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# CRITICAL POINT APPROACHES TO NONLINEAR SQUARE ROOT LAPLACIAN EQUATIONS

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*Abstract.* This work is devoted to the study of multiplicity results of solutions for a class of nonlinear equations involving the square root of the Laplacian. Indeed, we will use variational methods for smooth functionals, defined on reflexive Banach spaces, in order to achieve the existence of at least three solutions for the equations. Moreover, assuming that the nonlinear terms are nonnegative, we will prove that the solutions are nonnegative. Finally, by presenting an example, we will ensure the applicability of our results.

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## 1. INTRODUCTION

In the study ahead, we discuss the existence of multiple nontrivial weak solutions for the problem involving the square root of the Laplacian

$$\begin{cases} \mathcal{A}_{1/2}y = \varepsilon\gamma(\tau)g(y) & \text{in} \quad \mathcal{D}, \\ y = 0 & \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{1}$$

where $\mathcal{D}$ is an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary $\partial\mathcal{D}$, $\varepsilon > 0$ is a real parameter, $\gamma\colon \mathcal{D} \to \mathbb{R}$ is an $\mathrm{L}^\infty(\mathcal{D})$-function with $\operatorname{essinf}_{t\in\Omega}\gamma(t) > 0$, and $\mathcal{A}_{1/2}$ is a fractional operator which will be introduced in Section 2 (see [4–6]).

The fractions of the Laplacian, as in the square root of the Laplacian $\mathcal{A}_{1/2}$, are the infinitesimal generators of Lévy stable diffusion processes. They emerge in irregular diffusions in plasmas, flames propagation, population dynamics, geophysical fluid dynamics, and American options in finance.

Recently, elliptic equations involving fractional powers of the Laplacian have been under investigation from a physical point of view. Nonlocal operators play a major role in describing a set of phenomena. The recent paper of Cabré and Tan [4] can be cited here where the authors discuss the existence, nonexistence, and regularity of positive solutions of the problem (1) with power-type nonlinearities, along with a conjecture of Gidas–Spruck type and symmetry results of Gidas–Ni–Nirenberg type. Some earlier discussions can be seen in the papers [1–3, 9, 10, 12–14, 18, 19]. For instance, Molica Bisci et al. in [12] reviewed (1) concerning the existence of solutions. They established the existence of at least three weak solutions in $L^\infty$-bounded function space for certain values of $\varepsilon$ requiring the nonlinear term $g$ to be continuous, having a suitable growth, by using a variant of Caffarelli–Silvestre's extension method. Moreover, Ambrosio et al. in [1], explained the existence and nonexistence of weak solutions for (1). They established the existence of at least two nontrivial $L^1$-bounded weak solutions for large values of $\varepsilon$, requiring the nonlinear term $g$ to be continuous, superlinear at zero, and sublinear at infinity by using a suitable variant of the Caffarelli–Silvestre extension method.

The present article concentrates on this issue because it is clear that in (1), there are singularities in the term $\mathcal{A}_{1/2}(y)$, which leads to problems in the proofs. This article concerns existence results for (1). Our main result gives conditions ensuring the existence of at least two and infinitely many weak solutions for (1).

Throughout this paper, we denote by $\mathcal{G}$ the class of all continuous functions $g : \mathbb{R} \to \mathbb{R}$ that are superlinear at zero, i.e., $\lim_{t\to 0} \frac{g(t)}{t} = 0$ and sublinear at infinity, i.e., $\lim_{|t|\to\infty} \frac{g(t)}{t} = 0$.

The following theorem is the main result of this paper.

**Theorem 1.** *Let $g \in \mathcal{G}$.*

(a) *If $g$ is a positive-valued function, then* (1) *has at least two weak solutions.*
(b) *If $g$ is an odd function, then* (1) *has infinitely many weak solutions.*

This article has the following structure. In Section 2, necessary definitions, notations, and variational theorems are reviewed. In Section 3, a proof of the main results is presented. Finally, in Section 4, an example and several comments are given.

## 2. PRELIMINARIES

We give the following theorems (see [11, 16]) as our main tools to prove the results of Section 3.

**Theorem 2** ([11, Theorem 4.4])**.** *Let $\mathcal{X}$ be a Banach space and $\mathcal{M} : \mathcal{X} \to \mathbb{R}$ be a differentiable functional that is bounded below. If $\mathcal{M}$ satisfies the $(\mathrm{PS})_c$-condition with $c = \inf_{\mathcal{X}} \mathcal{M}$, then $\mathcal{M}$ has a minimum on $\mathcal{X}$.*

Obviously, the $(\mathrm{PS})_c$-condition for each $c \in \mathbb{R}$ is obtained by the $(\mathrm{PS})$-condition.

**Theorem 3** ([11, Theorem 4.10])**.** *Let $\mathcal{Q} \in C^1(\mathcal{X},\mathbb{R})$ satisfy the $(\mathrm{PS})$-condition. Assume that there exist $y_0, y_1 \in \mathcal{X}$ and a bounded neighborhood $\mathcal{D}$ of $y_0$ satisfying $y_1 \notin \mathcal{D}$ and $\inf_{z\in\partial\mathcal{D}} \mathcal{Q}(z) > \max\{\mathcal{Q}(y_0), \mathcal{Q}(y_1)\}$. Then there exists a critical point $y$ of $\mathcal{Q}$, i.e., $\mathcal{Q}'(y) = 0$, with $\mathcal{Q}(y) > \max\{\mathcal{Q}(y_0), \mathcal{Q}(y_1)\}$.*

**Theorem 4** ([16, Theorem 9.12]). *Let $\mathcal{X}$ be a real Banach space with infinite dimension, $\mathcal{Q} \in \mathrm{C}^1(\mathcal{X},\mathbb{R})$ be an even functional which satisfies the* (PS)*-condition, and $\mathcal{Q}(0)=0$. Assume that $\mathcal{X}=\mathcal{V}\bigoplus\mathcal{W}$, where $\mathcal{V}$ has infinite dimension, there exist $\beta>0$ and $e>0$ so that $\mathcal{Q}(y)\geq\beta$ for each $y\in\mathcal{W}$ with $\|y\|=e$ and for any subspace $\mathcal{Y}\subset\mathcal{X}$ with finite dimension, there is $d=d(\mathcal{Y})$ so that $\mathcal{Q}(y)\geq 0$ on $\mathcal{X}\setminus B_{d(\mathcal{Y})}$. Then $\mathcal{Q}$ has an unbounded sequence of critical values.*

Readers can see [7, 20], where Theorems 3 and 4 have been successfully used to guarantee multiple solutions of degenerate nonlocal problems and nonlinear impulsive differential equations, respectively. Reference can also be made to [21], where Theorem 4 has been used to guarantee the existence of infinitely many solutions for a boundary value problem.

To produce appropriate function spaces and apply the variational method to discuss the existence of solutions for (1), we introduce the following preliminaries, which have been initially given in [4], and which we will use in the proofs of our main results. Let $\mathcal{H}_0^{1/2}(\mathcal{D})$ be the space

$$\mathcal{H}_0^{1/2}(\mathcal{D}):=\left\{y\in\mathrm{L}^2(\mathcal{D}):\,y=\sum_{j=1}^{\infty}\kappa_j\mathcal{E}_j \quad\text{and}\quad \sum_{j=1}^{\infty}\kappa_j^2\varepsilon_j^{1/2}<\infty\right\}$$

with the norm

$$\|y\|_{\mathcal{H}_0^{1/2}(\mathcal{D})}:=\left(\sum_{j=1}^{\infty}\kappa_j^2\varepsilon_j^{1/2}\right)^{1/2},$$

where $\{\mathcal{E}_j,\varepsilon_j\}_{j\in\mathbb{N}}$ are the eigenfunctions and eigenvalues of the usual linear problem $\mathcal{A}_{1/2}y=\varepsilon g(y)$ in $\mathcal{D}$ with Dirichlet boundary condition $y=0$ on $\partial\mathcal{D}$. We note that

$$0<\varepsilon_1<\varepsilon_2<\varepsilon_3<\ldots<\varepsilon_j<\varepsilon_{j+1}<\ldots$$

and $\varepsilon_j\to\infty$ as $j\to\infty$. Moreover, we can suppose that the eigenfunctions $\{\mathcal{E}_j\}_{j\in\mathbb{N}}$ are normalized as

$$\int_{\mathcal{D}}|\nabla\mathcal{E}_j(\tau)|^2\mathrm{d}\tau=\varepsilon_j\int_{\mathcal{D}}|\mathcal{E}_j(\tau)|^2\mathrm{d}\tau=\varepsilon_j,\quad j\in N$$

and

$$\int_{\mathcal{D}}\nabla\mathcal{E}_i(\tau)\nabla\mathcal{E}_j(\tau)\mathrm{d}\tau=\varepsilon_j\int_{\mathcal{D}}\mathcal{E}_i(\tau)\mathcal{E}_j(\tau)\mathrm{d}\tau=0,\quad i\neq j.$$

*Remark* 1. The operator $\mathcal{A}_{1/2}$ of the Laplace operator $-\Delta$ in a bounded domain with zero-boundary conditions is given via the spectral decomposition, using the powers of the eigenvalues of the primary operator. Therefore, pursuant to primeval results on positive operators in $\mathcal{D}$, if $\{(\mathcal{E}_j,\varepsilon_j)\}_{j\in\mathbb{N}}$ are the eigenfunctions and eigenvalues of the usual linear Dirichlet problem $-\Delta y=\varepsilon g(y)$ in $\mathcal{D}$ with Dirichlet boundary condition $y=0$ on $\partial\mathcal{D}$, then $\{(\mathcal{E}_j,\varepsilon_j^{1/2})\}_{j\in\mathbb{N}}$ are the eigenfunctions and eigenvalues of the corresponding fractional problem $\{(\mathcal{E}_j,\varepsilon_j)\}_{j\in\mathbb{N}}$.

*Remark* 2. The fractional operator $\mathcal{A}_{1/2}$ should not be confused with the integro-differential operator $(-\Delta)^{1/2}$. In fact, these two operators, although determined in the same way, have distinct behaviors in eigenvalues, eigenfunctions and so on (see [15, 17] for more details).

*Remark* 3. The operator $\mathcal{A}_{1/2}$ is well defined on the space $\mathcal{H}_0^{1/2}(\mathcal{D})$, and it has the form $\mathcal{A}_{1/2}y = \sum_{j=1}^{\infty} \kappa_j^2 \varepsilon_j^{1/2} \mathcal{E}_j$, where $\kappa_j = \int_{\mathcal{D}} y(\tau)\mathcal{E}_j(\tau)\mathrm{d}\tau$.

*Remark* 4. Let $\mathcal{D}$ be a bounded domain. Set $C_{\mathcal{D}} := \{(\tau,\xi) : \tau \in \mathcal{D}, \xi > 0\} \subset \mathbb{R}_+^{n+1}$ and $\partial_L C_{\mathcal{D}} := \partial\mathcal{D} \times [0,\infty)$, and for a function $y \in \mathcal{H}_0^{1/2}(\mathcal{D})$, define the harmonic extension $E(y)$ to the cylinder $C_{\mathcal{D}}$ as the solution of $\operatorname{div}(\nabla E(y)) = 0$ in $C_{\mathcal{D}}$, $E(y) = 0$ on $\partial_L C_{\mathcal{D}}$ and $\operatorname{Tr}(E(y)) = y$ on $\mathcal{D}$, where $\operatorname{Tr}(E(y))(\tau) = E(y)(\tau,0)$ for all $\tau \in \mathcal{D}$. We note that $E(y)$ belongs to the Hilbert space

$$\begin{aligned}\mathcal{X}_0^{1/2}(\mathcal{D}) &:= \left\{ z \in \mathrm{L}^2(\mathcal{D}) : z = 0 \quad \text{on} \quad \partial_L C_{\mathcal{D}}, \int_{\mathcal{D}} |\nabla z(\tau,\xi)|^2 \mathrm{d}\tau\mathrm{d}\xi < \infty \right\} \\ &= \left\{ z \in \mathrm{L}^2(C_{\mathcal{D}}) : z = \sum_{j=1}^{\infty} \eta_j \mathcal{E}_j e^{-\varepsilon_j^{1/2}\xi} \quad \text{and} \quad \sum_{j=1}^{\infty} \eta_j^2 \varepsilon_j^{1/2} < \infty \right\},\end{aligned}$$

with the standard norm

$$\|z\|_{\mathcal{X}_0^{1/2}(\mathcal{D})} := \left( \int_{C_{\mathcal{D}}} |\nabla z(\tau,\xi)|^2 \mathrm{d}\tau\mathrm{d}\xi \right)^{\frac{1}{2}}$$

(see [4, Lemma 2.10] for more details). On the other hand, the trace operator $\operatorname{Tr} : \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \to \mathcal{H}_0^{1/2}(\mathcal{D})$, given by $\operatorname{Tr}(z)(\tau) := z(\tau,\xi)$ for all $\tau \in \mathcal{D}$, is a continuous map (see [4, Lemma 2.6]). Moreover,

$$\mathcal{H}_0^{1/2}(\mathcal{D}) := \left\{ y \in \mathrm{L}^2(\mathcal{D}) : y = \operatorname{Tr}(z) \quad \text{for some} \quad z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) \right\} \subset \mathcal{H}^{1/2}(\mathcal{D})$$

and $\|\operatorname{Tr}(z)\|_{\mathcal{H}_0^{1/2}(\mathcal{D})} \le \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}$. The embedding $j : \operatorname{Tr}(\mathcal{H}_0^{1/2}(C_{\mathcal{D}})) \hookrightarrow \mathrm{L}^p(\mathcal{D})$ is continuous for any $p \in [1, 2n/n-1]$, and it is compact whenever $p \in [1, 2n/n-1)$. Hence, if $p \in [1, 2n/n-1]$, then there is a constant $c_p > 0$ (depending on $p$, $n$ and the Lebesgue measure of $\mathcal{D}$) so that

$$\|\operatorname{Tr}(z)\|_{\mathrm{L}^p(\mathcal{D})} \le c_p \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} \tag{2}$$

for every $z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ (see [2–4]). We define the fractional operator $\mathcal{A}_{1/2}$ in $\mathcal{D}$ with

$$\mathcal{A}_{1/2}y(\tau) := -\lim_{\xi \to 0^+} \frac{\partial E(y)}{\partial \xi}(\tau,\xi) \quad \text{for all} \quad \tau \in \mathcal{D},$$

where $E(y) \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ is the extension of $y \in \mathcal{H}_0^{1/2}(\mathcal{D})$. Thus,

$$\mathcal{A}_{1/2}y(\tau) = \frac{\partial E(y)}{\partial \nu}(\tau) \quad \text{for all} \quad \tau \in \mathcal{D},$$

where $\nu$ is the unit outer normal to $C_{\mathcal{D}}$ at $\mathcal{D}\times\{0\}$.

A function $y=\mathrm{Tr}(z)\in\mathcal{H}_0^{1/2}(\mathcal{D})$ is a weak solution of (1) if $z\in\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ is a weak solution of

$$\begin{cases}-\operatorname{div}(\nabla z)=0 & \text{in} \quad C_{\mathcal{D}},\\ z=0 & \text{on} \quad \partial_L C_{\mathcal{D}},\\ \frac{\partial z}{\partial \nu}=\varepsilon\gamma(\tau)g(\mathrm{Tr}(\tau)) & \text{on} \quad \mathcal{D},\end{cases} \tag{3}$$

i.e.,

$$\int_{C_{\mathcal{D}}}\langle\nabla z,\nabla\vartheta\rangle \mathrm{d}\tau\mathrm{d}\xi=\varepsilon\int_{\Omega}\beta(\tau)f(\mathrm{Tr}(z)(\tau))\,\mathrm{Tr}(\vartheta)(\tau)\mathrm{d}\tau$$

for every $\vartheta\in\mathcal{X}_0^{\frac{1}{2}}(C_{\mathcal{D}})$.

Now, we consider the functionals

$$\mathcal{M}(z):=\frac{1}{2}\|z\|^2_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} \quad \text{and} \quad \mathcal{N}(z):=\int_{\mathcal{D}}\gamma(\tau)G(\mathrm{Tr}(z)(\tau))\mathrm{d}\tau \tag{4}$$

for all $z\in\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. Since

$$\lim_{\|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})}\to\infty}\mathcal{M}(z)=\infty,$$

this means that the functional $\mathcal{M}:\mathcal{X}_0^{1/2}(C_{\mathcal{D}})\to\mathbb{R}$ is coercive. Also, $\mathcal{M}$ and $\mathcal{N}$ are Fréchet differentiable and

$$\mathcal{M}'(z)\vartheta=\int_{C_{\mathcal{D}}}\langle\nabla z,\nabla\vartheta\rangle\mathrm{d}\tau\mathrm{d}\xi \quad \text{and} \quad \mathcal{N}'(z)\vartheta=\int_{\mathcal{D}}\gamma(\tau)f(\mathrm{Tr}(z)(\tau))\,\mathrm{Tr}(\vartheta)(\tau)\mathrm{d}\tau$$

for all $z,\vartheta\in\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. So, the critical points of the functional $I_\varepsilon:\mathcal{X}_0^{1/2}(C_{\mathcal{D}})\to\mathbb{R}$ defined by $I_\varepsilon(z):=\mathcal{M}(z)-\varepsilon\mathcal{N}(z)$ are exactly the weak solutions of (3). Thus, the traces of critical points of $I_\varepsilon$ are the weak solutions of (1).

*Remark* 5. The functional $I_\varepsilon$ is weakly lower semicontinuous on $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$. Indeed, the map $z\to\int_{\mathcal{D}}\gamma(\tau)G(\mathrm{Tr}(z)(\tau))\mathrm{d}\tau$ is continuous in the weak topology of $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$, and the map $z\to\int_{C_{\mathcal{D}}}|\nabla z(\tau,\xi)|^2\mathrm{d}\tau\mathrm{d}\xi$ is lower semicontinuous in the weak topology of $\mathcal{X}_0^{1/2}(C_{\mathcal{D}})$.

## 3. Proof of the Main Theorem

In this section, we prove Theorem 1. For this purpose, we first provide the following lemma.

**Lemma 1** ([1, Lemma 4.1])**.** *Let $g\in\mathcal{G}$. Then for every $\varepsilon>0$, the functional $I_\varepsilon$ satisfies the* (PS)*-condition.*

3.1. *Proof of Theorem 1(a)*

*Proof of Theorem 1(a).* In our case, it is clear that $I_\varepsilon(0)=0$. Lemma 1 gives that $I_\varepsilon$ satisfies the (PS)-condition.

*Step 1*

Since $g\in\mathcal{G}$, for $\varepsilon>0$ fixed, there exist two positive constants $\delta_1$ and $\delta_2$ such that $|f(t)|\leq\frac{\varepsilon}{\|\beta\|_{L^\infty(\Omega)}}|t|$ for all $t\in\mathbb{R}$ with $0<|t|<\delta_1\lambda$, and $|g(t)|\leq\frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}}|t|$ for all $t\in\mathbb{R}$ with $|t|>\delta_2\lambda$. Set $0<\delta_\lambda<\min\{1,\delta_1\lambda,\delta_2\lambda^{-1}\}$. Then

$$|g(t)|\leq\frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}}|t|\quad\text{for all}\quad t\in\mathbb{R}\quad\text{with}\quad 0<|t|<\delta_\lambda\quad\text{and}\quad |t|>\delta_2\lambda.$$

Besides, the function $h(t)=\frac{|g(t)|}{|t|}$ is continuous on $[\delta_\lambda,\delta_\lambda^{-1}]\cup[-\delta_\lambda^{-1},-\delta_\lambda]$, and therefore, there exists $0<m_\lambda<\frac{1-\lambda}{c_2^2\|\gamma\|_{L^\infty(\mathcal{D})}}$ such that $|h(t)|\leq m_\lambda$ for all $t\in[\delta_\lambda,\delta_\lambda^{-1}]\cup[-\delta_\lambda^{-1},-\delta_\lambda]$. Thus, we can say

$$|g(t)|\leq\left(\frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}}+m_\lambda\right)|t|\quad\text{for all}\quad t\in\mathbb{R}.$$

Now, since $\sup_{t\in\mathbb{R}}G(t)>0$ and by using (2) for $p=2$, we have

$$\begin{aligned}|\mathcal{N}(z)|&\leq\int_{\mathcal{D}}|\gamma(\tau)||G(\mathrm{Tr}(z)(\tau))|\mathrm{d}\tau\\&\leq\|\gamma\|_{L^\infty(\mathcal{D})}\int_{\mathcal{D}}\int_0^{\mathrm{Tr}(z)(\tau)}\left(\frac{\lambda}{\|\gamma\|_{L^\infty(\mathcal{D})}}+m_\lambda\right)|\xi|\mathrm{d}\xi\mathrm{d}\tau\\&\leq\frac{\lambda+m_\lambda\|\gamma\|_{L^\infty(\mathcal{D})}}{2}\|\mathrm{Tr}(z)(\tau)\|^2_{L^2(\mathcal{D})}\\&\leq\frac{\lambda+m_\lambda\|\gamma\|_{L^\infty(\mathcal{D})}}{2}c_2^2\|z\|^2_{X_0^{1/2}(C_{\mathcal{D}})}.\end{aligned}$$

Then, for any $y\in X$, by (4), we get

$$I_\varepsilon(y)\geq\frac{\|y\|^2_{X_0^{1/2}(C_{\mathcal{D}})}}{2}\left(1-\lambda-c_2^2m_\lambda\|\gamma\|_{L^\infty(\mathcal{D})}\right),\tag{5}$$

and we conclude that $I_\varepsilon$ is a coercive functional which is bounded below. Because $I_\varepsilon$ satisfies the (PS)-condition by Lemma 1, Theorem 2 yields that there is a minimum point $z_0$ of $I_\varepsilon$ on $X_0^{1/2}(C_{\mathcal{D}})$ and $0=I_\varepsilon(0)\geq I_\varepsilon(z_0)$ and $I'_\varepsilon(z_0)=0$.

*Step 2*

Since $z_0 = \min_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} I_\varepsilon$, there is $\mathcal{L} > 0$ large enough so that

$$I_\varepsilon(z_0) \leq 0 < \inf_{z \in \partial B_L} I_\varepsilon(z),$$

where $B_{\mathcal{L}} = \left\{ z \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}}) : \|z\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} < \mathcal{L} \right\}$. Now, we will show that there exists $z_1$ with $\|z_1\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} > \mathcal{L}$ such that $I_\varepsilon(z_1) < \inf_{\partial B_{\mathcal{L}}} I_\varepsilon(z)$. For this, let $\vartheta_1 \in \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and $z_1 = \kappa\vartheta_1$, $\kappa > 0$, where $\vartheta_1$ corresponding to $\varepsilon_1$ is the first eigenfunction of (3) and $\|\vartheta_1\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} = 1$. Since $\lim_{|t|\to\infty} \frac{g(t)}{t} = 0$, we can consider constants $a > 0$ and $\alpha > 2$ in a way that $G(t) \geq a|t|^\alpha$ for all $t \in \mathbb{R}$. Hence,

$$\begin{aligned} I_\varepsilon(z_1) &= (\mathcal{M} - \varepsilon\mathcal{N})(\kappa\vartheta_1) \\ &\leq \frac{1}{2}\|\kappa\vartheta_1\|^2_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} - \varepsilon \int_{\mathcal{D}} G(\kappa\vartheta_1(\tau))\mathrm{d}\tau \\ &\leq \frac{\kappa^2}{2} - \varepsilon\kappa^\alpha a \int_{\mathcal{D}} |\vartheta_1(\tau)|^\alpha \mathrm{d}\tau. \end{aligned}$$

So by $\alpha > 2$, there is sufficiently large $\kappa > \mathcal{L} > 0$ so that $I_\varepsilon(\kappa\vartheta_1) < 0$. Therefore, $\max\{I_\varepsilon(z_0), I_\varepsilon(z_1)\} < \inf_{z \in \partial B_{\mathcal{L}}} I_\varepsilon(z)$. Then, Theorem 3 with $\mathcal{X} := \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and $\vartheta := I_\varepsilon$ gives us the critical point $z^*$. So, $y_0$ and $y^*$ are two critical points of $I_\varepsilon$ and thus, are two weak solutions of (3). □

3.2. *Proof of Theorem 1(b)*

*Proof of Theorem 1(b).* In order to prove this part, we set $\mathcal{X} := \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ and consider the continuously Gâteaux-differentiable functional $I_\varepsilon$, that, by (4), is an even functional and $I_\varepsilon(0) = 0$. We continue the proof of this part in two steps to show that the functional $I_\varepsilon$ satisfies the assumptions of Theorem 4.

*Step 1*

The coercivity of $I_\varepsilon$ is achieved by (5), and this, together with the (PS)-condition, by the minimization theorem [11, Theorem 4.4], implies that the functional $I_\varepsilon$ has a minimum critical point $y$ with $I_\varepsilon(y) \geq \iota > 0$ and $\|y\|_{\mathcal{X}_0^{1/2}(C_{\mathcal{D}})} = \rho$ for sufficiently small $\rho > 0$.

*Step 2*

Let $\mathcal{K} \subset \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ be a finite-dimensional subspace. Since $\lim_{|t|\to\infty} \frac{g(t)}{t} = 0$, there exist constants $a > 0$ and $\alpha > 2$ so that $G(t) \geq a|t|^\alpha$ for all $t \in \mathbb{R}$. Now, for each $\kappa > 0$

and $y \in \mathcal{K} \setminus \{0\}$ with $\|y\|_{\mathcal{X}_0^{1/2}(C_\mathcal{D})} = 1$, we have

$$\begin{aligned} I_\varepsilon(\kappa y) &= (\mathcal{M} - \varepsilon\mathcal{N})(\kappa z) \\ &\leq \frac{1}{2}\|\kappa z\|^{2\alpha}_{\mathcal{X}_0^{1/2}(C_\mathcal{D})} - \varepsilon\int_\mathcal{D} G(\kappa z(\tau))d\tau \\ &\leq \frac{\kappa^2}{2}\|y\|^{2\alpha}_{\mathcal{X}_0^{1/2}(C_\mathcal{D})} - \varepsilon\kappa^\alpha a_1\int_\mathcal{D}|z(\tau)|^\alpha d\tau \to -\infty \quad \text{as} \quad \kappa \to \infty. \end{aligned}$$

This yields the existence of $\kappa_0$ so that $\|\kappa z\|_{\mathcal{X}_0^{1/2}(C_\mathcal{D})} > \rho$ and $I_\varepsilon(\kappa z) < 0$ for every $\kappa \geq \kappa_0 > 0$. Since $\mathcal{K}$ is a finite-dimensional subspace, there exists $\mathcal{R} = \mathcal{R}(\mathcal{K}) > 0$ such that $I_\varepsilon(z) \leq 0$ on $\mathcal{K} \setminus B_{\mathcal{R}(\mathcal{K})}$. Due to Theorem 4, the functional $I_\varepsilon(z)$ has infinitely many critical points, which are the weak solutions of (3). □

## 4. Examples and Remarks

Finally, we give an example and several concluding remarks.

*Example* 1. Let $\mathcal{D} = \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_1^2 + \tau_2^2 < 4\} \subset \mathbb{R}^2$ and $g(t) = t^2/(1+t^2)$ for all $t \in \mathbb{R}$. We see that $g(t) \geq 0$ for all $t \in \mathbb{R}$ is an odd function and $g \in \mathcal{G}$. Hence, the problem $\mathcal{A}_{1/2}y = \varepsilon y^2/(1+y^2)$ in $\mathcal{D}$ with boundary condition $y = 0$ on $\partial\mathcal{D}$, by applying Theorem 1, for every $\varepsilon > 0$, possesses infinitely many weak solutions in the space $\mathcal{H}_0^{1/2}(\mathcal{D})$.

*Remark* 6. For each $g \in \mathcal{G}$, the number $c_g := \max_{|t|>0} \frac{|g(t)|}{|t|}$ is well defined and strictly positive. Furthermore, the sublinear growth condition at infinity on the nonlinearity $g$ complements the classical Ambrosetti and Rabinowitz assumption.

*Remark* 7. Example 1 shows that our existence results for (1) in Theorem 1 is different from the existence results of Molica Bisci in [12]. This is because the function $g$ in [12] should satisfy the condition

$$|g(t)| \leq a_1 + a_2|t|^{q-1} \quad \text{for} \quad a_1, a_2 > 0,\ q \in \left(1, \frac{2n}{n-1}\right),\ t \in \mathbb{R}, \tag{6}$$

while in Example 1, $2n/(n-1) = 2$ and $g(t) = t^2/(1+t^2)$, and so $g$ does not satisfy (6).

*Remark* 8. We note that, if $g(0) \neq 0$, then Theorem 1(a) gives the existence of two nontrivial weak solutions for (1). If we do not consider the condition $g(0) \neq 0$, then the second solution $y_2$ of (1) can be trivial, but the problem has at least one nontrivial solution.

*Remark* 9. We notice that, by Remark 8, the solutions obtained in Example 1 are nonzero because $g(0) = 1 \neq 0$.

*Remark* 10. If $\limsup_{\xi\to 0^+} g(\xi)/|\xi| = \infty$ and $\liminf_{\xi\to 0^+} g(\xi)/|\xi| > -\infty$, then the second weak solution obtained by Theorem 1(a) can be nontrivial even in the case $g(0) = 0$. Indeed, let $\varepsilon > 0$ and let $\mathcal{M}$ and $\mathcal{N}$ be as given in (4). Due to Theorem 2 and Lemma 1, $I_\varepsilon = \mathcal{M} - \varepsilon\mathcal{N}$ has a critical point $y_\varepsilon$ that is a global minimum of $I_\varepsilon$. The function $y_\varepsilon$ cannot be trivial. Indeed, by the same argument as [8, Remark 3.3], we can prove that $\limsup_{\|y\|\to 0^+} \mathcal{M}(y)/\mathcal{N}(y) = \infty$. So, there is a sequence $\{\zeta_n\} \subset \mathcal{X}$ that converges strongly to zero, and for sufficiently large $n$, $I_\varepsilon(\zeta_n) = \mathcal{M}(\zeta_n) - \varepsilon\mathcal{N}(\zeta_n) < 0$. Since $y_\varepsilon$ is a global minimum of $I_\varepsilon$, we obtain $I_\varepsilon(y_\varepsilon) < 0$, so that $y_\varepsilon$ is not trivial.

*Remark* 11. If $g$ is nonnegative, then Theorem 1(a) is a bifurcation result, i.e., the pair $(0,0) \in \mathcal{H}_g^\varepsilon \subset \mathcal{H}_0^{1/2}(\mathcal{D}) \times \mathbb{R}$ with

$$\mathcal{H}_g^\varepsilon := \left\{ (y_\varepsilon, \varepsilon) \in \mathcal{H}_0^{1/2}(\mathcal{D}) \times (0,\infty) : \ y_\varepsilon \text{ is a nontrivial weak solution of (1)} \right\}.$$

Practically, by the proof of Theorem 1(a), $\|y_\varepsilon\|_{\mathcal{H}_0^{1/2}(\mathcal{D})} \to 0$ as $\varepsilon \to 0$. Hence, there exist two sequences $\{y_k\}$ in $\mathcal{H}_0^{1/2}(\mathcal{D})$ and $\{\varepsilon_k\}$ in $\mathbb{R}^+$ (here $y_k = y_{\varepsilon_k}$) such that $\varepsilon_k \to 0^+$ and $\|y_k\| \to 0$ as $k \to \infty$. Moreover, since $g$ is nonnegative, $\mathcal{M}(y) < 0$ for all $y \in \mathbb{R}$, and thus, the mapping $(0,\varepsilon^*) \ni \varepsilon \mapsto I_\varepsilon(y_\varepsilon)$ is strictly decreasing. Hence, for every $\varepsilon_1, \varepsilon_2 \in (0,\varepsilon^*)$ with $\varepsilon_1 \neq \varepsilon_2$, the weak solutions $y_{\varepsilon_1}$ and $y_{\varepsilon_2}$ ensured by Theorem 1(a) are different.

*Remark* 12. In Theorem 1, we looked for the points of the functional $I_\varepsilon$ naturally associated with (3). We note that, in general, $I_\varepsilon$ can be unbounded from below in $\mathcal{X}_0^{1/2}(C_\mathcal{D})$. Indeed, for example, in the case when $g(t) = 1 + |t|^{a-2}t$ for all $t \in \mathbb{R}$ with $a > 2$, for any fixed $z \in \mathcal{X}_0^{1/2}(C_\mathcal{D}) \setminus \{0\}$ and $\iota \in \mathbb{R}$, we obtain

$$\begin{aligned} I_\varepsilon(\iota z) &\le \frac{1}{2}\|\iota z\|^2_{\mathcal{X}_0^{\frac{1}{2}}(C_\mathcal{D})} - \varepsilon \int_\mathcal{D} \gamma(\tau) G(\mathrm{Tr}(\iota z(\tau)))\mathrm{d}\tau \\ &\le \frac{\iota^2}{2}\|z\|^2_{\mathcal{X}_0^{\frac{1}{2}}(C_\mathcal{D})} - \varepsilon\underline{\gamma}|\mathcal{D}| \left( \iota\|\mathrm{Tr}(z)\|_{\mathrm{L}^1(\mathcal{D})} + \frac{\iota^a}{a}\|\mathrm{Tr}(z)\|^a_{\mathrm{L}^a(\mathcal{D})} \right) \to -\infty \end{aligned}$$

as $\iota \to \infty$, where $\underline{\gamma} = \inf_{\tau\in\mathcal{D}} \gamma(\tau)$ and $|\mathcal{D}| = \int_\mathcal{D} \mathrm{d}\tau$. Hence, direct minimization cannot be used to find critical points of the functional $I_\varepsilon$.

*Remark* 13. All results of this paper are still valid if we consider the more general problem

$$\begin{cases} \mathcal{A}_{m/2} y = \varepsilon\gamma(\tau) g(y) & \text{in } \ \mathcal{D}, \\ y = 0 & \text{on } \ \partial\mathcal{D} \end{cases}$$

with $k \in (0,2)$ and $\mathcal{A}_{m/2}y(\tau) := -\kappa_m \lim_{z\to 0^+} z^{1-m}\frac{\partial E(y)}{\partial z}(\tau,z)$ for all $\tau \in \mathcal{D}$, where $y$ belongs to the space

$$\mathcal{H}_0^{m/2}(\mathcal{D}) := \left\{ y \in \mathrm{L}^2(\mathcal{D}) : \|y\|_{\mathcal{H}_0^{m/2}(\mathcal{D})} = \left( \sum_{j=1}^{\infty} a_j^2 \varepsilon_j^{m/2} \right)^{1/2} < \infty \right\}$$

and its $m$-harmonic extension $E_m(y)$ to the cylinder $C$ is the unique solution of the local problem

$$\begin{cases} -\operatorname{div}(y^{1-m}\nabla z) = 0 & \text{in} \quad C_{\mathcal{D}}, \\ E_m(y) = 0 & \text{on} \quad \partial_L C_{\mathcal{D}}, \\ \mathrm{Tr}(E_m(y)) = y & \text{on} \quad \mathcal{D}. \end{cases}$$

$E_m(y)$ as an extension lies in the space

$$\begin{aligned} \mathcal{X}_0^{m/2}(\mathcal{D}) &:= \left\{ z \in \mathrm{L}^2(\Omega) : z = 0 \text{ on } \partial_L C_{\mathcal{D}}, \int_{\mathcal{D}} \xi^{1-m} |\nabla z(\tau,\xi)|^2 \mathrm{d}\tau \mathrm{d}\xi < \infty \right\} \\ &= \left\{ z \in \mathrm{L}^2(C_{\mathcal{D}}) : z = \sum_{j=1}^{\infty} \eta_j \mathcal{E}_j e^{-\varepsilon_j^{m/2}\xi} \text{ and } \sum_{j=1}^{\infty} \eta_j^2 \varepsilon_j^{m/2} < \infty \right\} \end{aligned}$$

and the operator $E_m : \mathcal{H}_0^{1/2}(\mathcal{D}) \to \mathcal{X}_0^{1/2}(C_{\mathcal{D}})$ is an isometry.

*Remark* 14. Although we did not use the test function in the proof of our main results, in the approach of some manuscripts on this topic such as [12], $v_{\alpha_0}^{t_0} : \Omega \to \mathbb{R}$ as

$$v_{\alpha_0}^{t_0}(y) := \begin{cases} 0 & \text{if} \quad y \in \overline{\Omega} - B(y_0,t_0), \\ \frac{2\alpha_0}{t_0}(t_0 - |y-y_0|) & \text{if} \quad y \in B(y_0,t_0) - B(y_0,t_0/2), \\ \alpha_0 & \text{if} \quad y \in B(y_0,t_0/2) \end{cases}$$

as a class of test functions in $\mathcal{X}_0^{m/2}(\mathcal{D})$ is used.

*Remark* 15. If $g$ is an odd function, then we have the same result as Theorem 1(b) by placing the following conditions on $g$:

($\mathcal{G}_1$) there are constants $\rho > 0$ and $0 < \varepsilon R_1 < \frac{1}{2}\min\{1, m_0\}$ such that $G(y) \leq R_1|y|^2$, for all $y \in \mathbb{R}$ with $|y| \leq \rho$;

($\mathcal{G}_2$) there are constants $\rho_1 > 0$, $\delta_1 > 0$ and $\alpha_1 > \gamma$ such that $G(y) \geq \delta_1|y|^{\alpha_1}$ for all $y \in \mathbb{R}$ with $|y| \geq \rho$;

($\mathcal{G}_3$) there is a constant $\beta > 2$, $\gamma_2 \geq 0$ and $0 < \alpha_2 < 2$ such that $\nu G(\xi) - \xi g(\xi) \leq \gamma_2|y|^{\alpha_2}$.

*Remark* 16. In the future, it may be of interest to continue the research of this paper in this line, extending the study to the case that $g$ is not a continuous function, but discontinuous, starting from the case that $g$ is a function of vanishing mean oscillation.

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