previous results of existence and multiplicity of positive solutions for a class of p-Laplacian problems, $1 < p < +\infty$, of (1.1)-type, where the function a could be sign-changing or zero in some intervals. The main results in these last papers are obtained considering a monotonicity assumption involving the reaction term, i.e., the map

$$t \overset{\psi}{\mapsto} \frac{f(t)}{t^{p-1}} \tag{1.2}$$

is considered to be strictly decreasing in a suitable interval. Assumption (1.2) is crucial in several points of the proofs. First, studying the problem (1.1) with frozen nonlocal term, i.e.,

$$\begin{cases} -a(\alpha)\,\Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

a Díaz-Saá-type argument and (1.2) guarantee us the uniqueness of solution in a suitable interval. On the other hand, the building of a barrier from below of this unique solution, lies in the inverse of the ψ map. Finally, to come back to the nonlocal problem (1.1), a fixed point map is considered, also in this last point, exploiting the map ψ, one can achieve the right geometry to obtain the desired multiplicity result. In particular, in [15] the authors proved the existence of at least K pairs of positive solutions for problem (1.1), being K the number of positive bumps of the weight function a.

Throughout this work, when it is possible, we will maintain the sign-changing condition on the nonlocal term a. The main goals in this thesis are to prove original results on Carrier-type problems in the following directions:

- consider general driven operators as $p(x)$-Laplacian or Double Phase operator;
- increase the number of solutions;
- prove the nonexistence of solutions;
- consider different boundary conditions;
- remove any monotonicity assumption involving the reaction term.

All these objectives have been successfully pursued.

First, in Chapter 3, a (1.2)-type condition is maintained. For the existence of multiple solutions we obtain the following:

- in Section 3.1.4, we provide a multiplicity result for problem of (1.1)-type with Neumann boundary conditions, the results of this section are contained in the paper

  Á. Crespo-Blanco, G. Failla, B. Vassalo [37], *multiple solutions for a nonlocal Neumann p-Laplacian problem*, Nonlinear Anal. Real World Appl., 87, 2026;

- in Section 3.2, we generalize the results for the Dirichlet problem (1.1), considering a variable exponent framework. Furthermore, a wise use of the sub-super solutions method, allows us to consider a nonautonomous, but still with splitted variables, case. These results can be found in:

  P. Candito, G. Failla, R. Livrea [14], *Pairs of positive solutions for a Carrier $p(x)$-Laplacian type equation*, Mathematics, 2024, 12, 2441.

Moreover, it is possible to improve the multiplicity result for problem (1.1) in order to guarantee the existence of an arbitrary number of solutions.

- This result, can be found in Section 3.1.3 and in the work:

  G. Failla, L. Gasiński, J. R. Santos-Júnior [54], *Nonexistence and multiplicity results for p-Laplacian nonlocal problems*, Submitted.

We also prove two different nonexistence theorems for problem (1.1).

- These results are collected in Section 3.1.2, and in

  G. Failla, L. Gasiński, J. R. Santos-Júnior [54], *Nonexistence and multiplicity results for p-Laplacian nonlocal problems*, Submitted.

On the other hand, in Chapter 4, we remove all type of monotonicity assumption involving the reaction term. In this last chapter a deeper analysis of the problem reveals that, combining sub-super solutions method, variational tools, and truncation techniques with set-valued analysis and fixed point theory, it is possible to obtain new results for a very large class of nonlocal problems involving homogeneous (as p-Laplacian) and non-homogeneous (as double phase) operators. In particular, for multiplicity of solutions, we refer to

- Section 4.1.1, and the paper

  P. Candito, G. Failla, L. Gasiński, R. Livrea [13], *Multiple positive solutions for a quasilinear nonlocal problem via topological, variational and set-valued methods*, Submitted,

  for p-Laplacian problem. Furthemore, a novelty of this result lies in the possibility to have a fully nonautonomous reaction term.

- Section 4.1.4 and

  G. Failla, L. Gasiński [53], *Nonlocal Carrier's double phase problem*, Proc. Roy. Soc. Edinburgh Sect. A,

  for the double phase counterpart.

Also in this case, we provide a many solutions-type result, where we improve the number of solutions, see

- Section 4.1.2 and

  G. Failla, L. Gasiński [52], *Many solutions for a nonlocal p-Laplacian problem without monotonicity assumptions*, Submitted.

Finally,

- in Section 4.1.3 and again in

  G. Failla, L. Gasiński, J. R. Santos-Júnior [54], *Nonexistence and multiplicity results for p-Laplacian nonlocal problems*, Submitted,

  we prove a third nonexistence result for problem (1.1) without monotonicity conditions.

See the diagram below as guide:

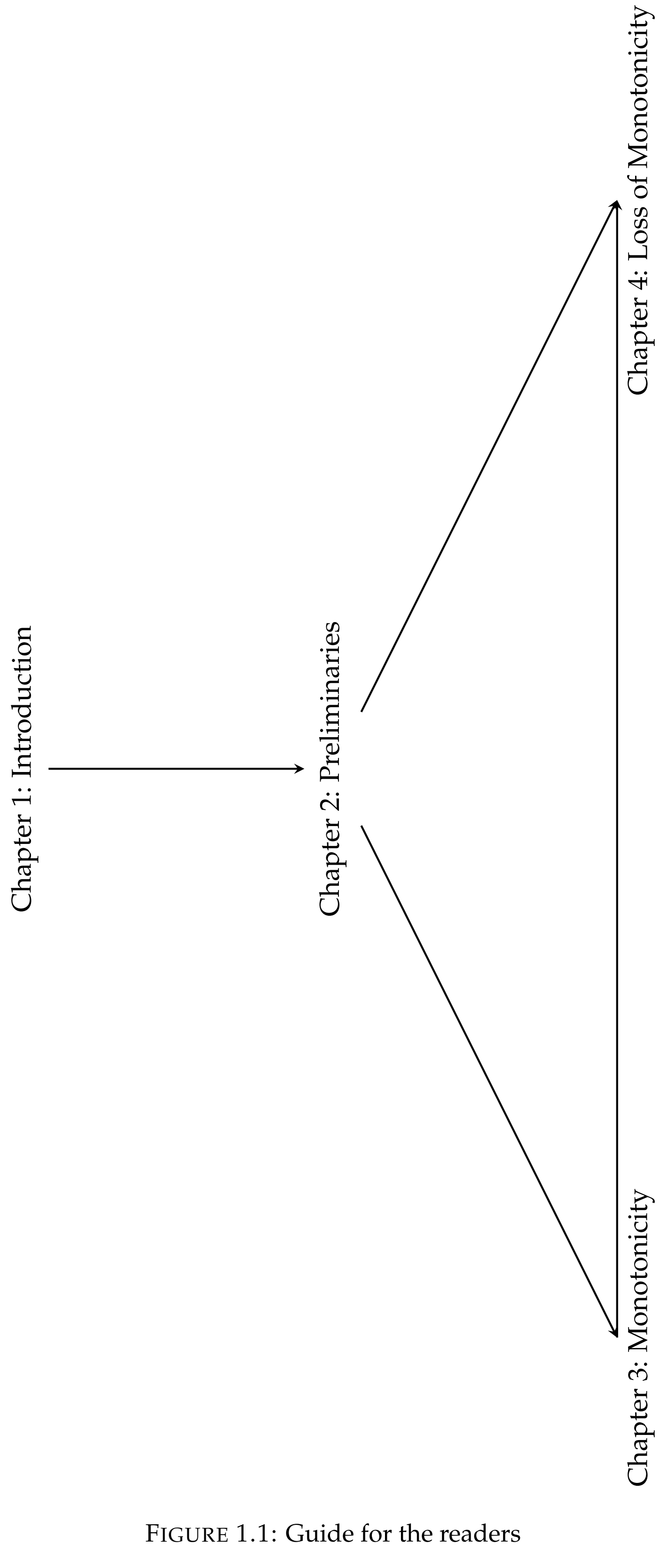

FIGURE 1.1: Guide for the readers

Chapter 2

Preliminaries

This chapter is devoted to introduce the main tools for the rest of the thesis. In the first part, without any claim to completeness, we introduce the setting spaces in which we will study our problems in Chapter 3 and Chapter 4. In particular, for constant exponent p-Laplacian problem, the natural setting space is the classical Sobolev space. The variable exponent problem brings the necessity to extend the classical Sobolev spaces with the variable exponent one. Moreover, to obtain results for double phase problems, we recall some notions on Sobolev-Musielak-Orlicz spaces. In each subsection we furnish a wide list of references for a complete overview on these topics. Next, we recall the main tools of sub-super solutions, variational and non-variational methods. In particular, we point out some useful properties of the main operators. Finally, a brief overview on set-valued analysis is provided. This last section will be fundamental in Chapter 4.

2.1 Functional setting spaces

For a deeper analysis on functional spaces we address the reader to the seminal books of Adams-Fournier [1] and Brézis [9] for constant exponent case, the books of Diening-Harjulehto-Hästö-Růžička [48], Rădulescu-Repovš [113] and Růžička [115] and the pivotal paper of Fan-Zhao [60] for variable exponent framework. The seminal works of Orlicz [109], Musielak [106] and the book of Harjulehto-Hästö [80], see also the recent survey of Chlebicka [31], complete the scenario with Sobolev-Musielak-Orlicz generalization. In particular, the constant exponent framework will be the natural setting for the p-Laplacian problems analyzed in Chapters 3 and 4. The variable exponent Sobolev space will be a key point in Section 3.2 for $p(x)$-Laplacian problems. Finally, the solutions of the double phase problem, studied in Section 4.1.4, will find their setting in Sobolev-Musielak-Orlicz spaces.

2.1.1 Constant exponent case

Following the classical notation, we will consider $L^0(\Omega)$ as the set of measurable functions $u : \Omega \to \mathbb{R}$ in which we identify two such a functions which differ on a zero Lebesgue-measure set. Let us consider $1 \leq p < +\infty$, the following space

$$L^p(\Omega) = \left\{ u \in L^0(\Omega) : \int_\Omega |u|^p dx < +\infty \right\}$$

is the classical Lebesgue space. On this space it is possible to consider the p-norm

$$\|u\|_p = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}}.$$

The normed space $\left(L^p(\Omega), \|\cdot\|_p\right)$ is a Banach space. Moreover, $L^p(\Omega)$ is reflexive for $1 \le p < +\infty$ and separable for $1 < p < +\infty$.

For $p = +\infty$, we define the space

$$L^\infty(\Omega) = \left\{ u \in L^0(\Omega) : \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| < +\infty \right\}$$

equipped with the sup-norm

$$\|u\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)| \quad \forall u \in L^\infty(\Omega).$$

For our scope, we introduce also the Sobolev spaces. Let $1 < p < +\infty$,

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega)\}$$

endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \|\nabla u\|_p \quad \forall u \in W^{1,p}(\Omega).$$

Also in this case, we have reflexive and separable Banach spaces. A useful space to study Dirichlet problems is $W_0^{1,p}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,p}(\Omega)}}$ equipped with the equivalent norm (see the Poincaré's inequality in Brézis [9, Corollary 9.19])

$$\|u\| = \|\nabla u\|_p \quad \forall u \in W_0^{1,p}(\Omega).$$

Finally, we recall the space

$$C_0^1(\Omega) = \left\{ u \in C^1(\Omega) : u = 0 \text{ on } \partial\Omega \right\}$$

of continuous functions with continuous derivative and the Hölder space

$$C^{1,\nu}(\Omega) = \left\{ u \in C^1(\Omega) : [\nabla u]_\nu < \infty \right\}$$

being

$$[\nabla u]_\nu = \sup_{x\neq y} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^\nu}$$

a seminorm, see Adams-Fournier [1, p. 10].

Embedding Theorems

The spaces defined above have embedding relations. Let

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \ge N, \end{cases} \tag{2.1}$$

be the critical Sobolev exponent. Let us recall the celebrated Sobolev's embedding Theorem (see, e.g., Brézis [9, Corollary 9.14]).

Theorem 1. *Suppose that $\Omega \subset \mathbb{R}^N$ be an open set with $\partial\Omega \in C^1$. Let $1 \le p \le +\infty$. Then, the following embedding are continuous:*

- $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ *if $p < N$;*

- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [p, +\infty)$ *if* $p = N$;
- $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ *if* $p > N$.

Thus, in particular when $1 < p < N$, we infer that

$$\|u\|_{p^*} \leq S_{p^*}\|u\| \quad \forall u \in W_0^{1,p}(\Omega)$$

being S_{p^*} the best constant in Sobolev embedding, see Talenti [121]. The following is the Rellich-Kondrachov's Theorem (see, e.g., Brézis [9, Theorem 9.16]).

Theorem 2. *Suppose that Ω be a bounded domain with $\partial\Omega \in C^1$. Let $p > 1$ Then, the following embedding are compact*

- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [1, p_*)$ *if* $p < N$;
- $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for all* $q \in [p, +\infty)$ *if* $p = N$;
- $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ *if* $p > N$.

Furthermore, the following compact embedding holds, see Adams-Fournier [1, Chapter 1],

$$C_0^{1,\nu}(\Omega) \hookrightarrow C_0^1(\Omega). \tag{2.2}$$

Finally, for our scope, we introduce the following notation for the positive cone

$$C_+(\Omega) = \left\{u \in C^1(\Omega) : u \geq 0\right\}$$

with its nonempty interior

$$\mathrm{int}\,(C_+) = \left\{u \in C_+(\Omega) : u(x) > 0 \text{ for all } x \in \Omega,\ \frac{\partial u}{\partial n} < 0\right\}$$

being n the outward normal derivative.

p-Laplacian

The prototype operator of the constant exponent case is the classical p-Laplace operator,

$$\Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \forall u \in W_0^{1,p}(\Omega).$$

For the sake of completeness, we recall some useful properties of this operator. In particular, the operator

$$\begin{aligned} -\Delta_p : W_0^{1,p}(\Omega) \quad &\rightarrow \quad W^{-1,p'}(\Omega) \\ \langle -\Delta_p u, v\rangle \quad &= \quad \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx \quad \forall u, v \in W_0^{1,p}(\Omega), \end{aligned}$$

fulfils the following lemma, see Carl-Le-Motreanu [18, Example 2.110] and Motreanu-Motreanu-Papageorgiou [104, Example 2.27c] for details.

Lemma 1. *Let $p > 1$ and let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with C^1-boundary. Then $-\Delta_p$ is a continuous, strictly monotone, and bounded operator. Moreover, it is of (S_+)-type, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $\limsup_{n\to+\infty}\langle -\Delta_p u_n, u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$.*

2.1.2 Variable exponent case

In a natural way, one can generalize the functional spaces introduced in the previous section when p is a function. Thus, let $p : \overline{\Omega} \to \mathbb{R}$ be a function in $C^{0,1}(\overline{\Omega})$ and set

$$1 < p_- = \min_{x \in \overline{\Omega}} p(x) \le p(x) \le p_+ = \max_{x \in \overline{\Omega}} p(x).$$

Let us introduce the so-called variable exponent Lebesgue spaces. Let $1 < p(x) < +\infty$ and set the modular function

$$\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

We recall a useful lemma on the modular function. The result can be found in Rădulescu-Repovš [113, p. 9].

Lemma 2. *Suppose that $p \in C(\overline{\Omega})$ with $p_- > 1$. Let $\{u_n\}, u \subseteq L^{p(x)}(\Omega)$, then, we have:*

1. $\|u\|_{L^{p(x)}} = \lambda \Leftrightarrow \rho_{p(x)}\left(\frac{u}{\lambda}\right) = 1$ *($\lambda > 0$);*
2. $\|u\|_{L^{p(x)}} < 1$ *(resp. =1,>1)* $\Leftrightarrow \rho_{p(x)}(u) < 1$ *(resp. =1, >1);*
3. $\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p_+} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p_-}$*;*
4. $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p_-} \le \rho_{p(x)}(u) \le \|u\|_{p(x)}^{p_+}$*;*
5. $\|u_n\|_{p(x)} \to 0$ *(resp. $\to +\infty$)* $\Leftrightarrow \rho_{p(x)}(u_n) \to 0$ *(resp. $\to +\infty$).*

The functional space

$$L^{p(x)}(\Omega) = \left\{ u \in L^0(\Omega) : \rho_{p(x)}(u) < +\infty \right\}$$

endowed with the Luxemburg norm (see Luxemburg [95])

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}$$

is a reflexive and separable Banach space. Incidentally, we recall that the Luxemburg norm is equivalent to the following Amemiya norm (see Musielak [106] and Nakano [107, p. 218])

$$\|u\|_A = \inf_{\lambda > 0} \left\{ \frac{1}{\lambda} \left(1 + \int_\Omega |\lambda u(x)|^{p(x)} dx \right) \right\}.$$

Clearly, one has $\|u\|_{p(x)} \le \|u\|_A \le 2\|u\|_{p(x)}$.

As in the constant case, we define the Sobolev space as

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm $\|u\|_{W^{1,p(x)}} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}$. The useful space $W_0^{1,p(x)}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,p(x)}}}$ is endowed with the equivalent norm (see Fan-Zhao [57, Proposition 2.5(iii)] for the Poincaré's inequality on variable exponent spaces)

$$\|u\|_{W_0^{1,p(x)}} = \|\nabla u\|_{p(x)}.$$

Embedding Theorems

In order to provide our results for variable exponent problems, we recall some embeddings relations also for variable exponent framework, see Edmunds-Rákosník [49, 50] and Fan-Shen-Zhao [57] for a complete overview on this topic.

Lemma 3. *Assume that $r, s \in C(\overline{\Omega})$ with $r_- > 1$, $s_- > 1$. Then, the following hold:*

- $W^{1,r(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ *is continuous if* $1 < s(x) \leq r^*(x)$ *for all* $x \in \overline{\Omega}$*;*
- $W^{1,r(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$ *is compact if* $1 < s(x) < r^*(x)$ *for all* $x \in \overline{\Omega}$*;*
- $W^{1,r(x)}(\Omega) \hookrightarrow W^{1,r_-}(\Omega)$.

$p(x)$-Laplacian

The model operator for variable exponent framework is represented by the $p(x)$-Laplace operator, i.e.,

$$\Delta_{p(x)} u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

The operator

$$\begin{aligned} -\Delta_{p(x)} : W_0^{1,p(x)}(\Omega) &\to \left(W_0^{1,p(x)}(\Omega)\right)^* \\ \langle -\Delta_{p(x)} u, v\rangle &= \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx \quad \forall u, v \in W_0^{1,p(x)}(\Omega), \end{aligned}$$

possesses the following properties, see Fan-Zhang [59, Theorem 3.1].

Lemma 4. *Let $p \in C(\overline{\Omega})$ be such that $p_- > 1$ and let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain. Then $-\Delta_{p(x)}$ is a continuous, strictly monotone, and bounded operator. Moreover, it is of (S_+)-type, i.e., if $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$ and $\limsup_{n\to+\infty}\langle -\Delta_{p(x)} u_n, u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.*

2.1.3 Sobolev-Musielak-Orlicz spaces

Next, to manage the solutions of the double-phase problem, we need to recall some notions on Musielak-Orlicz and Sobolev-Musielak-Orlicz spaces. Let us start considering the nonlinear Young function $\mathcal{H} : \Omega \times [0, +\infty) \to [0, +\infty)$ defined as

$$\mathcal{H}(x, t) = \mu(x) t^p + t^q \quad \forall (x, t) \in \Omega \times [0, +\infty)$$

being $1 < q < N$, $q < p$, $\frac{p}{q} < 1 + \frac{1}{N}$ and $\mu : \Omega \to [0, +\infty)$, $\mu \in C^{0,1}(\overline{\Omega})$. Since the weight function μ is not separated from zero, the map $\mathcal{H}$ shows an unbalanced growth, i.e., there exist $b_1, b_2 > 0$ constants such that

$$b_1|\xi|^q \leq \mathcal{H}(x, \xi) \leq b_2(1 + |\xi|^p) \text{ for a.a. } x \in \Omega, \text{ for all } \xi \in \mathbb{R},$$

thus, $\mathcal{H}$ is trapped between two different powers of $|\xi|$.

Remark 1. *Notice that the relations on the exponents q and p ensure that $q < p < q^*$, being q^* the critical Sobolev exponent introduced in (2.1), see Gasiński-Papageorgiou [66, Remark 2.1].*

Let us introduce the modular function also in this setting as

$$\rho_{\mathcal{H}}(u) = \int_\Omega \mathcal{H}(x, |u|)dx. \tag{2.3}$$

The so-called Musielak-Orlicz space is defined as

$$L^{\mathcal{H}}(\Omega) = \left\{u \in L^0(\Omega) : \rho_{\mathcal{H}}(u) < +\infty\right\}.$$

The usual norm on this space is the Luxemburg norm

$$\|u\|_{L^{\mathcal{H}}} = \inf\left\{\lambda > 0 : \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

Also in this setting, it is possible to consider two equivalent norms: the Amemiya norm (see again Musielak [106])

$$\|u\|_A = \inf_{\lambda>0}\left\{\frac{1}{\lambda}\left(1 + \rho_{\mathcal{H}}(\lambda u)\right)\right\}$$

and the so-called Orlicz norm (see Papageorgiou-Winkert [111, Remark 4.9.15])

$$\|u\|_O = \sup\left\{\int_\Omega uv dx : \rho_{\mathcal{H}^*}(v) \leq 1\right\}$$

being $\mathcal{H}^* : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ the Young conjugate function of $\mathcal{H}$, defined as

$$\mathcal{H}^*(x, u) = \sup_{v\geq 0}\left\{uv - \mathcal{H}(x, v)\right\} \text{ for all } x \in \Omega, \text{ for all } v \geq 0.$$

The equivalence follows from the relation

$$\|u\|_{L^{\mathcal{H}}} \leq \|u\|_O \leq \|u\|_{L^{\mathcal{H}}} \quad \forall u \in L^{\mathcal{H}}(\Omega).$$

For the modular function in (2.3), it is possible to provide some properties, see Hu-Papageorgiou [83, Proposition 4.159].

Lemma 5. *In our setting, let $u \in L^{\mathcal{H}}(\Omega)$. Then, one has*

1. *if $u \neq 0$ and $\lambda \in \mathbb{R}$, then, $\|u\|_{L^{\mathcal{H}}} = \lambda \Leftrightarrow \rho_{\mathcal{H}}\left(\frac{u}{\lambda}\right) = 1$;*
2. *$\|u\|_{L^{\mathcal{H}}} < 1 (= 1; > 1) \Leftrightarrow \rho_{\mathcal{H}}(u) < 1 (= 1; > 1)$;*
3. *if $\|u\|_{L^{\mathcal{H}}} < 1$, then $\|u\|_{L^{\mathcal{H}}}^p \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{L^{\mathcal{H}}}^q$;*
4. *if $\|u\|_{L^{\mathcal{H}}} > 1$, then $\|u\|_{L^{\mathcal{H}}}^q \leq \rho_{\mathcal{H}}(u) \leq \|u\|_{L^{\mathcal{H}}}^p$;*
5. *$\|u\|_{L^{\mathcal{H}}} \to 0$ if and only if $\rho_{\mathcal{H}}(u) \to 0$;*
6. *$\|u\|_{L^{\mathcal{H}}} \to +\infty$ if and only if $\rho_{\mathcal{H}}(u) \to +\infty$;*
7. *$\|u\|_{L^{\mathcal{H}}} \to 1$ if and only if $\rho_{\mathcal{H}}(u) \to 1$.*

Similarly as above, we have the Sobolev-Musielak-Orlicz spaces

$$W^{1,\mathcal{H}}(\Omega) = \left\{u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega)\right\}$$

with the norm

$$\|u\|_{W^{1,\mathcal{H}}} = \|u\|_{L^{\mathcal{H}}} + \|\nabla u\|_{L^{\mathcal{H}}}$$

and $W_0^{1,\mathcal{H}}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,\mathcal{H}}}}$ equipped with the equivalent norm (see Poincaré's inequality in Harjulehto-Hästö [80, Theorem 6.2.8, p. 130])

$$\|u\|_{W_0^{1,\mathcal{H}}} = \|\nabla u\|_{L^{\mathcal{H}}}.$$

In our setting, Colasuonno-Squassina [32, Proposition 2.18] ensures that $\|\cdot\|_{L^{\mathcal{H}}}$ is uniformly convex, then, by Milman-Pettis' Theorem (see, e.g., [9, Theorem 3.31]) the above spaces are reflexive.

Finally, for our convenience, we introduce the following weight Lebesgue space

$$L^p_\mu(\Omega) = \left\{ u \in L^0(\Omega) : \int_\Omega \mu(x)|u|^p dx < +\infty \right\}$$

and the related semi-norm

$$\|u\|_{p,\mu} = \left(\int_\Omega \mu(x)|u|^p dx \right)^{\frac{1}{p}}.$$

We point out that the following relation holds, see e.g., Gasiński-Winkert [73, eq. (2.2)]

$$\min\left\{ \|u\|^p_{L^{\mathcal{H}}(\Omega)}, \|u\|^q_{L^{\mathcal{H}}(\Omega)} \right\} \le \|u\|^p_{p,\mu} + \|u\|^q_q \le \max\left\{ \|u\|^p_{L^{\mathcal{H}}(\Omega)}, \|u\|^q_{L^{\mathcal{H}}(\Omega)} \right\}, \tag{2.4}$$

for all $u \in L^{\mathcal{H}}(\Omega)$.

Embedding Theorems

As for the previous cases, we recall some embedding relations also for Sobolev-Musielak-Orlicz spaces.

First, by Colasuonno-Squassina [32, Proposition 2.15] one has that

$$L^p(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega) \hookrightarrow L^p_\mu(\Omega) \cap L^q(\Omega)$$

with continuous embeddings.

Next, the following embeddings hold, see Crespo Blanco-Gasiński-Winkert [39, Proposition 2.2].

Theorem 3. *Assume that $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $1 < q < N$, $q < p < q^* = \frac{Nq}{N-q}$ and $0 \le \mu(\cdot) \in L^\infty(\Omega)$. Then,*

- *$L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ with continuous embedding for all $r \in [1, q]$;*
- *$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ with continuous embedding for all $r \in [1, q]$;*
- *$W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ with continuous embedding for $r \in [1, q^*]$ and compact for $r \in [1, q^*)$;*
- *$L^{\mathcal{H}}(\Omega) \hookrightarrow L^p_\mu(\Omega)$ continuously;*
- *$L^p(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ continuously.*

Double Phase operator

As stated at the begining of this section, we will provide results for double phase problems. For the sake of completeness, we recall also for this operator some valuable properties, see the seminal works of Crespo Blanco-Gasiński-Harjulehto-Winkert [38] and Liu-Dai [93] for more.

Let

$$\begin{aligned} L &: W_0^{1,\mathcal{H}}(\Omega) \to \left(W_0^{1,\mathcal{H}}(\Omega)\right)^* \\ \langle Lu, v\rangle &= \int_\Omega \left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right)\nabla v dx \quad \forall u, v \in W_0^{1,\mathcal{H}}(\Omega), \end{aligned}$$

be the double phase operator. The operator fulfils the following.

Lemma 6. *Assume that $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $1 < q < N$, $q < p < q^* = \frac{Nq}{N-q}$ and $0 \leq \mu(\cdot) \in L^\infty(\Omega)$. Then, L is bounded, continuous, strictly monotone and of (S_+)-type, i.e, if $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $\limsup_{n\to+\infty}\langle Lu_n, u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$.*

For our convenience, we will indicate with

$$-\Delta_p^\mu u - \Delta_q u = -\operatorname{div}\left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right) \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega)$$

the double phase operator.

2.2 Sub-super solutions and variational methods

In this section, we collect some useful tools. Let us start to recall that in calculus of variations there is a strong correlation between solutions of a partial differential equation and the critical points of a suitable function, called *energy functional*. Thus, in particular, on a Banach space consider a function $J : X \to \mathbb{R}$ associated to a differential problem. The critical points, i.e., the zeros of the Gâteaux derivative, $J'(u) = 0$, represent the (weak) solutions of the starting problem. For details on classical theory of calculus of variations we refer the reader to Costa [36], Dacorogna [41, 42], Evans [51, Chapter 8], Struwe [120], Willem [122] and the references therein. For the study of the elliptic problems considered in this thesis, we recall also the definitions of sub-solution, super-solution and weak solution, see Motreanu-Motreanu-Papageorgiou [104, Definition 8.2 and Definition 11.5]. Consider the classical elliptic problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

being $\Omega \subset \mathbb{R}^N$ a bounded domain with smooth boundary.

Definition 1. *We say that $\underline{u} \in W^{1,p}(\Omega)$ is a sub-solution for (2.5) if $\underline{u} \leq 0$ on $\partial\Omega$ and*

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla v dx \leq \int_\Omega f(x,\underline{u}) v dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega).$$

Similarly, we say that $\overline{u} \in W^{1,p}(\Omega)$ is a super-solution for (2.5) if $\overline{u} \geq 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla v dx \geq \int_\Omega f(x,\overline{u}) v dx \quad \forall v \in W_0^{1,p}(\Omega) \cap L_+^p(\Omega).$$

Finally, we call $u \in W_0^{1,p}(\Omega)$ a weak solution for (2.5), *if*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega f(x,u) v dx \quad \forall v \in W_0^{1,p}(\Omega).$$

Where we use the symbol $L_+^p(\Omega) = \{u \in L^p(\Omega) : u \geq 0\}$ and being $f(x,\cdot) \in L^r(\Omega)$ with $r \in (1,p')$ and p' the Hölder conjugate of p, i.e., p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Clearly, the above definition can be extended to variable exponent framework, see Fan [55], double phase setting, see Guarnotta-Livrea-Winkert [76] and also for elliptic problems with Neumann boundary conditions, see Motreanu-Sciammetta-Tornatore [105].

Furthermore, the main existence result in classical calculus of variations theory is the so-called *Direct Methods* Theorem or Tonelli-Weierstrass' Theorem, see Costa [36, Theorem 1.2] or Hu-Papageorgiou [83, Theorem 7.67].

Theorem 4. *Let X be a reflexive Banach space and let $J : X \to \mathbb{R}$ be a functional such that J is weakly sequentially lower semi-continuous, i.e., if $\{u_n\} \subset X$ is such that $u_n \rightharpoonup u$, then*

$$J(u) \leq \liminf_{n\to+\infty} J(u_n),$$

and coercive, i.e.,

$$J(u) \to +\infty \text{ as } \|u\| \to +\infty.$$

Then, J is bounded from below and there exists $u \in X$ such that

$$J(u) = \inf_X J.$$

Finally, a useful existence and uniqueness result is due to Minty-Browder, see Brézis [9, Theorem 5.16].

Theorem 5. *Let X be a reflexive Banach space and let $A : X \to X^*$ be a continuous nonlinear map such that A is strictly monotone, i.e.,*

$$\langle Au_1 - Au_2, u_1 - u_2 \rangle > 0$$

for all $u_1, u_2 \in X$, $u_1 \neq u_2$, and coercive, that is,

$$\lim_{\|u\|\to\infty} \frac{\langle Au, u\rangle}{\|u\|} = \infty.$$

Then, for every $f \in X^*$ there exists a unique solution $u \in X$ of the equation $Au = f$.

2.2.1 Díaz-Saá inequality and Picone's identity

One of the main tool to provide the uniqueness of solution for Dirichlet elliptic problems is the celebrated Díaz-Saá inequality, see the seminal paper of Brézis-Oswald [10] for the original idea in Laplacian case, and Díaz-Saá [45] for the p-Laplacian counterpart. We state the Díaz-Saá version of the inequality, see Díaz-Saá [45, Lemme 2].

Lemma 7. *For $i = 1,2$ let $w_i \in L^\infty(\Omega)$ be such that $w_i \geq 0$ a.e. in Ω, $w_i^{1/p} \in W^{1,p}(\Omega)$, $\Delta_p w_i^{1/p} \in L^\infty(\Omega)$ and $w_1 = w_2$ on $\partial\Omega$. Then, if $(w_i/w_j) \in L^\infty(\Omega)$ for $i \neq j$, $i,j = 1,2$,* *one has*

$$\int_\Omega \left(-\frac{\Delta_p w_1^{1/p}}{w_1^{(p-1)/p}} + \frac{\Delta_p w_2^{1/p}}{w_2^{(p-1)/p}} \right) (w_1 - w_2)\, dx \geq 0.$$

The previous result was extended to the non-smooth case by Mosconi [103, Theorem 1.1], where the main operator has a density function that is only strictly convex and p-positively homogeneous, but is not assumed to be C^1, thus, it follows that the related energy functional is only locally Lipschitz.

Moreover, in Section 3.2, we will use the Giacomoni-Takáč [74, Theorem 2.5] version of this inequality, which is set in a variable exponent framework. For the sake of completeness we recall also this result.

Lemma 8. *Let $r \in [1, +\infty)$ and $p : \Omega \to (1, +\infty)$ be a continuous function such that*

$$1 < p_- = \inf_\Omega p \leq p_+ = \sup_\Omega p < +\infty$$

and $1 \leq r \leq p_-$. Then, the following inequality holds

$$\int_\Omega \left(-\frac{\Delta_{p(x)} w_1}{w_1^{r-1}} + \frac{\Delta_{p(x)} w_2}{w_2^{r-1}} \right) (w_1^r - w_2^r)\, dx \geq 0$$

for all $w_1, w_2 \in W_0^{1,p(x)}(\Omega)$, such that $w_1 > 0$, $w_2 > 0$ a.e. in Ω and both $w_1/w_2, w_2/w_1 \in L^\infty(\Omega)$.

On the other hand, for Neumann and Robin elliptic problems, we recall the Picone's identity, see the works of Allegretto-Huang [2, Theorem 1.1] and Gasiński-Papageorgiou [68, Theorem 6.4.8] for general introduction and Fragnelli-Mugnai-Papageorgiou [63] for applications to Robin problems.

Lemma 9. *Let $p > 1$ and $v > 0$, $u \geq 0$ be two differentiable functions. Set*

$$L(u,v) = |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}\nabla u |\nabla v|^{p-2}\nabla v,$$

and

$$R(u,v) = |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v.$$

Then, $L(u,v) = R(u,v)$. Moreover, $L(u,v) \geq 0$ and $L(u,v) = 0$ a.e. in Ω if and only if $\nabla\left(\frac{u}{v}\right) = 0$ a.e. in Ω, i.e., $u = kv$ for some constant k in each component of Ω.

2.3 Set-valued analysis

We conclude the preliminaries section collecting some notions on set-valued maps. In particular, the lack of monotonicity conditions of (1.2)-type in Chapter 4 will require the use of these tools to provide our results. For a complete overview of these classic topics, we refer the interested reader to Carl-Le-Motreanu [18, Chapter 2], Denkowski-Migórski-Papageorgiou [44, Chapter 4], Gasiński-Papageorgiou [69, Chapters 3 and 7], Migórski-Ochal-Sofonea [101, Chapter 3] and the references therein.

Let X, Y be two nonempty sets. A map

$$\mathscr{S} : X \to 2^Y$$

is called set-valued map (or multifunction) with domain

$$\mathrm{dom}\,(\mathscr{S}) = \{x \in X : \mathscr{S}(x) \neq \emptyset\}\,.$$

The following definition on set-valued maps holds:

Definition 2. *Let X, Y be two topological spaces. Consider a set-valued map $\mathscr{S} : X \to 2^Y$.*

- *$\mathscr{S}$ is upper semi-continuous (briefly u.s.c.), if for every closed subset $C \subset Y$, the set*

$$\mathscr{S}^-(C) = \{x \in X : \mathscr{S}(x) \cap C \neq \varnothing\}$$

is closed in X. Alternatively, $\mathscr{S}$ is u.s.c., if for every open subset $O \subset Y$, the set

$$\mathscr{S}^+(O) = \{x \in X : \mathscr{S}(x) \subset O\}$$

is open in X.

- *$\mathscr{S}$ is lower semi-continuous (briefly l.s.c.), if for every closed subset $C \subset Y$, the set*

$$\mathscr{S}^+(C) = \{x \in X : \mathscr{S}(x) \subset C\}$$

is closed in X. Equivalently, $\mathscr{S}$ is l.s.c., if for every open subset $O \subset Y$, the set

$$\mathscr{S}^-(O) = \{x \in X : \mathscr{S}(x) \cap O \neq \varnothing\}$$

is open in X.

- *We say that $\mathscr{S}$ is continuous if it is u.s.c. and l.s.c.*

For our purposes, let us also recall the following characterization of l.s.c. set-valued functions, see e.g., Gasiński-Papageorgiou [69, Proposition 1.2.2], Migórski-Ochal-Sofonea [101, Proposition 3.9] and, for the proof, Fryszkowski [64, Proposition 20, p.59] or Klein-Thompson [89, Theorem 7.1.7].

Lemma 10. *Let X, Y be two metric spaces. Consider a set-valued map $\mathscr{S} : X \to 2^Y$. We say that $\mathscr{S}$ is l.s.c., if for each $u \in X$, and for each sequence $\{u_n\} \subset X$ such that $u_n \to u$ and $u^* \in \mathscr{S}(u)$, there exists $u_n^* \in \mathscr{S}(u_n)$ with $u_n^* \to u^*$ in Y.*

We recall the idea of selection for a set-valued map.

Definition 3. *A function $\Gamma : X \to Y$ is called selection for $\mathscr{S} : X \to 2^Y$, if $\Gamma(x) \in \mathscr{S}(x)$ for all $x \in X$.*

Finally, we would like to conclude this brief section by mentioning the following concepts on partially ordered sets, see Carl-Le-Motreanu [18, Definition 2.61, p. 27].

Definition 4. *Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. An element $b \in P$ is called lower bound of A if $b \leq x$ for each $x \in A$, and, in addition, if $b \in A$, we say that b is the smallest element of A. An element $x \in A$ is called a minimal element if there is no $y \neq x$ in A for which $y \leq x$. Clearly, every smallest element of A is a minimal element of A. A subset C of P is said downward direct if for each pair $x, y \in C$ there is $w \in C$ such that $w \leq x$ and $w \leq y$.*

The following is a consequence of the Kuratowski-Zorn Lemma, see, e.g., Brézis [9, Lemma 1.1] or Carl-Le-Motreanu [18, Theorem 2.62].

Lemma 11. *If in a partially ordered set P every chain has a lower bound then P possesses a minimal element. In particular, if P is downward directed, then P admits the smallest element.*

Chapter 3

Existence, nonexistence and multiplicity results via Díaz-Saá type assumptions

3.1 Constant exponent case

3.1.1 Dirichlet boundary conditions

This section is devoted to study the Carrier nonlocal problems introduced in Chapter 1, where the reaction term satisfies the following condition condition: the map

$$t \mapsto \frac{f(t)}{t^{p-1}} \tag{3.1}$$

is strictly decreasing.

Between 2019 and 2020, Gasiński-Santos Júnior [70, 71] provided new results on existence, nonexistence and multiplicity of positive solutions for the following nonlocal problem

$$\begin{cases} -a\left(\int_\Omega u^q dx\right)\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $\Omega \subseteq \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary and $q \geq 1$. One of the main novelty on these results is the degenerate condition on the weight function a, that is,

(H_a^1) let $a : [0, +\infty) \to [0, +\infty)$ be a continuous function. There exist positive numbers $0 \leq \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_K$ $(K \geq 1)$ such that $a(\alpha_k) = 0$ and $a > 0$ in (α_{k-1}, α_k),

thus, it is possible that the left-hand side goes to zero in several points. See Figure 3.1 below for a possible plot of the weight function a.

Moreover, the authors assume that, let $f \in C^1(\mathbb{R})$,

(H_f^1) there exists $t_* > 0$ such that $f(t_*) = 0$ and $f(t) > 0$ for all $t \in (0, t_*)$;

(H_1^1) the map $(0, t_*) \ni t \overset{\psi}{\mapsto} \frac{f(t)}{t}$ is strictly decreasing;

(H_2^1) $\alpha_K < t_*^q \int_\Omega e_1^q dx$, being e_1 the first eigenfunction of $\left(-\Delta, W_0^{1,2}(\Omega)\right)$ normalized in $L^\infty(\Omega)$-norm;

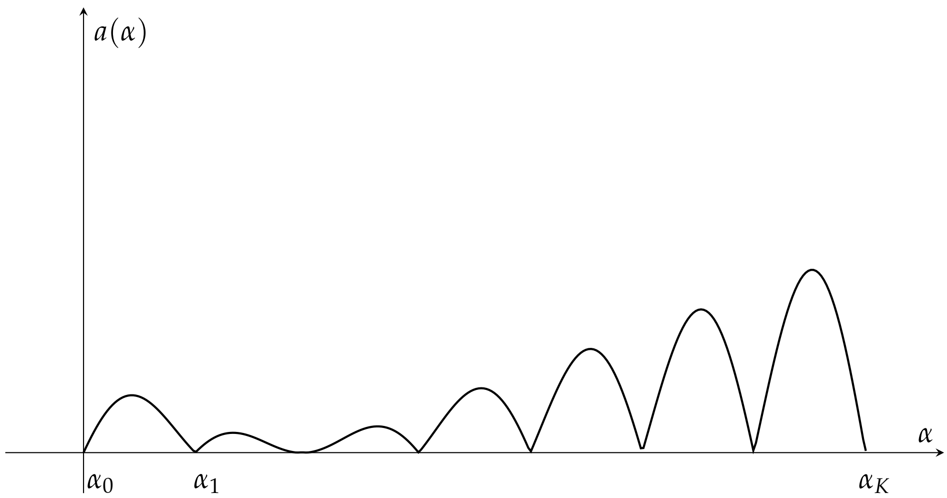

FIGURE 3.1: Example of weight function: $a(t) = \frac{1}{2}|(t-6)\sin(t)|$.

(H_3^1) $\max\limits_{t\in[0,\alpha_K]} a(t) < \frac{\gamma}{\lambda_{1,2}}$, where $\gamma = \lim\limits_{t\to 0^+} \frac{f(t)}{t}$ and $\lambda_{1,2}$ be the first eigenvalue of $\left(-\Delta, W_0^{1,2}(\Omega)\right)$;

(H_4^1) $\max\limits_{t\in[0,t_*]} f(t)t_*^{q-1} < \frac{1}{S_1}\sqrt{\frac{\lambda_{1,2}}{|\Omega|}} \max\limits_{t\in[\alpha_{k-1},\alpha_k]} [a(t)t]$ for all $k \in \{1, ..., K\}$, being S_1 the best constant in the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^1(\Omega)$ (see Talenti [121]).

Under the previous assumptions, the authors obtained the following multiplicity result, see Gasiński-Santos Júnior [70, Theorem 1.2].

Theorem 6. *Assume that* (H_a^1), (H_f^1), (H_1^1), (H_2^1), (H_3^1) *and* (H_4^1) *hold. Then, problem* (3.2) *admits at least* $2K$ *classical positive solutions with ordered in* L^q*-norms, i.e.,*

$$\alpha_{k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_k$$

for each $k \in \{1, ..., K\}$.

Moreover, in 2020, Candito-Gasiński-Livrea-Santos Júnior [15] improved the above result, since they studied the following more general problem

$$\begin{cases} -a\left(\int_\Omega u^q dx\right)\Delta_p u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, and

(H_f^2) $f : \mathbb{R} \to \mathbb{R}$ be a function such that there exists $t_* > 0$, $f(t_*) = 0$, $f \in C([0, t_*])$ and $f(t) > 0$ for all $t \in (0, t_*)$;

(H_1^2) the map $(0, t_*) \ni t \overset{\psi}{\mapsto} \frac{f(t)}{t^{p-1}}$ is strictly decreasing;

(H_2^2) $\alpha_{2K} < t_*^q \int_\Omega e_1^q dx$, being e_1 the first eigenfunction of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ normalized in $L^\infty(\Omega)$-norm;

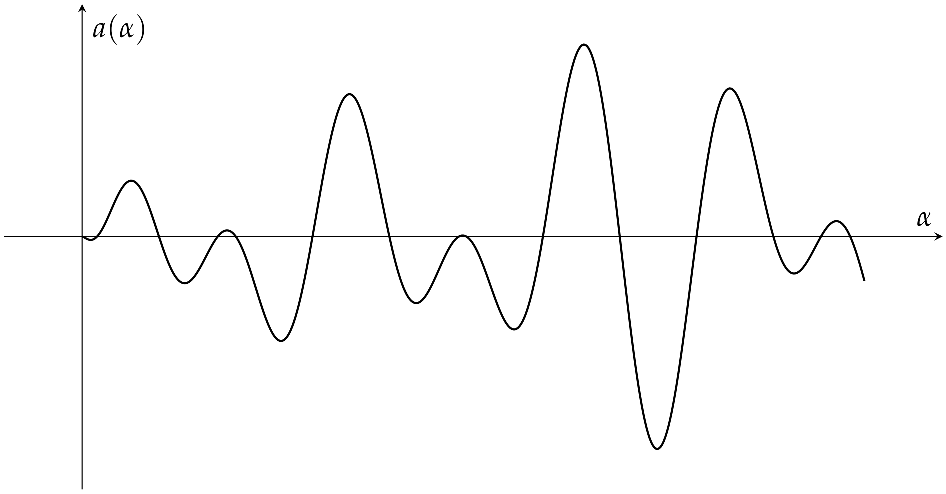

FIGURE 3.2: Example of weight function:
$a(t) = -2\sqrt{t}\sin(t)\cos(2\sqrt{t})$.

(H_3^2) $\max_{t\in[\alpha_{2k-1},\alpha_{2k}]} a(t) < \frac{\gamma}{\lambda_{1,p}}$, for all $k \in \{1, \dots, K\}$, where $\gamma = \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}}$ and $\lambda_{1,p}$ be the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$;

(H_4^2) $\max_{t\in[\alpha_{2k-1},\alpha_{2k}]} [t^p a(t)] > \frac{|\Omega|^p}{\lambda_{1,p}} t_*^{p(q-1)} \max_{t\in[0,t_*]} [tf(t)]$, for all $k \in \{1, \dots, K\}$.

In addition, to the best of our knowledge, this is the first paper in which the possibility of having a sign-changing weight function is considered, i.e.,

(H_a^2) let $a : [0, +\infty) \to \mathbb{R}$ be a continuous function. There exist positive numbers $0 \le \alpha_0 < \alpha_1 < \alpha_2 < \dots \le \alpha_{2K-1} < \alpha_{2K}$ ($K \ge 1$) such that $a(\alpha_{2k-1}) = 0 = a(\alpha_{2k})$ and $a > 0$ in $(\alpha_{2k-1}, \alpha_{2k})$, for all $k \in \{1, \dots, K\}$. See Figure 3.2.

The main result for p-Laplacian is the following, Candito-Gasiński-Livrea-Santos Júnior [15, Theorem 2.1]

Theorem 7. *Assume that (H_a^2), (H_f^2), (H_1^2), (H_2^2), (H_3^2) and (H_4^2) hold. Then, problem (3.3) admits at least* $2K$ *solutions belonging to* $\mathrm{int}(C_+)$. *Moreover, the solutions are ordered in* L^q*-norm, i.e.,*

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k}$$

for each $k \in \{1, ..., K\}$.

As stated in Chapter 1, these type of nonlocal problems does not possesses variational structure. This lack brings to develop new proof techniques.

Remark 2. *The main results were provided as follows:*

1. *First, consider a frozen nonlocal problem: in particular let* $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ *fixed, and consider the auxiliary problem*

$$\begin{cases} -a(\alpha)\Delta_p u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

2. *Let us consider the truncated reaction term*

$$\widehat{f}(s) = \begin{cases} f(0) & \text{if } s \leq 0, \\ f(s) & \text{if } 0 < s < t_*, \\ 0 & \text{if } s \geq t_*, \end{cases} \tag{3.5}$$

and the related problem

$$\begin{cases} -a\,(\alpha)\,\Delta_p u = \widehat{f}(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

3. *Direct methods of Calculus of Variations show that (3.6) possesses a positive solution* $u_\alpha \in \operatorname{int}(C_+)$. *Moreover, a Díaz-Saá-type argument [45, Lemme 2] (see Lemma 7 combined with the strictly monotonicity of the map* ψ *in* (H_1^2)*), permits to prove the uniqueness of* u_α *in* $[0, t_*]$.

4. *Again, from the strictly decreasing of* ψ*, it is possible to obtain a barrier control from below of the solution* u_α *produced above. In particular, it is possible to prove that*

$$\underline{u} = \psi^{-1}\left(\lambda_{1,p} a(\alpha)\right) e_1$$

is a positive function such that $u_\alpha \geq \underline{u}$.

5. *By steps 3 and 4, it is possible to well define the following one-dimensional map,*

$$\begin{aligned} \mathcal{P}_k &: (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R} \\ \mathcal{P}_k(\alpha) &= \int_\Omega u_\alpha^q dx. \end{aligned}$$

The main tool here is the following: each fixed point of the map $\mathcal{P}_k$ *is a solution of problem (3.3). In fact, notice that, if* $\mathcal{P}_k(\alpha) = \alpha$*, one has*

$$-a\left(\int_\Omega u_\alpha^q dx\right)\Delta_p u_\alpha = -a(\alpha)\Delta_p u_\alpha = f(u_\alpha) \text{ in } \Omega.$$

6. *Assumptions* (H_2^2) *and* (H_4^2) *allow us to ensures the existence of at least two fixed point for the map* $\mathcal{P}_k$ *for each positive bump of the weight function a.*

We can summarize the main techniques in the following scheme:

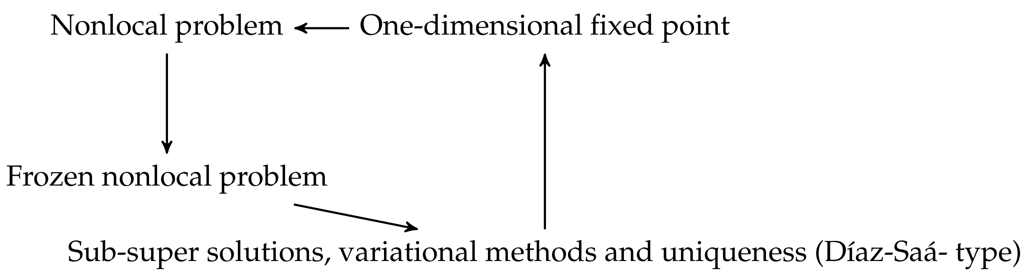

FIGURE 3.3: Proof path – Monotonicity

3.1.2 First nonexistence results for p-Laplacian problems

The first results that we will present in this section are nonexistence results. In particular, following the ideas developed for the Laplacian case by Gasiński-Santos Júnior [71], we obtain two nonexistence results for problem (3.3) (contained in Failla-Gasiński-Santos Júnior [54]). Considering a nonegative weight function a, see (H_a^1), and the main ideas is based on the following: due to the uniqueness of solution in $[0, t_*]$ obtained with a Díaz-Saá argument, one has that, if the fixed-point map $\mathcal{P}_k$ does not possesses fixed point, then problem (3.3) does not have any solutions with norm in $\left[\alpha_0^{1/q}, \alpha_K^{1/q}\right]$. Geometrically, we would to prove that either the fixed point map is strictly above or is strictly below the line

$$\operatorname{Fix}\left(\mathcal{P}_k\right) = \left\{\alpha \in (\alpha_{2k-1}, \alpha_{2k}) : \mathcal{P}_k(\alpha) = \alpha\right\}.$$

The first nonexistence result can be visualized in this Figure.

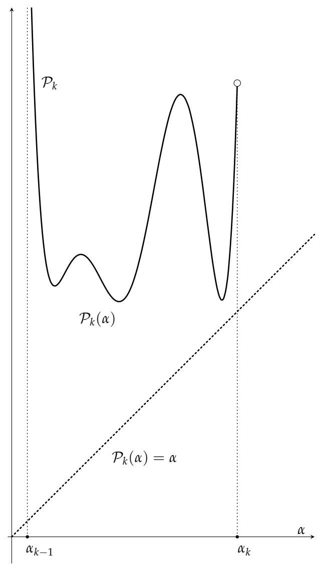

FIGURE 3.4: Nonexistence 1

Theorem 8. *Assume that (H_a^1), (H_f^2) and (H_1^2) hold. Moreover, if the following*

(H_6^2) *for all $\alpha \in (\alpha_0, \alpha_K)$,*

$$\alpha^{\frac{p-1}{q}} a(\alpha) < \frac{\|e_1\|_q^{p-1}}{\lambda_{1,p}} f\left(\frac{\alpha^{1/q}}{\|e_1\|_q}\right)$$

is true, then the problem (3.3) does not possesses any solution with L^q-norm in $\left[\alpha_0^{1/q}, \alpha_K^{1/q}\right]$.

Proof. Assumptions (H_a^1) and (H_f^2) lead to the well-posedness (see Remark 2) of the fixed point map $\mathcal{P}_k : (\alpha_{k-1}, \alpha_k) \to \mathbb{R}$, such that

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \quad \forall \alpha \in (\alpha_{k-1}, \alpha_k)$$

being u_α the unique solution of the frozen problem (3.6) in $[0, t_*]$. By the use of (H_6^2), we obtain that

$$\mathcal{P}_k(\alpha) > \alpha \quad \forall \alpha \in (\alpha_{k-1}, \alpha_k).$$

Indeed, by Remark 2, one has

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \geq \psi^{-1}\left(\lambda_{1,p} a(\alpha)\right)^q \int_\Omega e_1^q dx > \alpha$$

where the last inequality follows from (H_6^2) and the strictly decreasing of ψ, i.e.,

$$a(\alpha)\lambda_{1,p} < \psi\left(\frac{\alpha^{\frac{1}{q}}}{\|e_1\|_q}\right) \leq \lim_{t\to 0^+} \frac{f(t)}{t^{p-1}},$$

and the proof follows. □

The next result provide the symmetric situation, that is, if the map $\mathcal{P}_k$ is below the fixed point line.

Theorem 9. *Assume that* (H_a^1), (H_f^2), (H_1^2) *and* (H_3^2) *hold. If, the following relation is fulfilled*

(H_7^2) $\alpha_0 > t_*^q|\Omega|$

then, the problem (3.3) *does not possesses any solution with* L^q*-norm in* $\left[\alpha_0^{1/q}, \alpha_K^{1/q}\right]$.

Proof. By Remark 2, assumptions (H_a^1), (H_f^2) and (H_1^2) ensure that the fixed point map $\mathcal{P}_k$ is well defined. Moreover, notice that

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \leq t_*^q|\Omega| < \alpha_0 \leq \alpha \quad \forall \alpha \in (\alpha_{k-1}, \alpha_k),$$

and the proof is complete. □

Remark 3. *Notice that assumptions* (H_2^2) *and* (H_7^2) *are mutually independent. For the sake of clarity, see Figure* 3.5 *for a possible geometry of the fixed-point map in this setting.*

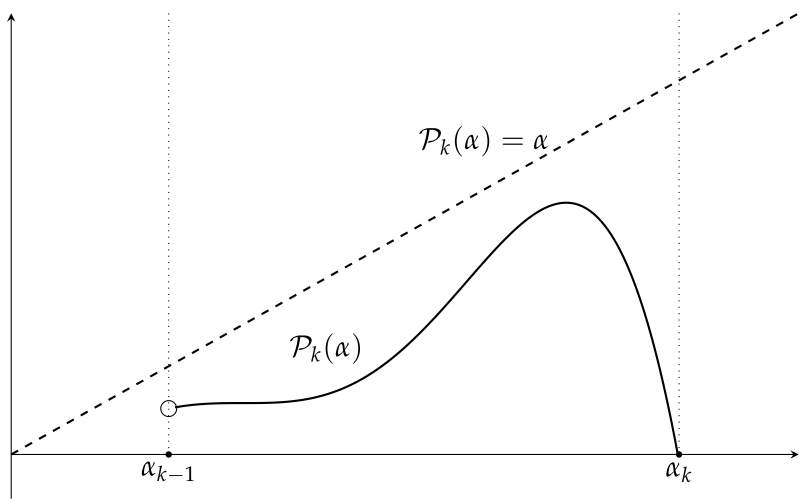

FIGURE 3.5: Nonexistence 2

3.1.3 Many solutions

It is possible to improve the multiplicity result imposing new conditions to obtain many fixed points of the map $\mathcal{P}_k$, that is, many solutions for problem (3.3). In particular, following the ideas introduced for the Laplacian case in Gasiński-Santos Júnior

[71], we obtain the existence of an arbitrary number of positive solutions for problem (3.3) in Failla-Gasiński-Santos Júnior [54]. Let us start to consider the following structure on the zeros of the weight function, for each $k \in \{1, \ldots, K\}$,

$$\alpha_{2k-1} < \bar{s}_{k,0} < s_{k,1} < \bar{s}_{k,1} < s_{k,2} < \ldots < s_{k,m_k} < \bar{s}_{k,m_k} < \alpha_{2k}$$

being $m_k \geq 1$ a switching points counter. Let us set

$$\widehat{K} = \sum_{k=1}^{K} (m_k + 1),$$

see Figure 3.6.

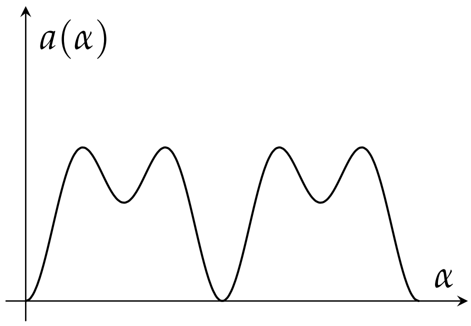

FIGURE 3.6: Example of weight function in many solutions case: $a(t) = \sin^2(\pi t) + \sin^2\left(\frac{\pi}{2} t\right)$

Then, the following multiplicity result holds.

Theorem 10. *Assume that* (H_a^2), (H_f^2), (H_1^2) *and* (H_2^2) *hold. Moreover, let*

$$(H_{10}^2) \quad s_{k,i}^{\frac{p-1}{q}} a(s_{k,i}) < \frac{\|e_1\|_q^{p-1}}{\lambda_{1,p}} f\left(\frac{s_{k,i}^{1/q}}{\|e_1\|_q}\right) \quad \forall k \in \{1, \ldots, K\} \text{ and } i \in \{1, \ldots, m_k\};$$

$$(H_{11}^2) \quad a(\bar{s}_{k,i})\bar{s}_{k,i}^{p} > \frac{|\Omega|}{\lambda_{1,p}} t_*^{p(q-1)} \max_{t \in [0,t_*]} [t f(t)] \quad \forall k \in \{1, \ldots, K\} \text{ and } i \in \{1, \ldots, m_k\}.$$

Then, (3.2) *admits at least* $2\widehat{K}$ *positive solutions.*

Proof. Fix $k \in \{1, \ldots, k\}$ and observe that (H_a^2), (H_f^2) and (H_1^2) imply that the fixed point map $\mathcal{P}_k$ is well defined (see Candito-Gasiński-Livrea-Santos Júnior [15, Proposition 2.2]). Moreover, by (H_2^2) and Candito-Gasiński-Livrea-Santos Júnior [15, Proposition 2.7], for all $k \in \{1, \ldots, K\}$, we have

$$\lim_{\alpha \to \alpha_{2k-1}^+} \mathcal{P}_k(\alpha) > \alpha_{2k-1} \text{ and } \lim_{\alpha \to \alpha_{2k}^-} \mathcal{P}_k(\alpha) > \alpha_{2k}. \tag{3.7}$$

Thus, by (H_3^2) and (H_{10}^2), and by strictly monotonicity of the ψ map, we get

$$\psi^{-1}\left(\lambda_{1,p} a(s_{k,i})\right) > \frac{s_{k,i}^{1/q}}{\|e_1\|_q} \quad \forall i \in \{1, ..., m_k\},$$

that is,

$$\mathcal{P}_k(s_{k,i}) = \int_\Omega u_{s_{k,i}}^q dx \geq \psi^{-1}\left(\lambda_{1,p} a(s_{k,i})\right)^q \|e_1\|_q^q > s_{k,i} \quad \forall i \in \{1, ..., m_k\}.$$

On the other hand, for $i \in \{1, ..., m_k\}$, considering the auxiliary problem,

$$\begin{cases} -\Delta_p u = u_{\bar{s}_{k,i}}^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

and following the proof of Candito-Gasiński-Livrea-Santos Júnior [15, Proposition 2.7], we obtain

$$\mathcal{P}_k(\bar{s}_{k,i}) \leq \left(\frac{|\Omega|}{a(\bar{s}_{k,i})\lambda_{1,p}} t_*^{p(q-1)} \max_{t\in[0,t_*]} [tf(t)] \right)^{\frac{1}{p}} < \bar{s}_{k,i} \quad \forall i \in \{0, ..., m_k\}$$

(the last inequality being the consequence of hypothesis (H_{11}^2)). By the Intermediate Mean Value Theorem, we conclude that $\mathcal{P}_k$ has at least $2(m_k + 1)$ fixed points, i.e., (3.3) has at least $2\widehat{K}$ positive solutions, ordered in L^q-norm. □

For the sake of clarity, we show a possible plot of the fixed point map $\mathcal{P}_k$ in this last case, see Figure 3.7.

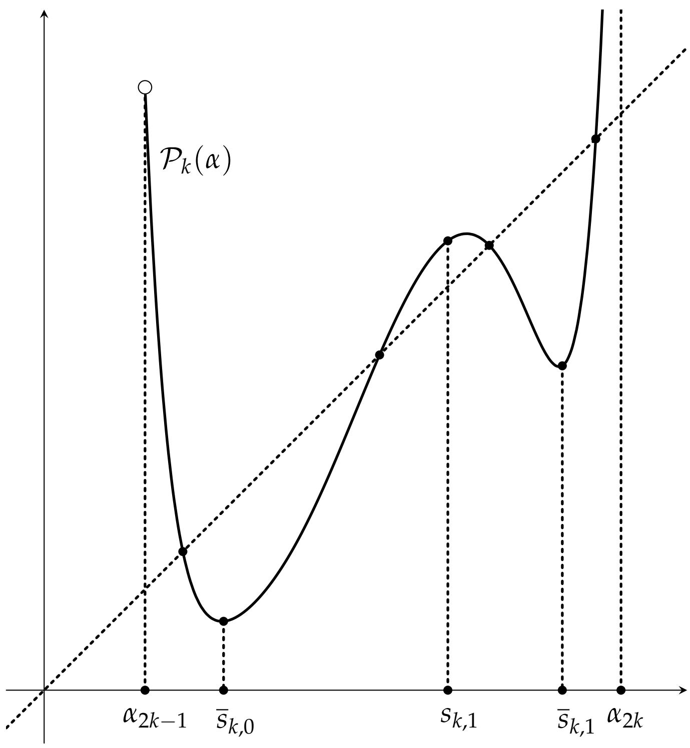

FIGURE 3.7: $\mathcal{P}_k$ map.

3.1.4 Neumann boundary conditions

Inspired by the paper of Candito-Gasiński-Livrea-Santos Júnior [15], in this section, we detail the results obtained in Crespo Blanco-Failla-Vassallo [37], where a new multiplicity result for Neumann nonlocal Carrier's problem is provided. Let us consider the elliptic degenerate problem

$$\begin{cases} a\left(\int_\Omega u^q dx\right)\left(-\Delta_p u + \mu(x)u^{p-1}\right) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = g(u) & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\Omega \in C^2$ and n be the outward normal.

For the best of our knowledge, this is the first work dealing with these type of boundary conditions. The choice of Neumann boundary conditions is due to the wide applicability of these problems, see, for instance, Mazumder [100, Chapters 2 and 7] for applications and numerical simulations in heat flux through boundary, the description of the flow for fluid (liquid flow) and electromagnetic (electric field) systems with Neumann boundary conditions driven by classical operator as the p-Laplacian.

Assume that

(H_μ) $\mu \in L^\infty(\Omega)$, $\underline{\mu} = \operatorname*{ess\,inf}_{x\in\Omega} \mu(x) > 0$ and the following relation holds:

$$(1-\mu(x))(p-2) \geq 0 \text{ for a.a. } x \in \Omega.$$

On the weight function a, we maintain the possible sign-changing property, i.e. we assume that (H_a^2) holds.

Throughout this section we consider the equivalent weight norm, see Maz'ja [99, Section 1.1.16],

$$\|u\| := \|u\|_\mu = \left(\int_\Omega \mu(x)|u|^p dx + \int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$$

on the Sobolev space $W^{1,p}(\Omega)$. In particular, we point out that this norm is such that

$$\|u\|_{W^{1,p}} = \|u\|_p + \|\nabla u\|_p \leq \tilde{c}\|u\|_\mu$$

being

$$\tilde{c} = \begin{cases} 2^{\frac{p-1}{p}}\underline{\mu}^{-\frac{1}{p}} & \text{if } 0 < \underline{\mu} < 1, \\ 2^{\frac{p-1}{p}} & \text{if } \underline{\mu} \geq 1. \end{cases} \tag{3.10}$$

To state our assumptions on the reaction term, we recall the following due to Gasiński-Papageorgiou [67, Proposition 2.1].

Proposition 1. *Assume that $\mu \in L^\infty(\Omega)$. Then, the following eigenvalue problem*

$$\begin{cases} -\Delta_p u + \mu(x)|u|^{p-2}u = \widehat{\lambda}_1(\mu)|u|^{p-2}u & \textit{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.11}$$

possesses a smallest eigenvalue $\widehat{\lambda}_1(\mu) \in \mathbb{R}$ which is simple, and has a corresponding L^p-normalized eigenfunction $\varphi_1 \in C^{1,\nu}(\overline{\Omega})$ for some $0 < \nu < 1$ with $\varphi_1(x) > 0$ for a.a. $x \in \overline{\Omega}$. Moreover, whenever $\mu(x) > 0$ for a.a. $x \in \Omega$ one has $\widehat{\lambda}_1(\mu) > 0$.

On the reaction term and on the boundary condition function, we assume that

(H_f^3) $f : \mathbb{R} \to \mathbb{R}$ be a function such that there exists $t_* > 0$ with $f(t_*) = 0$ and $f(t) > 0$ in $(0, t_*)$, $f \in C([0, t_*])$;

(H_1^3) the map $(0, t_*) \ni t \overset{\psi}{\mapsto} \frac{f(t)}{t^{p-1}}$ is strictly decreasing;

(H_{g1}^3) $g : \mathbb{R} \to \mathbb{R}$ be a function such that $g(t_*) = 0$ being t_* as in (H_f^3) and $g(t) \geq 0$ in $(0, t_*)$, $g \in C^{1,\nu}([0, t_*])$ for some $\nu \in (0, 1)$;

(H_{g2}^3) the map $(0, t_*) \ni t \mapsto \frac{g(t)}{t^{p-1}}$ is strictly decreasing;

Furthermore, consider the following

$$\gamma_f = \lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} \text{ and } \gamma_g = \lim_{t \to 0^+} \frac{g(t)}{t^{p-1}}.$$

Then, we assume that

(H_{g3}^3) $\gamma_g |\partial\Omega|_{\mathcal{H}^{N-1}} < \|\mu\|_1$;

(H_2^3) $\alpha_{2k} < t_*^q \int_\Omega e_1^q dx$ being $e_1 = \frac{\varphi_1}{\|\varphi_1\|_\infty}$ and φ_1 as in Proposition 1;

(H_3^3) $\max\limits_{\alpha \in [\alpha_{2k-1}, \alpha_{2k}]} a(\alpha) < \gamma_f \min\left\{ \frac{|\Omega|}{\|\mu\|_1}; \frac{1}{\widehat{\lambda}_1(\mu)} \right\}$ for all $k \in \{1, \ldots, K\}$;

(H_4^3) for each $k \in \{1, \ldots, K\}$ there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$a(\tau_k) \left(\tau_k^p - \vartheta |\partial\Omega|_{\mathcal{H}^{N-1}} \max_{t \in [0, t_*]} [t g(t)] \right) > \vartheta |\Omega| \max_{t \in [0, t_*]} [t f(t)]$$

being $\vartheta = 2^p \tilde{c}^p |\Omega|^{p-1} t_*^{p(q-1)}$.

Remark 4. *In the above assumptions appear the $(n-1)$-dimensional Hausdorff measure of the boundary of the domain, $|\partial\Omega|_{\mathcal{H}^{N-1}}$. Moreover, in the following, a crucial role will plays by the trace operator $\gamma : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ in*

$$\int_{\partial\Omega} G(\gamma(u))\gamma(v) d\sigma.$$

For the sake of simplicity, we will omit the symbol γ where it is clear. For a complete study on these topics we refer to Carl-Le-Motreanu [18, pp. 8-9] and Evans [51, pp. 257-261].

The main result for Neumann problem is achieved through variational and fixed-point arguments. Let us start to consider the following truncations for the functions f and g

$$\widehat{f}(t) = \begin{cases} f(0) & \text{if } s < 0, \\ f(t) & \text{if } 0 \leq s \leq t_*, \\ 0 & \text{if } s > t_*, \end{cases} \tag{3.12}$$

and

$$\widehat{g}(t) = \begin{cases} g(0) & \text{if } s < 0, \\ g(t) & \text{if } 0 \leq s \leq t_*, \\ 0 & \text{if } s > t_*. \end{cases} \tag{3.13}$$

Fixed $k \in \{1, \dots, K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, consider the following variational truncated problem

$$\begin{cases} a(\alpha)\left(-\Delta_p u + \mu(x)u^{p-1}\right) = \widehat{f}(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = \widehat{g}(u) & \text{on } \partial\Omega. \end{cases} \tag{3.14}$$

Due to the non-zero boundary conditions, according to Definition 1, we look for weak solutions for problem (3.14), i.e., functions $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &a(\alpha)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + a(\alpha)\int_\Omega \mu(x)|u|^{p-2}uv dx \\ &-\int_\Omega \widehat{f}(u)v dx - a(\alpha)\int_{\partial\Omega} \widehat{g}(u)v d\sigma \quad \forall v \in W^{1,p}(\Omega). \end{aligned}$$

Set

$$F(s) = \int_0^s \widehat{f}(t)dt \text{ and } G(s) = \int_0^s \widehat{g}(t)dt.$$

Then, we can prove that problem (3.14) possesses a unique solution for each $k \in \{1, \dots, K\}$ fixed. One of the main issue on Neumann boundary problems is the absence of Díaz-Saá-type arguments. The main technique to cover the uniqueness of the solution for the frozen problem (3.14) lies in the celebrated Picone's identity, see Lemma 9.

Theorem 11. *Assume that* (H_a^2), (H_f^3), (H_1^3), (H_{g1}^3), (H_{g2}^3) *and* (H_{g3}^3) *hold. Then, for each* $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$*, the problem* (3.14) *admits a unique solution* $u_\alpha \in [0, t_*]$*. Moreover,* $u_\alpha \in C^{1,\nu}(\overline{\Omega})$ *for some* $\nu \in (0,1)$*.*

Proof. First, notice that, the energy functional associated to (3.14), $J : W^{1,p}(\Omega) \to \mathbb{R}$ defined as

$$J(u) = \frac{a(\alpha)}{p}\|u\|^p - \int_\Omega F(u)dx - a(\alpha)\int_{\partial\Omega} G(u)d\sigma \quad \forall u \in W^{1,p}(\Omega),$$

is well-posed, coercive and sequentially weakly lower semi-continuous, then Theorem 4 ensures that there exists a global minimum $u_\alpha \in W^{1,p}(\Omega)$ of J, that is a weak solution for (3.14). Testing the weak formulation of u_α with $v = -u_\alpha^- \in W^{1,p}(\Omega)$ and $v = (u_\alpha - t_*)^+ \in W^{1,p}(\Omega)$, we infer that $0 \leq u_\alpha \leq t_*$. On the other hand, let $\xi_1 = \frac{1}{|\partial\Omega|}$ and $u = t\xi_1$ for t small enough. De l'Hôpital rule leads to the following

$$\begin{aligned} &\lim_{t\to 0^+} \frac{J(t\xi_1)}{t^p} \\ &= \lim_{t\to 0^+}\left[\frac{a(\alpha)\|\mu\|_1}{p|\partial\Omega|^p_{\mathcal{H}^{N-1}}} - \int_\Omega \frac{F(t\xi_1)}{(t\xi_1)^p}\xi_1^p dx - a(\alpha)\int_{\partial\Omega}\frac{F(t\xi_1)}{(t\xi_1)^p}\xi_1^p d\sigma\right] \\ &= \frac{a(\alpha)\|\mu\|_1}{p|\partial\Omega|^p_{\mathcal{H}^{N-1}}} - \frac{\gamma_f}{p}\frac{|\Omega|}{|\partial\Omega|^p_{\mathcal{H}^{N-1}}} - \frac{\gamma_g}{p}\frac{a(\alpha)}{|\partial\Omega|^{p-1}_{\mathcal{H}^{N-1}}} \\ &= \frac{1}{p|\partial\Omega|^p_{\mathcal{H}^{N-1}}}\left(a(\alpha)\|\mu\|_1 - \gamma_g|\Omega| - a(\alpha)\gamma_g|\partial\Omega|_{\mathcal{H}^{N-1}}\right). \end{aligned}$$

By (H^3_{g3}) and (H^3_3), we infer that

$$J(u_\alpha) \leq J(u) < 0,$$

that is, $u_\alpha \neq 0$. Furthermore, the maximum principle in Gasiński-Papageorgiou [68, Theorem 6.4.6] ensures that $u_\alpha > 0$. Finally, we prove that u_α is the unique solution of (3.14) such that $u_\alpha \in [0, t_*]$. For this scope, we exploit the Picone's identity, see Lemma 9. Assume that $v_\alpha > 0$ be a solution of (3.14) such that $u_\alpha \neq v_\alpha$ and $v_\alpha \in [0, t_*]$. For the sake of simplicity, we indicate with $u = u_\alpha$ and $v = v_\alpha$. Set

$$R(u,v) = |\nabla u|^p - |\nabla v|^{p-2}\nabla v \nabla \left(\frac{u^p}{v^{p-1}}\right). \tag{3.15}$$

We know that, if $v > 0$ and $u > 0$ are smooth function, then $R(u,v) \geq 0$, and $R(u,v) = 0$ if and only if $u = kv$, for some $k \in \mathbb{R}$. Notice that

$$\begin{aligned}
&\int_\Omega f(u)\frac{u^p - v^p}{u^{p-1}}dx \\
= \quad & -a(\alpha)\int_\Omega \Delta_p u \left(u - \frac{v^p}{u^{p-1}}\right)dx + a(\alpha)\int_\Omega \mu(x)(u^p - v^p)dx.
\end{aligned} \tag{3.16}$$

By Green's formula (see Casas-Fernández [21]), we get

$$\begin{aligned}
& a(\alpha)\int_\Omega \Delta_p u\left(u - \frac{v^p}{u^{p-1}}\right)dx \\
= \quad & -a(\alpha)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(u - \frac{v^p}{u^{p-1}}\right)dx \\
& +a(\alpha)\int_{\partial\Omega}|\nabla u|^{p-2}\left(u - \frac{v^p}{u^{p-1}}\right)\frac{\partial u}{\partial n}d\sigma \\
= \quad & -a(\alpha)\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(u - \frac{v^p}{u^{p-1}}\right)dx + a(\alpha)\int_{\partial\Omega} g(u)\left(u - \frac{v^p}{u^{p-1}}\right)d\sigma \\
= \quad & -a(\alpha)\|\nabla u\|_p^p + a(\alpha)\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\left(\frac{v^p}{u^{p-1}}\right)dx \\
& +a(\alpha)\int_{\partial\Omega} g(u)\left(u - \frac{v^p}{u^{p-1}}\right)d\sigma \\
= \quad & -a(\alpha)\|\nabla u\|_p^p + a(\alpha)\left[\int_\Omega |\nabla v|^p - |\nabla v|^p + |\nabla u|^{p-2}\nabla u\nabla\left(\frac{v^p}{u^{p-1}}\right)dx\right] \\
& +a(\alpha)\int_{\partial\Omega} g(u)\left(u - \frac{v^p}{u^{p-1}}\right)d\sigma \\
= \quad & -a(\alpha)\left[\|\nabla u\|_p^p - \|\nabla v\|_p^p + \int_\Omega R(v,u)dx\right] + a(\alpha)\int_{\partial\Omega} g(u)\left(u - \frac{v^p}{u^{p-1}}\right)d\sigma.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \int_\Omega f(u)\frac{u^p - v^p}{u^{p-1}}dx + a(\alpha)\int_{\partial\Omega} g(u)\left(u - \frac{v^p}{u^{p-1}}\right)d\sigma \\
= \quad & a(\alpha)\left[\|\nabla u\|_p^p - \|\nabla v\|_p^p + \int_\Omega R(v,u)dx\right] + a(\alpha)\int_\Omega \mu(x)(u^p - v^p)dx.
\end{aligned}$$

On the other hand, switching the roles of u and v, we obtain

$$\begin{aligned}
&\int_\Omega f(v)\frac{v^p-u^p}{v^{p-1}}dx + a(\alpha)\int_{\partial\Omega} g(v)\left(v-\frac{u^p}{v^{p-1}}\right)d\sigma \\
=\ & a(\alpha)\left[\|\nabla v\|_p^p - \|\nabla u\|_p^p + \int_\Omega R(u,v)dx\right] + a(\alpha)\int_\Omega \mu(x)(v^p-u^p)dx.
\end{aligned}$$

Then, combining the above identities and the monotonicity conditions on f and g, we have

$$\begin{aligned}
0 &> \int_\Omega \left(\frac{f(u)}{u^{p-1}} - \frac{f(v)}{v^{p-1}}\right)(u^p-v^p)dx \\
&\quad + a(\alpha)\int_{\partial\Omega}\left(\frac{g(u)}{u^{p-1}} - \frac{g(v)}{v^{p-1}}\right)(u^p-v^p)d\sigma \\
&= a(\alpha)\int_\Omega \left(R(v,u)+R(u,v)\right)dx \geq 0.
\end{aligned}$$

Since $a(\alpha) > 0$, we obtain a contradiction. Then, $u = v$. Moreover, a deeper analysis shows that $u_\alpha \in L^\infty(\Omega)$ and by (H^3_{g1}), one has $g \in C^{1,\beta}([0,t_*])$, thus by Lieberman [92, Theorem 2] (see also Lê [90]), we deduce that $u_\alpha \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. □

The main tool to come back from the frozen problem to our original problem, is the following one-dimensional fixed point map: let $\mathcal{P}_k : (\alpha_{2k-1},\alpha_{2k}) \to \mathbb{R}$ defined as

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \quad \forall \alpha \in (\alpha_{2k-1},\alpha_{2k}). \tag{3.17}$$

Notice that if $\alpha \in \mathrm{Fix}(\mathcal{P}_k) = \{\alpha \in (\alpha_{2k-1},\alpha_{2k}) : \mathcal{P}_k(\alpha) = \alpha\}$, then, one has

$$\begin{aligned}
& a\left(\int_\Omega u_\alpha^q dx\right)\left(-\Delta_p u_\alpha + \mu(x)u_\alpha^{p-1}\right) \\
=\ & a(\alpha)\left(\int_\Omega u_\alpha^q dx\right)\left(-\Delta_p u_\alpha + \mu(x)u_\alpha^{p-1}\right) = f(u_\alpha) \text{ in } \Omega.
\end{aligned}$$

Now, the following results permit us to guarantee a suitable geometry for the map $\mathcal{P}_k$ such that, for each positive bump of the weight a, we produce two different fixed points, see Figure 3.8 below. First, we prove the continuity of $\mathcal{P}_k$.

Proposition 2. *Assume that (H^2_a), (H_μ), (H^3_f), (H^3_1), (H^3_{g1}), (H^3_{g2}), (H^3_{g3}) and (H^3_3) hold. Then, the map $\mathcal{P}_k$ defined in (3.17) is continuous.*

Proof. Fix $k \in \{1,\dots,K\}$ and let $\{\alpha_n\} \subset (\alpha_{2k-1},\alpha_{2k})$ be a sequence such that $\alpha_n \to \alpha_* \in (\alpha_{2k-1},\alpha_{2k})$. Consider u_n be the sequence of minimum of the energy functional associated to α_n. Then, for all $n \in \mathbb{N}$, one has

$$\frac{a(\alpha_n)}{p}\|u_n\|^p - \int_\Omega F(u_n)dx - a(\alpha_n)\int_{\partial\Omega} G(u_n)d\sigma < 0,$$

i.e.,

$$\|u_n\|^p \leq \frac{p}{a(\alpha_n)}\left[F(t_*)|\Omega| + a(\alpha_n)G(t_*)|\partial\Omega|_{\mathcal{H}^{N-1}}\right].$$

Thus, since $a(\alpha_n) > 0$, we infer that $\{u_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$, and by Brézis [9, Theorem 4.9], one has

$$u_n \to u_* \text{ in } L^1(\Omega) \text{ and } u_n(x) \to u_*(x) \text{ a.e. in } \Omega,$$

for some $u_* \in W^{1,p}(\Omega)$. Moreover, for all $n \in \mathbb{N}$ and for all $v \in W^{1,p}(\Omega)$, one has

$$\begin{aligned} & a(\alpha_n) \int_\Omega |\nabla u_n|^{p-2} \nabla u \nabla v dx + a(\alpha_n) \int_\Omega \mu(x) u_n^{p-1} dx \\ & = \int_\Omega \widehat{f}(u_n) v dx + a(\alpha_n) \int_{\partial\Omega} \widehat{g}(u_n) v d\sigma, \end{aligned}$$

and, passing to the lim sup as $n \to +\infty$ and exploiting the boundedness of $\widehat{f}$ and $\widehat{g}$, we get

$$a(\alpha_*) \limsup_{n\to+\infty} \langle -\Delta_p u_n, u_n - u_* \rangle \leq 0.$$

The (S_+)-property of the operator ensures that $u_n \to u_*$ in $W^{1,p}(\Omega)$. Finally, we show that $u_* \neq 0$. Set

$$\bar{a} = \max_{k\in\{1,\dots,K\}} \max_{t\in[\alpha_{2k-1},\alpha_{2k}]} a(t)$$

and let $\varepsilon \in \left(0, \gamma_f - \frac{\|\mu\|_1}{|\Omega|} \bar{a}\right)$. Clearly, ε is well posed by (H_3^3). Consider the function

$$y_f = \psi^{-1} \left(\frac{\|\mu\|_1}{|\Omega|} a(\alpha) + \varepsilon \right) \in W^{1,p}(\Omega)$$

being ψ as in (H_1^3). Furthermore, (H_1^3) leads to

$$F(t) = \int_0^t \widehat{f}(s) dx = \int_0^t \frac{s^{p-1}}{s^{p-1}} \widehat{f}(s) ds \geq \frac{\widehat{f}(t)}{t^{p-1}} \int_0^t s^{p-1} ds = \frac{1}{p} t \widehat{f}(t).$$

Thus, since under our assumptions g and $a(\alpha)$ are positive functions, we have

$$\begin{aligned} & \frac{I_{k,\alpha}(y_f)}{\left(\psi^{-1}\left(\frac{\|\mu\|_1}{|\Omega|} a(\alpha) + \varepsilon\right)\right)^p} \\ = \; & \frac{a(\alpha)\|\mu\|_1}{p} - \int_\Omega \frac{F_*(y_f)}{\left(\psi^{-1}\left(\frac{\|\mu\|_1}{|\Omega|} a(\alpha) + \varepsilon\right)\right)^p} dx \\ & -a(\alpha) \int_{\partial\Omega} \frac{G_*(y_f)}{\left(\psi^{-1}\left(\frac{\|\mu\|_1}{|\Omega|} a(\alpha) + \varepsilon\right)\right)^p} d\sigma \\ \leq \; & \frac{a(\alpha)\|\mu\|_1}{p} - \int_\Omega \frac{F_*(y_f)}{\left(\psi^{-1}\left(\frac{\|\mu\|_1}{|\Omega|} a(\alpha) + \varepsilon\right)\right)^p} dx \\ \leq \; & \frac{a(\alpha)\|\mu\|_1}{p} - \frac{1}{p} \int_\Omega \frac{f_*(y_f)}{y_f^{p-1}} dx \\ = \; & \frac{\|\mu\|_1 a(\alpha)}{p} - \frac{\|\mu\|_1 a(\alpha)}{p} - \frac{\varepsilon}{p}|\Omega| = -\frac{\varepsilon}{p}|\Omega|. \end{aligned}$$

That is,

$$J(u_n) \leq -\frac{\varepsilon|\Omega|}{p} \left[\psi^{-1} \left(\frac{\|\mu\|_1}{|\Omega|} a(\alpha_n) + \varepsilon \right) \right] < 0,$$

and passing to the limit as $n \to +\infty$, we infer

$$J(u_*) \leq -\frac{\varepsilon|\Omega|}{p}\left[\psi^{-1}\left(\frac{\|\mu\|_1}{|\Omega|}a(\alpha_*)+\varepsilon\right)\right] < 0.$$

This shows that $u_* \neq 0$ and the proof is complete. □

Next, we build a control of the unique solution u_α of the frozen problem. This will be crucial to obtain the fixed points of $\mathcal{P}_k$.

Lemma 12. *Assume that (H_a^2), (H_f^3), (H_1^3), (H_{g1}^3), (H_{g2}^3) and (H_{g3}^3) hold. For $k \in \{1,\dots,K\}$ and $\alpha \in (\alpha_{2k-1},\alpha_{2k})$, set*

$$0 < \underline{u} = \psi^{-1}\left(\hat{\lambda}_1(\mu)a(\alpha)\right)u_1 \leq t_*$$

being u_1 the first eigenfunction as in Proposition 1 normalized in L^∞-norm. Then, $\underline{u}$ is a sub-solution for the frozen Neumann problem

$$\begin{cases} a(\alpha)\left(-\Delta_p u + \mu(x)u^{p-1}\right) = f(u) & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial n} = g(u) & \text{on } \partial\Omega. \end{cases} \tag{3.18}$$

Proof. Since ψ is a strictly decreasing map and $\|u_1\|_\infty = 1$, we infer that

$$\hat{\lambda}_1(\mu)a(\alpha) = \frac{f\left(\psi^{-1}(\hat{\lambda}_1(\mu)a(\alpha))\right)}{\left(\psi^{-1}\left(\hat{\lambda}_1(\mu)a(\alpha)\right)\right)^{p-1}} \leq \frac{f(\underline{u})}{\underline{u}^{p-1}}$$

that is,

$$\begin{aligned} & a(\alpha)\left(-\Delta_p\underline{u} + \mu(x)\underline{u}^{p-1}\right)\\ = \; & a(\alpha)\psi^{-1}(\hat{\lambda}_1(\mu)a(\alpha))^{p-1}\left(-\Delta_p u_1 + \mu(x)u_1^{p-1}\right)\\ = \; & a(\alpha)\hat{\lambda}_1(\mu)\psi^{-1}(\hat{\lambda}_1(\mu)a(\alpha))^{p-1}u_1^{p-1}\\ = \; & a(\alpha)\hat{\lambda}_1(\mu)\underline{u}^{p-1} \leq f(\underline{u}). \end{aligned}$$

Consider the Carathéodory functions

$$\widetilde{f}(x,t) = \begin{cases} \widehat{f}(t) & \text{if } \; t > \underline{u}(x),\\ \widehat{f}(\underline{u}(x)) & \text{if } \; t \leq \underline{u}(x), \end{cases} \tag{3.19}$$

and

$$\widetilde{g}(x,t) = \begin{cases} \widehat{g}(t) & \text{if } \; t > \underline{u}(x),\\ \widehat{g}(\underline{u}(x)) & \text{if } \; t \leq \underline{u}(x), \end{cases} \tag{3.20}$$

and the C^1-functional

$$\widetilde{J}(u) = \frac{a(\alpha)}{p}\|u\|^p - \int_\Omega \widetilde{F}(x,u)dx - a(\alpha)\int_{\partial\Omega}\widetilde{G}(x,u)d\sigma$$

being $\widetilde{F}(x,t) = \int_0^t \widetilde{f}(x,s)ds$ and $\widetilde{G}(x,t) = \int_0^t \widetilde{g}(x,s)ds$. $\widetilde{J}$ is coercive, then by Tonelli-Weierstrass' Theorem (see Theorem 4), there exists a global minimum $v_\alpha \in W^{1,p}(\Omega)$ of $\widetilde{J}$, such that $0 \leq v_\alpha \leq t_*$. Clearly, $v_\alpha \neq 0$. Indeed, if we consider $0 < t\underline{u} \leq \underline{u}$ with

$t \in (0,1)$ in Ω,

$$\widetilde{J}(v_\alpha) \le \widetilde{J}(t\underline{u}) = t\left(\frac{t^{p-1}a(\alpha)\|\underline{u}\|^p}{p} - \int_\Omega \widehat{f}(\underline{u})\underline{u}dx - a(\alpha)\int_{\partial\Omega}\widehat{g}(\underline{u})\underline{u}d\sigma\right) < 0,$$

for t small enough. Choosing as test functions $v = -v_\alpha^-$ and $v = (v_\alpha - t_*)^+$ in $W^{1,p}(\Omega)$, and, again by strong maximum principle in Gasiński-Papageorgiou [68, Theorem 6.4.6], we infer that $0 < v_\alpha \le t_*$. Moreover, choosing $v = (\underline{u} - v_\alpha)^+ \in W^{1,p}(\Omega)$ as test function, we deduce that

$$\begin{aligned} & a(\alpha)\langle -\Delta_p v_\alpha, (\underline{u}-v_\alpha)^+\rangle + a(\alpha)\langle \mu(x)v_\alpha^{p-1}, (\underline{u}-v_\alpha)^+\rangle \\ = & \int_\Omega \widetilde{f}(x,v_\alpha)(\underline{u}-v_\alpha)^+dx + a(\alpha)\int_{\partial\Omega}\widetilde{g}(x,v_\alpha)(\underline{u}-v_\alpha)^+d\sigma \\ \ge & \; a(\alpha)\langle -\Delta_p \underline{u}, (\underline{u}-v_\alpha)^+\rangle + a(\alpha)\langle \mu(x)\underline{u}^{p-1}, (\underline{u}-v_\alpha)^+\rangle. \end{aligned}$$

Thus, combining the monotonicity of the operator and (3.19)-(3.20), we get $v_\alpha \ge \underline{u}$ and

$$-a(\alpha)\Delta_p v_\alpha + a(\alpha)\mu(x)v_\alpha^{p-1} = \widetilde{f}(x,v_\alpha) = \widehat{f}(v_\alpha) = f(v_\alpha).$$

□

Now, we are ready to show as the assumptions (H_2^3) and (H_4^3) lead to the existence of at least two fixed points for the map $\mathcal{P}_k$ in each positive bump of a.

Proposition 3. *Assume that (H_a^2), (H_μ), (H_f^3), (H_1^3), (H_{g1}^3), (H_{g2}^3), (H_{g3}^3), (H_2^3), (H_3^3) and (H_4^3) hold. Then, for each $k \in \{1,\dots,K\}$ the map $\mathcal{P}_k$ has at least two fixed points.*

Proof. Fix $k \in \{1,\dots,K\}$ and notice that, by Lemma 12, we have

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \ge \psi^{-1}\left(\hat{\lambda}_1(\mu)a(\alpha)\right)^q \int_\Omega u_1^q dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}).$$

Thus, by (H_2^3), the following hold

$$\lim_{\alpha\to\alpha_{2k-1}^+} \mathcal{P}_k(\alpha) \ge \lim_{\alpha\to\alpha_{2k-1}^+} \psi^{-1}\left(\hat{\lambda}_1(\mu)a(\alpha)\right)^q \int_\Omega u_1^q dx = t_*^q \int_\Omega u_1^q dx > \alpha_{2k-1},$$

and

$$\lim_{\alpha\to\alpha_{2k}^-} \mathcal{P}_k(\alpha) \ge \lim_{\alpha\to\alpha_{2k}^-} \psi^{-1}\left(\hat{\lambda}_1(\mu)a(\alpha)\right)^q \int_\Omega u_1^q dx = t_*^q \int_\Omega u_1^q dx > \alpha_{2k},$$

being $u_1 = \frac{e_1}{\|e_1\|_\infty}$ and e_1 as in Proposition 1.

On the other hand, we prove that, for each $k \in \{1,\dots,K\}$ there exists at least one $\alpha_0 \in (\alpha_{2k-1}, \alpha_{2k})$, such that

$$\mathcal{P}_k(\alpha_0) < \alpha_0.$$

For each fixed $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, we can consider the auxiliary problem

$$\begin{cases} -\Delta_p u + \mu(x)^{\frac{2}{p}} u^{p-1} = u_\alpha^{q-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.21}$$

being u_α the unique solution of (3.14). Furthermore, by Minty-Browder's Theorem, see Theorem 5, the problem (3.21) possesses a unique positive solution w_α. Choosing $v = u_\alpha \in W^{1,p}(\Omega)$ as test function in the weak formulation of w_α, and by Hölder's inequality, we infer

$$\begin{aligned}
\mathcal{P}_k(\alpha) &= \int_\Omega u_\alpha^q dx = \int_\Omega |\nabla w_\alpha|^{p-2}\nabla w_\alpha \nabla u_\alpha dx + \int_\Omega \mu(x) w_\alpha^{p-1} u_\alpha dx \\
&\overset{(H_\mu)}{\leq} \int_\Omega |\nabla w_\alpha|^{p-1}|\nabla u_\alpha| dx + \int_\Omega \mu(x)^{\frac{1}{p}} w_\alpha^{p-1} \mu(x)^{\frac{1}{p}} u_\alpha dx \\
&\overset{\text{Hölder}}{\leq} \|\nabla w_\alpha\|_p^{p-1} \|\nabla u_\alpha\|_p + \|\mu^{\frac{1}{p}} w_\alpha\|_p^{p-1} \|\mu^{\frac{1}{p}} u_\alpha\|_p \qquad (3.22) \\
&\leq \left(\|\nabla w_\alpha\|_p^{p-1} + \|\mu^{\frac{1}{p}} w_\alpha\|_p^{p-1}\right)\left(\|\nabla u_\alpha\|_p + \|\mu^{\frac{1}{p}} u_\alpha\|_p\right) \\
&\leq 2^{\frac{1}{p}} \|w_\alpha\|^{p-1} 2^{\frac{1}{p'}} \|u_\alpha\| \\
&= 2\|w_\alpha\|^{p-1}\|u_\alpha\|.
\end{aligned}$$

Recall that u_α possesses a variational structure, that is,

$$\begin{aligned}
&a(\alpha)\int_\Omega |\nabla u_\alpha|^p dx + a(\alpha)\int_\Omega \mu(x) u_\alpha^p dx \qquad (3.23) \\
&= \int_\Omega \widehat{f}(u_\alpha) u_\alpha dx + a(\alpha)\int_{\partial\Omega} \widehat{g}(u_\alpha) u_\alpha d\sigma.
\end{aligned}$$

Thus,

$$\|u_\alpha\| \leq \left[\frac{|\Omega|}{a(\alpha)} \max_{t\in[0,t_*]}[tf(t)] + |\partial\Omega|_{\mathcal{H}^{N-1}} \max_{t\in[0,t_*]}[tg(t)]\right]^{\frac{1}{p}}. \qquad (3.24)$$

On w_α we can compute that Hölder's inequality and assumption (H_μ) lead to

$$\begin{aligned}
\|w_\alpha\|^p &= \int_\Omega \mu(x)|w_\alpha|^p dx + \int_\Omega |\nabla w_\alpha|^p dx \\
&\overset{(H_\mu)}{\leq} \int_\Omega \mu(x)^{\frac{2}{p}} |w_\alpha|^p dx + \int_\Omega |\nabla w_\alpha|^p dx \\
&\overset{(3.21)}{=} \int_\Omega u_\alpha^{q-1} w_\alpha dx \\
&\overset{\text{Hölder}}{\leq} \left(\int_\Omega u_\alpha^{(q-1)p'} dx\right)^{\frac{1}{p'}} \left(\int_\Omega w_\alpha^p dx\right)^{\frac{1}{p}} \\
&\leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \|w_\alpha\|_p \\
&\leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \left(\|w_\alpha\|_p + \|\nabla w_\alpha\|_p\right) \\
&\leq \tilde{c}|\Omega|^{\frac{1}{p'}} t_*^{q-1} \|w_\alpha\|,
\end{aligned}$$

being $\tilde{c}$ as in (3.10). Then,

$$\|w_\alpha\|^{p-1} \leq \tilde{c}|\Omega|^{\frac{1}{p'}} t_*^{q-1}. \qquad (3.25)$$

Combining (3.24) and (3.25) with (3.22), we achieve

$$\begin{aligned}
\mathcal{P}_k(\alpha) &\leq 2\|w_\alpha\|^{p-1}\|u_\alpha\| \\
&\leq 2\tilde{c}|\Omega|^{\frac{1}{p'}} t_*^{q-1} \left[\frac{|\Omega|}{a(\alpha)} \max_{t\in[0,t_*]}[tf(t)] + |\partial\Omega|_{\mathcal{H}^{N-1}} \max_{t\in[0,t_*]}[tg(t)]\right]^{\frac{1}{p}}.
\end{aligned}$$

Finally, assumption (H_4^3) guarantees us that there exists at least one $\alpha_0 \in (\alpha_{2k-1}, \alpha_{2k})$ such that,

$$\mathcal{P}_k(\alpha_0) < \alpha_0. \qquad (3.26)$$

The continuity of the $\mathcal{P}_k$ map, see Proposition 2, leads to the existence of at least two fixed points, see Figure 3.8. □

Then, our main theorem reads as follows.

Theorem 12. *Assume that* (H_a^2), (H_f^3), (H_1^3), (H_2^3), (H_3^3), (H_4^3), (H_{g1}^3) *and* (H_{g2}^3) *hold. Then, the problem* (3.9) *admits at least* K *pairs of positive smooth solutions* $u_{k,1}, u_{k,2} \in C^{1,\nu}(\overline{\Omega})$, *for some* $\nu \in (0,1)$. *Moreover, the solutions are ordered in* L^q*-norm, i.e.,*

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k}$$

for all $k \in \{1 \ldots, K\}$.

Proof. The proof follows showing that the fixed point map admits at least two fixed point for each positive bump of the weight function a. In particular, our hypotheses lead to Proposition 3, that ensures the right geometry for the $\mathcal{P}_k$ map. Thus, for each $k \in \{1, \ldots, K\}$, we provide the existence of at least two positive smooth solutions $u_{k,1}, u_{k,2} \in C^{1,\nu}(\overline{\Omega})$, for some $\nu \in (0,1)$, for problem (3.9). Furthermore, the geometry of the fixed point map also provides the following order: for each $k \in \{1, \ldots, K\}$, one has

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k}.$$

□

3.2 Variable exponent case

This section is devoted to improve the previous existence and multiplicity results for a variable exponent nonlocal degenerate elliptic problem. Moreover, for the best of our knowledge, this result, contained in Candito-Failla-Livrea [14], is the first one in which the reaction term may be non autonomous but maintaining the separation of variables. One of the main issues here is the lack of homogeneity of the driven operator, that forces us to produce new ideas for the control of the fixed point map. In particular, here we introduce the sub-super solutions method to obtain a control from above and from below of the (unique) solution of the frozen-truncated problem. The main problem of this section is the following

$$\begin{cases} -a\left(\displaystyle\int_\Omega u^q dx\right)\Delta_{p(x)}u = \beta(x)f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.27}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $\partial\Omega \in C^2$, $q \geq 1$ and the main operator is the $p(x)$-Laplacian, i.e.,

$$\Delta_{p(x)}u = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

In particular, we assume that the weight function fulfils (H_a^2), that is, a could be sign-changing. On the nonlinearity, we assume that (H_f^2) holds. Moreover, we assume the following structure:

(p) $p : \overline{\Omega} \to \mathbb{R}$ be a C^1-function such that

(i) $1 < p_- = \min_{x\in\overline{\Omega}} p(x) \le p(x) \le p_+ = \max_{x\in\overline{\Omega}} p(x) < N$;

(ii) there exists a vector $\ell \in \mathbb{R}^N \setminus \{0\}$ such that for all $x \in \Omega$, the function $p(x+t\ell)$ is monotone for $t \in \Sigma = \{t : x+t\ell \in \Omega\}$.

(H_β) $\beta \in L^\infty(\Omega)$ be such that $\underline{\beta} = \operatorname{ess\,inf}_{x\in\Omega} \beta(x) > 0$;

(H_1^4) $r \in [1, p_-]$. The map $(0, t_*) \ni t \overset{\psi_r}{\mapsto} \frac{f(t)}{t^{r-1}}$ is strictly decreasing;

(H_2^4) $\alpha_{2K} < \left(\frac{t_*}{\|\varphi_1\|_\infty}\right)^q \int_\Omega \varphi_1^q dx$, being φ_1 the first eigenfunction of $\left(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$;

(H_3^4) $\max_{\alpha\in[\alpha_{2k-1},\alpha_{2k}]} a(\alpha) < \frac{\beta\gamma\|\varphi_1\|_\infty^{r-1}}{\lambda_{1,p(x)}\max\left\{1,\|\varphi_1\|_\infty^{p_+-1}\right\}}$ for each $k \in \{1,\dots,K\}$, being $\gamma = \lim_{t\to 0^+} \frac{f(t)}{t^{r-1}}$ and $\lambda_{1,p(x)}$ the first eigenvalue of $\left(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$.

Finally, set

$$\alpha_* = \left[\frac{2}{\sqrt{\pi}}\left(1-\frac{1}{p_+}\right)\left(1+\frac{1}{N}\right)\Gamma\left(1+\frac{N}{2}\right)^{\frac{1}{N}}\right]^q |\Omega|^{1+\frac{q}{N}}$$

being Γ the Euler's Gamma function. Assume that

(H_4^4) for all $k \in \{1,\dots,K\}$ such that $\alpha_* < \alpha_{2k-1}$, there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ satisfying one of the following: either

(i) $a(\tau_k)p_- > \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma\left(1+\frac{N}{2}\right)\right)^{\frac{1}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)$;

or

(ii)

$$\tau_k^{\frac{p_--1}{q}} a(\tau_k) > \left(\frac{(N+1)(p_+-1)}{p_+}\right)^{p_--1} \left(\frac{2^{p_-}\left(\Gamma(1+\frac{N}{2})\right)^{\frac{p_-}{N}}}{p_-(N\sqrt{\pi})^{p_-}}\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)\right) |\Omega|^{\frac{p_-}{N}+\frac{p_--1}{q}}.$$

Remark 5. *Some comments on the above setting are necessary.*

- *We point out that the assumption (p)(ii) is due to Fan [58, Theorem 3.3] (see also Fan [56], Hu-Papageorgiou [83, Theorem 4.177(b)], and Rădulescu-Repovš [113, p. 56]). This monotonicity on the p function guarantees us that the first eigenvalue $\lambda_{1,p(x)}$ of $\left(-\Delta_{p(x)}, W_0^{1,p(x)}(\Omega)\right)$ is strictly positive.*

- *A careful reading of the assumption (H_4^4) shows that the constant*

$$C_0 = \frac{\left(\Gamma\left(1+\frac{N}{2}\right)\right)^{\frac{1}{N}}}{N\sqrt{\pi}} \tag{3.28}$$

appears, this is the best constant in the Sobolev embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$. We refer the reader to the seminal work of Talenti [121] and the book of Kesavan [87, p. 46], for a complete overview on the best constant embedding and its relations with geometrical aspect as the isoperimetrical inequality in $\mathbb{R}^N$.

The main result of this section is the following

Theorem 13. *Assume that* (p), (H_a^2), (H_f^2), (H_β), (H_1^4), (H_2^4), (H_3^4) *and* (H_4^4) *hold. Then, for all* $k \in \{1, \dots, K\}$ *such that* $\alpha_* < \alpha_{2k-1}$, *the problem* (3.27) *admits at least* $K - k + 1$ *pairs of positive solutions* $(u_{k,1}, u_{k,2}) \in \mathrm{int}(C_+)$ *ordered in* L^q*-norm, i.e.,*

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k} \quad \forall k \in \{1, \dots, K\}.$$

Following the ideas introduced for the p-Laplacian, we freeze the nonlocal term in the positive bumps of the weight function. Fix $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and consider

$$\begin{cases} -a(\alpha)\Delta_{p(x)} u = \beta(x) f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.29}$$

The main idea is to build a suitable fixed point map with a geometry that permits us to obtain the desired number of fixed points. To this scope, we control the solutions of the problem (3.29) truncating the reaction term with appropriate sub- and super-solutions. We refer the reader to Chapter 2 for definition of sub-solution and super-solution. Clearly, the constant function

$$\overline{u} = t_*$$

is a super-solution of (3.29). To build a barrier from below, consider the following lemma.

Lemma 13. *Assume that* (p), (H_a^2), (H_f^2), (H_β), (H_1^4) *and* (H_3^4) *hold. For each* $k \in \{1, \dots, K\}$ *and* $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, *set*

$$k_\alpha = \frac{a(\alpha)\lambda_{1,p(x)} \max\left\{1, \|\varphi_1\|_\infty^{p_+-1}\right\}}{\|\varphi_1\|_\infty^{r-1}},$$

and

$$\frac{\|\varphi_1\|_\infty}{m} \leq c \leq \|\varphi_1\|_\infty,$$

being $m \geq \frac{\|\varphi_1\|_\infty}{t_*}$. *Consider* $\psi_r^{-1} : (0, \gamma) \to (0, t_*)$ *the inverse function of* ψ_r. *Then, the function*

$$\underline{u} = \begin{cases} \psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right) \frac{\varphi_1}{\|\varphi_1\|_\infty} & \text{if } \|\varphi_1\|_\infty \geq \psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right); \\ c\frac{\varphi_1}{\|\varphi_1\|_\infty} & \text{otherwise,} \end{cases} \tag{3.30}$$

is a sub-solution for (3.29).

Proof. Clearly, by the strictly decreasing of ψ_r stated in assumption (H_1^4) and observing that, by (H_3^4) one has $\frac{k_\alpha}{\underline{\beta}} < \gamma$, the inverse function $\psi_r^{-1} : (0, \gamma) \to (0, t_*)$ is well-posed. Let $\gamma < +\infty$, similarly it follows for $\gamma = +\infty$. By definition of ψ_r (see (H_1^4)), we have

$$\frac{k_\alpha}{\underline{\beta}} = \psi_r\left(\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\right) = \frac{f\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\left(\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\right)^{r-1}} \leq \frac{f(\underline{u})}{\underline{u}^{r-1}},$$

thus,

$$k_\alpha \underline{u}^{r-1} \le \underline{\beta} f(\underline{u}) \le \beta(x) f(\underline{u}) \quad \text{for a.a. } x \in \Omega.$$

Now, we analyze two cases, let $v \in W_0^{1,p(x)}(\Omega) \cap L_+^{p(x)}(\Omega)$.

Case 1: If $\|\varphi_1\|_\infty \ge \psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)$, i.e., $\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty} \le 1$, arguing by duality, we get

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla v dx \\
= \;& a(\alpha)\int_\Omega \left|\nabla\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\varphi_1\right)\right|^{p(x)-2} \nabla\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\varphi_1\right)\nabla v dx \\
= \;& a(\alpha)\int_\Omega \left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{p(x)-1} |\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla v dx \\
\le \;& a(\alpha)\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1} \int_\Omega |\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla v dx \\
= \;& a(\alpha)\lambda_{1,p(x)}\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1} \int_\Omega \varphi_1^{p(x)-1} v dx \\
= \;& a(\alpha)\lambda_{1,p(x)}\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1} \int_\Omega \frac{\|\varphi_1\|_\infty^{p(x)-1}}{\|\varphi_1\|_\infty^{p(x)-1}}\varphi_1^{p(x)-1} v dx \\
\le \;& a(\alpha)\lambda_{1,p(x)}\left(\frac{\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)}{\|\varphi_1\|_\infty}\right)^{r-1} \max\{1, \|\varphi_1\|_\infty^{p_+-1}\}\int_\Omega \frac{\varphi_1^{p(x)-1}}{\|\varphi_1\|_\infty^{p(x)-1}} v dx \\
\le \;& \frac{a(\alpha)\lambda_{1,p(x)}\max\{1, \|\varphi_1\|_\infty^{p_+-1}\}}{\|\varphi_1\|_\infty^{r-1}}\int_\Omega \left(\psi_r^{-1}\left(\frac{k_\alpha}{\underline{\beta}}\right)\frac{\varphi_1}{\|\varphi_1\|_\infty}\right)^{r-1} v dx \\
= \;& \frac{a(\alpha)\lambda_{1,p(x)}\max\{1, \|\varphi_1\|_\infty^{p_+-1}\}}{\|\varphi_1\|_\infty^{r-1}}\int_\Omega \underline{u}^{r-1} v dx \\
= \;& \int_\Omega k_\alpha \underline{u}^{r-1} v dx \le \int_\Omega \beta(x) f(\underline{u}) v dx.
\end{aligned}$$

Case 2: If $\|\varphi_1\|_\infty < \psi_r^{-1}\left(\frac{k_\alpha}{\beta}\right)$,

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla v dx \\
= \ & a(\alpha)\int_\Omega \left|\nabla\left(\frac{c}{\|\varphi_1\|_\infty}\right)\varphi_1\right|^{p(x)-2}\nabla\left(\frac{c}{\|\varphi_1\|_\infty}\varphi_1\right)\nabla v dx \\
= \ & a(\alpha)\int_\Omega \left|\left(\frac{c}{\|\varphi_1\|_\infty}\right)\right|^{p(x)-1}|\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla v dx \\
\leq \ & a(\alpha)\left(\frac{c}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega |\nabla\varphi_1|^{p(x)-2}\nabla\varphi_1\nabla v dx \\
= \ & a(\alpha)\lambda_{1,p(x)}\left(\frac{c}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega \varphi_1^{p(x)-1} v dx \\
= \ & a(\alpha)\lambda_{1,p(x)}\left(\frac{c}{\|\varphi_1\|_\infty}\right)^{r-1}\int_\Omega \frac{\|\varphi_1\|_\infty^{p(x)-1}}{\|\varphi_1\|_\infty^{p(x)-1}}\varphi_1^{p(x)-1} v dx \\
\leq \ & \frac{a(\alpha)\lambda_{1,p(x)}\max\{1,\|\varphi_1\|_\infty^{p_+-1}\}}{\|\varphi_1\|_\infty^{r-1}}\int_\Omega c^{r-1}\left(\frac{\varphi_1}{\|\varphi_1\|_\infty}\right)^{p(x)-1} v dx \\
\leq \ & k_\alpha\int_\Omega c^{r-1}\left(\frac{\varphi_1}{\|\varphi_1\|_\infty}\right)^{r-1} v dx \\
= \ & \int_\Omega k_\alpha \underline{u}^{r-1} v dx \leq \int_\Omega \beta(x) f(\underline{u}) v dx.
\end{aligned}$$

Then, in both cases, $\underline{u}$ is a sub-solution for (3.29). □

Next, bearing in mind that to obtain solutions of problem (3.27) we will associate a fixed-point problem to the frozen one, we truncate the reaction term in the frozen problem to manage its solutions. Let us consider the Carathéodory function

$$\hat{f}(x,t) = \begin{cases} f(\underline{u}) & \text{if } t < \underline{u}(x); \\ f(t) & \text{if } \underline{u}(x) \leq t \leq \overline{u}(x); \\ f(\overline{u}) & \text{if } t \geq \overline{u}(x), \end{cases} \tag{3.31}$$

and the associated frozen truncated problem

$$\begin{cases} -a(\alpha)\Delta_{p(x)} u = \beta(x)\hat{f}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

The following existence result holds.

Proposition 4. *Assume that (p), (H_a^2), (H_β), (H_1^4) and (H_3^4) hold. Then, there exists a unique solution $u_\alpha \in \operatorname{int}(C_+)$ of (3.32). Moreover, each solution of (3.32) is such that $\underline{u} \leq u_\alpha \leq \overline{u}$, that is, u_α solves (3.29).*

Proof. Fix $k \in \{1, \dots, K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$. Let

$$J(u) = a(\alpha)\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \int_\Omega \beta(x)F(x,u)dx \quad \forall u \in W_0^{1,p(x)}(\Omega)$$

be the energy functional associated to (3.32), where $F(x,s) = \int_0^s \hat{f}(x,t)dt$. By Lemma 2 and (3.31) one has that J is coercive and weakly lower semi-continuous (see, e.g., Rădulescu-Repovš [113, Lemma 3] or Diening-Harjulehto-Hästö-Růžička [48, Chapter 13, Section 2]). Then, Tonelli-Weierstrass' Theorem (see Theorem 4) ensures that there exists $u_\alpha \in W_0^{1,p(x)}(\Omega)$ global minimum of J, that is a weak solution of (3.32). Thus, in particular, u_α fulfils

$$a(\alpha)\int_\Omega |\nabla u_\alpha|^{p(x)-2}\nabla u_\alpha \nabla v dx = \int_\Omega \beta(x)\hat{f}(x,u)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega). \tag{3.33}$$

Consider the functions $v_1 = (u-\underline{u})^-$ and $v_2 = (u-\overline{u})^+$, $v_1, v_2 \in W_0^{1,p(x)}(\Omega)$ (see Motreanu-Motreanu-Papageorgiou [104, Remark 1.35]). By definition of sub-solution, one has

$$a(\alpha)\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} \nabla v dx \leq \int_\Omega \beta(x)\hat{f}(x,\underline{u})v dx, \quad \forall v \in W_0^{1,p(x)}(\Omega)\cap L_+^{p(x)}(\Omega). \tag{3.34}$$

Choosing $v = v_1$ and combining (3.31) and the strictly monotonicity of $-\Delta_{p(x)}$, we infer

$$\int_\Omega \left(|\nabla \underline{u}|^{p(x)-2}\nabla \underline{u} - |\nabla u_\alpha|^{p(x)-2}\nabla u_\alpha\right)\nabla v_1 dx \leq 0.$$

That is, the measure of the sub-level $\{u_\alpha < \underline{u}\}$ is zero, thus $u_\alpha \geq \underline{u}$. Similarly, follows that $u_\alpha \leq \overline{u}$ combining the super-solution and the test function $v = v_2$. Then, a deeper analysis of the truncation (3.31) shows that, each solution of (3.32) is a solution of (3.29). Finally, we prove that u_α is the unique solution such that $\underline{u} \leq u_\alpha \leq \overline{u}$. For this scope, we exploit the strictly monotonicity of the map ψ_r (see (H_1^4)). In particular, a Díaz-Saá-type argument is possible also for variable exponent framework, see Lemma 8. Suppose that there exists $v_\alpha \neq u_\alpha$ solution of (3.29), such that $\underline{u} \leq v_\alpha \leq \overline{u}$, then, we have

$$\begin{aligned} 0 &\leq \int_\Omega \left(-\frac{\Delta_{p(x)}u_\alpha}{u_\alpha^{r-1}} + \frac{\Delta_{p(x)}v_\alpha}{v_\alpha^{r-1}}\right)(u_\alpha^r - v_\alpha^r)\,dx \\ &= \int_\Omega \beta(x)\left(\frac{f(u_\alpha)}{u_\alpha^{r-1}} - \frac{f(v_\alpha)}{v_\alpha^{r-1}}\right)(u_\alpha^r - v_\alpha^r)\,dx < 0, \end{aligned}$$

a clear contradiction, thus $u_\alpha \equiv v_\alpha$. Furthermore, our solution $u_\alpha \in L^\infty(\Omega)$, thus Lieberman [92, Theorem 1] (see also Gasiński-Papageorgiou [69, Theorem 1.5.6]) ensures that $u_\alpha \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. The maximum principle of Fan [55, Proposition 3.1] (see also Pucci-Serrin [112] and Zhang [126]) guarantees that $u_\alpha \in \text{int}(C_+)$. □

Now, we are ready to prove the multiplicity results of this section. First, we consider the map $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ defined as

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}),$$

being u_α the unique solution obtained in the previous proposition. We highlight that, each fixed point of $\mathcal{P}_k$ is a solution of (3.27), in fact, let $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ such that $\mathcal{P}_k(\alpha) = \alpha$, then

$$-a\left(\int_\Omega u_\alpha^q dx\right)\Delta_{p(x)}u_\alpha = -a(\alpha)\Delta_{p(x)}u_\alpha = \beta(x)f(u_\alpha) \quad \text{in } \Omega. \tag{3.35}$$

Thus, the following propositions permit us to build the right geometry to ensure the existence of fixed points for $\mathcal{P}_k$. For the sake of clarity, we furnish a possible plot for

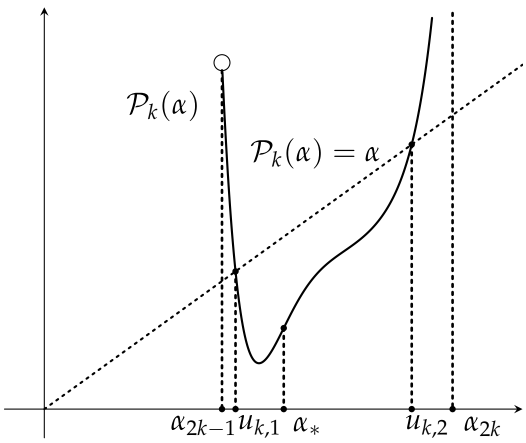

FIGURE 3.8: A possible plot for $\mathcal{P}_k(\alpha)$

the fixed-point map $\mathcal{P}_k$ (see Figure 3.8).

First, we prove that the map $\mathcal{P}_k$ is continuous.

Proposition 5. *Assume that* (p), (H_a^2), (H_β), (H_1^4) *and* (H_3^4) *hold. Then, for every* $k \in \{1, \dots, K\}$, *the map* $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ *defined as*

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k})$$

is continuous.

Proof. Let $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ be such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and let $\{u_n\}$ be the associated sequence. By Proposition 4, $\underline{u}_n \leq u_n \leq \overline{u}$ and u_n is a global minimum of J. Our aim is to prove that

$$\mathcal{P}_k(\alpha_n) \to \mathcal{P}_k(\alpha)$$

as $n \to +\infty$. Let us start to prove that $\{u_n\}$ is a bounded sequence in $W_0^{1,p(x)}(\Omega)$. Clearly, by Lemma 2, if $\rho_{p(x)}(\nabla u_n) \leq 1$, we have $\|u_n\| \leq 1$. Then, it is not restrictive to assume that $\rho_{p(x)}(\nabla u_n) > 1$. One has

$$\begin{aligned} & \frac{a(\alpha_n)}{p_+}\|u_n\|^{p_-} - \int_\Omega \beta(x)F(x,u_n)dx \\ \leq \quad & a(\alpha_n)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \int_\Omega \beta(x)F(x,u_n)dx \leq 0 \end{aligned}$$

where the last inequality holds since u_n is a global minimum of J for each $n \in \mathbb{N}$ and $J(0) = 0$. Thus, we infer

$$\frac{a(\alpha_n)}{p_+}\|u_n\|^{p_-} \leq \|\beta\|_\infty F(x,t_*)|\Omega| < +\infty.$$

That is, $\{u_n\}$ is a bounded sequence. Then Banach-Aloauglu-Bourbaki and Kakutani's Theorems, see Brézis [9, Theorem 3.16 and Theorem 3.17], ensure that $u_n \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$, and by Brézis [9, Theorem 4.9], we obtain $u_n \to u$ in $L^1(\Omega)$ and $u_n(x) \to u(x)$ a.e. in Ω for some $u \in W_0^{1,p(x)}(\Omega)$. Moreover, u_n is a weak solution of

(3.32), i.e.,

$$a(\alpha_n)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n \nabla v dx = \int_\Omega \beta(x)\hat{f}(x,u_n)v dx \quad \forall v \in W_0^{1,p(x)}(\Omega)$$

then, choosing $v = u_n - u$ as test function and passing to the limit we deduce

$$a(\alpha)\limsup_{n\to+\infty}\langle -\Delta_{p(x)}u_n, u_n - u\rangle \le 0.$$

The (S_+)-property of the main operator (see Lemma 4), allows us to infer that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Moreover, since $\underline{u}_n \le u_n \le \overline{u}$, and by the continuity of a and ψ_r, passing to the limit, we obtain that $\lim_{n\to+\infty} \underline{u}_n = \underline{u} > 0$, then $u \neq 0$. Finally, by Lebesgue Dominate Convergence's Theorem (see, e.g., Brézis [9, Theorem 4.2]) and by the uniqueness of solution provided in Proposition 4, we obtain that $\mathcal{P}_k(\alpha_n) \to \mathcal{P}_k(\alpha)$. □

Next, we obtain that if α is close to the boundary of the interval $(\alpha_{2k-1}, \alpha_{2k})$, the map $\mathcal{P}_k(\alpha) > \alpha$.

Proposition 6. *Assume that (p), (H_a^2), (H_β), (H_1^4), (H_2^4) and (H_3^4) hold. Then, for each $k \in \{1,\dots,K\}$, follows that*

$$\lim_{\alpha\to\alpha_{2k-1}^+} \mathcal{P}_k(\alpha) > \alpha_{2k-1} \text{ and } \lim_{\alpha\to\alpha_{2k}^-} \mathcal{P}_k(\alpha) > \alpha_{2k}.$$

Proof. From Lemma 13, we know that $u_\alpha \ge \underline{u}$, then

$$\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \ge \int_\Omega \underline{u}^q dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}).$$

On the other hand, one has, by (3.30)

$$\int_\Omega \underline{u}^q dx = \begin{cases} \left(\dfrac{\psi_r^{-1}\left(\frac{k_\alpha}{\beta}\right)}{\|\varphi_1\|_\infty}\right)^q \|\varphi_1\|_q^q & \text{if } \|\varphi_1\|_\infty \ge \psi_r^{-1}\left(\frac{k_\alpha}{\beta}\right); \\ \left(\dfrac{c}{\|\varphi_1\|_\infty}\right)^q \|\varphi_1\|_q^q & \text{otherwise.} \end{cases} \tag{3.36}$$

Thus, passing to the limit and by (H_2^4), the proof is complete. □

On the other hand, we obtain the existence of at least one point $\alpha_* \in (\alpha_{2k-1}, \alpha_{2k})$, such that

$$\mathcal{P}_k(\alpha_*) < \alpha_*.$$

Proposition 7. *Assume that (p), (H_a^2), (H_β), (H_1^4), (H_3^4) and (H_4^4) hold. Then, for each $k \in \{1,\dots,K\}$ there exists $\alpha_* \in (\alpha_{2k-1}, \alpha_{2k})$ such that $\mathcal{P}_k(\alpha_*) < \alpha_*$.*

Proof. Let u_{α_*} be a solution of (3.32). By (H_a^2), for each $\alpha_* \in (\alpha_{2k-1}, \alpha_{2k})$ one has $a(\alpha_*) > 0$. Clearly, we have

$$-\Delta_{p(x)}u_{\alpha_*} = \frac{1}{a(\alpha_*)}\beta(x)f(u_{\alpha_*}) \le \frac{\|\beta\|_\infty}{a(\alpha_*)}\max_{t\in[0,t_*]} f(t).$$

Set

$$M_{\alpha_*} = \frac{\|\beta\|_\infty}{a(\alpha_*)}\max_{t\in[0,t_*]} f(t),$$

and consider the elliptic problem

$$\begin{cases} -\Delta_{p(x)} w = M_{\alpha_*} & \text{in} \quad \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.37}$$

Moreover, consider the following constants:

$$C^* = \frac{(N+1)(2C_0)^{(p_-)'}}{(p_+)'(p_-)^{\frac{(p_-)'}{p_-}}} |\Omega|^{\frac{(p_-)'}{N}}, \tag{3.38}$$

and

$$C_* = \frac{(N+1)(2C_0)^{(p_+)'}}{(p_+)'(p_-)^{\frac{(p_+)'}{p_+}}} |\Omega|^{\frac{(p_+)'}{N}}, \tag{3.39}$$

with C_0 as in (3.28). Notice that, these constants depend only on the data of the problem.

Let w be the unique solution (by Minty-Browder's Theorem, see Theorem 5) of (3.37), by Fan [55, Lemma 2.1], one has

$$\|w\|_\infty \leq \begin{cases} C^* M_{\alpha_*}^{\frac{1}{p_- - 1}} & \text{if} \quad M_{\alpha_*} \geq \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}; \\ C_* M_{\alpha_*}^{\frac{1}{p_+ - 1}} & \text{if} \quad M_{\alpha_*} < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}. \end{cases} \tag{3.40}$$

On the other hand, the comparison principle of Fan [55, Proposition 2.3] ensures that, since $u_\alpha \leq w$ on $\partial\Omega$ and

$$-\Delta_{p(x)} u_\alpha = \frac{1}{a(\alpha_*)} \beta(x) f(u_{\alpha_*}) \leq M_{\alpha_*} = -\Delta_{p(x)} w$$

then, $u_\alpha \leq w$ in Ω.

We dived our study in two cases:

Case 1: Let us assume that $M_{\alpha_*} < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}$, then

$$\begin{aligned} \|w\|_\infty &\leq C_* M_{\alpha_*}^{\frac{1}{p_+ - 1}} \\ &< \frac{(N+1)(2C_0)^{(p_+)'}}{(p_+)'(p_-)^{\frac{(p_+)'}{p_+}}} |\Omega|^{\frac{(p_+)'}{N}} \left(\frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0} \right)^{\frac{1}{p_+ - 1}} \\ &= \frac{(N+1)(p_+ - 1)}{p_+} \frac{2^{\frac{p_+}{p_+ - 1}} 2^{\frac{-1}{p_+ - 1}} C_0^{\frac{p_+}{p_+ - 1}} C_0^{\frac{-1}{p_+ - 1}} |\Omega|^{\frac{p_+}{N(p_+ - 1)}} |\Omega|^{\frac{-1}{N(p_+ - 1)}}}{(p_-)^{\frac{1}{p_+ - 1}} (p_-)^{\frac{-1}{p_+ - 1}}} \\ &= \frac{(N+1)(p_+ - 1)}{p_+} 2C_0 |\Omega|^{\frac{1}{N}} \\ &= \frac{(N+1)(p_+ - 1)}{p_+} 2|\Omega|^{\frac{1}{N}} \frac{(\Gamma(1 + \frac{N}{2}))^{\frac{1}{N}}}{N\sqrt{\pi}}. \end{aligned}$$

This computation implies

$$\begin{aligned}\mathcal{P}_k(\alpha) &= \int_\Omega u_\alpha^q dx \le \|w\|_\infty^q |\Omega| \\ &< \left(\frac{(N+1)(p_+-1)}{p_+}\right)^q \left[2\frac{\Gamma(1+\frac{N}{2})^{\frac{1}{N}}}{N\sqrt{\pi}}\right]^q |\Omega|^{\frac{q+N}{N}}.\end{aligned}$$

Assumption (H_4^4)(i) guarantees us that

$$\alpha_* > \left(\frac{(N+1)(p_+-1)}{p_+}\right)^q \left[2\frac{\Gamma(1+\frac{N}{2})^{\frac{1}{N}}}{N\sqrt{\pi}}\right]^q |\Omega|^{\frac{q+N}{N}}. \tag{3.41}$$

Moreover, by definition of M_{α_*}, one has

$$\frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_*)} < \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0},$$

that is,

$$a(\alpha_*)p_- > \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma\left(1+\frac{N}{2}\right)\right)^{\frac{1}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t). \tag{3.42}$$

Case 2: Let us assume that $M_{\alpha_*} \ge \frac{p_-}{2|\Omega|^{\frac{1}{N}} C_0}$, then

$$\begin{aligned}\|w\|_\infty &\le C^* M_{\alpha_*}^{\frac{1}{p_--1}} \\ &= \frac{(N+1)(2C_0)^{(p_-)'}}{(p_+)'(p_-)^{\frac{(p_-)'}{p_-}}} |\Omega|^{\frac{(p_-)'}{N}} \left(\frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_*)}\right)^{\frac{1}{p_--1}}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{P}_k(\alpha) = \int_\Omega u_\alpha^q dx \le \|w\|_\infty^q |\Omega| \\ \le \left(\frac{(N+1)(p_+-1)}{p_+} \frac{2^{\frac{p_-}{p_--1}} C_0^{\frac{p_-}{p_--1}}}{(p_-)^{\frac{1}{p_--1}}} |\Omega|^{\frac{p_-}{N(p_--1)}} |\Omega|^{\frac{1}{q}} \left(\frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_*)}\right)^{\frac{1}{p_--1}}\right)^q \\ = \left(\frac{(N+1)(p_+-1)}{p_+}\left[\frac{2^{p_-} C_0^{p_-} |\Omega|^{\frac{qp_-+N(p_--1)}{qN}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_*)p_-}\right]^{\frac{1}{p_--1}}\right)^q \\ = \left(\frac{(N+1)(p_+-1)}{p_+}\right)^q \left[\frac{2^{p_-}(\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}} \|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{p_-(N\sqrt{\pi})^{p_-} a(\alpha_*)}\right]^{\frac{q}{p_--1}} |\Omega|^{\frac{qp_-+N(p_--1)}{N(p_--1)}}.\end{aligned}$$

In order to ensure that $\mathcal{P}_k(\alpha_*) < \alpha_*$, we have

$$a(\alpha_*)\alpha_*^{\frac{p_- -1}{q}} > \left(\frac{(N+1)(p_+-1)}{p_+}\right)^{p_- -1}\left(\frac{2^{p_-}(\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}}}{p_-(N\sqrt{\pi})^{p_-}}\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)\right)|\Omega|^{\frac{qp_-+N(p_- -1)}{qN}},$$

that follows by (H_4^4)(ii). Furthermore, since again by definition of M_{α_*},

$$\frac{\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{a(\alpha_*)} \geq \frac{p_-}{2|\Omega|^{\frac{1}{N}}C_0},$$

i.e.,

$$a(\alpha_*)p_- \leq \frac{2}{N\sqrt{\pi}}\left(|\Omega|\Gamma\left(1+\frac{N}{2}\right)\right)^{\frac{1}{N}}\|\beta\|_\infty \max_{t\in[0,t_*]} f(t), \tag{3.43}$$

and one has

$$\begin{aligned}
\alpha_* &> \left(\frac{(N+1)(p_+-1)}{p_+}\right)^q \left[\frac{2^{p_-}(\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}}\|\beta\|_\infty \max_{t\in[0,t_*]} f(t)}{p_-(N\sqrt{\pi})^{p_-}a(\alpha_*)}\right]^{\frac{q}{p_- -1}} |\Omega|^{\frac{qp_-+N(p_- -1)}{N(p_- -1)}} \\
&\geq \left(\frac{(N+1)(p_+-1)}{p_+}\right)^q \left[\frac{2^{p_-}(\Gamma(1+\frac{N}{2}))^{\frac{p_-}{N}}}{p_-(N\sqrt{\pi})^{p_-}}\frac{p_-}{2|\Omega|^{\frac{1}{N}}C_0}\right]^{\frac{q}{p_- -1}} |\Omega|^{\frac{qp_-+N(p_- -1)}{N(p_- -1)}} \\
&= \left(\frac{(N+1)(p_+-1)}{p_+}\right)^q \left[2\frac{\Gamma(1+\frac{N}{2})^{\frac{1}{N}}}{N\sqrt{\pi}}\right]^q |\Omega|^{\frac{q+N}{N}},
\end{aligned}$$

which is guaranteed by hypothesis (H_4^4). □

Finally, we are ready to prove our main result for variable exponent framework.

Proof of Theorem 13. First, by Proposition 5, the map $\mathcal{P}_k$ is continuous. Proposition 6 and Proposition 7 provide the right geometry to obtain at least two fixed points (see again Figure 3.8) for the map $\mathcal{P}_k(\alpha)$ for each positive bump of the weight function a (see (H_a^2)). The proof follows from (3.35). □

Remark 6. *We emphasize that, when $p_- = p = p_+$, that is , when we come back to constant exponent case, we obtain an original result also for p-Laplacian, since assumption (H_4^4) is different with respect to (H_4^2) obtained in Candito-Gasiński-Livrea-Santos Júnior [15].*

Chapter 4

Existence, nonexistence and multiplicity of solutions without monotonicity-type assumptions

4.1 Lack of monotonicity

The aim of this chapter is to remove any monotonicity assumption connected to the reaction term. As pointed out in Chapter 3, the strictly decreasing of the map

$$(0, t_*) \ni t \overset{\psi}{\mapsto} \frac{f(t)}{t^{p-1}}$$

(see, for instance, assumption (H_1^2) in Chapter 3), is crucial in several points of the proof's results. In particular, it is essential to apply a Díaz-Saá-type argument to obtain the uniqueness of solution, crucial to well define the fixed point map $\mathcal{P}_k$. Moreover, through the inverse of the map ψ, it is possible to produce a lower barrier of the frozen problem. This control became fundamental to provide the right geometry of the fixed point map.

The main question of this last chapter is the following: is it possible to remove any monotonicity assumption involving the reaction term?

Here, we address the issue and we provide a general result of multiplicity of positive solutions for Carrier-type problem driven by p-Laplacian. Furthermore, we produce a result for the celebrated double phase operator. For the best of our knowledge, this is the first result in literature that explore the Carrier version of the double phase problem.

However, the lack of monotonicity causes many difficulties. As we will prove in the following sections, to obtain our results we need to arrange the ideas arose in Chapter 3 with new techniques. In particular, we combine variational and sub-super solutions methods with set-valued analysis. Finally, again a suitable one-dimensional fixed point problem will permits us to cover the troubles encountered in this new setting.

Throughout this chapter, we will assume the sign-changing condition on the weight function a. For reader's convenience, we recall this assumption here:

(H_a^2) let $a : [0, +\infty) \to \mathbb{R}$ be a continuous function. There exist positive numbers $0 \leq \alpha_0 < \alpha_1 < \alpha_2 < \ldots \leq \alpha_{2K-1} < \alpha_{2K}$ $(K \geq 1)$ such that $a(\alpha_{2k-1}) = 0 = a(\alpha_{2k})$ and $a > 0$ in $(\alpha_{2k-1}, \alpha_{2k})$, for all $k \in \{1, \ldots, K\}$.

The new ideas developed in this chapter can by summarized in the following scheme:

$$
\begin{tikzcd}
\text{Nonlocal problem} \arrow[d] & & \text{One-dimensional fixed point} \arrow[ll] \\
\text{Frozen nonlocal problem} \arrow[dr] & & \text{Set-valued Analysis (smallest solution)} \arrow[u] \\
& \text{Sub-super solutions and variational methods} \arrow[ur] &
\end{tikzcd}
$$

FIGURE 4.1: Proof path – No monotonicity

4.1.1 The p-Laplacian case

First, we remove the monotonicity assumption of (1.2)-type, in the p-Laplacian case. We provide a multiplicity result for the following problem

$$
\begin{cases} -a\left(\int_\Omega u^q dx\right)\Delta_p u = f(x,u) & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}
$$

Notice that, this is the first time in which, in addition to removing the monotonicity assumption, the fully nonautonomous case is considered, i.e., without any special split assumptions as in Section 3.2. The results presented here are contained in the works Candito-Failla-Gasiński-Livrea [13] for existence and multiplicity of solutions, and Failla-Gasiński [52] for the increase in the number of solutions.

We set our problem in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$. We consider again a possible sign-changing weight function a as in assumption (H_a^2). Furthermore, on the reaction term we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function fulfilling the following:

(H_f^5) there exists $t_* > 0$ such that $f(x,t_*) = 0$ for a.a. $x \in \Omega$;

(H_1^5) for each compact interval $J \subset (0,t_*)$, one has that

$$
\operatorname*{ess\,inf}_{x\in\Omega} f(x,t) \geq \rho_J
$$

for each $t \in J$ and some $\rho_J > 0$ constant;

(H_2^5) there exists a constant $M > 0$ such that

$$
\operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,t_*]} f(x,t) \leq M;
$$

(H_3^5) let $\lambda_{1,p}$ be the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$, and

$$
\liminf_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} > \lambda_{1,p} \max_{k\in\{1,\dots,K\}} \max_{\alpha\in[\alpha_{2k-1},\alpha_{2k}]} a(\alpha)
$$

uniformly for a.a. $x \in \Omega$.

Moreover, to obtain a suitable geometry for the fixed point map $\mathcal{P}_k$, we assume that the following relations, between the weight function a and the reaction term f, hold.

(H_4^5) For each $k \in \{1, \dots, K\}$ there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$\tau_k^p a(\tau_k) > \frac{|\Omega|^p}{\lambda_{1,p}} t_*^{p(q-1)} \operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,t_*]} [tf(x,t)];$$

(H_5^5) $\alpha_{2K} < t_*^q \|e_1\|_q^q$ being e_1 the first eigenfunction of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ normalized in L^∞-norm.

Notice that the assumption (H_1^5) is the generalization of the positivity, $f(t) > 0$ in $(0, t_*)$ in the nonatonomous case. Moreover, if we came back to $f(x,t) = f(t)$, assumption (H_2^5) is free.

One of the key points of this new framework is a clever construction of the sub-solution for the frozen problem. Then, let us start freezing the nonlocal term: let $k \in \{1, \dots, K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, and consider the variational problem

$$\begin{cases} -a(\alpha)\Delta_p u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The following lemma holds.

Lemma 14. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5) and (H_3^5) hold. Then, problem (4.2) possesses a positive sub-solution $\underline{u} \in C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$ such that $0 < \underline{u} \le t_*$.*

Proof. Fix $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$. By definition of limit inferior, assumption (H_3^5) gives us the following structure: there exist $\Omega_0 \subset \Omega$, with $|\Omega_0| = 0$ and $\delta > 0$ such that

$$f(x,t) > t^{p-1}\lambda_{1,p} \max_{\alpha\in[\alpha_{2k-1},\alpha_{2k}]} a(\alpha) \quad \forall (x,t) \in \Omega\setminus\Omega_0 \times (0,\delta]. \tag{4.3}$$

Fix $x \in \Omega \setminus \Omega_0$ and put

$$T(x,\alpha) := \left\{ t \in [0, t_*] : f(x,t) = t^{p-1}\lambda_{1,p} a(\alpha) \right\}. \tag{4.4}$$

Clearly, by (4.3) and the continuity of $f(x,\cdot)$, one has $T(x,\alpha)$ is non-empty and compact. In particular, its minimum is attained, and we set

$$\widehat{t}(x,\alpha) = \min T(x,\alpha). \tag{4.5}$$

Again, by (4.3), we have $\widehat{t}(x,\alpha) > \delta$. Finally, setting

$$t(\alpha) = \inf_{x\in\Omega\setminus\Omega_0} \widehat{t}(x,\alpha), \tag{4.6}$$

one has

$$0 < \delta \le t(\alpha) \le t_*. \tag{4.7}$$

Next, let us consider the classical eigenvalue p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda_{1,p}|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.8}$$

It is well known that problem (4.8) possesses a positive solution $\varphi_1 \in C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$, see Motreanu-Motreanu-Papageorgiou [104, Chapter 9 Section 2] and Lieberman [92, Theorem 1]. Let us define

$$\underline{u} = t(\alpha)e_1 \tag{4.9}$$

being $e_1 = \frac{\varphi_1}{\|\varphi_1\|_\infty}$. Clearly, by (4.7), we get that $\underline{u} > 0$ in Ω. Moreover, one has that

$$0 < \underline{u} \leq \sup_{\Omega} \underline{u} = t(\alpha)\|e_1\|_\infty = t(\alpha) \leq t_*$$

and, since by construction, $t(\alpha)$ is the minimum positive solution of the equation $f(x,t) = t^{p-1}\lambda_{1,p}a(\alpha)$, by (H_3^5), we infer

$$a(\alpha)\lambda_{1,p}\left(t(\alpha)e_1\right)^{p-1} \leq f(x, t(\alpha)e_1),$$

i.e.,

$$a(\alpha)\lambda_{1,p}\underline{u}^{p-1} \leq f(x, \underline{u}).$$

Thus, combining the above computations, we obtain

$$\begin{aligned}
-a(\alpha)\Delta_p \underline{u} &= -a(\alpha)\left(\frac{t(\alpha)}{\|\varphi_1\|_\infty}\right)^{p-1}\Delta_p\varphi_1 \\
&= a(\alpha)\lambda_{1,p}\left(\frac{t(\alpha)}{\|\varphi_1\|_\infty}\right)^{p-1}\varphi_1^{p-1} \\
&= a(\alpha)\lambda_{1,p}\underline{u}^{p-1} \leq f(x,\underline{u}),
\end{aligned}$$

that is, $\underline{u}$ is a sub-solution for problem (4.2). □

We highlight that the previous lemma permits us to obtain a well-order pair $(\underline{u}, \overline{u})$ of sub- and super-solutions for problem (4.2). In particular

$$0 < \underline{u} = t(\alpha)e_1 \leq t_* = \overline{u}.$$

Hence, following the notation in Motreanu-Motreanu-Papageorgiou [104, p. 307], we consider the symbol

$$[\underline{u}, \overline{u}] = \left\{u \in W_0^{1,p}(\Omega) : \underline{u}(x) \leq u(x) \leq \overline{u}(x) \text{ for a.a. } x \in \Omega\right\}.$$

For the sake of clarity, we refer to Figure 4.2 for the building of sub-solution.

Next, we truncate the reaction term to obtain a control from above and from below of some set of solutions. Let us consider the Carathéodory function

$$\widehat{f}(x,t) = \begin{cases} f(x,\underline{u}) & \text{if } t < \underline{u}(x); \\ f(x,t) & \text{if } \underline{u}(x) \leq t \leq \overline{u}(x); \\ f(x,\overline{u}) & \text{if } t \geq \overline{u}(x), \end{cases} \tag{4.10}$$

and the associated truncated variational elliptic problem

$$\begin{cases} -a(\alpha)\Delta_p u = \widehat{f}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.11}$$

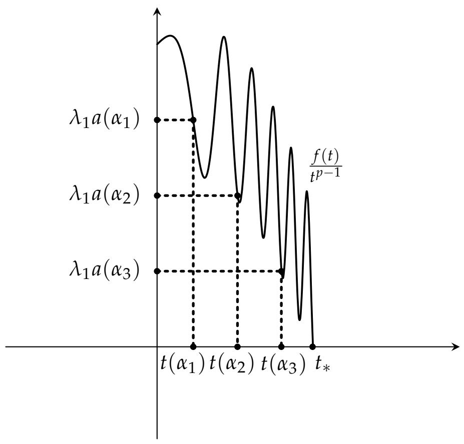

FIGURE 4.2: Idea of building sub-solutions

Arguing as in Proposition 4, the following holds.

Proposition 8. *Assume that* (H_a^2), (H_f^5), (H_1^5), (H_2^5), *and* (H_3^5) *hold. Then, the problem* (4.11) *admits at least one positive solution* $u_\alpha \in C_0^{1,\nu}(\overline{\Omega})$ *such that* $\underline{u} \leq u \leq \overline{u}$.

Clearly, since we removed any monotonicity assumption of (1.2)-type, we can not ensures the uniqueness of the solution u_α with a Díaz-Saá-type argument. Thus, to define a suitable fixed point map and come back to the solutions of problem (4.1), we must find a new path. The main ideas for a possible proof, come from the work of Liu-Motreanu-Zeng [94] (see also the evolution of these ideas in Baldelli-Guarnotta [6], Candito-Livrea-Moussaoui [16], Guanotta-Marano [77], and Guarnotta-Marano-Motreanu [78]) where the authors, involving a suitable set-valued map, obtained solutions for a class of convective problems.

The aim of the following propositions is to obtain the unique smallest solution for (4.11). This unique positive solution will be the starting point to guarantee the existence and multiplicity results for problem (4.1). Let us consider the set-valued map $\mathscr{S} : (\alpha_{2k-1}, \alpha_{2k}) \to 2^{C_0^1(\overline{\Omega})}$ defined as

$$\mathscr{S}(\alpha) = \left\{ u \in C_0^1(\overline{\Omega}) : u \text{ solves } (4.11) \right\} \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}).$$

Clearly, by Proposition 8, we infer that

$$\mathscr{S}(\alpha) \subseteq [\underline{u}, \overline{u}]. \tag{4.12}$$

The following propositions permits us to build a suitable geometry for the set-valued map $\mathscr{S}$. First, we prove that $\mathscr{S}$ maps convergent sequences of $(\alpha_{2k-1}, \alpha_{2k})$ into sequences with convergent subsequences in $C_0^1(\overline{\Omega})$.

Proposition 9. *Assume that* (H_a^2), (H_f^5), (H_1^5), (H_2^5), *and* (H_3^5) *hold. Let* $\alpha_n \in (\alpha_{2k-1}, \alpha_{2k})$ *be a sequence such that* $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$. *Set* $u_n \in \mathscr{S}(\alpha_n)$. *Then, there exists* $u \in \mathscr{S}(\alpha)$ *such that* $u_n \to u \in \mathscr{S}(\alpha)$.

Proof. Consider α_n and u_n be sequences that satisfy the statement. Since $u_n \in \mathscr{S}(\alpha_n)$, one has that u_n solves

$$\begin{cases} -a(\alpha_n)\,\Delta_p u_n = \widehat{f}(x,u) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.13}$$

for each $n \in \mathbb{N}$. Combining Proposition 8 and Lieberman [92, Theorem 1] (see also Gasiński-Papageorgiou [69, Theorems 1.5.5 and 1.5.6]) and since $a(\alpha_n) > 0$ for each $n \in \mathbb{N}$, we get $u_n \in C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. By compact embedding $C_0^{1,\nu}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ (see (2.2)), we know that, up to subsequence, $u_n \to u$ in $C_0^1(\overline{\Omega})$. Passing to the limit in (4.13), we infer

$$\begin{cases} -a(\alpha)\,\Delta_p u = \widehat{f}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.14}$$

this completes the proof. □

Next, to prove the existence of the (unique) smallest solution, we show in the following results, that $\mathscr{S}$ is l.s.c. and downward direct.

Proposition 10. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5), and (H_3^5) hold. Then, $\mathscr{S}$ is l.s.c.*

Proof. We prove the statement exploiting the characterization in Lemma 10. Let $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ be such that $\alpha_n \to \alpha$ in $(\alpha_{2k-1}, \alpha_{2k})$ and let $v \in \mathscr{S}(\alpha)$. For each $n \in \mathbb{N}$, consider the problem

$$\begin{cases} -a(\alpha_n)\Delta_p u = \widehat{f}(x,v) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$FT^v_{\alpha_n}$}$$

Minty-Browder Theorem's (see Theorem 5) ensures that problem ($FT^v_{\alpha_n}$) possesses a unique positive solution $u_n^0 \in C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. Arguing as in Proposition 9, the sequence $\{u_n^0\}$ is bounded in $C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$. By compact embedding of $C_0^{1,\nu}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ (see (2.2)), one has that $\{u_n^0\}$ is relatively compact in $C_0^1(\overline{\Omega})$. Then, up to subsequence $\{u_{n_k}^0\}$ of $\{u_n^0\}$, we have $u_{n_k}^0 \to u \in C_0^1(\overline{\Omega})$ as $k \to +\infty$, where u is clearly the unique solution of

$$\begin{cases} -a(\alpha)\Delta_p u = \widehat{f}(x,v) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, since $v \in \mathscr{S}(\alpha)$, by the definition of $\mathscr{S}(\alpha)$, we have that v is a solution of

$$\begin{cases} -a(\alpha)\Delta_p v = \widehat{f}(x,v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, it follows that $u = v$. Since each subsequence $\{u_{n_k}^0\}$ of $\{u_n^0\}$ converges to the same limit v, then

$$\lim_{n \to +\infty} u_n^0 = v.$$

Again, for each $n \in \mathbb{N}$ consider the problem

$$\begin{cases} -a(\alpha_n)\Delta_p u = \widehat{f}(x,u_n^0) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$FT^{u_n^0}_{\alpha_n}$}$$

As above, there exists a unique solution u_n^1 of ($FT_{\alpha_n}^{u_n^0}$), for each $n \in \mathbb{N}$, such that

$$\lim_{n\to+\infty} u_n^1 = v.$$

By iterating this process, for any $k \geq 1$, we get a sequence $\{u_n^k\}$, such that u_n^k is a solution of

$$\begin{cases} -a(\alpha_n)\Delta_p u_n^k = \widehat{f}(x, u_n^{k-1}) & \text{in } \Omega, \\ u_n^k > 0 & \text{in } \Omega, \\ u_n^k = 0 & \text{on } \partial\Omega, \end{cases} \tag{$FT_{\alpha_n}^{u_n^{k-1}}$}$$

and

$$\lim_{n\to+\infty} u_n^k = v \quad \forall k \in \mathbb{N}. \tag{4.15}$$

Fix $n \geq 1$, arguing as in Proposition 9, since $\{u_n^k\}$ is relatively compact in $C_0^1(\overline{\Omega})$, then, up to subsequence, we have that

$$u_n^k \to u_n \text{ as } k \to +\infty,$$

that is, it solves

$$\begin{cases} -a(\alpha_n)\Delta_p u_n = \widehat{f}(x, u_n) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{$FT_{\alpha_n}^{u_n}$}$$

Arguing as in Proposition 9, by (4.15), and double limit lemma (see, e.g., Gasiński-Papageorgiou [68, Proposition A.2.35]), it follows that $u_n \to v$ in $C_0^1(\overline{\Omega})$. Thus, by Lemma 10, the proof is achieved. □

For the downward directness of the set-valued map, first we prove the following classical result, see Carl-Le-Motreanu [18, Remark 3.61] and Liu-Motreanu-Zeng [94, Lemma 10].

Lemma 15. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5), and (H_3^5) hold. Let $\overline{u}_1, \overline{u}_2 \in W^{1,p}(\Omega)$ be two super-solutions of (4.11), then $\overline{u} = \min\{\overline{u}_1, \overline{u}_2\} \in W^{1,p}(\Omega)$ is also a super-solution for (4.11).*

Proof. Let $\overline{u}_1, \overline{u}_2 \in W^{1,p}(\Omega)$ be two super-solutions for (4.11). Let $\varepsilon > 0$ and consider the following Lipschitz continuous truncation function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$, defined by

$$\eta_\varepsilon(t) = \begin{cases} 0 & \text{if} \quad t \leq 0; \\ \frac{t}{\varepsilon} & \text{if} \quad 0 < t < \varepsilon; \\ 1 & \text{if} \quad t \geq \varepsilon. \end{cases} \tag{4.16}$$

Notice that, by Ziemer [130, Theorem 2.1.11] (see also Adams-Fournier [1, Lemma 8.34] and Marcus-Mizel [98]), one has $\eta_\varepsilon(\overline{u}_2 - \overline{u}_1) \in W^{1,p}(\Omega)$, and, by Lebesgue Dominate Convergence Theorem (see e.g., Brézis [9, Theorem 4.2]), we get

$$\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)(x) \to \chi_{\{\overline{u}_1 < \overline{u}_2\}}(x) \text{ for a.a. } x \in \Omega \text{ as } \varepsilon \to 0^+, \tag{4.17}$$

where χ_E is the characteristic function of the set E. Moreover, by chain rule, we observe that

$$\nabla\eta_\varepsilon(\overline{u}_2 - \overline{u}_1)v = v\nabla\eta_\varepsilon(\overline{u}_2 - \overline{u}_1) + \eta_\varepsilon(\overline{u}_2 - \overline{u}_1)\nabla v \quad \forall v \in C_0^\infty(\Omega)_+.$$

Now, by definition of super-solution (see Definition 1), we have

$$a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla \varphi_1 dx \geq \int_\Omega \widehat{f}(x,\overline{u}_1)\varphi_1 dx \quad \forall \varphi_1 \in W_0^{1,p}(\Omega)\cap L_+^p(\Omega),$$

and

$$a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla \varphi_2 dx \geq \int_\Omega \widehat{f}(x,\overline{u}_2)\varphi_2 dx \quad \forall \varphi_2 \in W_0^{1,p}(\Omega)\cap L_+^p(\Omega).$$

Choosing $\varphi_1 = \eta_\varepsilon(\overline{u}_2 - \overline{u}_1)v$ and $\varphi_2 = (1-\eta_\varepsilon(\overline{u}_2 - \overline{u}_1))v$ for $v \in C_0^\infty(\Omega)_+$, and adding the two previous inequalities, we get

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla\left[\eta_\varepsilon(\overline{u}_2-\overline{u}_1)v\right]dx \\
& +a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla\left[(1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1))v\right]dx \\
\geq\ & \int_\Omega \widehat{f}(x,\overline{u}_1)\eta_\varepsilon(\overline{u}_2-\overline{u}_1)v dx + \int_\Omega \widehat{f}(x,\overline{u}_2)(1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1))v dx.
\end{aligned}$$

By direct computation, we have

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla\left[\eta_\varepsilon(\overline{u}_2-\overline{u}_1)v\right]dx \\
=\ & a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla \eta_\varepsilon(\overline{u}_2-\overline{u}_1)\cdot v dx \\
& +a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla v \cdot \eta_\varepsilon(\overline{u}_2-\overline{u}_1) dx \\
=\ & \frac{a(\alpha)}{\varepsilon}\int_{\{0<\overline{u}_2-\overline{u}_1<\varepsilon\}} |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla(\overline{u}_2-\overline{u}_1)\cdot v dx \\
& +a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla v \cdot \eta_\varepsilon(\overline{u}_2-\overline{u}_1) dx.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla\left[(1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1))v\right]dx \\
=\ & a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla(1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1))\cdot v dx \\
& +a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla v \cdot (1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1)) dx \\
=\ & -\frac{a(\alpha)}{\varepsilon}\int_{\{0<\overline{u}_2-\overline{u}_1<\varepsilon\}} |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla(\overline{u}_2-\overline{u}_1)\cdot v dx \\
& +a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla v \cdot (1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1)) dx.
\end{aligned}$$

Putting the above computation together, gives

$$\begin{aligned}
& a(\alpha)\int_\Omega |\nabla \overline{u}_1|^{p-2}\nabla \overline{u}_1 \nabla v \cdot \eta_\varepsilon(\overline{u}_2-\overline{u}_1) dx \\
& +a(\alpha)\int_\Omega |\nabla \overline{u}_2|^{p-2}\nabla \overline{u}_2 \nabla v \cdot (1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1)) dx \\
\geq\ & \int_\Omega \widehat{f}(x,\overline{u}_1)\eta_\varepsilon(\overline{u}_2-\overline{u}_1)v dx + \int_\Omega \widehat{f}(x,\overline{u}_2)(1-\eta_\varepsilon(\overline{u}_2-\overline{u}_1))v dx.
\end{aligned}$$

By (4.17), passing to the limit as $\varepsilon \to 0^+$, we get

$$\begin{aligned} & a(\alpha) \int_{\{\overline{u}_1 < \overline{u}_2\}} |\nabla \overline{u}_1|^{p-2} \nabla \overline{u}_1 \nabla v dx + a(\alpha) \int_{\{\overline{u}_1 \geq \overline{u}_2\}} |\nabla \overline{u}_2|^{p-2} \nabla \overline{u}_2 \nabla v dx \\ \geq & \int_{\{\overline{u}_1 < \overline{u}_2\}} \widehat{f}(x, \overline{u}_1) v dx + \int_{\{\overline{u}_1 \geq \overline{u}_2\}} \widehat{f}(x, \overline{u}_2) v dx \end{aligned}$$

for all $v \in C_0^\infty(\Omega)_+$. Recall that, since $\overline{u}_1, \overline{u}_2 \in W^{1,p}(\Omega)$, then $\overline{u} = \min\{\overline{u}_1, \overline{u}_2\} \in W^{1,p}(\Omega)$ (see Carl-Le-Motreanu [18, Lemma 2.87]). Moreover, $\overline{u} \geq 0$ on $\partial\Omega$, and

$$\nabla \overline{u}(x) = \begin{cases} \nabla \overline{u}_1(x) & \text{for a.a. } x \in \{\overline{u}_1 < \overline{u}_2\}, \\ \nabla \overline{u}_2(x) & \text{for a.a. } x \in \{\overline{u}_1 \geq \overline{u}_2\}. \end{cases} \tag{4.18}$$

Thus, we obtain that

$$a(\alpha) \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla v dx \geq \int_\Omega \widehat{f}(x, \overline{u}) v dx \quad \forall v \in C_0^\infty(\Omega)_+.$$

Finally, since $C_0^\infty(\Omega)_+$ is dense in $W_0^{1,p}(\Omega)_+$ (see Chapter 2), then we can choose $v \in W_0^{1,p}(\Omega) \cap L^p(\Omega)_+$ and we conclude the proof. □

Proposition 11. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5), and (H_3^5) hold. Then, the set $\mathscr{S}(\alpha)$ is downward direct.*

Proof. Fix $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and let $u_1, u_2 \in \mathscr{S}(\alpha)$. Set $\widetilde{u} = \min\{u_1, u_2\}$. Notice that, since u_1, u_2 are solutions of (4.11) they are also super-solutions of (4.11). Then, by Lemma 15, $\widetilde{u}$ is a super-solution for (4.11). Let us consider the following truncation

$$f_*(x,s) = \begin{cases} \widehat{f}(x, \underline{u}) & \text{if} \quad s < \underline{u}(x); \\ \widehat{f}(x, s) & \text{if} \quad \underline{u}(x) \leq s \leq \widetilde{u}(x); \\ \widehat{f}(x, u) & \text{if} \quad s > \widetilde{u}(x), \end{cases} \tag{4.19}$$

and the associated problem

$$\begin{cases} -a(\alpha)\Delta_p u = f_*(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{FTT_α}$$

Now, arguing as in Proposition 8, there exists u solution of (FTT_α), such that $\underline{u} \leq u \leq \widetilde{u}$. Then, u is obviously a solution of (4.11), and $u \in \mathscr{S}(\alpha)$. Thus, for each $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, one has that $\mathscr{S}(\alpha)$ is partially ordered and downward direct. □

Now, we are ready to obtain the existence of the unique smallest solution of (4.11).

Proposition 12. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5), and (H_3^5) hold. Then, (4.11) admits the smallest solution $u_\alpha^* \in [\underline{u}, \overline{u}]$.*

Proof. Let $\mathcal{C} \subset \mathscr{S}(\alpha)$ be a chain, by Hu-Papageorgiou [82, Lemma 3.10], there exists $\{u_n\} \subset \mathcal{C}$ such that

$$\lim_{n \to +\infty} u_n = \inf \mathcal{C}.$$

On the other hand, u_n solves (4.11) for each $n \in \mathbb{N}$. Arguing as in Proposition 9, up to subsequence if necessary, there exists $v \in C_0^1(\overline{\Omega})$ such that $u_n \to v$ in $C_0^1(\overline{\Omega})$ and it maintain $v \geq \underline{u}$, that is $v = \inf \mathcal{C}$. The Kuratowski-Zorn Lemma (see Lemma 11) completes the proof. □

The following results will help us to obtain a suitable geometry for the fixed point map.

Proposition 13. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5), and (H_3^5) hold. Then, the function $\Gamma : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ such that*

$$\Gamma(\alpha) = u_\alpha^* \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k})$$

is well defined and continuous.

Proof. First, notice that, by the uniqueness of u_α^*, the function Γ is well defined. Moreover, Γ is a selection of $\mathscr{S}$. Let $\alpha_n \in (\alpha_{2k-1}, \alpha_{2k})$ such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$. For each $n \in \mathbb{N}$, one has that $u_n^* = \Gamma(\alpha_n)$ solves

$$\begin{cases} -a(\alpha_n)\Delta_p u_n^* = \widehat{f}(x, u_n^*) & \text{in } \Omega, \\ u_n^* > 0 & \text{in } \Omega, \\ u_n^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.20}$$

Arguing as in Proposition 9, up to subsequence, $u_n^* \to u$ in $C_0^1(\overline{\Omega})$ and $u \in \mathscr{S}(\alpha)$. By Lemma 10, there exists $\{v_n\} \subseteq C_0^1(\overline{\Omega})$ such that $v_n \in \mathscr{S}(\alpha_n)$ for all $n \in \mathbb{N}$ and $v_n \to \Gamma(\alpha) \in \mathscr{S}(\alpha)$. By minimality of Γ, we infer

$$\Gamma(\alpha) = u_\alpha^* \leq \lim_{n \to +\infty} u_n^* = \lim_{n \to +\infty} \Gamma(\alpha_n) \leq \lim_{n \to +\infty} v_n = \Gamma(\alpha).$$

Thus, $\lim\limits_{n \to +\infty} \Gamma(\alpha_n) = \Gamma(\alpha)$. □

Therefore, the use of set-valued methods allowed us to remove any monotonicity assumption involving the reaction term (as in (1.2)), but maintain the existence of the unique smallest solution for the frozen problem. Thus, we can define the map $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ such that

$$\mathcal{P}_k(\alpha) = \|\Gamma(\alpha)\|_q^q \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}).$$

The final part of this section is devoted to realize a suitable geometry to guarantee the existence of at least two fixed points for each positive bump of the weight function a. As in Chapter 3, we have that each fixed point of this map is a solution for problem (4.1). Indeed,

$$-a\left(\int_\Omega (u_\alpha^*)^q dx\right) \Delta_p u_\alpha^* = -a(\alpha)\Delta_p u_\alpha^* = f(x, u_\alpha^*) \text{ in } \Omega. \tag{4.21}$$

Let us start proving that $\mathcal{P}_k$ is a continuous map.

Proposition 14. *Assume that (H_a^2), (H_f^5), (H_1^5), (H_2^5), and (H_3^5) hold. Then the map $\mathcal{P}_k$ is continuous.*

Proof. Let $\alpha_n \in (\alpha_{2k-1}, \alpha_{2k})$ be a sequence such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$. Since $\Gamma(\alpha_n) = u_n^*$ is a solution of (4.11) for each $n \in \mathbb{N}$, and Γ is a continuous selection for $\mathscr{S}$ (see Proposition 13), by (4.12), we have $u_n \in [\underline{u}, \overline{u}]$. Lebesgue Dominate Convergence Theorem (see, e.g., Brézis [9, Theorem 4.2]) ensures that

$$\lim_{n \to +\infty} \mathcal{P}_k(\alpha_n) = \lim_{n \to +\infty} \|\Gamma(\alpha_n)\|_q^q = \mathcal{P}_k(\alpha),$$

and the proof follows. □

Next, we prove that, if α approach the boundary of $(\alpha_{2k-1}, \alpha_{2k})$, then $\mathcal{P}_k(\alpha) > \alpha$.

Proposition 15. *Assume that* (H_a^2), (H_f^5), (H_1^5), (H_2^5), (H_3^5) *and* (H_5^5) *hold. Then, for each* $k \in \{1, \ldots, K\}$, *the following limits are true:*

$$\lim_{\alpha \to \alpha_{2k-1}^+} \mathcal{P}_k(\alpha) > \alpha_{2k-1}, \text{ and } \lim_{\alpha \to \alpha_{2k}^-} \mathcal{P}_k(\alpha) > \alpha_{2k}. \tag{4.22}$$

Proof. First, we show that $t(\alpha) \to t_*$ as α goes to the boundary of $(\alpha_{2k-1}, \alpha_{2k})$. Let $\Omega_0 \subset \Omega$, $T^*(x, \alpha)$, $\widehat{t}(x, \alpha)$ and $t(\alpha)$, for $x \in \Omega \setminus \Omega_0$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, be as in the proof of Lemma 14. We show that

$$\lim_{\alpha \to \alpha_{2k-1}^+} t(\alpha) = \lim_{\alpha \to \alpha_{2k}^-} t(\alpha) = t_*. \tag{4.23}$$

Assume, by contradiction, that (4.23) is false, then there exist $\widetilde{t} \in (0, t_*)$ and a sequence $\{\beta_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$, with $\beta_n \to \alpha_{2k}$ (the case $\beta_n \to \alpha_{2k-1}$ follows similarly), such that $t(\beta_n) \to \widetilde{t}$ as $n \to +\infty$. By (4.6) and (4.7), for every $n, h \in \mathbb{N}^+$, there exists $x_h^{(n)} \in \Omega \setminus \Omega_0$ such that

$$t(\beta_n) \leq \widehat{t}(x_h^{(n)}, \beta_n) \leq t(\beta_n) + \frac{1}{h}.$$

Thus, $\lim_{n \to +\infty} \widehat{t}(x_n^{(n)}, \beta_n) = \widetilde{t}$, and there exists a compact interval $J \subset (0, t_*)$ such that $\widehat{t}(x_n^{(n)}, \beta_n) \in J$ for all $n \in \mathbb{N}^+$. Now, using (H_3^5), for each $n \in \mathbb{N}^+$, we have

$$\widehat{t}^{p-1}(x_n^{(n)}, \beta_n) \lambda_{1,p} a(\beta_n) = f(x_n^{(n)}, \widehat{t}(x_n^{(n)}, \beta_n)) \geq \operatorname*{ess\,inf}_{x \in \Omega} f(x, \widehat{t}(x_n^{(n)}, \beta_n)) \geq \nu_J.$$

Passing to the limits in the previous inequality, we obtain

$$\nu_J > 0 = \lambda_{1,p} \widetilde{t}^{p-1} a(\alpha_{2k}) = \lim_{n \to +\infty} \lambda_{1,p} \widehat{t}^{p-1}(x_n^{(n)}, \beta_n) a(\beta_n) \geq \nu_J$$

a contradiction. This proves (4.23).

Now, by Lemma 14, for all $k \in \{1, ..., K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, we have

$$\mathcal{P}_k(\alpha) = \int_\Omega (u_\alpha^*)^q dx \geq \int_\Omega \underline{u}^q dx = t(\alpha)^q \int_\Omega e_1^q dx.$$

Thus, passing to the limits and using (H_5^5) and (4.23), we have

$$\lim_{\alpha \to \alpha_{2k-1}^+} \mathcal{P}_k(\alpha) \geq \lim_{\alpha \to \alpha_{2k-1}^+} t(\alpha)^q \int_\Omega e_1^q dx = t_*^q \int_\Omega e_1^q dx > \alpha_{2K} \geq \alpha_{2k-1}, \tag{4.24}$$

and

$$\lim_{\alpha \to \alpha_{2k}^-} \mathcal{P}_k(\alpha) \geq \lim_{\alpha \to \alpha_{2k}^-} t(\alpha)^q \int_\Omega e_1^q dx = t_*^q \int_\Omega e_1^q dx > \alpha_{2K} \geq \alpha_{2k}. \tag{4.25}$$

□

Finally, we provide the existence of at least one point $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$, for each $k \in \{1, \ldots, K\}$, such that the fixed point map is low enough.

Proposition 16. *Assume that* (H_a^2), (H_f^5), (H_1^5), (H_2^5), (H_3^5) *and* (H_4^5) *hold. Then, for all* $k \in \{1, \dots, K\}$, *there exists* $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ *such that* $\mathcal{P}_k(\tau_k) < \tau_k$.

Proof. Fix $k \in \{1, \dots, K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, and consider the following auxiliary problem

$$\begin{cases} -\Delta_p w = (u_\alpha^*)^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.26}$$

where u_α^* is the unique smallest positive solution obtained in Proposition 12. Minty-Browder's Theorem (see Theorem 5) ensures that there exists a unique positive solution w of (4.26). Moreover, by variational formulation of w, i.e.,

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla v dx = \int_\Omega (u_\alpha^*)^{q-1} v dx \quad \forall v \in W_0^{1,p}(\Omega), \tag{4.27}$$

choosing $v = u_\alpha^*$ as test function in (4.27) and by Hölder's inequality, we have

$$\begin{aligned} \mathcal{P}_k(\alpha) &= \int_\Omega (u_\alpha^*)^q dx = \int_\Omega |\nabla w|^{p-2} \nabla w \nabla u_\alpha^* dx \leq \int_\Omega |\nabla w|^{p-1} |\nabla u_\alpha^*| dx \\ &\leq \left(\int_\Omega |\nabla w|^{(p-1)p'} dx \right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla u_\alpha^*|^p dx \right)^{\frac{1}{p}} = \|w\|^{p-1} \|u_\alpha^*\|. \end{aligned} \tag{4.28}$$

Now, since u_α^* is a solution of (4.11), we infer

$$a(\alpha) \int_\Omega |\nabla u_\alpha^*|^p dx = \int_\Omega \widehat{f}(x, u_\alpha^*) u_\alpha^* dx.$$

Thus,

$$\|u_\alpha^*\| \leq \left(\frac{|\Omega|}{a(\alpha)} \operatorname*{ess\,sup}_{(x,t) \in \Omega \times [0, t_*]} [t f(x,t)] \right)^{\frac{1}{p}}. \tag{4.29}$$

On the other hands, since w solves (4.26) and every solution of (4.11) lies in $[\underline{u}, \overline{u}]$ (see Proposition 8 and (4.12)), by Hölder's inequality and the variational characterization of the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ (see Motreanu-Motreanu-Papageorgiou [104, Chapter 9]), choosing $v = w$ in (4.27), we have

$$\begin{aligned} \|w\|^p &= \int_\Omega |\nabla w|^p dx = \int_\Omega (u_\alpha^*)^{q-1} w dx \leq \left(\int_\Omega (u_\alpha^*)^{(q-1)p'} dx \right)^{\frac{1}{p'}} \left(\int_\Omega w^p dx \right)^{\frac{1}{p}} \\ &\leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \|w\|_p \leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \lambda_{1,p}^{-\frac{1}{p}} \|w\|. \end{aligned}$$

Thus, we get

$$\|w\|^{p-1} \leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \lambda_{1,p}^{-\frac{1}{p}}. \tag{4.30}$$

Now, combining (4.28), (4.29) and (4.30), we obtain

$$\mathcal{P}_k(\alpha) \leq |\Omega| t_*^{q-1} \left(\frac{1}{\lambda_{1,p} a(\alpha)} \operatorname*{ess\,sup}_{(x,t) \in \Omega \times [0, t_*]} [t f(x,t)] \right)^{\frac{1}{p}}.$$

Assumption (H_4^5) ensures that there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$\tau_k^p a(\tau_k) > \frac{|\Omega|^p}{\lambda_{1,p}} t_*^{p(q-1)} \operatorname*{ess\,sup}_{(x,t) \in \Omega \times [0, t_*]} [t f(x,t)],$$

that is,

$$\mathcal{P}_k(\tau_k) < \tau_k$$

and the proof is complete. □

Then, we are ready to prove our main result for p-Laplacian problem.

Theorem 14. *Assume that* (H_a^2), (H_f^5), (H_1^5), (H_2^5), (H_3^5), (H_4^5) *and* (H_5^5) *hold. Then, the problem* (4.1) *admits at least* K *pairs of positive solutions ordered in* L^q*-norm.*

Proof. First, observe that, by (4.21), it is enough to prove that the fixed point map $\mathcal{P}_k$ admits at least two fixed points for each positive bump of the weight function a. Proposition 14 guarantees us that the map is continuous, and by Proposition 15 and Proposition 16, we realize a suitable geometry (see again Figure 3.8) for the fixed-point map. Then, there exist K pairs $(u_{k,1}, u_{k,2})$ of positive solutions of (4.1). Moreover, by construction of the map $\mathcal{P}_k$, we obtain

$$\alpha_{2k-1} < \int_\Omega u_{k,1}^q dx < \int_\Omega u_{k,2}^q dx < \alpha_{2k} \quad \forall k \in \{1, \dots, K\}.$$

□

We present an example of application of the previous theorem. For the sake of simplicity we drop the x-dependence.

Example 1. *Set* $N = 2$, $\Omega = B(0,1)$ *be the ball of radius one centred in zero,* $K = 2$, $p = 4$, $q = 1$, *and*

$$\alpha_0 = \alpha_1 = 0 < \alpha_2 = 2 < \alpha_3 = 4 < \alpha_4 = 6.$$

Moreover, let

$$f(t) = c - t - \sin^2(2ct)$$

being $c > 1 + \frac{6}{\|e_1\|_1}$, *see Figure 4.3 below. Notice that*

$$\lim_{t \to 0^+} \frac{f(t)}{t^3} = +\infty.$$

Finally, consider

$$p(t) = t^2(t-2)(t-4)(6-t), \quad a(t) = Ap(t),$$

with $A > \frac{\pi^4}{72\lambda_{1,p}} c^2$. *Then, Theorem 14 ensures that problem*

$$\begin{cases} -a\left(\int_\Omega u dx\right) \Delta_4 u = c - u - \sin^2(2cu) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{E}$$

possesses at least four positive solutions, u_1, u_2, u_3, u_4, *ordered as follows*

$$0 < \int_\Omega u_1 dx < \int_\Omega u_2 dx < 2 < 4 < \int_\Omega u_3 dx < \int_\Omega u_4 dx < 6.$$

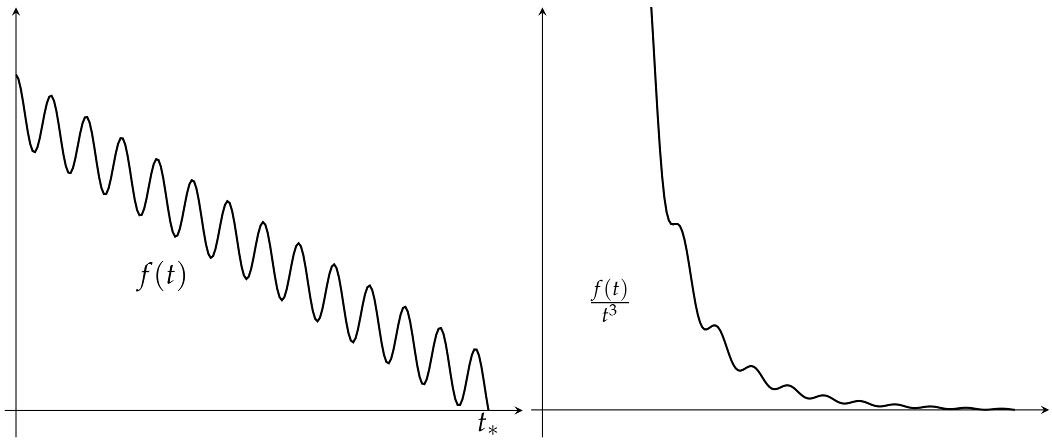

FIGURE 4.3: Possible plot for the functions $f(t)$ and $\frac{f(t)}{t^3}$ in problem (E)

Finally, we would highlight two different assumptions that allow us to build the same geometry for the fixed point map. Consider the following number

$$R = \max_{x\in\overline{\Omega}} d(x, \partial\Omega)$$

being $d : \overline{\Omega} \to \mathbb{R}$ the distance function

$$d(x) = \min_{y\in\partial\Omega} |x - y| \quad \forall x \in \Omega.$$

Assume that

$(H_3^5)'$ for each $k \in \{1, ..., K\}$,

$$\liminf_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} > \frac{2^p(2^N - 1) \max\limits_{\alpha\in[\alpha_{2k-1},\alpha_{2k}]} a(\alpha)}{R^p \left[1 + 2^{N+p}\left(\beta(N, p+1) - \beta_{\frac{1}{2}}(N, p+1)\right)\right]}$$

uniformly w.r.t. $x \in \Omega$ and where β and $\beta_{\frac{1}{2}}$ are the Euler's beta function

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y+1)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

with Γ the Euler's Gamma function, and

$$\beta_{\frac{1}{2}}(x,y) = \int_0^{\frac{1}{2}} t^{x-1}(1-t)^{y-1}dt$$

the incomplete beta function, respectively, see Arkfen-Harris-Weber [4, Chapter 13].

Then, the following holds.

Lemma 16. *$(H_3^5)'$ implies (H_3^5).*

Proof. By the variational characterization of $\lambda_{1,p}$, see Motreanu-Motreanu-Papageorgiou [104, Chapter 9], one has

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|^p_{L^p(\Omega)}}{\|u\|^p_{L^p(\Omega)}}. \tag{4.31}$$

Let $x_0 \in \Omega$ be such that $d(x_0, \partial\Omega) = R$, consider $B(x_0, R) \subseteq \Omega$, fix $\eta > 0$, and put

$$\phi(x) = \begin{cases} 0 & \text{if } \ x \in \Omega \setminus B(x_0, R), \\ \frac{2\eta}{R}(R - |x - x_0|) & \text{if } \ x \in B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right), \\ \eta & \text{if } \ x \in B\left(x_0, \frac{R}{2}\right). \end{cases} \tag{4.32}$$

Clearly, $\phi \in W_0^{1,p}(\Omega)$ and its derivative is

$$\nabla\phi(x) = \begin{cases} 0 & \text{if } \ x \in \Omega \setminus B(x_0, R), \\ \frac{2\eta}{R} & \text{if } \ x \in B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right), \\ 0 & \text{if } \ x \in B\left(x_0, \frac{R}{2}\right). \end{cases} \tag{4.33}$$

One can compute that

$$\begin{aligned} \|\phi\|^p_{W_0^{1,p}(\Omega)} &= \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left|\frac{2\eta}{R}\right|^p dx = \left(\frac{2\eta}{R}\right)^p \left|B(x_0, R) \setminus B\left(x_0, \frac{R}{2}\right)\right| \\ &= \left(\frac{2\eta}{R}\right)^p \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)} R^N \left(1 - \frac{1}{2^N}\right) \\ &= \frac{2^p \eta^p \pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)} \left(1 - \frac{1}{2^N}\right) R^{N-p}. \end{aligned} \tag{4.34}$$

Moreover, one has

$$\begin{aligned} \|\phi\|^p_{L^p(\Omega)} &= \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} \left|\frac{2\eta}{R}(R - |x - x_0|)\right|^p dx + \int_{B\left(x_0,\frac{R}{2}\right)} \eta^p dx \\ &= \frac{\eta^p \pi^{\frac{N}{2}}}{\Gamma\left(1+\frac{N}{2}\right)} \left(\frac{R}{2}\right)^N + \left(\frac{2\eta}{R}\right)^p \int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} |R - |x - x_0||^p \, dx. \end{aligned}$$

We calculate the following

$$\int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} |R - |x - x_0||^p \, dx.$$

Passing to the N-dimensional spherical coordinates (see Adams-Fournier [1, p. 121]), i.e., let $r = |x - x_0|$, where the determinant of the Jacobian matrix is

$$\det(J) = r^{N-1} \sin^{N-2}(\theta_1) \sin^{N-3}(\theta_2) ... \sin(\theta_{N-2}) dr d\theta_1 d\theta_2 ... d\theta_{N-1},$$

being $\theta_1, \ldots, \theta_{N-2}$ in $[0, \pi[$ and θ_{N-1} in $[0, 2\pi[$, one has,

$$\int_{B(x_0,R)\setminus B\left(x_0,\frac{R}{2}\right)} |R - |x - x_0||^p \, dx$$

$$= \int_{\frac{R}{2}}^{R} \int_0^{\pi} \int_0^{\pi} \cdots \int_0^{2\pi} (R-r)^p r^{N-1} \sin^{N-2}(\theta_1) \sin^{N-3}(\theta_2) \cdots \sin(\theta_{N-2}) dr d\theta_1 d\theta_2 \ldots d\theta_{N-1}$$

$$= \int_{\frac{R}{2}}^{R} (R-r)^p r^{N-1} dr \int_0^{\pi} \int_0^{\pi} \cdots \int_0^{2\pi} \sin^{N-2}(\theta_1) \sin^{N-3}(\theta_2) \cdots \sin(\theta_{N-2}) d\theta_1 d\theta_2 \ldots d\theta_{N-1}.$$

Notice that,

$$\int_0^{\pi} \int_0^{\pi} \cdots \int_0^{2\pi} \sin^{N-2}(\theta_1) \sin^{N-3}(\theta_2) \cdots \sin(\theta_{N-2}) d\theta_1 d\theta_2 \ldots d\theta_{N-1} = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)},$$

that is, the volume of the ball of radius one. Furthermore,

$$\int_{\frac{R}{2}}^{R} (R-r)^p r^{N-1} dr = R^{N+p} \left(\beta(N, p+1) - \beta_{\frac{1}{2}}(N, p+1) \right).$$

Indeed,

$$\int_{\frac{R}{2}}^{R} (R-r)^p r^{N-1} dr = \int_0^{R} (R-r)^p r^{N-1} dr - \int_0^{\frac{R}{2}} (R-r)^p r^{N-1} dr.$$

In the first integral, by substitution $r = Rt$, one has

$$\int_0^{R} (R-r)^p r^{N-1} dr = \int_0^{1} (R - Rt)^p R^{N-1} t^{N-1} R dt$$

$$= R^{N+p} \int_0^{1} (1-t)^p t^{N-1} dt = R^{N+p} \beta(N, p+1).$$

Again, for $r = Rt$, in the second integral, one has

$$\int_0^{\frac{R}{2}} (R-r)^p r^{N-1} dr = \int_0^{\frac{1}{2}} (R - Rt)^p R^{N-1} t^{N-1} R dt$$

$$= R^{N+p} \int_0^{\frac{1}{2}} (1-t)^p t^{N-1} dt = R^{N+p} \beta_{\frac{1}{2}}(N, p+1).$$

Then,

$$\int_{\frac{R}{2}}^{R} (R-r)^p r^{N-1} dr = R^{N+p} \left(\beta(N, p+1) - \beta_{\frac{1}{2}}(N, p+1) \right).$$

Putting all together, we infer

$$\|\phi\|^p_{L^p(\Omega)} = \frac{\eta^p \pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} \left(\frac{R}{2}\right)^N + \left(\frac{2\eta}{R}\right)^p \frac{\pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} R^{N+p} \left(\beta(N, p+1) - \beta_{\frac{1}{2}}(N, p+1) \right),$$

i.e.,

$$\|\phi\|^p_{L^p(\Omega)} = \frac{\eta^p \pi^{\frac{N}{2}}}{\Gamma\left(1 + \frac{N}{2}\right)} \left(\frac{R}{2}\right)^N \left[1 + 2^{N+p} \left(\beta(N, p+1) - \beta_{\frac{1}{2}}(N, p+1) \right) \right]. \tag{4.35}$$

Finally, from (4.34) and (4.35), in view of (4.31), we obtain the following estimate on $\lambda_{1,p}$,

$$\lambda_{1,p} \leq \frac{\|\nabla\phi\|^p_{L^p(\Omega)}}{\|\phi\|^p_{L^p(\Omega)}} = \frac{2^p(2^N-1)}{R^p\left[1+2^{N+p}\left(\beta(N,p+1)-\beta_{\frac{1}{2}}(N,p+1)\right)\right]}.$$

Thus, assumption $(H_3^5)'$ implies (H_3^5). □

Finally, consider the following.

$(H_4^5)'$ For each $k \in \{1, ..., K\}$ there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$\tau_k^{\frac{p-1}{q}} a(\tau_k) > \left(\frac{(N+1)p}{p-1}\right)^{p-1}\left(\frac{2}{N\sqrt{\pi}p^{\frac{1}{p}}}\right)^p \Gamma\left(1+\frac{N}{2}\right)^{\frac{p}{N}} |\Omega|^{\frac{Np+qp-N}{qN}} \operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,t_*]} f(x,t).$$

Theorem 15. *Assume that* (H_a^2), (H_f^5), (H_1^5), (H_2^5), $(H_3^5)'$, $(H_4^5)'$ *and* (H_5^5) *hold. Then, the conclusion of Theorem 14 holds.*

Proof. Clearly, by Lemma 16, assumption $(H_3^5)'$ implies (H_3^5). We point out that, assumption (H_4^5) plays a crucial role to realize the suitable geometry for the fixed point map, in particular, to conclude our theorem, we need to show that assumption $(H_4^5)'$ realizes the geometry condition provided in Proposition 16, i.e., for every $k \in \{1, \ldots, K\}$ there exists $\tilde{\alpha} \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$\mathcal{P}_k(\tilde{\alpha}) < \tilde{\alpha}. \tag{4.36}$$

For this scope, we exploit the comparison principle, see Fan [55, Lemma 2.1]. Fix $k \in \{1, \ldots, K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and consider the following auxiliary problem

$$\begin{cases} -\Delta_p z = \widehat{M} & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.37}$$

being $\widehat{M} = \frac{1}{a(\alpha)} \operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,t_*]} f(x,t)$. This problem admits a unique positive solution $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (see Costa [36, Theorem 1.2] and Gasiński-Papageorgiou [68, Theorem 1.5.5]). Furthermore, we deduce that

$$-\Delta_p u_\alpha^* = \frac{1}{a(\alpha)} f(x, u_\alpha^*) \leq \widehat{M} = -\Delta_p z. \tag{4.38}$$

Then, since $u_\alpha^* = z$ on $\partial\Omega$ because they are solutions of Dirichlet problems, and $-\Delta_p u_\alpha^* \leq -\Delta_p z$, Fan [55, Proposition 2.3] ensures that $u_\alpha^* \leq z$ in Ω. Again by Fan [55, Lemma 2.1], we obtain the following estimate on the L^∞-norm of z,

$$\|z\|_{L^\infty(\Omega)} \leq \widetilde{C}\widehat{M}^{\frac{1}{p-1}}, \tag{4.39}$$

where

$$\widetilde{C} = \frac{(N+1)p}{p-1}\left(\frac{2}{N\sqrt{\pi}p^{\frac{1}{p}}}\right)^{\frac{p}{p-1}}\left(|\Omega|\Gamma\left(1+\frac{N}{2}\right)\right)^{\frac{p}{N(p-1)}}. \tag{4.40}$$

Hence, exploiting the above computations, we get

$$\begin{aligned}
\mathcal{P}_k(\alpha) &= \int_\Omega (u_\alpha^*)^q\,dx \le \|z\|^q_{L^\infty(\Omega)}|\Omega| \le \widetilde{C}^q \widehat{M}^{\frac{q}{p-1}}|\Omega| \\
&= \left(\frac{(N+1)p}{p-1}\right)^q \left(\frac{2}{N\sqrt{\pi}p^{\frac{1}{p}}}\right)^{\frac{qp}{p-1}} \Gamma\left(1+\frac{N}{2}\right)^{\frac{qp}{N(p-1)}} \\
&\quad |\Omega|^{\frac{Np+qp-N}{N(p-1)}} \left(\frac{1}{a(\alpha)} \operatorname*{ess\,sup}_{(x,t)\in\Omega\times[0,t_*]} f(x,t)\right)^{\frac{q}{p-1}}.
\end{aligned}$$

Then, assumption $(H_4^5)'$ produces the geometry in (4.36). The proof follows as in Theorem 14. $\square$

Remark 7. *We stress that the assumptions $(H_3^5)'$ and $(H_4^5)'$ involve only numerical data relating to the geometry of the problem, and the explicit dependence on the first eigenvalue $\lambda_{1,p}$ is not required.*

Remark 8. *We observe that in the number $\widetilde{C}$ in (4.40), appears the best constant of the continuous embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$, which is related to the isoperimetrical inequality on $\mathbb{R}^N$ see (3.28) and Kesavan [87, p. 46].*

4.1.2 Many solutions for p-Laplacian

It is possible to improve Theorem 14 at least in the autonomous case, i.e., $f(x,t) = f(t)$ to obtain a free number of solutions for problem (4.1), we refer to Failla-Gasiński [52] for the main result. In particular, we remove any monotonicity assumption in the results gained in Section 3.1.3. Consider again the following structure, for each $k \in \{1,\dots,K\}$,

$$\alpha_{2k-1} < \bar{s}_{k,0} < s_{k,1} < \bar{s}_{k,1} < s_{k,2} < \ldots < s_{k,m_k} < \bar{s}_{k,m_k} < \alpha_{2k}$$

being $m_k \ge 1$ a switching points counter, see Figure 3.6. As in Section 3.1.3, set

$$\widehat{K} = \sum_{k=1}^{K}(m_k+1).$$

Clearly, in the autonomous case, we can collect the assumptions (H_f^5), (H_1^5), (H_2^5) and (H_3^5) as follows.

$(\widehat{H}_f^5)$ Let $f:\mathbb{R}\to\mathbb{R}$ be a function such that there exists $t_*>0$ such that $f(t_*)=0$, $f\in C([0,t_*])$ and $f(t)>0$ in $(0,t_*)$. Moreover,

$$\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}} > \lambda_{1,p}\max_{k\in\{1\dots,K\}}\max_{\alpha\in[\alpha_{2k-1},\alpha_{2k}]} a(\alpha).$$

Now, we state the following assumptions.

(H_6^5) $s_{k,i} < t(s_{k,i})^q \int_\Omega e_1^q dx \quad \forall k\in\{1,\dots,K\}, i\in\{1,\dots,m_k\}$;

(H_7^5) $a(\bar{s}_{k,i})\bar{s}_{k,i}^p > \frac{|\Omega|^p}{\lambda_{1,p}} t_*^{p(q-1)} \max_{t\in[0,t_*]}[tf(t)] \quad \forall k\in\{1,\dots,K\}, i\in\{1,\dots,m_k\}$, being

$$t(\alpha) = \min \left\{ t \in [0, t_*] : f(t) = \lambda_{1,p} t^{p-1} a(\alpha) \right\}.$$

Then, the following holds.

Theorem 16. *Assume that* (H_a^2), $(\widehat{H}_f^5)$, (H_5^5), (H_6^5) *and* (H_7^5) *hold. Then, the problem*

$$\begin{cases} -a\left(\int_\Omega u^q dx\right) \Delta_p u = f(u) & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.41}$$

possesses at least $2\widehat{K}$ *positive solutions ordered in* L^q*-norm.*

Proof. Clearly, by assumptions (H_a^2), $(\widehat{H}_f^5)$ and (H_5^5) and Proposition 14-Proposition 15 ensure that $\mathcal{P}_k$ is well defined, continuous and the limits on the boundary are great enough.

Next, by (H_6^5), we infer that

$$\mathcal{P}_k(s_{k,i}) > s_{k,i} \quad \forall k \in \{1, ..., K\},\ i \in \{1, ..., m_k\};$$

indeed, since $u^*_{s_{k,i}} \geq \underline{u}$, we obtain that

$$\mathcal{P}_k(s_{k,i}) = \int_\Omega (u^*_{s_{k,i}})^q dx \geq t(s_{k,i})^q \int_\Omega e_1^q dx. \tag{4.42}$$

On the other hand, assumption (H_7^5) permits us to have

$$\mathcal{P}_k(\bar{s}_{k,i}) < \bar{s}_{k,i} \quad \forall k \in \{1, ..., K\},\ i \in \{1, ..., m_k\}.$$

Fix $k \in \{1, ..., K\}$ and $i \in \{1, ..., m_k\}$. For $\bar{s}_{k,i} \in (\alpha_{2k-1}, \alpha_{2k})$ as in Proposition 16, consider the following auxiliary problem

$$\begin{cases} -\Delta_p w = (u^*_{\bar{s}_{k,i}})^{q-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.43}$$

being $u^*_{\bar{s}_{k,i}}$ the unique smallest positive solution of (4.2) such that $u^*_{\bar{s}_{k,i}} \geq \underline{u}$. Clearly, Minty-Browder's Theorem (see Theorem 5) ensures that there exists a unique positive solution w of (4.43). By variational formulation of w, one has

$$\int_\Omega |\nabla w|^{p-2} \nabla w \nabla v dx = \int_\Omega \left(u^*_{\bar{s}_{k,i}}\right)^{q-1} v dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.44}$$

Choosing $v = u^*_{\bar{s}_{k,i}}$ as test function in (4.44) and by Hölder's inequality, we have

$$\begin{aligned} \mathcal{P}_k(\bar{s}_{k,i}) &= \int_\Omega (u^*_{\bar{s}_{k,i}})^q dx = \int_\Omega |\nabla w|^{p-2} \nabla w \nabla u^*_{\bar{s}_{k,i}} dx \leq \int_\Omega |\nabla w|^{p-1} |\nabla u^*_{\bar{s}_{k,i}}| dx \\ &\leq \left(\int_\Omega |\nabla w|^{(p-1)p'} dx\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla u^*_{\bar{s}_{k,i}}|^p dx\right)^{\frac{1}{p}} = \|w\|^{p-1} \|u^*_{\bar{s}_{k,i}}\|. \end{aligned} \tag{4.45}$$

Since w is a solution of (4.43), Hölder's inequality and the variational characterization of the first eigenvalue of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$, ensure that choosing $v = w$ in

(4.44), one gets

$$\begin{aligned}\|w\|^p &= \int_\Omega |\nabla w|^p dx = \int_\Omega (u^*_{\bar{s}_{k,i}})^{q-1} w dx \leq \left(\int_\Omega (u^*_\alpha)^{(q-1)p'} dx\right)^{\frac{1}{p'}} \left(\int_\Omega w^p dx\right)^{\frac{1}{p}} \\ &\leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \|w\|_p \leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \lambda_{1,p}^{-\frac{1}{p}} \|w\|.\end{aligned}$$

Thus, we infer

$$\|w\|^{p-1} \leq |\Omega|^{\frac{1}{p'}} t_*^{q-1} \lambda_{1,p}^{-\frac{1}{p}}. \tag{4.46}$$

On the other hand, $u^*_{\bar{s}_{k,i}}$ solves (4.2), then

$$a(\bar{s}_{k,i}) \int_\Omega |\nabla u^*_{\bar{s}_{k,i}}|^p dx = \int_\Omega f(u^*_{\bar{s}_{k,i}}) u^*_{\bar{s}_{k,i}} dx,$$

thus,

$$\|u^*_{\bar{s}_{k,i}}\| \leq \left(\frac{|\Omega|}{a(\bar{s}_{k,i})} \max_{t\in[0,t_*]} [t f(t)]\right)^{\frac{1}{p}}. \tag{4.47}$$

Combining (4.46) and (4.47) in (4.45), we conclude that

$$\mathcal{P}_k(\bar{s}_{k,i}) \leq |\Omega| t_*^{q-1} \left(\frac{1}{\lambda_{1,p} a(\bar{s}_{k,i})} \max_{t\in[0,t_*]} [t f(t)]\right)^{\frac{1}{p}}.$$

The proof follows as in Theorem 14 (see also Theorem 10 and Figure 3.7). □

4.1.3 Another nonexistence result for p-Laplacian problems

We present a third nonexistence result, see Failla-Gasiński-Santos Júnior [54]. In contrast to the theorems proved in Section 3.1.2, here we do not require any monotonicity assumption involving the reaction term but we remove the zero of the nonlinearity. On the weight function a we assume the hypothesis (H_a^1). In particular, let us start to consider the torsion problem

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.48}$$

By Minty-Browder's Theorem (see Theorem 5) and classical regularity theory (Lieberman [92, Theorem 1]), there exists a unique positive solution $\varphi_t \in C_0^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$ for the problem (4.48). Then, our theorem reads as follows.

Theorem 17. *Assume that (H_a^1) holds. Moreover, suppose that*

(H_8^5) *there exists some constant $\widetilde{\psi} > 0$ such that $f(t) \geq \widetilde{\psi}$ for all $t \geq 0$;*

(H_9^5) $\alpha^{\frac{p-1}{q}} a(\alpha) < \widetilde{\psi} \|\varphi_t\|_q^{p-1}$ *for all $\alpha \in (\alpha_0, \alpha_K)$.*

Then, the problem (3.3) does not admits any positive solution.

Proof. First, we point out that, by (H_8^5), problem (3.3) does not possesses solution u such that $\int_\Omega u^q dx = \alpha_k$ with α_k zero of the weight function a. In fact, if it is not true, we get

$$0 = a\left(\int_\Omega u^q dx\right) \Delta_p u = f(u) \geq \widetilde{\psi} > 0$$

a contradiction. Let us assume that u be a solution of (3.3) such that $\int_\Omega u^q dx \neq \alpha_k$. A straightforward computation shows that

$$-a\left(\int_\Omega u^q dx\right)\Delta_p u = f(u) \geq \widetilde{\psi} = \widetilde{\psi}\cdot 1 = -\widetilde{\psi}\Delta_p\varphi_t = -\Delta_p\left(\widetilde{\psi}^{\frac{1}{p-1}}\varphi_t\right).$$

On the other hand, since u is a solution of (3.3), one has

$$-a\left(\int_\Omega u^q dx\right)\Delta_p u = -\Delta_p\left(a\left(\int_\Omega u^q dx\right)^{\frac{1}{p-1}} u\right).$$

Combining the above computations, we infer

$$-\Delta_p\left(a\left(\int_\Omega u^q dx\right)^{\frac{1}{p-1}} u\right) \geq -\Delta_p\left(\widetilde{\psi}^{\frac{1}{p-1}}\varphi_t\right)$$

that we can read as

$$a\left(\int_\Omega u^q dx\right)^{\frac{1}{p-1}} u \geq \widetilde{\psi}^{\frac{1}{p-1}}\varphi_t.$$

Finally, elevating to the q power and integrating on Ω, follows that

$$a\left(\int_\Omega u^q dx\right)^{\frac{q}{p-1}}\int_\Omega u^q dx \geq \widetilde{\psi}^{\frac{q}{p-1}}\|\varphi_t\|_q^q.$$

Clear in contrast with (H_9^5). $\square$

Let us conclude this section with a computable example of nonexistence.

Example 2. *Let us assume that $\Omega = B(0,R) \subset \mathbb{R}^N$ be the ball centered in zero with radius R and $\partial\Omega = S^{N-1}(0)$ be its boundary. Consider the torsion problem*

$$\begin{cases} -\Delta_p u = 1 & \text{in } B(0,R), \\ u = 0 & \text{on } S^{N-1}(0). \end{cases} \tag{4.49}$$

It is well known (see, e.g., Bueno-Ercole [11]) that, for all $p > 1$, problem (4.49) admits a unique positive (radial) solution

$$\varphi_t(r) = \frac{p-1}{p}N^{\frac{1}{p-1}}\left(R^{\frac{p}{p-1}} - r^{\frac{p}{p-1}}\right),$$

being $r = |x| \leq R$, $\varphi_t \in C^{1,\nu}(\overline{\Omega})$ for some $\nu \in (0,1)$.

Fix $p = 4, q = 1$, $N = 3$, $R = 1$, $K = 2$ and $\alpha_0 = 0 < \alpha_1 = 1 < \alpha_2 = 2$. Consider $a(t) = \sin^2(\pi t)$ and $f(t) = \cosh(t) + \widetilde{\psi}$, being

$$\widetilde{\psi} > \frac{64}{81}\frac{\max\limits_{t\in[0,2]}\left[t^3\sin^2(\pi t)\right]}{\left(\int\limits_{B(0,1)}\left(1-|x|^{\frac{3}{4}}\right)^3 dx\right)^{\frac{1}{3}}} = \frac{64}{81}\sqrt[3]{\frac{4\pi}{105}}\max_{t\in[0,2]}\left[t^3\sin^2(\pi t)\right] \approx 1.45.$$

Then, by Theorem 17, the problem

$$\begin{cases} -\sin^2\left(\pi \int\limits_{B(0,1)} udx\right)\Delta_4 u = \cosh(u) + \widetilde{\psi} & \textit{in } B(0,1), \\ \qquad\qquad\qquad u > 0 & \textit{in } B(0,1), \\ \qquad\qquad\qquad u = 0 & \textit{on } S^2(0), \end{cases} \tag{4.50}$$

does not admits any positive solution.

4.1.4 The Double-phase case

In this last section we prove a multiplicity result for a double phase nonlocal Carrier problem. The main theorem have been proven in Failla-Gasiński [53]. We highlight that for the best of our knowledge, this is the first result for Carrier's type problem driven by such a operator in Musielak-Orlicz space. In this new setting a lot of issues arise. In particular, we point out that the double phase operator is a non homogeneous operator, there are no regularity results in term of global Hölder regularity and only few results on eigenvalue problem are available. A careful reading of the previous section shows that, at least for p-Laplacian problem, the above mentioned ingredients are fundamental. This lack of regularity for the double phase operator, force us to find new proofs paths. On the other hand, in recent years, the double phase operator has been the subject of numerous publications. This wide production is also due to the large range of problems modelled by this operator. We refer the interested readers to the works of Bahrouni-Rădulescu-Repovš [5] for transonic flow models, Benci-D'Avenia-Fortunato-Pisani [7] that provide applications in quantum physics field, Bonheure-D'Avenia-Pomponio [8] for investigation on the Born-Infeld equation in electromagnetism, image restoration and denoising have been studied in Charkaoui-Ben Loghfyry-Zeng [23] and Harjulehto-Hästö [79], Cherfils-Il'yasov [24] that produced models on reaction-diffusion system, and the seminal works of Marcellini [96, 97] and Zhikov [127, 128, 129] that dealing with elasticity theory to describe the behaviour of composite materials made of two different materials under deformation. See also the two recent surveys due to Mingione-Rădulescu [102], and Papageorgiou [110].

The main problem in this section is the following: consider the double phase operator

$$\mathcal{G}(u) = \operatorname{div}\left(\mu(x)|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u\right) \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega)$$

and the elliptic nonlocal degenerate problem

$$\begin{cases} -a\left(\int_\Omega u^r + \nu(x)u^s dx\right)\mathcal{G}(u) = f(u) & \text{in } \Omega, \\ \qquad\qquad u > 0 & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.51}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with Lipschitz boundary $\partial\Omega$. First, we assume the following:

$(H_\mathcal{G})$ let $1 < q < N$, $q < p$, $\frac{p}{q} < 1 + \frac{1}{N}$, $r, s \geq 1$, $\mu : \Omega \to [0, +\infty)$, with $\mu \in C^{0,1}(\overline{\Omega})$ and $\nu : \Omega \to [0, +\infty)$, $\nu \in C(\Omega)$.

Notice that the weight function μ is not separated from zero, this issue force the setting space in Sobolev-Musielak-Orlicz spaces, we refer the interest reader to the book of Harjulehto-Hästö [80] and the recent paper Crespo Blanco-Gasiński-Harjulehto-Winkert [38] as well as Section 2.1.3 of Chapter 2 in this thesis for an overview on this setting space and on this class of operators. Moreover, we highlight that assumption $(H_{\mathcal{G}})$ implies that $q < p < q^*$ being $q^* = \frac{Nq}{N-q}$ the critical Sobolev exponent (see (2.1)), see, e.g., Gasiński-Papageorgiou [66, Remark 2.1].

Again, we maintain assumption (H_a^2) on the nonlocal sign-changing weight function a, and we state our assumptions on the reaction term: let $f : \mathbb{R} \to \mathbb{R}$ be a function fulfilling the following.

(H_f^6) There exists $t_* > 0$ such that $f(t_*) = 0$, $f \in C([0, t_*])$ and $f(t) > 0$ for all $t \in (0, t_*)$.

(H_1^6) Let

$$\lim_{t \to 0^+} f(t) > \max_{k \in \{1 \dots, K\}} \max_{\alpha \in [\alpha_{2k-1}, \alpha_{2k}]} a(\alpha).$$

Next, due to the absence of eigenvalue results on this type of problems (see Colasuonno - Squassina [32] for details), we need to build a sub-solution for our problem in a new way. Let us consider the torsion problem

$$\begin{cases} -\mathcal{G}(u) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{T}$$

Clearly, problem (T) admits a unique positive solution (see Theorem 5) $\varphi_t \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ (see Ho-Winkert [81, Theorem 4.2] and Crespo Blanco-Winkert [40, Theorem 3.1]). The following assumptions ensure the correct correlation between the weight function a and the reaction term f.

(H_2^6) Assume that $t_* \leq \|\varphi_t\|_\infty$ and $\alpha_{2K} < t_*^r \|e_t\|_r^r + t_*^s \|e_t\|_s^s \min_{\Omega} \nu$, being $e_t = \frac{\varphi_t}{\|\varphi_t\|_\infty}$;

(H_3^6) for each $k \in \{1, \dots, K\}$ there exists $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ such that

$$\tau_k^q a(\tau_k) > \frac{|\Omega|^q}{\lambda_{1,q}} \left(t_*^{r-1} + t_*^{s-1} \max_{\Omega} \nu \right)^q \max_{t \in [0, t_*]} [t f(t)],$$

where $\lambda_{1,q}$ is the first eigenvalue of $\left(-\Delta_q, W_0^{1,q}(\Omega) \right)$.

The mathematical issues for double phase problem can be summarized as follows: on one hand, building a sub-solution for an appropriate frozen double phase problem is challenging, as we have lost both the homogeneity of the operator and an appropriate variational formulation of the first eigenvalue. On the other hand, the lack of global regularity results for this type of operators, makes it more difficult to prove the existence of the smallest solution.

Let us start to recover the existence of a suitable sub-solution for the frozen nonlocal counterpart of (4.51), i.e.,

$$\begin{cases} -a(\alpha)\,\mathcal{G}(u) = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.52}$$

Lemma 17. *Assume that* (H_a^2), $(H_{\mathcal{G}})$, (H_f^6), (H_1^6) *and* (H_2^6) *hold. Then,* (4.52) *admits a positive sub-solution* $\underline{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.

Proof. First, let us consider the Torsion problem (T). Let $\varphi_t \in W_0^{1,\mathcal{H}}(\overline{\Omega}) \cap L^\infty(\Omega)$ be its unique positive solution and consider the set

$$t(\alpha) = \min\left\{t \in [0, t_*] : f(t) = a(\alpha)\right\}. \tag{4.53}$$

Put $\underline{u} = t(\alpha)e_t$ being $e_t = \frac{\varphi_t}{\|\varphi_t\|_\infty}$. The structure of $t(\alpha)$ allows us to infer that $f(t(\alpha)) \leq f(t)$ for all $t \in (0, t(\alpha))$. Moreover, by (H_2^6), we have $t(\alpha) \leq t_* \leq \|\varphi_t\|_\infty$. that is, $\underline{u} = t(\alpha)e_t \leq t(\alpha)$. Thus, we obtain

$$\begin{aligned} -a(\alpha)\mathcal{G}(\underline{u}) &\leq -a(\alpha) \min\left\{\left(\frac{t(\alpha)}{\|\varphi_t\|_\infty}\right)^{p-1}; \left(\frac{t(\alpha)}{\|\varphi_t\|_\infty}\right)^{q-1}\right\}\mathcal{G}(\varphi_t) \\ &\leq \quad a(\alpha) = f(t(\alpha)) \leq f(\underline{u}). \end{aligned}$$

Thus, $\underline{u} \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ is a sub-solution of (4.52). □

Remark 9. *Arguing as in Proposition 15, we can show that* $t(\alpha) \to t_*$ *as* $\alpha \to \alpha_{2k-1}^+$ *and* $\alpha \to \alpha_{2k}^-$. *Moreover, the pair* $(\underline{u}, \overline{u})$ *being* $\underline{u}$ *as in the previous Lemma and* $\overline{u} = t_*$, *represent a well ordered pair of sub-super solutions.*

By the use of the above remark we can consider the following truncation on the reaction term

$$\widehat{f}(x,s) = \begin{cases} f(\underline{u}(x)) & \text{if} \quad s < \underline{u}(x); \\ f(s) & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x); \\ f(\overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{4.54}$$

To build the fixed point map, we require the existence of the unique smallest solution for the frozen truncated problem

$$\begin{cases} -a\,(\alpha)\,\mathcal{G}(u) = \widehat{f}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.55}$$

A classical argument shows that the associated energy functional $J : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ defined as

$$J(u) = \frac{a(\alpha)}{p}\|\nabla u\|_{p,\mu}^p + \frac{a(\alpha)}{q}\|\nabla u\|_q^q - \int_\Omega F(x,u)dx \quad \forall u \in W_0^{1,\mathcal{H}}(\Omega),$$

being $F(x,t) = \displaystyle\int_0^t \widehat{f}(x,s)ds$, possesses a global minimum (see argument in Proposition 4). Thus, the set of solutions of the problem (4.55) is not empty and its solutions u_α lies in $[\underline{u}, \overline{u}]$. Furthermore, arguing as for problem (T) above, we get $u_\alpha \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$. Such considerations permit us to well define the following set-valued map $\mathscr{S} : (\alpha_{2k-1}, \alpha_{2k}) \to 2^{W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)}$ such that

$$\mathscr{S}(\alpha) = \left\{u \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega) : u \text{ solves } (4.55)\right\} \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}),$$

with

$$\mathscr{S}(\alpha) \subseteq [\underline{u}, \overline{u}].$$

The lack of $C_0^{1,\nu}$-regularity does not allow us to argue as in Proposition 9. Therefore, we prove the following.

Proposition 17. *Assume that* (H_a^2), $(H_{\mathcal{G}})$, (H_f^6), (H_1^6) *and* (H_2^6) *hold. Then the map* $\mathscr{S}$ *maps convergent sequences in* $(\alpha_{2k-1}, \alpha_{2k})$ *into sequences with convergent subsequences in* $W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $\{\alpha_n\} \subset (\alpha_{2k-1}, \alpha_{2k})$ be a sequence such that $\alpha_n \to \alpha \in (\alpha_{2k-1}, \alpha_{2k})$ and $u_n \in \mathscr{S}(\alpha_n)$. We prove that, up to a subsequence $u_n \to u \in \mathscr{S}(\alpha)$. By definition of $\mathscr{S}$, one has that, for all $n \in \mathbb{N}$, u_n is a solution of

$$\begin{cases} -a(\alpha_n)\mathcal{G}(u_n) = \widehat{f}(x, u_n) & \text{in } \Omega, \\ u_n > 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.56}$$

Notice that, $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\mathcal{H}}(\Omega)$. Indeed

$$\begin{aligned} &a(\alpha_n) \int_\Omega \left(\mu(x)|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n\right) \nabla v dx \\ = &\int_\Omega \widehat{f}(x, u_n) v dx \quad \forall v \in W_0^{1,\mathcal{H}}(\Omega). \end{aligned}$$

That is, by (2.4) and (H_f^6),

$$a(\alpha_n) \int_\Omega \left(\mu(x)|\nabla u_n|^{p-2}\nabla u_n + |\nabla u_n|^{q-2}\nabla u_n\right) \nabla v dx \le |\Omega| \max_{t\in[0,t_*]} [tf(t)] \tag{4.57}$$

thus $\|u_n\| < +\infty$ for all $n \in \mathbb{N}$. Then, up to a subsequence, we have $u_n \rightharpoonup u$ in $W_0^{1,\mathcal{H}}(\Omega)$ and $u_n \to u$ in $L^{\mathcal{H}}(\Omega)$ (remember that $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^q(\Omega)$ compactly for $q < p^*$ and $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ continuously, see (2.4) and Colasuonno-Squassina [32, Proposition 2.18]). Then, we obtain

$$\lim_{n\to+\infty} \langle \mathcal{G}(u_n) - \mathcal{G}(u), u_n - u \rangle = 0, \tag{4.58}$$

and by the (S_+)-property of $\mathcal{G}$ (see Section 2.1.3), $u_n \to u$ in $W_0^{1,\mathcal{H}}(\Omega)$. Moreover, by the Lebesgue Dominate Convergence Theorem (see e.g., Brézis [9, Theorem 4.2]), passing to the limit in (4.56), u is a solution of

$$\begin{cases} -a(\alpha)\mathcal{G}(u) = \widehat{f}(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.59}$$

i.e., $u \in \mathscr{S}(\alpha)$. □

The previous proposition allows us to cover the convergence issues arise from the lack of Hölder regularity. Now, arguing as in Propositions 10-11 we can prove that $\mathscr{S}$ is l.s.c. and downward direct. Then, there exists the smallest solution $u_\alpha^* \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ and $u_\alpha^* \in [\underline{u}, \overline{u}]$. Furthermore, we can define the fixed point map $\mathcal{P}_k : (\alpha_{2k-1}, \alpha_{2k}) \to \mathbb{R}$ such that

$$\mathcal{P}_k(\alpha) = \int_\Omega \left((u_\alpha^*)^r + \nu(x)(u_\alpha^*)^s\right) dx \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k}).$$

Following the notation of the p-Laplacian case, we set $\Gamma(\alpha) = u^*_\alpha$. Arguing as in Proposition 14, one can prove that the map

$$\mathcal{P}_k(\alpha) = \|\Gamma(\alpha)\|_r^r + \|\Gamma(\alpha)\|_{s,\nu}^s \quad \forall \alpha \in (\alpha_{2k-1}, \alpha_{2k})$$

is continuous. Moreover the following properties hold.

Proposition 18. *Assume that* (H_a^2), $(H_{\mathcal{G}})$, (H_f^6), (H_1^6) *and* (H_2^6) *hold. Then,*

$$\lim_{\alpha \to \alpha_{2k-1}^+} \mathcal{P}_k(\alpha) > \alpha_{2k-1} \text{ and } \lim_{\alpha \to \alpha_{2k}^-} \mathcal{P}_k(\alpha) > \alpha_{2k}. \tag{4.60}$$

Proof. We show the first limit, the second being analogous. First, by Lemma 17, for all $k \in \{1, ..., K\}$ and $\alpha \in (\alpha_{2k-1}, \alpha_{2k})$, we have

$$\begin{aligned} \mathcal{P}_k(\alpha) &= \int_\Omega ((u^*_\alpha)^r + \nu(x)(u^*_\alpha)^s)\, dx \\ &\geq \int_\Omega \underline{u}^r dx + \min_{\overline{\Omega}} \nu \int_\Omega \underline{u}^s dx = t(\alpha)^r \|e_t\|_r^r + t(\alpha)^s \min_{\overline{\Omega}} \nu \|e_t\|_s^s. \end{aligned} \tag{4.61}$$

By the definition of $t(\alpha)$ (see (4.53)) and by hypotheses (H_f^6)-(H_1^6), we obtain that

$$\lim_{\alpha \to \alpha_{2k-1}^+} t(\alpha) = \lim_{\alpha \to \alpha_{2k}^-} t(\alpha) = t_*. \tag{4.62}$$

Indeed, by assumptions (H_f^6)-(H_1^6), the map $t \mapsto f(t)$ is strictly positive in $(0, t_*)$, it possesses a zero value at $t = t_*$ and

$$\lim_{t \to 0^+} f(t) > \max_{s \in [\alpha_{2k-1}, \alpha_{2k}]} a(s) \geq a(\alpha).$$

Arguing by contradiction, assume that $t(\alpha) \not\to t_*$ as $\alpha \to \alpha_{2k-1}^+$. Recall that $a(\alpha) \to 0$ as $\alpha \to \alpha_{2k-1}^+$ and then, there exists an accumulation point $\bar{t} < t_*$ of $t(\alpha)$, i.e., there exists a sequence $\{\alpha_n\} \subseteq [\alpha_{2k-1}, \alpha_{2k}]$, such that $\alpha_n \to \alpha_{2k-1}^+$, $t(\alpha_n) \to \bar{t}$ (obviously $\bar{t} > 0$). But, by the continuity of f and a, this means that $f(t) = \lim_{n \to +\infty} a(\alpha_n) = 0$ (see (4.53)). Thus, $f(\bar{t}) = 0$, a contradiction to hypotheses (H_f^6).

Finally, passing to the limit in (4.61) as $\alpha \to \alpha_{2k-1}^+$, by (H_2^6), we get

$$\lim_{\alpha \to \alpha_{2k-1}^+} \mathcal{P}_k(\alpha) \geq t_*^r \|e_t\|_r^r + t_*^s \min_{\overline{\Omega}} \nu \|e_t\|_s^s > \alpha_{2K}.$$

□

We prove the existence of at least one mid point τ_k in $(\alpha_{2k-1}, \alpha_{2k})$ such that $\mathcal{P}_k(\tau_k) < \tau_k$ for all $k \in \{1, \ldots, K\}$. In particular, we move our problem to the q-Laplacian case exploiting the continuous embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$.

Proposition 19. *Assume that* (H_a^2), $(H_{\mathcal{G}})$, (H_f^6), (H_1^6), (H_2^6) *and* (H_3^6) *hold. Then, for each* $k \in \{1, ..., K\}$, *there exists* $\tau_k \in (\alpha_{2k-1}, \alpha_{2k})$ *such that*

$$\mathcal{P}_k(\tau_k) < \tau_k.$$

Proof. Consider the auxiliary problem

$$\begin{cases} -\Delta_q w = (u_\alpha^*)^{r-1} + \nu(x)(u_\alpha^*)^{s-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.63}$$

being $u_\alpha^* \in W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$ the unique minimal solution of (4.55). By Minty-Browder's Theorem (see Theorem 5), there exists a unique positive solution $w \in W_0^{1,q}(\Omega) \cap L^\infty(\Omega)$ of (4.63). Recalling that the embedding $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$ is continuous, choosing $v = u_\alpha$ as a test function in the weak formulation of (4.63) and applying the Hölder's inequality, we infer

$$\begin{aligned} \mathcal{P}_k(\alpha) &= \int_\Omega ((u_\alpha^*)^r + \nu(x)(u_\alpha^*)^s)\, dx \\ &= \int_\Omega |\nabla w|^{q-2}\nabla w \nabla u_\alpha^* dx \\ &\leq \int_\Omega |\nabla w|^{q-1}|\nabla u_\alpha^*| dx \\ &\leq \left(\int_\Omega |\nabla w|^{(q-1)q'} dx\right)^{\frac{1}{q'}} \left(\int_\Omega |\nabla u_\alpha^*|^q dx\right)^{\frac{1}{q}} dx \\ &= \|\nabla w\|_q^{q-1} \|\nabla u_\alpha^*\|_q. \end{aligned} \tag{4.64}$$

Note that, since u_α^* solves (4.55), then, for $v = u_\alpha^*$,

$$a(\alpha)\int_\Omega (\mu(x)|\nabla u_\alpha^*|^p + |\nabla u_\alpha^*|^q)\, dx = \int_\Omega f(u_\alpha^*) u_\alpha^* dx.$$

That is, one has

$$\|\nabla u_\alpha^*\|_q^q \leq \|\nabla u_\alpha^*\|_{p,\mu}^p + \|\nabla u_\alpha^*\|_q^q \leq \frac{|\Omega|}{a(\alpha)} \max_{t\in[0,t_*]} [tf(t)], \tag{4.65}$$

and

$$\|\nabla u_\alpha^*\|_q \leq \left(\frac{|\Omega|}{a(\alpha)} \max_{t\in[0,t_*]} [tf(t)]\right)^{\frac{1}{q}} \tag{4.66}$$

holds. On the other hand, since w is a weak solution of (4.63), choosing $v = w$ as test function and by Hölder's inequality, we have

$$\begin{aligned} \|\nabla w\|_q^q &= \int_\Omega |\nabla w|^q dx \\ &= \int_\Omega \left((u_\alpha^*)^{r-1} + \nu(x)(u_\alpha^*)^{s-1}\right) w dx \\ &\leq \left(\int_\Omega \left((u_\alpha^*)^{r-1} + \nu(x)(u_\alpha^*)^{s-1}\right)^{q'} dx\right)^{\frac{1}{q'}} \left(\int_\Omega |w|^q dx\right)^{\frac{1}{q}} \\ &\leq \left(t_*^{r-1} + t_*^{s-1} \max_{\overline{\Omega}} \nu\right) |\Omega|^{\frac{1}{q'}} \|w\|_q \\ &\leq \frac{|\Omega|^{1/q'}}{\lambda_{1,q}^{1/q}} (t_*^{r-1} + t_*^{s-1} \max_{\overline{\Omega}} \nu) \|\nabla w\|_q, \end{aligned}$$

that is,

$$\|\nabla w\|_q^{q-1} \leq \frac{|\Omega|^{1/q'}}{\lambda_{1,q}^{1/q}} \left(t_*^{r-1} + t_*^{s-1} \max_{\overline{\Omega}} \nu \right). \tag{4.67}$$

Finally, putting together (4.66) and (4.67) in (4.64), we have

$$\begin{aligned}
&\mathcal{P}_k(\alpha) \leq \|\nabla w\|_q^{q-1} \|\nabla u_\alpha^*\|_q \\
\leq\ & |\Omega|^{\frac{1}{q'}} \left(t_*^{r-1} + t_*^{s-1} \max_{\overline{\Omega}} \nu \right) \left(\frac{|\Omega|}{a(\alpha)\lambda_{1,q}} \max_{t\in[0,t_*]} [tf(t)] \right)^{\frac{1}{q}} \\
=\ & |\Omega| \left(t_*^{r-1} + t_*^{s-1} \max_{\overline{\Omega}} \nu \right) \left(\frac{\max_{t\in[0,t_*]} [tf(t)]}{a(\alpha)\lambda_{1,q}} \right)^{\frac{1}{q}}.
\end{aligned}$$

The proof follows by (H_3^6). □

Finally, from the continuity of the map $\mathcal{P}_k$, combining Propositions 18-19, and arguing as in Theorem 14, the following holds.

Theorem 18. *Assume that (H_a^2), $(H_\mathcal{G})$, (H_f^6), (H_1^6), (H_2^6) and (H_3^6) hold. Then, the problem (4.51) admits at least K pairs of positive solutions $(u_{k,1}, u_{k,2})$ lying in $W_0^{1,\mathcal{H}}(\Omega) \cap L^\infty(\Omega)$.*

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