Neutrosophic set connected mappings

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Abstract. In this paper, the concept of neutrosophic connectedness between neutrosophic sets in neutrosophic topological spaces has been introduced. It is shown that a neutrosophic topological space is neutrosophic connected if and only if it is neutrosophic connected between every pair of its nonempty neutrosophic sets. Further, a new class of mappings called neutrosophic set connected mappings has been defined. It is shown that the class of all neutrosophic continuous mappings is a subclass all neutrosophic set-connected mappings. Several properties and characterizations of neutrosophic set-connected mappings in neutrosophic topological spaces have been studied.

Keywords: Neutrosophic sets; neutrosophic connectedness; neutrosophic connectedness between neutrosophic sets and neutrosophic set-connected mappings.

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1. **Introduction**

The fusion of technology and generalized forms of classical sets is very useful to solve many real world complex problems which involve the vague and uncertain information. A classical set is defined by its characteristic function from universe of discourse to two point set {0,1}. Classical set theory is insufficient to handle the complex problems involving vague and uncertain information. To handle the vagueness and uncertainty of complex problems, Zadeh [18] in 1965, created fuzzy sets as a generalization of classical sets which characterised by membership function from universe of discourse to closed interval [0,1]. After the occurrence of fuzzy sets most of the mathematical structure have been extended to fuzzy sets. In 1968, Chang [4] used fuzzy sets to create fuzzy topological spaces and extended some topological concepts to fuzzy sets. In 1986, Atanassov [3] gave a generalization of fuzzy set called Intuitionistic fuzzy set characterized by a membership degree and a non-membership degree that satisfies the case in which the sum of its membership degree and a non-membership degree is less than or equal to one. In 1997, Coker [5], proposed the notion of intuitionistic fuzzy topological spaces and studied some analog versions of classical topology such as continuity and compactness. In 1999 , Smarandache [15] created the concept of neutrosophic sets as an extension of intuitionistic fuzzy sets and developed the theory of neutrosophy. Smarandache [**?**, 15],Salama and Alblowi [13] , Lupiáñez [10] and others presented some more properties of neutrosophic sets. In 2008, Lupiáñez [10] introduced the neutrosophic topology as an extension of intuitionistic fuzzy topology. Since 2008 many authors such as Lupiáñez [10, 11], Salama et.al. [13, 14] Karatas and Cemil [9], Acikgoz and his coworkers [1, 2], Dhavaseelan et.al. [6–8], Parimala et.al. [12], and others extended various topological notions to neutrosophic sets and studied in neutrosophic topological spaces. Recently Acikgoz and his coworkers [2] iniated the study of connectedness in neutrosophic topology. Connectedness between sets and set connected mappings are the important topic of research in Topology. Till the date these concepts are not studied in neutrosophic topology. Therefore, to fill this gap, the present paper introduces the concept of connectedness between neutrosophic sets and studied some of its properties in neutrosophic topological spaces. Further, neutrosophic set-connected mappings are defined and some theorems related to its characterizations and properties are established.

2. Preliminaries

Definition 2.1. [15] A Neutrosophic set (**NS**) in X is a structure

$$A = \{< x, \varrho_A(x), \varpi_A(x), \sigma_A(x) >: x \in X\}$$

where $\varrho_A : X \to]^-0, 1^+[$, $\varpi A : X \to]^-0, 1^+[$, and $\sigma_A : X \to]^-0, 1^+[$ denotes the membership, indeterminacy, and non-membership of A satisfies the condition if $^-0 \leq \varrho_A(x) + \varpi_A(x) + \sigma_A(x) \leq 3^+,\ \forall\ x \in X$.

In the real life applications in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the closed interval [0,1] and sum of membership, indeterminacy, and nonmembership degrees of each element of universe of discourse lies between 0 and 3. The family of all **NS**s over X will be denoted by $\mathbf{N(X)}$.

Definition 2.2. [15] A neutrosophic set $A = \{< x, \varrho_A(x), \varpi_A(x), \sigma_A(x) >: x \in X\}$ is said to be

(a) Universal neutrosophic set if $\varrho_A(x) = 1$, $\varpi_A(x) = 1$, and $\sigma_A(x) = 0$, $\forall x \in X$. It is denoted by $\tilde{X}$.

(b) Empty neutrosophic set if $\varrho_A(x) = 0$, $\varpi_A(x) = 0$, and $\sigma_A(x) = 1$, $\forall x \in X$. It is denoted by $\tilde{\Phi}$.

Definition 2.3. [15–17] Let $A = \{< x, \varrho_A(x), \varpi_A(x), \sigma_A(x) >: x \in X\}$, $A_1 = \{< x, \varrho_{A_1}(x), \varpi_{A_1}(x), \sigma_{A_1}(x) >: x \in X\}$, and $A_2 = \{< x, \varrho_{A_2}, \varpi_{A_2}(x), \sigma_{A_2}(x) >: x \in X\}$ be **NS**s over X. Then the subset, equality, union, intersection and complement operations over $\mathbf{N(X)}$ are defined as follow:

(a) $A_1 \subset A_2 \Leftrightarrow \varrho_{A_1}(x) \leq \varrho_{A_2}(x)$, $\varpi_{A_1}(x) \leq \varpi_{A_2}(x)$, and $\sigma_{A_1} \geq \sigma_{A_2}$, $\forall\ x \in X$.

(b) $A_1 = A_2 \Leftrightarrow \varrho_{A_1}(x) = \varrho_{A_2}(x)$, $\varpi_{A_1}(x) = \varpi_{A_2}(x)$, and $\sigma_{A_1}(x) = \sigma_{A_2}(x)$, $\forall\ x \in X$.

(c) $A_1 \cup A_2 = \{< x, \varrho_{A_1}(x) \vee \varrho_{A_2}(x),\ \varpi_{A_1}(x) \vee \varpi_{A_2}(x), \sigma_{A_1}(x) \wedge \sigma_{A_2}(x) >: x \in X\}$.

(d) $A_1 \cap A_2 = \{< x, \varrho_{A_1}(x) \wedge \varrho_{A_2}(x),\ \varpi_{A_1}(x) \wedge \varpi_{A_2}(x), \sigma_{A_1}(x) \vee \sigma_{A_2}(x) >: x \in X\}$.

(e) $A^c = \{< x, \sigma_A(x), \varpi_A(x), \varrho_A(x) >: x \in X\}$.

Definition 2.4. [15–17] Let $\{A_i : i \in \Lambda\}$ be an arbitrary family of **NS***s* in X. Then

(a) $\cup A_i = \{< x, \vee\varrho_{A_i}(x),\ \vee\varpi_{A_i}(x), \wedge\sigma_{A_i}(x) >: x \in X\}$.

(b) $\cap A_i = \{< x, \wedge\varrho_{A_i}(x),\ \wedge\varpi_{A_i}(x), \vee\sigma_{A_1}(x) >: x \in X\}$.in

Example 2.5. Let $X = \{x_1, x_2, x_3\}$ and **NS**s A, B, C over X are defined as follows:

$$A = \{< x_1, 0.8, 0.2, 0.7 >, < x_2, 0.7, 0.5, 0.6 >, < x_3, 0.6, 0.4, 0.9 >\}$$

$$B = \{< x_1, 0.9, 0.3, 0.4 >, < x_2, 0.8, 0.5, 0.3 >, < x_3, 0.7, 0.5, 0.3 >\}$$

$$C = \{< x_1, 0.6, 0.5, 0.3 >, < x_2, 0.9, 0.4, 0.5 >, < x_3, 0.8, 0.5, 0.6 >\}.$$

Then,

(i) $A \subset B$, but $A \not\subseteq C$.

(ii) $A \cup B = \{< x_1, 0.9, 0.3, 0.4 >, < x_2, 0.8, 0.5, 0.3 >, < x_3, 0.7, 0.5, 0.3 >\}$.

(iii) $B \cap C = \{< x_1, 0.6, 0.3, 0.4 >, < x_2, 0.8, 0.4, 0.5 >, < x_3, 0.7, 0.5, 0.6 >\}$.

(iv) $A^c = \{< x_1, 0.7, 0.2, 0.8 >, < x_2, 0.6, 0.5, 0.7 >, < x_3, 0.9, 0.4, 0.6 >\}$.

Definition 2.6. $[10, 13]$ A subfamily Γ of $\mathbf{N(X)}$ is called a Neutrosophic topology (**NT**) on X if:

(a) $\tilde{\Phi}, \tilde{X} \in \Gamma$.

(b) $G_i \in \Gamma,\ \forall\ i \in \Lambda \Rightarrow \cup_{i\in\Lambda} G_i \in \Gamma$.

(c) $G_1, G_2 \in \Gamma \Rightarrow G_1 \cap G_2 \in \Gamma$.

If Γ is a **NT** on X then the structure (X, Γ) is called a neutrosophic topological space (**NTS**) over X and the members of Γ are called neutrosophic open (**NO**) sets.The complement of **NO** set is called neutrosophic closed (**NC**).

Definition 2.7. [8] Let X be a nonempty set. If r, t, s are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$, then the neutrosophic set $x_{r,t,s}$ defined by

$$x_{r,t,s}(x_p) = \begin{cases} (r, t, s), & \text{if } x = x_p, \\ (0, 0, 1), & \text{if } x \neq x_p. \end{cases}$$

is called a neutrosophic point. The point $x_p \in X$ is called the support of $x_{r,t,s}$, and r denotes the degree of membership value, t the degree of indeterminacy and s the degree of non-membership value of $x_{r,t,s}$.

Definition 2.8. [1] a nuetrosophic point $x_{r,t,s}$ is said to be quasi-coincident (q-coincident, for short) with F, denoted by $x_{r,t,s}qF$ iff $x_{r,t,s} \nsubseteq F^c$. If $x_{r,t,s}$ is not quasi-coincident with F, we denote by $x_{r,t,s}\tilde{q}F$.

Definition 2.9. [1] a **NS** F is a neutrosophic topological space (X, Γ) is said to be a q-neighbourhood of a neutrosophic point $x_{r,t,s}$ if there exists a neutrosophic open set G such that $x_{r,t,s}qG \subset F$.

Definition 2.10. [1] a **NS** G is said to be quasi-coincident (q-coincident, for short) with F, denoted by GqF if $G \nsubseteq F^c$. If G is not quasi coincident with F, we denote by $G\tilde{q}F$.

Definition 2.11. [1] a nuetrosophic point $x_{r,t,s}$ of a **NTS** (X, Γ)is said to be interior point of a **NS** F if there exists a neutrosophic open q-neighbourhood G of $x_{r,t,s}$ such that $G \subset F$. The union of all interior points of F is called the interior of F and denoted by Int(F).

Definition 2.12. [1] a nuetrosophic point $x_{r,t,s}$ of a **NTS** (X, Γ) is said to be cluster point of a **NS** F if every neutrosophic open q-neighbourhood G of$x_{r,t,s}$ is q-coincident with F. The union of all cluster points of F is called the closure of F. It is denoted by Cl(F).

Definition 2.13. [1] Let (X, Γ) be a **NTS** and $Y \subset X$. Then the family $\Gamma_Y = \{G \cap Y : G \in \Gamma\}$ is called the neutrosophic relative topology on Y and the pair (Y, Γ_Y) is called neutrosophic sub space of (X, Γ).

Lemma 2.14. *Let (Y, Γ_Y) be a neutrosophic subspace of a* **NTS** *(X, Γ) and F be a neutrosophic open set in Y. If $Y \in \Gamma$ then $F \in \Gamma$.*

Lemma 2.15. *Let (X,Γ) be a* **NTS** *and (Y, Γ_Y) be a neutrosophic subspace of (X, Γ), then a neutrosophic closed set F_Y of Y is neutrosophic closed in X if and only if Y is neutrosophic closed in X.*

Definition 2.16. [1] A **NTS** (X,Γ) said to be neutrosophic connected, if $\nexists$ proper neutrosophic open sets U and V in (X,Γ) such that that $A\tilde{q}B$ and $A^c\tilde{q}B^c$.

Theorem 2.17. *[1] A* **NTS** *(X,Γ)is neutrosophic connected if and only if it has no proper neutrosophic clopen (neutrosophic closed and neutrosophic open) set.*

Example 2.18. Let X = {a,b} be universe of discourse and the **NS**s U, A and B on X are defined as follows:

$$U = \{\langle a, 0.4, 0.3, 0.8\rangle, \langle b, 0.3, 04, 0.9\rangle\}$$

$$V = \{\langle a, 0.6, 0.5, 0.5\rangle, \langle b, 0.5, 0.7, 0.8\rangle\}$$

Let Γ = {$\tilde{\Phi}$, $\tilde{X}$, U, V } be a neutrosophic topology on X, then **NTS** (X,Γ) is neutrosophic connected

Definition 2.19. [1] Consider that f is a mapping from X to Y.

(a) Let $A \in \mathbf{N}(\mathbf{X})$ with membership function $\varrho_A(x)$, indeterminacy function $\varpi_A(x)$ and non-membership function $\sigma_A(x)$. The image of A under f, written as f(A), is a neutrosophic set of Y whose membership function, indeterminacy function and non-membership function are defined as

$$\varrho_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\varrho_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & \text{if } f^{-1}(y) = \phi, \end{cases}$$

$$\varpi_{f(A)}(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \{\varpi_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & iff^{-1}(y) = \phi, \end{cases}$$

$$\sigma_{f(A)}(y) = \begin{cases} \inf\limits_{x \in f^{-1}(y)} \{\sigma_A(x)\}, & \text{if } f^{-1}(y) \neq \phi, \\ 0, & iff^{-1}(y) = \phi, \end{cases}$$

$\forall\ , y \in Y$. Where $f^{-1}(y) = \{x : f(x) = y\}$.

(b) Let $B \in \mathbf{N}(\mathbf{Y})$ with membership function $\varrho_B(y)$, indeterminacy function $\varpi_B(y)$ and non-membership function $\sigma_B(y)$. Then, the inverse image of B under f, written as $f^{-1}(B)$ is a neutrosophic set of X whose membership function, indeterminacy function and non-membership function are respectively defined as:

$$\varrho_{f^{-1}(B)}(x) = \varrho_B(f(x)),$$

$$\varpi_{f^{-1}(B)}(x) = \varpi_B(f(x)),$$

$$\sigma_{f^{-1}(B)}(x) = \sigma_B(f(x)).$$

$\forall\ x \in X$.

Definition 2.20. [14] A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is said to be neutrosophic continuous if $f^{-1}(G) \in \Gamma$ for every **NS** $G \in \vartheta$.

Example 2.21. Let $X = \{a, b\}$, $Y = \{p, q\}$ and the neutrosophic sets U , V and W are defined as follows:

$$U = \{\langle a, 0.3, 0.5, 0.6\rangle, \langle b, 0.4, 0, 5, 0.6\rangle\}$$

$$V = \{\langle a, 0.4, 0.6, 0.5\rangle, \langle b, 0.5, 0.7, 0.4\rangle\}$$

$$W = \{\langle p, 0.3, 0.5, 0.6\rangle, \langle q, 0.4, 0.5, 0.6\rangle\}$$

Let $\Gamma = \{\tilde{\Phi},\ \tilde{X},\ \text{U},\ \text{V}\ \}$ and $\vartheta = \{\tilde{\Phi},\ \tilde{Y},\ \text{W}\ \}$ be neutrosophic topologies on X and Y respectively. Then the mapping $f : (X, \Gamma) \to (Y, \vartheta)$ defined by f(a)= p and f(b)= q is neutrosophic continuous.

3. Connectedness between neutrosophic sets

Definition 3.1. A **NTS** (X ,Γ) is said to be neutrosophic connected between **NS**s A and B if there is no neutrosophic clopen set F in X such that A ⊂ F and F$\tilde{q}$B.

Theorem 3.2. *A* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B if and only if there is no neutrosophic clopen set F in X such that $A \subset F \subset B^c$.*

Proof. Follows from Definition 3.1. □

Example 3.3. Let X = {a,b} be universe of discourse and the **NS**s U, A and B on X are defined as follows:

$$U = \{\langle a, 0.4, 0.5, 0.2\rangle, \langle b, 0.6, 04, 0.7\rangle\}$$

$$A = \{\langle a, 0.6, 0.5, 0.5\rangle, \langle b, 0.5, 04, 0.8\rangle\}$$

$$B = \{\langle a, 0.4, 0.5, 0.7\rangle, \langle b, 0.5, 04, 0.9\rangle\}$$

Let $\Gamma = \{\tilde{\Phi}\ ,\ \tilde{X},\ \text{U}\ \}$ be a neutrosophic topology on X, then **NTS** (X,Γ) is neutrosophic connected between the **NS**s A and B, but A$\tilde{q}B$.

Theorem 3.4. *If a* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B, then $A \neq \tilde{\Phi} \neq B$.*

Proof. If any **NS** A = $\tilde{\Phi}$ then A is a neutrosophic clopen set over X such that A ⊂ B and A$\tilde{q}$B and hence (X,Γ) can not be neutrosophic connected between **NS**s A and B, which is a contradiction. □

Theorem 3.5. *If a* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B and if $A \subset C$ and $B \subset D$ then (X,Γ) is neutrosophic connected between* **NS***s C and D.*

Proof. Suppose **NTS** (X,Γ) is not neutrosophic connected between **NS**s C and D then there is a neutrosophic clopen set F over X such that C ⊂ F and F$\tilde{q}$D. Clearly A ⊂ F. Now we claim that F$\tilde{q}$B. If FqB then FqD a contradiction. Consequently, (X,Γ) is not neutrosophic connected between **NS**s A and B. □

Theorem 3.6. *A* **NTS** *(X,Γ) is neutrosophic connected between* **NS***s A and B if and only if (X,Γ) is neutrosophic connected between* **NS***s Cl(A) and Cl(B).*

Proof. Necessity : Follows from Theorem 3.5.

Sufficiency : Suppose **NTS** (X,Γ) is not neutrosophic connected between **NS**s A and B, then there exists neutrosophic clopen set F in X such that A ⊂ F and F $\tilde{q}$ B. Since F is neutrosophic closed, Cl(A) ⊂ Cl(F) = F. Clearly, by Definition 2.10, F $\tilde{q}$ B ⇔ F ⊂ B^c. Therefore F = Int(F) ⊂ Int(B^c) = $(Cl(B))^c$. Hence, F $\tilde{q}$ Cl(B) and (X,Γ) is not neutrosophic connected between **NS**s Cl(A) and Cl(B). □

Theorem 3.7. *If A and B are two* **NS***s in a* **NTS** *(X,Γ) and A q B, then (X,Γ) is neutrosophic connected between A and B.*

Proof. If F is any neutrosophic clopen set over X such that A ⊂ F, then AqB ⇒ FqB. □

Remark 3.8. The converse of Theorem 3.7 need not be true . For **NTS** (X,Γ) of Example 3.3 is neutrosophic connected between A and B but $A\tilde{q}B$.

Theorem 3.9. *If a* **NTS** *(X,Γ) is neither neutrosophic connected between F and F_0 nor neutrosophic connected between F and F_1 then it is not neutrosophic connected between F and $F_0 \cup F_1$.*

Proof. Since a **NTS** (X,Γ) is not neutrosophic connected between F and F_0, there is a neutrosophic clopen set G_0 in X such that $F \subset G_0$ and $G_0\tilde{q}F_0$. Also since (X,Γ) is not neutrosophic connected between F and F_1 there exists a neutrosophic clopen set G_1 in X such that F ⊂ G_1 and $G_1\tilde{q}F_1$. Put G = $G_0 \cap G_1$. Since any intersection of neutrosophic closed sets is neutrosophic closed, G is neutrosophic closed. Again intersection of finite family of neutrosophic open sets is neutrosophic open, G is neutrosophic open. Therefore G is neutrosophic clopen set over X such that $F \subset G$ and $G\tilde{q}F_0 \cup F_1$. If $GqF_0 \cup F_1$,then GqF_0 or GqF_1 a contradiction. Hence, (X,Γ) is not neutrosophic connected between F and $F_0 \cup F_1$. □

Theorem 3.10. *A* **NTS** *(X,Γ) is neutrosophic connected if and only if it is neutrosophic connected between every pair of its nonempty* **NS***s.*

Proof. Let F and G be a pair of nonempty **NS**s in X. Suppose (X,Γ) is not neutrosophic connected between F and G. Then there is a neutrosophic clopen set H in X such that $F \subset H$ and $G\tilde{q}G$. Since F and G are nonempty it follows that H is a nonempty neutrosophic proper clopen set in X. Hence, (X,Γ) is not neutrosophic connected.

Conversely, suppose that (X,Γ) is not neutrosophic connected. Then there exists a nonempty proper **NS** H in X which is both neutrosophic open and neutrosophic closed. Consequently, (X,Γ) is not neutrosophic connected between H and H^c. Thus, (X,Γ) is not neutrosophic connected between arbitrary pair of its nonempty **NS**s. □

Remark 3.11. If a **NTS** (X,Γ) is neutrosophic connected between a pair of its **NS**s, then it is not necessary that it is neutrosophic connected between each pair of its **NS**s and hence it is not necessarily neutrosophic connected.

Example 3.12. Let X = {a,b} be an universe set, and the neutrosophic sets U,V,A,B,C,D,E over X are defined as follows:

$$U = \{\langle a, 0.2, 0.5, 0.7\rangle, \langle b, 0.3, 0, 5, 0.6\rangle\}$$

$$V = \{\langle a, 0.7, 0.5, 0.2\rangle, \langle b, 0.6, 0.5, 0.3\rangle\}$$

$$A = \{\langle a, 0.4, 0.5, 0.5\rangle, \langle b, 0.2, 0.5, 0.6\rangle\}$$

$$B = \{\langle a, 0.5, 0.5, 0.4\rangle, \langle b, 0.4, 0.5, 0.5\rangle\}$$

$$C = \{\langle a, 0.1, 0.5, 0.8\rangle, \langle b, 0.2, 0.5, 0.7\rangle\}$$

$$D = \{\langle a, 0.3, 0.5, 0.4\rangle, \langle b, 0.3, 0.5, 0.4\rangle\}$$

Let Γ = {$\tilde{\Phi}$, $\tilde{X}$, U, V} be a neutrosophic topology over X. Then the **NTS** (X, Γ) is neutrosophic connected between the **NS**s A and B but it is not neutrosophic connected between C and D. Also the **NTS** (X, Γ) is not neutrosophic connected.

Theorem 3.13. *Let (Y,Γ_Y,E) be a neutrosophic subspace of a* **NTS** *(X,Γ). If (Y,Γ_Y) is neutrosophic connected between the* **NS***s F and G over Y, then* **NTS** *(X,Γ) is neutrosophic connected between F and G.*

Proof. Suppose **NTS** (X,Γ) is not neutrosophic connected between **NS**s F and G,then there is neutrosophic clopen set H in X such that $F \subset H$ and $H\tilde{q}G$. Then $Y \cap H$ is neutrosophic clopen set in Y such that $F \subset H \cap Y$ and $H \cap Y\ \tilde{q}G$. Consequently, (Y,Γ_Y) is not neutrosophic connected between F and G, a contradiction. □

Theorem 3.14. *Let (Y,Γ_Y) be a neutrosophic clopen subspace of a* **NTS** *(X,Γ) and F, G are* **NS**s *of Y. If (X,Γ) is neutrosophic connected between F and G then (Y,Γ_Y) is neutrosophic connected between F and G.*

Proof. Suppose (Y,Γ_Y) is not neutrosophic connected between F and G. Then there is neutrosophic clopen set H of (Y,Γ_Y) such that F $\subset$ H and $H\tilde{q}G$. Since, (Y,Γ_Y) is neutrosophic clopen in (X,Γ), by Lemma 2.14 and Lemma 2.15 H is neutrosophic clopen set of (X,Γ) such that $F \subset H$ and $H\tilde{q}G$. Consequently, (X,Γ) is not neutrosophic connected between F and G, a contradiction. □

4. Neutrosophic set-connected mappings

Definition 4.1. A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is said to be neutrosophic set-connected provided, if **NTS**(X, Γ) is neutrosophic connected between **NS**s F and G then neutrosophic subspace $(f(X), \vartheta_{f(X)})$ is neutrosophic connected between $f(F)$ and $f(G)$ with respect to neutrosophic relative topology.

Theorem 4.2. *A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is neutrosophic set-connected if and only if $f^{-1}(F, K)$ is a neutrosophic clopen set over X for any neutrosophic clopen set H of ($f(X)$,$\vartheta_{f(X)}$).*

Proof. Necessity : Let f be neutrosophic set-connected and H is a neutrosophic clopen set in $(f(X), \vartheta_{f(X)})$. Suppose $f^{-1}(H)$ is not neutrosophic clopen in (X ,Γ). Then (X ,Γ) is neutrosophic connected between $f^{-1}(H)$ and $(f^{-1}(H))^c$. Therefore, $(f(X), \vartheta_{f(X)})$ is neutrosophic connected between $f(f^{-1}(H))$ and $f((f^{-1}(H))^c)$ because f is neutrosophic set-connected. But, $f(f^{-1}(H)) = H \cap (f(X) = H$ and $f((f^{-1}(H))^c = H^c$ imply that H is not neutrosophic clopen in $(f(X), \vartheta_{f(X)})$, a contradiction. Hence, $f^{-1}(H)$ is neutrosophic clopen in (X ,Γ).

Sufficiency : Let (X ,Γ) be neutrosophic connected between F and G. If $(f(X), \vartheta_{f(X)})$ is not neutrosophic-connected between $f(F)$ and $f(G)$ then there exists a neutrosophic clopen set H in $(f(X), \vartheta_{f(X)})$ such that $f(F) \subset H \subset (f(G))^c$. By hypothesis, $f^{-1}(H)$ is neutrosophic clopen set over X and $F \subset f^{-1}(H) \subset G^c$. Therefore, (X ,$\Gamma$) is not neutrosophic connected between F and G. This is a contradiction. Hence, f is neutrosophic set-connected. □

Theorem 4.3. *Every neutrosophic continuous mapping f : (X ,Γ) $\to$ (Y,ϑ) is a neutrosophic set-connected.*

Proof. It is obvious. □

Remark 4.4. The converse of Theorem 4.3 need not be true.

Example 4.5. Let $X = \{a, b\}$ and $Y = \{p, q\}$. The neutrosophic sets U and V are defined as follows:

$$U = \{\langle a, 0.3, 0.5, 0.6\rangle, \langle b, 0.4, 0, 5, 0.5\rangle\}$$
$$V = \{\langle p, 0.4, 0.5, 0.6\rangle, \langle q, 0.5, 0.5, 0.4\rangle\}$$

Let $\Gamma = \{\tilde{\Phi}, \tilde{X}$, U } and $\vartheta = \{\tilde{\Phi}$, $\tilde{Y}$, V } are neutrosophic topologies on X and Y respectively. Then the mapping $f : (X, \Gamma) \to (Y, \vartheta)$ defined by f(a)=p and f(b)= q is neutrosophic set-connected but it is not neutrosophic continuous.

Theorem 4.6. *Every mapping $f : (X, \Gamma) \to (Y, \vartheta)$ such that $(f(X), \vartheta_{f(X)})$ is neutrosophic connected is neutrosophic set-connected mapping.*

Proof. Let $(f(X), \vartheta_{f(X)})$ be neutrosophic connected. Then by Theorem 2.17, no nonempty proper **NS** of $(f(X), \vartheta_{f(X)})$ which is neutrosophic clopen. Hence, f is neutrosophic set-connected. □

Theorem 4.7. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a neutrosophic set-connected mapping. If (X ,Γ) is neutrosophic connected , then $(f(X), \vartheta_{f(X)})$ is a neutrosophic connected sub space of (Y,ϑ).*

Proof. Suppose $(f(X), \vartheta_{f(X)})$ is not neutrosophic connected in (Y,ϑ). Then by Theorem 2.17, there is a nonempty proper neutrosophic clopen set G of $(f(X), \vartheta_{f(X)})$. Since f is neutrosophic set-connected, $f^{-1}(G)$ is a nonempty proper neutrosophic clopen set over X. Consequently, (X, Γ) is not neutrosophic connected. □

Theorem 4.8. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a neutrosophic set-connected mapping and F be a neutrosophic set over X such that $f(F)$ is neutrosophic clopen set of $(f(X), \vartheta_{f(X)})$. Then $f/F : F \to (Y, \vartheta)$ is a neutrosophic set-connected mapping.*

Proof: Let F be neutrosophic connected between G and H. Then by Theorem 3.13, (X, Γ) is neutrosophic connected between G and H. Since f is neutrosophic set-connected, $(f(X), \vartheta_{f(X)})$ is neutrosophic connected between $f(G)$ and $f(H)$. Now, since $f(F)$ is neutrosophic clopen set of $(f(X), \vartheta_{f(X)})$, it follows by Theorem 3.14 that $f(F)$ is neutrosophic connected between $f(G)$ and $f(H)$.

Theorem 4.9. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a neutrosophic set-connected surjection. Then every neutrosophic clopen set H of (Y, ϑ) is neutrosophic connected if $f^{-1}(H)$ is neutrosophic connected in (X ,Γ). In particular, if (X ,Γ) is neutrosophic connected then (Y,ϑ) is neutrosophic connected.*

Proof. By Theorem 4.8 $f/f^{-1}(H) : f^{-1}(H) \to (Y, \vartheta)$ is neutrosophic set-connected. And, since $f^{-1}(H)$ is neutrosophic connected by Theorem 4.7, $f/f^{-1}(H)[f^{-1}(H)] = H$ is neutrosophic connected. □

Theorem 4.10. *Let $f : (X, \Gamma) \to (Y, \vartheta)$ be a surjective neutrosophic set-connected and $g : (Y, \vartheta) \to (Z, \eta)$ a neutrosophic set-connected mapping. Then $gof : (X, \Gamma) \to (Z, \eta)$ is neutrosophic set-connected.*

Proof. Let H be a neutrosophic clopen set in g(Y). Then $g^{-1}(H)$ is neutrosophic clopen over Y $= f(X)$ and so f^{-1} (g^{-1}(H)) is neutrosophic clopen in (X ,Γ). Now (gof)(X) $= g$ (Y) and $(gof)^{-1}$ (H) $= f^{-1}$ (g^{-1}(H)) is neutrosophic clopen in (X ,Γ). Hence, gof is neutrosophic set connected. □

Definition 4.11. A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is said to be neutrosophic weakly continuous if for each neutrosophic point $x_{r,t,s} \in X$ and each neutrosophic open set G over Y containing $f(x_{r,t,s})$, there exists a neutrosophic open set F over X containing $x_{r,t,s}$ such that f(F) ⊂ Cl(G).

Theorem 4.12. *A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is neutrosophic weakly continuous if and only if for each neutrosophic open set H in Y, $f^{-1}(H) \subset Int(f^{-1}(Cl(H)))$.*

Proof. Necessity : Let H be a neutrosophic open set over Y and $x_{r,t,s} \in f^{-1}(H)$, then $f(x_{r,t,s}) \in H$. Therefore, there exists a neutrosophic open set F in X such that $x_{r,t,s} \in F$ and $f(F) \subset Cl(H)$. Hence, $x_{r,t,s} \in F \subset f^{-1}(Cl(H))$ and $x_{r,t,s} \in Int(f^{-1}(Cl(H)))$ since F is neutrosophic open.

Sufficiency : Let $x_{r,t,s} \in X$ and $f(x_{r,t,s}) \in H$. Then $x_{r,t,s} \in f^{-1}(H) \subset Int(f^{-1}(Cl(H)))$. Let $F = Int(f^{-1}(Cl(H)))$ then F is neutrosophic open set containing $x_{r,t,s}$ and $f(F) = f(Int(f^{-1}(Cl(H)))) \subset f(f^{-1}(Cl(H))) \subset Cl(H)$. Hence,f is neutrosophic weakly continuous. □

Theorem 4.13. *If a* **NTS** *space (X ,Γ) is neutrosophic connected and $f : (X, \Gamma) \to (Y, \vartheta)$ is a neutrosophic weakly continuous surjection, then (Y,ϑ) is neutrosophic connected.*

Proof. Follows from Theorem 4.12 and Theorem 2.17. □

Theorem 4.14. *A mapping $f : (X, \Gamma) \to (Y, \vartheta)$ is neutrosophic weakly continuous, then $Cl(f^{-1}(H)) \subset (f^{-1}(Cl(H))$ for each neutrosophic open set H over Y*

Proof. Suppose there exists a neutrosophic point $x_{r,t,s} \in Cl(f^{-1}(H))$ but $x_{r,t,s} \notin f^{-1}(Cl(H))$. Then $f(x_{r,t,s}) \notin (Cl(H))$. Therefore , there exists a neutrosophic open q-neighbourhood G of $f(x_{r,t,s})$ such that $G\tilde{q}H$. Since H is neutrosophic open set in Y, we have $H\tilde{q}Cl(G)$. Again, f is neutrosophic weakly continuous, there exists a neutrosophic open set F in X containing $x_{r,t,s}$ such that $f(F) \subset Cl(G)$. Thus, we obtain $f(F)\tilde{q}H$. On the other hand, since $x_{r,t,s} \in Cl(f^{-1}(H))$, we have $Fqf^{-1}(H)$ and hence, $f(F)qH$. Thus we have a contradiction. Hence $Cl(f^{-1}(H)) \subset (f^{-1}(Cl(H))$. □

Theorem 4.15. *If a neutrosophic surjection f : (X ,Γ) → (Y,ϑ) is neutrosophic weakly continuous , then f is neutrosophic set-connected.*

Proof. Let H be any neutrosophic clopen set over Y. Since H is neutrosophic closed, we have Cl(H) = H. Thus, by Theorem 4.12, we obtain $f^{-1}(H) \subset Int(f^{-1}(H))$. This shows that $f^{-1}(H)$ is neutrosophic open set in X. Moreover , by Theorem 4.14, we obtain $Cl(f^{-1}(H)) \subset f^{-1}(H)$. This shows that $f^{-1}(H)$ is a neutrosophic closed set in X. Since f is neutrosophic surjection, by Theorem 4.2, we obtain that f is a neutrosophic set-connected mapping. □

Remark 4.16. The converse of Theorem 4.15 is not true.

Example 4.17. Let $X = \{a, b\}$ and $Y = \{p, q\}$. The neutrosophic sets U and V are defined as follows:

$$U = \{\langle a, 0.3, 0.5, 0.6\rangle, \langle b, 0.4, 0, 5, 0.6\rangle\}$$

$$V = \{\langle p, 0.4, 0.5, 0.5\rangle, \langle q, 0.3, 0.5, 0.5\rangle\}$$

Let $\Gamma = \{\tilde{\Phi}$,$\tilde{X}$, U } and $\vartheta = \{\tilde{\Phi}$, $\tilde{Y}$, V } are neutrosophic topologies on X and Y respectively. Consider a mapping $f : (X, \Gamma) \to (Y, \vartheta)$ defined by f(a)= p and f(b)= q. Clearly,$\tilde{\Phi}$, $\tilde{Y}$ are the only neutrosophic clopen sets of Y and their inverse images Φ, $\tilde{Y}$ are neutrosophi clopen sets in X. Hence By Theorem 4.2 f is neutrosophic set-connected . But it is not neutrosophic weakly continuous.

Definition 4.18. A **NTS** (X, Γ) is said to be neutrosophic extremally disconnected if the closure of every neutrosophic open set of X is neutrosophic open in X.

Theorem 4.19. *Let (Y,ϑ) be a neutrosophic extremally disconnected space . If a mapping* $f : (X, \Gamma) \to (Y, \vartheta)$ *is neutrosophic set-connected, then f is neutrosophic weakly continuous.*

Proof. Let $x_{r,t,s}$ be a neutrosophic point of X and G be any neutrosophic open set in Y containing $f(x_{r,t,s})$. Since (Y,ϑ,) is neutrosophic extremally disconnected, Cl(G) is a neutrosophic clopen set in Y. Thus $Cl(G) \cap f(X)$ is neutrosophic clopen set in the neutrosophic subspace $(f(X), \vartheta_{f(X)})$. Put $f^{-1}(Cl(G) \cap f(X)) = F$. Then, since f is neutrosophic set-connected, it follows from Theorem 4.2 that F is neutrosophic clopen set over X. Therefore, F is a neutrosophic open set containing $x_{r,t,s}$ in X such that $f(F) \subset Cl(G)$. This implies that f is neutrosophic weakly continuous. □

Theorem 4.20. *Let (Y, ϑ) be a neutrosophic extremally disconnected space. A neutrosophic surjection $f : (X, \Gamma) \to (Y, \vartheta)$ is neutrosophic set-connected if and only if f is neutrosophic weakly continuous.*

Proof. It follows from Theorem 4.15 and Theorem 4.19. □

5. Conclusions

Connectedness is an important and major area of topology and it can give many relationships between other scientific areas and mathematical models. The notion of connectedness captures the idea of hanging-togetherness of image elements in an object by assigning a strength of connectedness to every possible path between every possible pair of image elements. This paper, introduces the notion of neutrosophic connectedness between neutrosophic sets in neutrosophic topological spaces. It is shown that a neutrosophic topological space is neutrosophic connected if and only if it is neutrosophic connected between every pair of its nonempty neutrosophic sets. Further two new classes of neutrosophic mappings called neutrosophic set connected and neutrosopic weakly continuous mappings have been introduced . It is shown that the class of neutrosophic set connected (respt. neutrosophic weakly continuous) mappings properly contains the class of all neutrosophic continuous mappings. Several properties and characterizations of neutrosophic set connected and neutrosophic weakly continuous mappings have been studied. Hope that the concepts and results established in this paper will help researcher to enhance and promote the further study on neutrosophic topology to carry out a general framework for the development of information systems.

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