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CONSUMPTION BASED OTHER-REGARDING PREFERENCES IN GENERAL EQUILIBRIUM: EXISTENCE

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Abstract. Following the works of Kranich and Mercier Ythier, this paper addresses the existence problem in a general equilibrium model featuring other-regarding preferences and allowing for transfers of goods between individuals. Our main contribution is to identify and discuss several economic and mathematical issues present in the model proposed in [12] and to provide a consistent proof of the existence of equilibrium. Specifically, we define an auxiliary concept of equilibrium, construct a generalized game associated with the economy under consideration, demonstrate the existence of an equilibrium in this game - shown to be an auxiliary equilibrium for the economy - and finally prove that these results imply the existence of the originally defined economic equilibrium.

1. **Introduction.** Following the works [7] and [12], the present paper studies the existence problem in a general equilibrium model with other-regarding preferences and allowing for transfers of goods between individuals through a generalized game approach.

The standard general equilibrium model analyzes an abstract economic environment in which households own goods or commodities and trade them to maximize their utility from consumption. A fundamental goal of the analysis is to show existence of the so-called equilibrium prices, i.e., prices at which agents exchange goods according to their individual interests, and resources are sufficient to meet aggregate demand without any waste of desirable commodities. While the foundations of general equilibrium theory trace back to Walras in the late 19th century, it was only in 1954 that [1] and [10] provided a mathematically rigorous treatment of the model. They showed that under some conditions, for any economy, equilibria exist and associated distributions of resources are efficient.

The general equilibrium model is a highly simplified version of the real economic systems. Since the 1960s, many efforts have been made to enrich the Arrow-Debreu-McKenzie model by introducing "imperfections" that capture more realistic economic features.

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The model presented in this paper contributes to this broader effort by exploring the role of utility functions that depend not only on personal consumption but also on the welfare of others - measured, for example, by their wealth or consumption.[1]

In the well-known general equilibrium textbook [2], a proof of existence is presented for a model with production and preferences described by utility functions. Paper [13] provides a proof of existence in an exchange economy model where preferences are other-regarding (or interdependent), price-dependent and not necessarily transitive or complete.

While it is well recognized that households may care about the well-being of others, a simple yet often overlooked implication of this assumption is that such concern can manifest as altruistic behavior—such as voluntarily transferring goods or resources to other households. Consistent with this natural observation, a distinct strand of the literature not only acknowledges the importance of other-regarding preferences but also considers a natural consequence: the possibility of transferring resources to individuals one cares about. The first work to explore this is [7], which presents a pure exchange model with a finite number of goods and households. This model introduces two key departures from the standard framework: (i) each household's utility depends not only on personal consumption but also on the wealth of others, and (ii) households can promise transfers to others, under the assumption that these promises are binding. Later, [12] proposed a model incorporating transfers, where other-regarding preferences are based on others' consumption utility.

The contribution of our paper is to point out and discuss some economic and mathematical problems contained in the model presented in [12] and to present a consistent proof of existence of equilibrium. The main problems in [12] are as follows.

It is well known that the quasi-concavity of utility functions is essential for applying standard existence theorems, yet this assumption becomes problematic when the utility function includes arguments representing bads - for instance, the consumption of people one dislikes. In [12], quasi-concavity is assumed even with respect to others' utility levels, under the strong assumption that every household likes every other household. We address this issue by noting that if a household dislikes another, it will simply choose not to transfer any resources to that individual. A formal treatment of this argument is given in Remark 2.1 below.

The analysis of the household maximization problems does not acknowledge the fact that, under the assumptions made in the paper, the constraint set could be empty - see the discussion at the beginning of Section 4 for a detailed account of that crucial point.

Although a proof of equilibrium existence can be obtained by imposing an ad hoc upper bound on the size of possible transfers, [12] attempts to avoid this assumption by imposing a restrictive condition on an endogenous object - a methodologically questionable approach. In contrast, our proof does adopt the ad hoc upper bound, and we compare our assumptions with those in [12] in Appendix 9, where we show that the two sets of assumptions are logically non-comparable.

In a model with other-regarding preferences and transfers, it is crucial to provide a clear and well-motivated definition of equilibrium. The standard and natural choice is to use consumption and transfers as the agents' choice variables. However,

[1]For a review on the existence/nonexistence problem in general equilibrium and a more detailed analysis of the literature on other-regarding preferences in general equilibrium, see [5].

without sufficient justification, [12] reformulates the household maximization problems in terms of excess demands and transfers. In contrast, we present the model using consumption and transfers and then demonstrate the equivalence between our definition and both the one used in [12] and a third formulation we provide. We believe that our equivalent characterizations clarify the definition of equilibrium.

The structure of the paper is as follows. Section 2 introduces the model. Section 3 presents various notions of equilibrium and proves their equivalence. In Section 4, we address a fundamental issue: the potential non-existence of solutions to the maximization problem due to the "robust" possibility of emptiness of the constraint set. To address this, Section 5 introduces an auxiliary concept of equilibrium and a generalized game associated with the economy. We prove that this game admits an equilibrium that also qualifies as an auxiliary equilibrium of the original economy and show that this implies the existence of the original equilibrium. In Section 6, we present an example of our model applied to a two-household, single-good Cobb-Douglas economy and describe some simple properties of the equilibria. Appendix A provides proofs of the equivalence between the different equilibrium definitions, while Appendix B compares our assumptions to those made in the seminal work by Mercier Ythier.

2. **Set-up of the model.** We consider an economy with C different commodities, denoted by $c \in \{1, ..., C\} := \mathcal{C}$ and H households, denoted by $h \in \{1, ..., H\} := \mathcal{H}$. For any $h \in \mathcal{H}$, $X_h \subseteq \mathbb{R}^C$ is the consumption set whose elements are $x_h = (x_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $x_h^c \in \mathbb{R}$ denotes the consumption of good c by household h; $E_h \subseteq \mathbb{R}^C$ is the endowment set with generic element $e_h = (e_h^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$, where $e_h^c \in \mathbb{R}$ denotes the amount of good c owned by household h. Households can transfer vector of goods to other households: $t_{hh'}^c \in \mathbb{R}$ denotes the transfer of good c from household h to household h', and $t_{hh'} := (t_{hh'}^c)_{c \in \mathcal{C}} \in \mathbb{R}^C$. Commodities can be exchanged with other commodities at exchange ratios described by a price vector p belonging to a price set $P \subseteq \mathbb{R}^C$. For "physical/biological" reasons, we assume non-negativity of the consumption. Using a standard "survival assumption", we endow households with a strictly positive vector of goods. For institutional reasons (households are not allowed to "steal" goods), we assume non-negativity of the transfer vectors. Consistently with a monotonicity assumption with respect the consumption of goods we are going to impose on the utility function, we restrict prices to belong to $\mathbb{R}_+^C \backslash \{0\}$.

Summarizing, we assume that for any $h \in \mathcal{H}$, $X_h = \mathbb{R}_+^C$, $E_h = \mathbb{R}_{++}^C$ and $P = \mathbb{R}_+^C \backslash \{0\}$.

Moreover, we assume that the utility function of each household depends on its own consumption as well as the consumption of other households. To proceed with our argument, we need to define the set of individuals that household h likes. Specifically, we assume that for any $h \in \mathcal{H}$, there exists an exogenously given partition of $\mathcal{H} \backslash \{h\}$ in the two sets $\mathcal{B}_h$ and $\mathcal{B}_h^\backslash$ such that household $h's$ utility is strictly increasing in $x_{h'}$ for any $h' \in \mathcal{B}_h$ and decreasing in $x_{h'}$ for any $h' \in \mathcal{B}_h^\backslash$. We refer to $\mathcal{B}_h$ and $\mathcal{B}_h^\backslash$ as the sets of individuals that household h likes and dislikes, respectively.

Assumption u1. *For any $h \in \mathcal{H}$, there exists a set $\mathcal{B}_h \subseteq \mathcal{H} \backslash \{h\}$ with cardinality B_h and an associated utility function*

$$u_h : \mathbb{R}_+^C \times \mathbb{R}_+^{CB_h} \to \mathbb{R}, \quad \left(x_h, (x_{h'})_{h' \in \mathcal{B}_h}\right) \longmapsto u_h\left(x_h, (x_{h'})_{h' \in \mathcal{B}_h}\right) \tag{1}$$

which is strictly increasing.

Remark 2.1. This assumption captures the notion that households derive utility not only from their own consumption bundles but also from the consumption profiles of a selected subset of other agents in the economy. The exclusion of the consumption of agents in $\mathcal{B}_h^{\setminus}$ - those whom household h dislikes - from the domain of the utility function is without loss of generality: if household h derives disutility from another agent's consumption, it will not engage in transfers toward that agent. Hence, from a modeling perspective focused on characterizing equilibrium existence, the omission of disliked agents' consumption from the utility representation does not affect the generality of the results. This modeling choice is rigorously justified and formally established within the alternative framework developed in [5], and the core arguments of that proof extend directly to the present setting.

To apply Proposition 4.1 below, we also make the following assumption - which is discussed after the statement of that Proposition.

Assumption u2. *The function u_h is concave and there exists $L \in \mathbb{R}_{++}$ such that u_h is L-Lipschitz continuous.*

Moreover, it can be proved that, without imposing an artificial upper bound on transfers, equilibria may fail to exist - see for example [5], Section 4. Therefore, we introduce the following Assumption.

Assumption t1. *For any $h \in \mathcal{H}$, there exists an upper bound $k_h = (k_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{CB_h}_{++}$ such that $t_h \leq k_h$.*

For any $h \in \mathcal{H}$, we define $\mathcal{B}_{\to h} = \{h' \in \mathcal{H} : h \in \mathcal{B}_{h'}\}$, the set of households who like h, $\mathcal{B} = (\mathcal{B}_h)_{h \in \mathcal{H}} \in \times_{h \in \mathcal{H}} \mathcal{P}(\mathcal{H} \backslash \{h\})$, where $\mathcal{P}(\mathcal{H} \backslash \{h\})$ is the family of all subsets of $\mathcal{H} \backslash \{h\}$, $B_h = \# \mathcal{B}_h$, $B = \sum_{h \in \mathcal{H}} B_h$, $B_{\to h} = \# \mathcal{B}_{\to h}$. Moreover

$$t_h = (t_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{CB_h},$$

$$t_{\setminus h} = \left(t_{\mathcal{B}_{h'}}\right)_{h' \neq h} = ((t_{h'h''})_{h'' \in \mathcal{B}_{h'}})_{h' \in \mathcal{H} \backslash \{h\}} \in \mathbb{R}^{C \sum_{h' \neq h} B_{h'}}$$

$$t_{\to h} = \sum_{h' \in \mathcal{B}_{\to h}} t_{h'h}, \quad t_{h \to} = \sum_{h' \in \mathcal{B}_h} t_{hh'}, \quad t = (t_h, t_{\setminus h}) = (t_h)_{h \in \mathcal{H}} \in \mathbb{R}^{CB},$$

$$t_{\to h' \neg h} = \sum_{h'' \in \mathcal{B}_{\to h'} \backslash \{h\}} t_{h''h'},$$

where the last symbol denotes the total transfer to h' excluding the transfer from h to h'. That variable is useful when we write the maximization problem for household h and we need to make clear that $t_{hh'}$ is a variable chosen by household h.

A distinguishing feature of other-regarding preferences *with the possibility of transfers* is that household h knows she can influence $x_{h'}$ by choosing $t_{hh'}$. In solving her maximization problem, household h takes as given the choices of other households $(x^*_{h'}, t^*_{h'})_{h' \neq h}$, as well as their expectations $t^e_{hh'}$ regarding her transfers. However, she also knows that she may choose a transfer vector $t_{hh'}$ that differs from what household h' expects. Therefore, from household h 's viewpoint, the consumption vector of household h' is

$$x^*_{h'} - t^e_{hh'} + t_{hh'},$$

i.e., what h expects h' to consume, minus what h' expects to receive from h, plus what h decides to transfer to h'.

For any $n \in \mathbb{N}$, $x, y \in \mathbb{R}^n$ such that $x << y$, we define $[x, y] = \{z \in \mathbb{R}^n : x \leq z \leq y\}$.[2]

We can now present the definition of equilibrium we are going to show existence of.

Definition 2.2. An economy is $\mathcal{E} := (e_h, u_h, k_h)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} \left(\mathbb{R}^C_{++} \times \mathcal{U} \times \mathbb{R}^{CB_h}_{++}\right) :$

$= \mathbb{E}$, where $\mathcal{U}$ is the set of utility functions satisfying Assumption u1 and u2.

Remark 2.3. Since households choices are invariant with respect to strictly positive scalar multiplications of the price vector - see maximization problem below - we are going to restrict the price set to $\Delta := \left\{p \in \mathbb{R}^C_+ : \sum_{c\in\mathcal{C}} p^c = 1\right\}$.

Definition 2.4. The vector $(x^*, t^*, t^e, p^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \Delta$ is an **allocation**-transfer-price equilibrium for the economy $(e_h, u_h, k_h)_{h\in\mathcal{H}} \in \mathbb{E}$,

if

1. household h maximizes, i.e., for any $h \in \mathcal{H}$, for given $\mathcal{E} \in \mathbb{E}$, $p^* \in \Delta$, $t^*_{\backslash h} \in \times_{h'\in\mathcal{H}\backslash\{h\}} [0, k_{h'}]$, $(x^*_{h'})_{h'\in\mathcal{B}_h} \in \mathbb{R}^{CB_h}_+$, $t^e_h \in [0, k_h]$, $(x^*_h, t^*_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ solves

$$\begin{array}{ll} \max_{(x_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} & u_h\left(x_h, \ (x^*_{h'} - t^e_{hh'} + t_{hh'})_{h'\in\mathcal{B}_h}\right) \\ \text{s.t.} & (x_h, t_h) \in \Gamma_h(p^*, t^*_{\backslash h}, (x^*_{h'})_{h'\in\mathcal{B}_h}, t^e_h), \end{array} \quad (2)$$

where

$$\Gamma_h : \Delta \times \left(\times_{h'\in\mathcal{H}\backslash\{h\}} [0, k_{h'}]\right) \times \mathbb{R}^{CB_h}_+ \times [0, k_h] \longrightarrow\longrightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h},$$

$$\begin{array}{lll} (p^*, t^*_{\backslash h}, (x^*_{h'})_{h'\in\mathcal{B}_h}, t^e_h) \mapsto\mapsto & \{ & (x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \\ & & -p^*\left(x_h - (e_h + t^*_{\to h} - t_{h\to})\right) \geq 0 \\ & & x_h \geq 0, \\ & & t_h \geq 0, \\ & & t_h \leq k_h, \\ & & x^*_{h'} - t^e_{hh'} + t_{hh'} \geq 0, \text{ for any } h' \in \mathcal{B}_h \quad \} \end{array}$$

2. Markets clear, i.e., $\sum_{h\in\mathcal{H}} (x^*_h - e_h) = 0$.

3. Expectations are fulfilled, i.e., $t^e = t^*$.

For any $h \in \mathcal{H}$, define $x_{\mathcal{B}_h} = (x_{h'})_{h'\in\mathcal{B}_h}$.

Remark 2.5. Note that the last constraint in the definition of Γ_h arises directly from the definition of the utility function's domain, which requires consumption vectors to be non-negative. Typically, conditions ensuring that choice variables lie within the domain of the objective function are not explicitly reiterated in the definition of the constraint set. However, in some of the literature on the model we are analyzing, this convention has led to that requirement being overlooked. For this reason, we choose to state it explicitly in the definition of Γ_h.

The reason why the above definition contains no "market clearing conditions for transfers" is explained in the result below, whose proof is straightforward.

[2]Following standard notation, for vectors $y := (y_i)^n_{i=1}, z := (z_i)^n_{i=1} \in \mathbb{R}^n$: $y \geq z$ means that for $i \in \{1, ..., n\}$, $y_i \geq z_i$; $y >> z$ means that for $i \in \{1, ..., n\}$, $y_i > z_i$ and $y > z$ means that $y \geq z$ but $y \neq z$.

Proposition 2.6. *For any* $t \in \mathbb{R}^{CB}$,

$$\sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_h} t_{hh'} = \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_{\to h}} t_{h'h}. \tag{3}$$

3. **Equivalent definitions of equilibrium.** In this section we introduce two formally different definitions of equilibrium and we state they are equivalent to Definition 2.4.

Following [12], we introduce the definition of excess demand which takes into account the presence of the transfers, i.e., we define, for any $h \in \mathcal{H}$,

$$z_h = x_h - e_h - \sum_{h'\in\mathcal{B}_{\to h}} t_{h'h} + \sum_{h'\in\mathcal{B}_h} t_{hh'} = x_h - e_h - t_{\to h} + t_{h\to} \in \mathbb{R}^C.$$

We can then give a definition of equilibrium in terms of that newly introduced variable.

Definition 3.1. $(z^*, t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB}_+ \times \Delta$ is an **excess demand**-transfer-price equilibrium for the economy $\mathcal{E} \in \mathbb{E}$ if

(i) households maximize, i.e., $\forall h \in \mathcal{H}$, for given $(z^*_{h'})_{h'\in\mathcal{B}_h} \in \mathbb{R}^{CB_h}$, $p^* \in \Delta$, $t^*_{\backslash h} \in \times_{h'\in\mathcal{H}\backslash\{h\}}[0, k_{h'}], e \in \mathbb{R}^{CH}_{++}$ and $u_h \in \mathcal{U}$, (z^*_h, t^*_h) solves the problem

$$\begin{array}{ll} \max_{(z_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} & u_h(z_h + e_h + t^*_{\to h} - t_{h\to}, (z^*_{h'} + e_{h'} + t_{hh'} + t^*_{\to h'\neg h} - t^*_{h'\to})_{h'\in\mathcal{B}_h}) \\ \text{s.t.} & (z_h, t_h) \in \Psi_h(p^*, t^*_{\backslash h}, (z^*_{h'})_{h'\in\mathcal{B}_h}), \end{array} \tag{4}$$

where

$$\Psi_h : \Delta \times (\times_{h'\in\mathcal{H}\backslash\{h\}}[0, k_{h'}]) \times \mathbb{R}^{CB_h} \to\to \mathbb{R}^C \times \mathbb{R}^{CB_h}_+,$$

$$(p^*, t^*_{\backslash h}, (z^*_{h'})_{h'\in\mathcal{B}_h})$$

$$\begin{array}{lll} \mapsto\mapsto & \{ & (z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}_+ : \\ & & -p^* z_h \geq 0 \\ & & z_h + e_h + t^*_{\to h} - t_{h\to} \geq 0 \\ & & t_h \geq 0 \\ & & t_h \leq k_h \\ & & z^*_{h'} + e_{h'} + t_{hh'} + t^*_{\to h'\neg h} - t^*_{h'\to} \geq 0 \text{ for any } h' \in \mathcal{B}_h \} \end{array}$$

(ii) z^* satisfies the market clearing conditions $\sum_{h\in\mathcal{H}} z^*_h = 0$.

Proposition 3.2. *1. If* $(x^*, t^*, t^e, p^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \Delta$ *is an* ***allocation****-transfer-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$*, then* $(z^*, t^*, p^*) := ((x^*_h - e_h - t^*_{\to h} + t^*_{h\to})_{h\in\mathcal{H}}), t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB}_+ \times \Delta$ *is an* ***excess demand****-transfer-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$.

2. If $(z^*, t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB}_+ \times \Delta$ *is an* ***excess demand****-transfer-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$*, then* $(x^*, t^*, t^e, p^*) := ((z^*_h + e_h + t^*_{\to h} - t^*_{h\to})_{h\in\mathcal{H}}, t^*, t^*, p^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \Delta$ *is an* ***allocation****-transfer-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$.

Proof. See Appendix A. □

Since we recognize that the very definition of equilibrium for consumption-based other-regarding preferences can be difficult to grasp, we provide a more detailed explanation of the concept. As said before, in solving her maximization problem, household h takes as given the values $x^*_{h'}$ and $t^e_{hh'}$ and chooses the transfer $t_{hh'}$. From household h's viewpoint, the consumption of household h' is $x^*_{h'} - t^e_{hh'} + t_{hh'}$, where the first two terms are taken for given by household h, while the third term is her choice. Defining

$$y_{hh'} := x^*_{h'} - t^e_{hh'},$$

we interpret $y_{hh'}$ as a kind of "minimum consumption" that household h' can secure regardless of household h's decisions. In line with these remarks, we introduce a third definition of equilibrium, which we prove to be equivalent to the previous two.

Definition 3.3. The vector $(x^*, t^*, y^*, p^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \Delta$ is an **allocation-transfer-minimum consumption-price** equilibrium for the economy $\mathcal{E} \in \mathbb{E}$.

if 1. for any $h \in \mathcal{H}$, household h maximizes, i.e., for given $\mathcal{E} \in \mathbb{E}$, $p^* \in \Delta, t^*_{\setminus h} \in \times_{h' \in \mathcal{H}\setminus\{h\}} [0, k_{h'}]$, $(\, y^*_{hh'})_{h' \in \mathcal{B}_h} \in \mathbb{R}^{CB_h}_+$,

$(x^*_h, t^*_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ solves the problem

$$\begin{array}{ll} \max_{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}} & u_h(x_h, (y^*_{hh'} + t_{hh'})_{h' \in \mathcal{B}_h}) \\ \text{s.t.} & (x_h, t_h) \in \Phi_h(p^*, t^*_{\setminus h}, (\, y^*_{hh'})_{h' \in \mathcal{B}_h}), \end{array} \tag{5}$$

where

$$\begin{array}{ll} \Phi_h : \Delta \times \left(\times_{h' \in \mathcal{H}\setminus\{h\}} [0, k_{h'}]\right) \times \mathbb{R}^{CB_h} \longrightarrow\!\!\!\!\longrightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h}, & \\ \Phi_h(p^*, t^*_{\setminus h}, (\, y^*_{hh'})_{h' \in \mathcal{B}_h}) \mapsto\!\!\!\!\mapsto & \{(x_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \\ & -p^* \left(x_h - (e_h + t^*_{\to h} - t_{h\to})\right) \geq 0 \\ & x_h \geq 0 \\ & t_h \geq 0 \\ & t_h \leq k_h, \\ & y^*_{hh'} + t_{hh'} \geq 0, \quad \text{for any } \; h' \in \mathcal{B}_h\} \end{array}$$

2. Markets clear, i.e., $\sum_{h \in \mathcal{H}} (x^*_h - e_h) = 0$.
3. Expectations are fulfilled, i.e., for any $h \in \mathcal{H}$, $h' \in \mathcal{B}_h$, $y^*_{hh'} = x^*_{h'} - t^*_{hh'}$.

The following proposition states the equivalence between Definitions 3.3 and 2.4, whose proof is similar to the proof of Proposition 3.2 and therefore omitted.

Proposition 3.4. *1. If* $(x^*, t^*, t^e, p^*) \in \mathbb{R}^{CH}_+ \times \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \Delta$ *is an* ***allocation****-transfer-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$*, then* $(x^*, t^*, y^*, p^*) := (x^*, t^*, ((x^*_{h'} - t^*_{hh'})_{h' \in \mathcal{B}_h})_{h \in \mathcal{H}}, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB} \times \mathbb{R}^{CB} \times \Delta$ *is an allocation-transfer-* ***minimum consumption****-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$*.*

2. If $(x^*, t^*, y^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB} \times \mathbb{R}^{CB} \times \Delta$ *is an allocation-transfer-* ***minimum consumption****-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$*, then* $(x^*, t^*, t^e, p^*) := (((y^*_{hh'} - t^*_{hh'})_{h' \in \mathcal{B}_h})_{h \in \mathcal{H}}, t^*, t^*, \; p^*) \in \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \mathbb{R}^{CB}_+ \times \Delta$ *is an* ***allocation****-transfer-price equilibrium for the economy* $\mathcal{E} \in \mathbb{E}$*.*

4. **Emptyness of the constraint set and extension of the utility functions.**

It is easy to see that for many values of the variables taken as given by household h, the constraint set may be empty: in particular, household h might be unable to deliver the excessively high transfer levels expected by another household h'. To illustrate this, consider an economy with three households who like everybody else and a single commodity. Assume that, for any $h \in \{1, 2, 3\}$, $k_h = 4$. We now describe the constraint set of the maximization problem for household $h = 1$. For given $e_1, e_2, e_3 \in \mathbb{R}_{++}$, $t_2, t_3 \in \mathbb{R}^2_+$, $p = 1$, we have

$$\begin{aligned}
&\Gamma_1(p, t_{21}, t_{23}, t_{31}, t_{32}, x_2, x_3, t_1^e) = \{\ (x_1, t_{12}, t_{13}) \in \mathbb{R}^3 : \\
&x_1 + t_{12} + t_{13} \leq e_1 + t_{21} + t_{31}, \quad x_1 \geq 0, \\
&t_{12}, t_{13} \in [0, 4]\,, \\
&x_2 - t_{12}^e + t_{12} \geq 0, \quad x_3 - t_{13}^e + t_{13} \geq 0\ \}\,.
\end{aligned}$$

Indeed, the budget constraint and the non-negativity constraint on household 2's consumption may not be satisfied if t_{12}^e is large and household 1's budget constraint does not allow her to choose t_{12} to compensate. For example, if $e_1 = 1$, $t_{21} = t_{31} = 0$, $x_2 = 1$, $t_{12}^e = 3$, then

$$\begin{aligned}
&x_1 + t_{12} + t_{13} \leq 1 \text{ and } t_{12} \leq 1; \\
&1 - 3 + t_{12} \geq 0 \text{ and } t_{12} \geq 2.
\end{aligned}$$

Our existence result implies that those kinds of extreme expectations, under which the constraint set is empty, do not arise in equilibrium.

The way we choose to show equilibria is presented below. We introduce a fictitious utility function that extends the utility defined in (1), in order to formally "allow negative consumption". We then construct a game associated with "the original economy with the extended utility function"; we show that the game has an equilibrium and finally that equilibrium is an equilibrium of the "true" economy, i.e., equilibrium consumption is indeed positive. The key result used to extend the utility function is as follows.

Proposition 4.1. *(see [11] and Exercise 8.3.4, page 399 in [3]) Let A be a convex subset of a normed space X and let $L \in \mathbb{R}_{++}$. If $g : A \to \mathbb{R}$ is an L-Lipschitz concave function then it admits an L-Lipschitz concave extension G to the whole X.* [3]

Assumption u2 requires that households' utility functions be L-Lipschitz continuous and concave, allowing us to apply Proposition 4.1. We acknowledge that these conditions—based on a cardinalistic approach—are rather strong. In many standard models, continuity and quasi-concavity are sufficient to guarantee the existence of equilibria. However, as discussed above, extending utility functions appears to be unavoidable in our framework. It is also well known that assumptions more compatible with an ordinalistic approach, namely continuity and quasi-concavity, do not, by themselves, ensure the existence of an extension that preserves these properties (see [3], page 7).

Proposition 4.1 and Assumption u2 allow to give the following definition which is used in the proof of existence in next section.

Definition 4.2. For any $h \in \mathcal{H}$, a L-Lipschitz continuous and concave extension of the utility function u_h is defined as follows

$$G_h : \mathbb{R}^C \times \mathbb{R}^{CB_h} \to \mathbb{R}, \quad \left(x_h, (x_{h'})_{h' \in \mathcal{B}_h}\right) \longmapsto G_h\left(x_h, (x_{h'})_{h' \in \mathcal{B}_h}\right) \tag{6}$$

[3]Such an extension G can be defined by the formula

$$G(x) = \sup_{y \in A}\left[g(y) - L\|x - y\|\right], \qquad x \in X.$$

5. **Existence of equilibria.** The main result of the paper is to prove, under Assumptions u1, u2 and t1, the existence of an allocation-transfer-price equilibrium for any economy $\mathcal{E} \in \mathbb{E}$.

To facilitate the comparison with the proof in [12], we introduce the definition of equilibrium in terms of excess demands (see Definition 5.3 below).

Our proof strategy proceeds through the following steps, each corresponding to a subsection of the current section.

1. Preliminaries and definition: we begin with some preliminary remarks and introduce the notion of an excess demand–transfer–price equilibrium, defined using the extended utility function and an upper bound on excess demand (see Definition 5.3.).

2. Generalized game and existence: we define a generalized game associated with the economy and show that it admits a Nash equilibrium (see Proposition 5.13).

3. Equilibrium with upper bound: we demonstrate that the Nash equilibria of the generalized game are such that consumption vectors are non-negative, implying that they correspond to equilibria of the original model with an upper bound on excess demand (see Proposition 5.17).

4. Removing the upper bound: using a standard argument, we show that any equilibrium with an upper bound on excess demand is also an equilibrium without it (see Proposition 5.21). Finally, by applying Proposition 5.21, we obtain the desired existence result.

5.1. **Preliminary remarks and the auxiliary equilibrium.** For reasons explained in the proofs of Remark 5.1, Proposition 5.15 and Lemma 5.12, we introduce the following notation.

$$k' := \sum_{c\in\mathcal{C}} \sum_{h\in\mathcal{H}} \left(e_h^c + \sum_{h'\in\mathcal{B}_{\to h}} k_{h'h}^c \right) + 1 \in \mathbb{R}_{++}. \tag{7}$$

Remark 5.1. For any $p \in \Delta$, any $h \in \mathcal{H}$, any $h' \in \mathcal{B}_h$ any $t = \left((t_{hh'})_{h'\in\mathcal{B}_h} \right)_{h\in\mathcal{H}} \in \times_{h\in\mathcal{H}} \left(\times_{h'\in\mathcal{B}_h} [0, k_{hh'}] \right)$, $w_h(p,t) := p\left(e_h + t_{\to h} - t^*_{h\to}\right) < k'$. Indeed, using the facts $p \in \Delta$ and for any h, h' with $h' \neq h$, $t_{h'h} \leq k_{h'h}$ and $t_{hh'} \geq 0$, we have

$$w_h(p,t) = p\left(e_h + t_{\to h} - t^*_{h\to}\right) \leq p\left(e_h + \textstyle\sum_{h'\in\mathcal{B}_{\to h}} k_{h'h}\right) \leq$$

$$\textstyle\sum_{c\in\mathcal{C}} \left(e_h^c + \sum_{h'\in\mathcal{B}_{\to h}} k_{h'h}^c\right) < \sum_{h\in\mathcal{H}} \sum_{c\in\mathcal{C}} \left(e_h^c + \sum_{h'\in\mathcal{B}_{\to h}} k_{h'h}^c\right) < k'.$$

Remark 5.2. To use Debreu's Theorem - see Theorem 5.6 below - we need to restrict z_h to belong to a compact set. Observe that there is a lower bound on z_h which follows from the constraint $z_h + (e_h + t_{\to h} - t_{h\to}) \geq 0$.

Indeed, defined $\overline{k} \in \mathbb{R}^C_{++}$ as done below

$$t_{\to h} = \sum_{h'\in\mathcal{B}_{\to h}} t_{h'h} \leq \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_{\to h}} k_{h'h} := \overline{k},$$

we get

$$0 \leq z_h + (e_h + t_{\to h} - t_{h\to}) \quad \Rightarrow \quad z_h \geq -e_h - t_{\to h} + t_{h\to} \overset{t_{h\to}\geq 0,\ -t_{\to h} > -\overline{k}}{\geq} -e_h - \overline{k}.$$

We also need to add an *artificial* upper bound on z_h. That bound has to be large enough in order to allow to use a standard trick - see Proposition 5.21.

Defined[4]

$$t_{h\rightarrow} = \sum_{h'\in\mathcal{B}_h} t_{hh'} \leq \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_h} k_{hh'} := \overline{k},$$

we impose the following upper bound constraint

$$z_h = x_h - e_h - t_{\rightarrow h} + t_{h\rightarrow} \leq k'\mathbf{1}_C{}^{5} + \overline{k} := \overline{\overline{k}}.$$

Summarizing, we construct a constraint set in which we impose that z_h belongs to the compact set described by the inequalities

$$-e_h - \overline{k} \leq z_h \leq \overline{\overline{k}}.$$

For future use, observe that $\overline{\overline{k}} >> k'\mathbf{1}_C >> r := \sum_{h\in\mathcal{H}} e_h$.

Definition 5.3. $(z^*, t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB} \times \Delta$ is an **excess demand**-transfer-price equilibrium with upper bound and extended utility function for the economy $\mathcal{E}' := (G_h, e_h, k_h)_{h\in\mathcal{H}} \in \mathcal{U}' \times R^{CH}_{++} \times R^{CB_h}_{++} := \mathbb{E}'$, where $\mathcal{U}'$ is the set of the extended utility functions above defined, if

(i) households maximize, i.e.,

$\forall h \in \mathcal{H}$, for given $(z^*_{h'})_{h'\in\mathcal{B}_h} \in \mathbb{R}^{CB_h}$, $p^* \in \Delta$, $t^*_{\setminus h} \in \times_{h'\in\mathcal{H}\setminus\{h\}}[0, k_{h'}], e \in \mathbb{R}^{CH}_{++}$ and $G_h \in \mathcal{U}'$, (z^*_h, t^*_h) solves the problem

$$\max_{(z_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} G_h(z_h + e_h + t^*_{\rightarrow h} - t_{h\rightarrow}, (z^*_{h'} + e_{h'} + t_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow})_{h'\in\mathcal{B}_h})$$

$$\text{s.t. } (z_h, t_h) \in \widetilde{\Psi}_h(p^*, t^*_{\setminus h}),$$

where

$$\widetilde{\Psi}_h : \Delta \times (\times_{h'\in\mathcal{H}\setminus\{h\}}[0, k_{h'}]) \rightarrow\rightarrow \mathbb{R}^C \times \mathbb{R}^{CB_h}_+,$$

$$(p^*, t^*_{\setminus h}) \mapsto\mapsto \left\{ \begin{array}{l} (z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \\ -p^* z_h \geq 0 \\ z_h + e_h + t^*_{\rightarrow h} - t_{h\rightarrow} \geq 0 \\ t_h \geq 0 \\ t_h \leq k_h \\ z_h \geq -e_h - \overline{k} \\ z_h \leq \overline{\overline{k}} \end{array} \right\}.$$

(ii) z^* satisfies the market clearing conditions, $\sum_{h\in\mathcal{H}} z^*_h = 0$.

5.2. **The generalized game associated with the economy.** We begin by reviewing established results on generalized games, including a standard existence theorem. Next, we formulate a generalized game for our economy and present the associated notion of Nash equilibrium. After presenting some key results, Proposition 5.13 proves the existence of a Nash equilibrium for that generalized game.

Definition 5.4. ([8], page 339) Given $n \in \mathbb{N}$, an n-player generalized game is a triple $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$, where for any $i \in \{1, ..., n\}$,

1. A_i is a set of strategies or actions with generic element a_i;
2. $C_i : A_{\setminus i} := \times_{j\in\{1,...,n\}\setminus\{i\}} A_j \longrightarrow\longrightarrow A_i$, $\quad a_{\setminus i} = (a_j)_{j\neq i} \mapsto\mapsto C_i\left(a_{\setminus i}\right)$ is a constraint set-valued function;

[4]Observe that $\sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_{\rightarrow h=}} k_{h'h} := \sum_{h\in\mathcal{H}} \sum_{h'\in\mathcal{B}_h} k_{hh'} := \overline{k}$ because of the same idea of the proof of Proposition 2.6.

[5]$\mathbf{1}_C$ is the vector of $\mathbb{R}^C$ of all ones.

3. $\pi_i : A := \times_{i\in\{1,..,n\}} A_i \longrightarrow \mathbb{R}$, $a \mapsto u_i(a)$ is a utility or payoff function.

Definition 5.5. A Nash equilibrium for the generalized game $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$ is an n-tuple of actions $a^* := (a_i^*)_{i=1}^n \in A$ such that for any $i \in \{1, ..., n\}$, a_i^* solves the following problem. For given $a_{\setminus i}^* := \left(a_j^*\right)_{j\in\{1,...,n\}\setminus\{i\}} \in A_{\setminus i}$,

$$\max_{a_i \in A_i} \pi_i(a_i, a_{\setminus i}^*) \qquad \text{s.t.} \qquad a_i \in C_i(a_{\setminus i}^*).$$

Theorem 5.6. *([4]. See also [8], page 340) Let a generalized game $\mathcal{G} = \{A_i, C_i, \pi_i\}_{i=1}^n$ be given. If for any $i \in \{1, ..., n\}$,*

1. there exists $n_i \in \mathbb{N}$ such that A_i is a nonempty, compact, convex subset of $\mathbb{R}^{n_i}$;

2. C_i is a non-empty valued, convex valued, lower hemicontinuous and upper hemicontinuous set-valued function;

3. π_i is a continuous function and for any $a_{\setminus i} \in A_{\setminus i}$, the function $\pi_i(\cdot, a_{\setminus i}) : A_i \longrightarrow \mathbb{R}$, $a_i \mapsto \pi_i(a_i, a_{\setminus i})$ is quasi-concave,

then $\mathcal{G}$ has a Nash equilibrium.

We now define the generalized game associated with an economy $\mathcal{E}$ we are going to use.

Definition 5.7. There are $n = 1 + H$ players. For each player $h \in \{0, 1..., H\}$, we describe below the appropriate definition of the triple of 1. set of actions, 2. constraint set-valued functions and 3. utility functions.

1.

$$\begin{array}{rcll} A_0 & = & \Delta \subseteq \mathbb{R}^C & \\ A_h & = & Z_h \times T_h \subseteq \mathbb{R}^{C+B_h\cdot C} & \quad \text{for any } h \in \mathcal{H} \end{array}$$

where for any $h \in \mathcal{H}$,

$$Z_h = \left\{ z_h \in \mathbb{R}^C : -e_h - \overline{k} \le z_h \le \overline{\overline{k}} \right\} \quad \text{and} \quad T_h = \left\{ t_h \in \mathbb{R}^{CB_h} : 0 \le t_h \le k_h \right\}.$$

2.

$$\begin{array}{ll} C_0 : \times_{h\in\mathcal{H}} A_h \to\to A_0 & \\ C_0 : \left(\times_{h\in\mathcal{H}} (Z_h \times T_h)\right) \to\to \Delta, & (z,t) \mapsto\mapsto \Delta \end{array}$$

$$\begin{array}{ll} C_h : A_0 \times \left(\times_{h'\in\mathcal{H}\setminus\{h\}} A_{h'}\right) \to\to A_h & \\ \widehat{\Psi}_h : \Delta \times \left(\times_{h'\in\mathcal{H}\setminus\{h\}} (Z_{h'} \times T_{h'})\right) \to\to Z_h \times T_h, & (p, (z_{h'}, t_{h'})_{h'\in\mathcal{H}\setminus\{h\}}) \mapsto\mapsto \widetilde{\Psi}_h\left(p, t_{\setminus h}\right) \end{array}$$

3.

$$\begin{array}{l} \pi_0 : A_0 \times (\times_{h\in\mathcal{H}} A_h) \to \mathbb{R} \\ \pi_0 : \Delta \times (\times_{h\in\mathcal{H}} (Z_h \times T_h)) \to \mathbb{R} \\ \left(p, (z_h, t_h)_{h\in\mathcal{H}}\right) \to p \cdot \sum_{h\in\mathcal{H}} z_h \end{array}$$

$$\begin{array}{l} \pi_h : A_0 \times (\times_{h\in\mathcal{H}} A_h) \to \mathbb{R} \\ \pi_h : \Delta \times (\times_{h\in\mathcal{H}} (Z_h \times T_h)) \to \mathbb{R} \\ \widetilde{G}_h : \Delta \times (\times_{h\in\mathcal{H}} (Z_h \times T_h)) \to \mathbb{R} \\ \left(p, (z_h, t_h)_{h\in\mathcal{H}}\right) \mapsto G_h \left(\begin{array}{l} (z_h + e_h + t_{\to h} - t_{h\to}), \\ (z_{h'} + e_{h'} + t_{hh'} + t_{\to h'\neg h} - t_{h'\to})_{h'\in\mathcal{B}_h} \end{array} \right) \end{array}$$

Definition 5.8. A Nash equilibrium for the Generalized Game associated with an economy $\mathcal{E}' \in \mathbb{E}'$, as presented in Definition 5.7, is a vector $\left(p^*, (z_h^*, t_h^*)_{h\in\mathcal{H}}\right) \in$

$\Delta \times \left(\times_{h\in\mathcal{H}} \left(Z_h \times T_h\right)\right)$ such that

$$\begin{array}{l} \text{for given } \left(z_h^*, t_h^*\right)_{h\in\mathcal{H}}, \\ p^* \text{ solves } \quad \max_{p\in\Delta} \quad p\cdot\sum_{h\in\mathcal{H}} z_h^*, \end{array}$$

and for any $h \in \mathcal{H}$, for given p^* and $\left(z_{h'}^*, t_{h'}^*\right)_{h'\in\mathcal{H}\setminus\{h\}}$, $\left(z_h^*, t_h^*\right)$ solves

$$\begin{array}{l} \max_{(z_h,t_h)\in(Z_h\times T_h)} \; G_h\left(\begin{array}{l} \left(z_h + e_h + t_{\to h}^* - t_{h\to}^*\right), \\ \left(z_{h'}^* + e_{h'} + t_{hh'} + t_{\to h'\neg h}^* - t_{h'\to}^*\right)_{h'\in\mathcal{B}_h} \end{array}\right) \\ \text{s.t.} \\ \left(z_h, t_h\right) \in \widehat{\Psi}_h\left(p^*, \left(z_{h'}^*, t_{h'}^*\right)_{h'\in\mathcal{H}\setminus\{h\}}\right) = \widetilde{\Psi}_h\left(p^*, t_{\setminus h}^*\right). \end{array}$$

Before proving the existence of a Nash equilibrium for the generalized game presented in Definition 5.7, we need the following preliminary results, whose proof is straightforward.

Lemma 5.9. *Let X_1, X_2 and Y be metric spaces and $\varphi : X_1 \to\to Y$, $x_1 \longmapsto\longmapsto \varphi(x_1)$ and $\psi : X_1 \times X_2 \to\to Y$, $(x_1, x_2) \longmapsto\longmapsto \varphi(x_1)$ be set valued functions.*

Then if φ satisfies any of the properties listed below, then ψ does as well: 1. non-empty valued; 2. convex valued; 3. closed; 4. compact valued; 5. lower hemi-continuous; 6. upper hemi-continuous.

To prove crucial properties of the set-valued function $\widetilde{\Psi}_h$ which are presented in Lemma 5.12 below, we need the following definition and proposition.

Definition 5.10. Let a nonempty, convex subset X of $\mathbb{R}^n$ and a function $f : X \to \mathbb{R}$ be given. We say that f is

- locally nonsatiated if $\forall x \in X$ and $\forall \varepsilon > 0$, $\exists x' \in B(x, \varepsilon) \cap X$ such that $f(x') > f(x)$;
- nonsatiated if $\forall x \in X$ $\exists x' \in X$ such that $f(x') > f(x)$;
- semistrictly quasi-concave if for any $x, y \in X$ and any $\lambda \in (0, 1)$, $f(x) > f(y) \Rightarrow f((1-\lambda)x + \lambda y) > f(y)$.

Proposition 5.11. *(see [5]) Let a subset Π of $\mathbb{R}^p$ and a function $f : \Pi \times \mathbb{R}^C \longrightarrow \mathbb{R}^m$, $x \mapsto (f_j(\pi, x))_{j=1}^m$ be given (with $f_j : \Pi \times \mathbb{R}^C \longrightarrow \mathbb{R}$). Let also the following set valued function be given.*

$$\begin{array}{l} B : \Pi \longrightarrow\longrightarrow \mathbb{R}^C, \quad \pi \mapsto\mapsto \left\{x \in \mathbb{R}^C : f(\pi, x) \geq 0\right\}; \\ \widetilde{B} : \Pi \longrightarrow\longrightarrow \mathbb{R}^C, \quad \pi \mapsto\mapsto \left\{x \in \mathbb{R}^C : f(\pi, x) >> 0\right\}. \end{array}$$

I.) If $\widetilde{B}$ is non-empty valued, f is continuous and for any $j = 1, ...m$ and for any $\pi \in \Pi$, either $f_j(\pi, \cdot)$[6] is nonsatiated and semistrictly quasi-concave, or $f_j(\pi, \cdot)$ is locally nonsatiated and quasi-concave, then B is non-empty valued convex valued, closed graph and lower hemicontinuous.

II.) If in addition either a. B is compact valued or b. $Im(B)$ is contained in a compact set, then B is upper hemicontinuous.

Lemma 5.12. *For any $h \in \mathcal{H}$, the set-valued function $\widetilde{\Psi}_h$, defined in Definition 5.3, is non-empty valued; convex valued; closed; compact valued and $Im(\widetilde{\Psi}_h) \subseteq Z_h \times T_h$; lower hemi-continuous and upper hemi-continuous.*

[6] The function $f_j(\pi, \cdot)$, for any $j = 1, ..., m$, is defined on $\mathbb{R}^C$ with values in $\mathbb{R}$.

Proof. Define

$$\widetilde{\widetilde{\Psi}}_h : \Delta \times \mathbb{R}^{CB_h} \to\to \mathbb{R}^C \times \mathbb{R}^{CB_h},$$
$$\widetilde{\widetilde{\Psi}}_h \left(p, t_{\setminus h}\right) = \{(z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \begin{array}{l} -pz_h > 0 \\ z_h + e_h + t_{\to h} - t_{h\to} >> 0 \\ t_h >> 0 \\ t_h << k_h \\ z_h >> -e_h - \overline{k} \\ z_h << \overline{\overline{k}} \end{array} \}. \tag{8}$$

and

$$f : Z_h \times T_h \times \Delta \times T_{\setminus h} \to \mathbb{R} \times \mathbb{R}^C \times \mathbb{R}^C \times \mathbb{R}^C \times \mathbb{R}^{CB_h} \times \mathbb{R}^{CB_h}$$

$$\left((z_h, t_h), \left(p, t_{\setminus h}\right)\right) \longmapsto \text{ Left Hand Side of inequalities in (8).}$$

To get the desired result, we have to check that the Assumptions of Proposition 5.11 are satisfied.

More precisely, we have to check that

1. $\widetilde{\widetilde{\Psi}}_h$ is nonempty valued;
2. f is continuous and for any $j = 1, ...m$, fixed $(p, t_{\setminus h}) \in \Delta \times T_{\setminus h}$, $f_j(\cdot, (p, t_{\setminus h}))$ is locally nonsatiated and quasi-concave;
3. either $\widetilde{\Psi}_h$ is compact valued or b. $\text{Im}(\widetilde{\Psi}_h)$ is contained in a compact set.

1. Take

$$\widetilde{z}_h = -\frac{e_h}{2H} << 0, \quad \text{for any } c \in \mathcal{C}, \text{ for any } h' \in \mathcal{B}_h, \ \ \widetilde{t}^c_{hh'} = \min\left\{\frac{k^c_{hh'}}{2}, \frac{e^c_h}{H}\right\} \in (0, k^c_{hh'}),$$

then, all inequalities in the definition of $\widetilde{\widetilde{\Psi}}_h$ are clearly satisfied but the second one, which is verified below:

$$\widetilde{z}_h + e_h + \sum_{h' \in \mathcal{B}_{\to h}} \widetilde{t}_{h'h} - \sum_{h' \in \mathcal{B}_h} \widetilde{t}_{hh'} \geq -\frac{e_h}{2H} + e_h - \sum_{h' \in \mathcal{B}_h} \frac{e_h}{H} = \frac{e_h}{H}(-\frac{1}{2} + H - B_h) >> 0,$$

since $B_h \leq H - 1$ and then $-\frac{1}{2} + H - B_h \geq \frac{1}{2}$.

2. f is clearly continuous and any component function of f for fixed $(p, t_{\setminus h})$ is affine and not constant, a fact which implies the desired assumptions.

3. Since $\widetilde{\Psi}_h$ is defined in terms of weak inequalities via continuous function, it is closed valued. Moreover, $\text{Im}(\widetilde{\Psi}_h) \subseteq Z_h \times T_h$, where $Z_h \times T_h$ is a compact set. Since closed subsets of compact sets are compact, the desired result follows. □

Proposition 5.13. *For any economy $\mathcal{E}' \in \mathbb{E}'$, the generalized game*

$$\left((\Delta, (Z_h \times T_h)_{h\in\mathcal{H}}), \left(C_0, (\widehat{\Psi}_h)_{h\in\mathcal{H}}\right), (\pi_0, (G_h)_{h\in\mathcal{H}})\right)$$

presented above has a Nash equilibrium (p^*, z^*, t^*).

Proof. We show that the Assumptions of Theorem 5.6 are verified.

1. there exists $n_i \in \mathbb{N}$ such that A_i is a nonempty, compact, convex subset of $\mathbb{R}^{n_i}$.

$A_0 = \Delta$ satisfies the needed assumptions.

For any $h \in \mathcal{H}$, $Z_h \times T_h$ satisfies the needed assumptions by definition.

3. π_i is continuous and for any $a_{\setminus i} \in A_{\setminus i}$, the function $\pi_i\left(\cdot, a_{\setminus i}\right) : A_i \to \mathbb{R}$, $a_i \mapsto \pi_i\left(a_i, a_{\setminus i}\right)$ is quasi-concave.

For given $(z_h)_{h\in\mathcal{H}}$, π_0 is linear in p and therefore concave and quasi-concave.

By assumption, G_h is concave. We can show that $\widetilde{G}_h : \Delta \times (\times_{h \in \mathcal{H}} (Z_h \times T_h)) \to \mathbb{R}$,

$$(p, z, t) \mapsto G_h \left(\begin{array}{l} (z_h + e_h + t_{\to h} - t_{h\to}), \\ (z_{h'} + e_{h'} + t_{hh'} + t_{\to h' \neg h} - t_{h' \to})_{h' \in \mathcal{B}_h} \end{array} \right)$$

is concave in (z_h, t_h).

2. C_i is a non-empty value, convex valued, lower hemicontinuous and upper hemicontinuous set-valued function.

By definition of Δ and since $C_0 : (\times_{h \in \mathcal{H}} (Z_h \times T_h)) \to \Delta, (z, t) \mapsto\mapsto \Delta$, the desired results follow. Indeed, C_0 is the constant set valued function and Δ is a compact nonempty set.

Verification of the needed properties for $\widehat{\Psi}_h$ goes through two steps: 1. If $\widetilde{\Psi}_h$ has the desired properties, then $\widehat{\Psi}_h$ has the desired properties - see Lemma 5.9; 2. $\widetilde{\Psi}_h$ has the desired properties - see Lemma 5.12. □

5.3. **Equilibria of the game and equilibria with upper bound on excess demand.** Having established the existence of equilibria in a suitably defined generalized game - formulated using the extended utility function - our next objective is to show that these equilibrium allocations lie within the domain of the original (unextended) utility function. This result is established in Lemma 5.14. Furthermore, after proving Walras' Law, we demonstrate that the equilibria of the generalized game are equilibria in the original economy with an artificial upper bound on excess demand. To this end, we introduce some preliminary definitions.

Define $\xi_h : \Delta \times (\times_{h \in \mathcal{H}} (Z_h \times T_h)) \to \mathbb{R}^C_+ \times \mathbb{R}^{B_h}$, such that

$$\begin{array}{l} (p, z_h, z_{\setminus h}, t_h, t_{\setminus h}) \mapsto \\ \left((z_h + e_h + t_{\to h} - t_{h \to}), (z_{h'} + e_{h'} + t_{hh'} + t_{\to h' \neg h} - t_{h' \to})_{h' \in \mathcal{B}_h} \right). \end{array} \tag{9}$$

For any $\left(z^*_{\setminus h} = (z^*_{h'})_{h' \in \mathcal{H} \setminus \{h\}}, p^*, t^*_{\setminus h} \right) \in \mathbb{R}^{C(H-1)} \times \Delta \times T_{\setminus h}$,

$$\begin{array}{l} \xi_{h | \left(z^*_{\setminus h}, p^*, t^*_{\setminus h} \right)} := \xi_h \left(\cdot ; z^*_{\setminus h}, p^*, t^*_{\setminus h} \right) : Z_h \times T_h \to \mathbb{R}^C \times \mathbb{R}^{CB_h}, \\ (z_h, t_h) \mapsto \xi_h \left((z_h, t_h) ; z^*_{\setminus h}, p^*, t^*_{\setminus h} \right). \end{array}$$

Lemma 5.14. *If (z^*, t^*, p^*) is a Nash equilibrium as presented in Definition 5.8, then for any $h \in \mathcal{H}$,*

1. for any $h' \in \mathcal{B}_h$,

$$z^*_{h'} + e_{h'} + t^*_{hh'} + t^*_{\to h' \neg h} - t^*_{h' \to} \geq 0.$$

2.

$$\left(G_h \circ \xi_{h | \left(z^*_{\setminus h}, p^*, t^*_{\setminus h} \right)} \right)_{| \widetilde{\Psi}_h \left(p^*, t^*_{\setminus h} \right)} = \left(u_h \circ \xi_{h | \left(z^*_{\setminus h}, p^*, t^*_{\setminus h} \right)} \right)_{| \widetilde{\Psi}_h \left(p^*, t^*_{\setminus h} \right)}.$$

Proof. 1. Observe that for any $h \in \mathcal{H}$, since $\widehat{\Psi}_h \left(p^*, (z^*_{h'}, t^*_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \right) := \widetilde{\Psi}_h \left(p^*, t^*_{\setminus h} \right)$, then

$$z^*_{h'} + e_{h'} + t_{hh'} + t^*_{\to h' \neg h} - t^*_{h' \to} \geq 0$$

2. Observe that $G_{h | \mathbb{R}^C_+ \times \mathbb{R}^{CB}_+} = u_h$, and from 1. above, we have

$$\xi_{h | \left(z^*_{\setminus h}, p^*, t^*_{\setminus h} \right)} \left(\widetilde{\Psi}_h (p^*, t^*_{\setminus h}) \right) \subseteq \mathbb{R}^C_+ \times \mathbb{R}^{CB}_+$$

and then

$$\left(G_h \circ \xi_{h|\left(z^*_{\setminus h}, p^*, t^*_{\setminus h}\right)}\right)_{|\widetilde{\Psi}_h\left(p^*, t^*_{\setminus h}\right)} = \left(u_h \circ \xi_{h|\left(z^*_{\setminus h}, p^*, t^*_{\setminus h}\right)}\right)_{|\widetilde{\Psi}_h\left(p^*, t^*_{\setminus h}\right)}.$$

□

Proposition 5.15. *If (z^*, p^*, t^*) is a Nash equilibrium for the generalized game presented in Definition 5.7, then*

$$p^* z^*_h = 0.$$

Proof. Since by assumption $(z^*_h, t^*_h) \in \widetilde{\Psi}(p^*, t^*_{\setminus h})$, then $p^* z^*_h \leq 0$. Suppose our claim does not hold, i.e.,

$$p^* z^*_h < 0. \tag{10}$$

Then

$$p^*(z^*_h + e_h + t^*_{\to h} - t^*_{h\to}) \overset{(10)}{<} p^*(e_h + t^*_{\to h} - t^*_{h\to}) = w_h\left(p^*, t^*\right) \tag{11}$$

Since $p^* \in \Delta$, then we can define $\mathcal{C}^+ = \{c \in \mathcal{C} : p^c > 0\} \neq \emptyset$. We then distinguish the following two cases.

Case a. There exists $\widetilde{c} \in \mathcal{C}^+$ such that $z^{*\widetilde{c}}_h < \overline{\overline{k}}$;

Case b. For any $c \in \mathcal{C}^+$, $z^{*c}_h = \overline{\overline{k}}$.

Case a. Define $z^{**}_h = (z^{**c}_h)_{c\in\mathcal{C}}$ such that $z^{**c}_h =$

$$\begin{cases} z^{*c}_h & \text{if} \quad c \neq \widetilde{c} \\ z^{*\widetilde{c}}_h + \frac{1}{2}\min\left\{\frac{w_h(p^*,t^*) - p^*(z^*_h + e_h + t^*_{\to h} - t^*_{h\to})}{p^{*\widetilde{c}}},\ \overline{\overline{k}}\right\} > z^{*\widetilde{c}}_h & \text{if} \quad c = \widetilde{c}, \end{cases}$$

where the strictly inequality follows from the fact that

$$w_h(p^*, t^*) - p^*\left(z^*_h + e_h + t^*_{\to h} - t^*_{h\to}\right) \overset{(10)}{>} 0$$

and $\overline{\overline{k}} \geq 0$. Below, we show that $(z^{**}_h, t^*_h) \in \widetilde{\Psi}_h(p^*, t^*_{\setminus h})$ and gives a higher utility than (z^*_h, t^*_h), a fact that contradicts the assumption that (z^*_h, t^*_h) is a solution to household h maximization problem.

$(z^{**}_h, t^*_h) \in \widetilde{\Psi}_h\left(p^*, t^*_{\setminus h}\right)$:

Observe that by assumption, $-p^* z^*_h \geq 0$ and $z^*_h + e_h + t^*_{\to h} - t^*_{h\to} \geq 0$. Then,

$$\begin{aligned} &p^* z^{**}_h = \\ &p^* z^*_h + p^{*\widetilde{c}}\left(\frac{1}{2}\min\left\{\frac{w_h(p^*,t^*) - p^*(z^*_h + e_h + t^*_{\to h} - t^*_{h\to})}{p^{*\widetilde{c}}},\ \overline{\overline{k}}\right\}\right) \leq \\ &p^* z^*_h + \frac{1}{2}\left(w_h\left(p^*, t^*\right) - p^*\left(z^*_h + e_h + t^*_{\to h} - t^*_{h\to}\right)\right) \overset{(11)}{<} p^* z^*_h \leq 0 \end{aligned}$$

and

$$\begin{aligned} &z^{**}_h + e_h + t^*_{\to h} - t^*_{h\to} = z^*_h + e_h + t^*_{\to h} - t^*_{h\to} + \\ &+\left(\frac{1}{2}\min\left\{\frac{w_h(p^*,t^*) - p^*(z^*_h + e_h + t^*_{\to h} - t^*_{h\to})}{p^{*\widetilde{c}}}, \overline{\overline{k}}\right\}\right) \geq 0 \end{aligned}$$

The other inequalities are obviously satisfied.

The statement about higher utility follows from the fact that

$$z^{**}_h + e_h + t^*_{\to h} - t^*_{h\to} > z^*_h + \ e_h + t^*_{\to h} - t^*_{h\to}.$$

Case b. This case cannot hold. Assume it does. Then, we get the following contradiction.

$$k' \overset{\text{Remark 5.1}}{>} w_h(p^*, t^*) \overset{(11)}{>} p^*\left(z_h^* + e_h + t_{\to h}^* - t_{h\to}^*\right) \overset{Assumption \text{ Case b}}{=}$$
$$p^*\left(\overline{\overline{k}} + t_{\to h}^* - t_{h\to}^*\right) \overset{(a)}{\geq} p^*(\overline{\overline{k}} - \overline{k}) \overset{(b)}{\geq} \sum_{c\in\mathcal{C}} p^{*c} k' \mathbf{1}_C = k'$$

where (a) follows from the facts that $-\sum_{h'\in\mathcal{B}_h} t_{hh'}^* \geq -\overline{k}$ and that $\sum_{h'\in\mathcal{B}_h} t_{h'h}^* \geq 0$, and (b) from $\overline{\overline{k}} = k'\mathbf{1}_C + \overline{k}$. □

After establishing in Lemma 5.14 that equilibria defined via the extension G_h lie within the domain of the original utility function u_h, we can now proceed to define equilibria with an upper bound on excess demand using u_h directly. The definition follows the structure of Definition 5.3.

Definition 5.16. $(z^*, t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB} \times \Delta$ is an **excess demand**-transfer-price equilibrium with upper bound for the economy $\mathcal{E} \in \mathbb{E}$ if

(i) households maximize, i.e.,

$\forall h \in \mathcal{H}$, for given $(z_{h'}^*)_{h'\in\mathcal{B}_h} \in \mathbb{R}_+^{CB_h}$, $p^* \in \Delta$, $t_{\setminus h}^* \in \times_{h'\in\mathcal{H}\setminus\{h\}}[0, k_{h'}], e \in \mathbb{R}_{++}^{CH}$ and $u_h \in \mathcal{U}$, (z_h^*, t_h^*) solves the problem

$$\max_{(z_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}}$$

$$u_h(z_h + e_h + t_{\to h}^* - t_{h\to}, (z_{h'}^* + e_{h'} + t_{hh'} + t_{\to h'\neg h}^* - t_{h'\to}^*)_{h'\in\mathcal{B}_h})$$

$$\text{s.t. } (z_h, t_h) \in \overline{\Psi}_h(p^*, t_{\setminus h}^*, (z_{h'}^*)_{h'\in\mathcal{B}_h}),$$

where

$$\overline{\Psi}_h(p^*, t_{\setminus h}^*, (z_{h'}^*)_{h'\in\mathcal{B}_h}) := \left\{ \begin{array}{l} (z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h} : \\ -p^* z_h \geq 0 \\ z_h + e_h + t_{\to h}^* - t_{h\to} \geq 0, \\ t_h \geq 0 \\ t_h \leq k_h \\ z_h \geq -e_h - \overline{k} \\ z_h \leq \overline{\overline{k}}, \\ z_{h'}^* + e_{h'} + t_{hh'} + t_{\to h'\neg h}^* - t_{h'\to}^* \geq 0 \text{ for any } h' \in \mathcal{B}_h \end{array} \right\}.$$

(ii) markets clear.

Proposition 5.17. *If (z^*, p^*, t^*) is a Nash equilibrium for the generalized game presented in Definition 5.7, then it is an excess demand equilibrium with upper bound and $p^* \gg 0$.*

Proof. By definition of Nash equilibrium, each player is maximizing. Therefore, for player $h = 0$, we have that

$$\text{for any } p \in \Delta, \quad p^* \cdot \sum_{h\in\mathcal{H}} z_h^* \geq p \cdot \sum_{h\in\mathcal{H}} z_h^*. \tag{12}$$

From Definition 5.16, we want to show that

for given $p^* \in \Delta$, $z_{\setminus h}^* \in \mathbb{R}^{CB_h}$, and $t_{\setminus h}^* \in \times_{h'\in\mathcal{H}\setminus\{h\}}[0, k_{h'}]$, (z_h^*, t_h^*) solves

$$\max_{(z_h,t_h)\in\mathbb{R}^C\times\mathbb{R}^{CB_h}} \left(u_h \circ \xi_{h|\left(z_{\setminus h}^*, p^*, t_{\setminus h}^*\right)}\right)(z_h, t_h) \quad \text{s.t.}$$
$$(z_h, t_h) \in \overline{\Psi}_h(p^*, t_{\setminus h}^*, (z_{h'}^*)_{h'\in\mathcal{B}_h}).$$

By assumption and from Definition 5.8, for any $h \in \mathcal{H}$, we have that

$$\begin{array}{l} \text{for given } p^* \in \Delta \quad \text{and} \quad (z^*_{h'}, t^*_{h'}) \in \times_{h' \in \mathcal{H}\setminus\{h\}} (Z_{h'} \times T_{h'}), \quad (z^*_h, t^*_h) \ \text{ solves} \\ \max_{(z_h, t_h) \in (Z_h \times T_h)} \quad \left(G_h \circ \xi_{h|\left(z^*_{\setminus h}, p^*, t^*_{\setminus h}\right)} \right)(z_h, t_h) \quad \text{s.t.} \quad (z_h, t_h) \in \widetilde{\Psi}_h(p^*, t^*_{\setminus h}). \end{array} \tag{13}$$

Then, the desired result holds true because the objective functions restricted to the constraint sets of the two problems are the same, thanks to Lemma 5.14.2, and the constraint $z^*_{h'} + e_{h'} + t_{hh'} + t^*_{\to h' \neg h} - t^*_{h' \to} \geq 0$, for any $h' \in \mathcal{B}_h$, which is contained in the definition of $\overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$, but not in $\widetilde{\Psi}_h(p^*, t^*_{\setminus h})$ is satisfied at (z^*_h, t^*_h) from Lemma 5.14.1.

Then, households maximize. We are left with checking market clearing. From (13), we get that

$$0 \geq \sum_{h \in \mathcal{H}} p^* z^*_h. \tag{14}$$

From (14) and (12), we then get

$$\text{for any } p \in \Delta, \qquad 0 \geq p^* \cdot \textstyle\sum_{h \in \mathcal{H}} z^*_h \geq p \cdot \sum_{h \in \mathcal{H}} z^*_h. \tag{15}$$

For any $c \in \mathcal{C}$, define $p(c) = \left(p(c)^{c'}\right)_{c' \in \mathcal{C}}$ such that

$$p(c)^{c'} = \begin{cases} 1 & \text{if} \qquad c' = c \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $p(c) \in \Delta$. Then from (15), we get $0 \geq \sum_{h \in \mathcal{H}} z^{*c}_h$, and therefore,

$$\sum_{h \in \mathcal{H}} z^*_h \leq 0. \tag{16}$$

Let us now show that $p^* \gg 0$. Suppose our claim is false and without loss of generality, assume that $p^{*1} = 0$. Then, from strict monotonicity of u_h in x_h and also in z_h (and since $z^*_h + e_h + t^*_{\to h} - t^*_{h \to} \in \operatorname{dom} u_h$), we would have for any $h \in \mathcal{H}$, $z^{*1}_h = \overline{\overline{k}}$. Then, $\sum_{h \in \mathcal{H}} z^{*1}_h = H\overline{\overline{k}} \overset{Definition\ \overline{\overline{k}}}{>} Hr^1 > r^1$, contradicting (16).

From Proposition 5.15, we have $p^* \sum_{h \in \mathcal{H}} z^*_h = 0$. Since $p^* >> 0$, from (16), we also have $\sum_{h \in \mathcal{H}} z^*_h = 0$. □

5.4. **Existence of equilibria without upper bound on excess demand.** In this section, we prove that the artificial upper bound on excess demand can be removed, as is commonly done for the consumption vector in the existing literature. Specifically, we demonstrate that any solution to the problem with the upper bound is also a solution to the problem without it. The basic idea of the proof is as follows. Suppose, for contradiction, that the claim is false: that is, let z^* be a solution to the problem with the upper bound, but suppose there exists a point z that yields higher utility than z^* and violates the upper bound. Since z^* lies strictly below the upper bound by construction, we can form a convex combination of z and z^* that satisfies the upper-bound constraint. Due to the semistrict quasiconcavity of the utility function, this convex combination yields strictly higher utility than z^*, contradicting its optimality.

Proposition 5.18. *If X is a convex metric space and $u : X \to \mathbb{R}$ is continuous, then u is semistrictly quasi-concave and nonsatiated if and only if u is quasi-concave and locally nonsatiated.*

Proof. See, for example, [14] Corollary 42, page 15. □

Remark 5.19. Since u_h is strictly increasing in $(x_h, (x_{h'})_{h' \in \mathcal{B}_h})$, then u_h is locally nonsatiated; since u_h is concave in $(x_h, (x_{h'})_{h' \in \mathcal{B}_h})$, then u_h is quasi-concave. Therefore, u_h is semistrictly quasi-concave in $(x_h, (x_{h'})_{h' \in \mathcal{B}_h})$.

Definition 5.20. For any $h \in \mathcal{H}$,

$$\xi^{\setminus}_{\mathcal{B}_h} : \times_{h' \in \mathcal{B}_h} Z_{h'} \times T \to \mathbb{R}^{B_h C},$$

$$\left(z_{\setminus h}, t_h, t_{\setminus h}\right) \mapsto \left(z_h + e_{h'} + t_{hh'} + t_{\to h' \neg h} - t_{h' \to}\right)_{h' \in \mathcal{B}_h}$$

Proposition 5.21. *For any economy $\mathcal{E} \in \mathbb{E}$, an excess demand equilibrium with upper bound is an equilibrium according to Definition 3.1 (without the upper bound on excess demand).*

Proof. Let (z^*, t^*, p^*) be an equilibrium with upper bound on excess demand. We want to show that

if

(a) $(z^*_h, t^*_h) \in \overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$ and

(b) for any $(z_h, t_h) \in \overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$,

$u_h \left(z^*_h + e_h + t^*_{\to h} - t^*_{h \to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, t^*_h, t^*_{\setminus h}\right)\right) \geq u_h \left(z_h + e_h + t^*_{\to h} - t^*_{h \to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, t_h, t^*_{\setminus h}\right)\right)$

then

$$\begin{array}{ll} (1) & (z^*_h, t^*_h) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}), \text{ and} \\ (2) & \begin{array}{l} \text{for any } (z_h, t_h) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}), \\ u_h \left(z^*_h + e_h + t^*_{\to h} - t^*_{h \to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, t^*_h, t^*_{\setminus h}\right)\right) \\ \geq u_h \left(z_h + e_h + t^*_{\to h} - t^*_{h \to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, t_h, t^*_{\setminus h}\right)\right). \end{array} \end{array} \tag{17}$$

Since $\overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \subseteq \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$, conclusion (17.1) follows from assumption (a).

Observe that, for any $h \in \mathcal{H}$,

$$z^*_h + e_h + t^*_{\to h} - t^*_{h \to} \leq z^*_h + \sum_{h \in \mathcal{H}} e_h + t^*_{\to h} << z^*_h + \overline{\overline{k}} + \overline{k} \leq 2\overline{\overline{k}} + \overline{k},$$

then

$$0 \leq z^*_h << 2\overline{\overline{k}} + \overline{k} - e_h - t^*_{\to h} + t^*_{h \to} \leq 2(\overline{\overline{k}} + \overline{k}). \tag{18}$$

Since the set

$$\left\{(z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h C} : z_h << 2(\overline{\overline{k}} + \overline{k})\right\}$$

is open, from (18), and contains (z^*_h, t^*_h), then there exists $\delta^* \in \mathbb{R}_{++}$ such that

$$\begin{array}{l} \beta\left(\left(z^*_h, t^*_h\right), \delta^*\right) := \left\{(z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h C} : d\left(\left(z^*_h, t^*_h\right), (z_h, t_h)\right) < \delta^*\right\} \\ \subseteq \\ \left\{(z_h, t_h) \in \mathbb{R}^C \times \mathbb{R}^{B_h C} : z_h << 2(\overline{\overline{k}} + \overline{k})\right\}. \end{array}$$

Defined $\left[-e_h - \overline{k}, +\infty\right) = \left\{z_h \in \mathbb{R}^C : z_h \geq -e_h - \overline{k},\right\}$, we then have

$$\beta\left(\left(z^*_h, t^*_h\right), \delta^*\right) \cap \left(\left[-e_h - \overline{k}, +\infty\right) \times [0, k_h]\right) \subseteq \left[-e_h - \overline{k}, 2(\overline{\overline{k}} + \overline{k})\right] \times [0, k_h] \tag{19}$$

Now suppose conclusion (17.2) does not hold, i.e.,

$$\exists\ \left(\widetilde{z}_h, \widetilde{t}_h\right) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \setminus \overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \text{ such that}$$

$$u_h\left(\widetilde{z}_h + e_h + t^*_{\to h} - \widetilde{t}_{h\to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, \widetilde{t}_h, t^*_{\setminus h}\right)\right) >$$
$$u_h\left(z^*_h + e_h + t^*_{\to h} - t^*_{h\to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, t^*_h, t^*_{\setminus h}\right)\right). \tag{20}$$

Then

$$\left(\widetilde{z}_h, \widetilde{t}_h\right) \neq \left(z^*_h, t^*_h\right). \tag{21}$$

Since $\left(z^*_h, t^*_h\right), \left(\widetilde{z}_h, \widetilde{t}_h\right) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$ and $\Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$ is convex, then

$$\forall \lambda \in (0,1), \quad \left(\widehat{z}_h, \widehat{t}_h\right)(\lambda) := (1-\lambda)\left(z^*_h, t^*_h\right) + \lambda\left(\widetilde{z}_h, \widetilde{t}_h\right) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}). \tag{22}$$

From the semistrict quasi-concavity of u_h (see Definition 5.10), (20) and (22), we have, that for any $\lambda \in (0,1)$,

$$u_h\left((1-\lambda) z^*_h + \lambda \widetilde{z}_h + e_h + t^*_{\to h} - ((1-\lambda)t^*_{h\to} + \lambda \widehat{t}_{h\to}), \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, (1-\lambda)t^*_h + \lambda \widetilde{t}_h, t^*_{\setminus h}\right)\right)$$
$$> u_h\left(z^*_h + e_h + t^*_{\to h} - t^*_{h\to}, \xi^{\setminus}_{\mathcal{B}_h}\left(z^*_{\setminus h}, t^*_h, t^*_{\setminus h}\right)\right). \tag{23}$$

Now, $\|\left(\widehat{z}_h, \widehat{t}_h\right) - \left(z^*_h, t^*_h\right)\| = \lambda \cdot \left\|\left(z^*_h, t^*_h\right) - \left(\widetilde{z}_h, \widetilde{t}_h\right)\right\| < \delta^*$ if and only if $\lambda < \frac{\delta^*}{\left\|\left(z^*_h, t^*_h\right) - \left(\widetilde{z}_h, \widetilde{t}_h\right)\right\|} \in \mathbb{R}_{++}$, where $\left\|\left(z^*_h, t^*_h\right) - \left(\widetilde{z}_h, \widetilde{t}_h\right)\right\| > 0$ from (21). Then, for any $\lambda \in \left(0, \frac{\delta}{\left\|\left(z^*_h, t^*_h\right) - \left(\widetilde{z}_h, \widetilde{t}_h\right)\right\|}\right)$, and using (22), we have that

$$\left(\widehat{z}_h, \widehat{t}_h\right)(\lambda) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \cap \beta\left(\left(z^*_h, t^*_h\right), \delta^*\right) \cap \left(\left[-e_h - \overline{k}, +\infty\right) \times [0, k_h]\right), \tag{24}$$

where the last intersection follows from the fact that the constraints $z_h \geq -e_h - \overline{k}$, and $t_h \in [0, k_h]$ are part of the definition of $\Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$. Observe that

$$\overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) = \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \cap \left(\left[-e_h - \overline{k}, 2(\overline{\overline{k}} + \overline{k})\right] \times [0, k_h]\right). \tag{25}$$

Then,

$$\Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \cap \beta\left(\left(z^*_h, t^*_h\right), \delta^*\right) \cap \left(\left[-e_h - \overline{k}, +\infty\right) \times [0, k_h]\right) \overset{(19)}{\subseteq}$$
$$\subseteq \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}) \cap \left(\left[-e_h - \overline{k}, 2(\overline{\overline{k}} + \overline{k})\right] \times [0, k_h]\right) \overset{(25)}{=} \overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}). \tag{26}$$

From (24) and (26), we have

$$\left(\widehat{z}_h, \widehat{t}_h\right) \in \overline{\Psi}_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h' \in \mathcal{B}_h}). \tag{27}$$

(27) and (23) contradict assumption (b). □

We can then get the desired result of the paper.

Theorem 5.22. *For any economy $\mathcal{E} \in \mathbb{E}$, for any $h \in \mathcal{H}$ such that $e_h \in \mathbb{R}^C_{++}$ and under Assumptions u1, u2 and t1, then an allocation-transfer-price equilibrium $(x^*, t^*, t^e, p^*) \in \mathbb{R}^{CH}_+ \times \Delta \times T$ exists and $p^* >> 0$.*

Proof. From Proposition 5.13 a generalized Nash equilibrium exists. From Proposition 5.17, an excess demand-transfer-price equilibrium with upper bound on excess demand exists. From Proposition 5.21, then an excess demand-transfer-price equilibrium without upper bound exists as well. Finally, Proposition 3.2 provides the desired result. □

6. **A Cobb-Douglas economy.** In this section, we analyze a two-household, one-good, Cobb-Douglas economy, and we assume that the utility function of household 1 is as follows.

$$u_1 : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}, \qquad (x_1, x_2) \mapsto \log x_1 + \beta_1 \log x_2,$$

where $\beta_1 \in \mathbb{R}_{++}$ represents how much household 1 cares about household 2. Symmetric definition applies to household 2. We also assume that the upper bound on household h's transfer is equal to $e_h + k$, with $k \in \mathbb{R}_{++}$. Observe that since there is only one good available, we can normalize its price to 1.

Assuming both households are endowed with one unit of the available commodity, we represent the space of economies in the positive orthant, where the variables on the axes indicate the extent to which each household values the other. We compute equilibria for all possible parameterizations of such economies and confirm that equilibria exist even when households value each other "too much"—a result made possible by the presence of an upper bound on transfers. Furthermore, we show that only in a very small (indeed, closed and measure-zero) subset of the economy space can there exist infinitely many equilibria. These equilibria differ in terms of transfers but correspond to a unique consumption level.

We can then specialize Definition 2.4 as follows.

Definition 6.1. The vector $(x^*, t^*, t^e) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ is an allocation-transfer equilibrium for the economy $\mathcal{E} :=(e_1, e_2, \beta_1, \beta_2, k) \in \mathbb{R}^5_{++}$ if

1. (x_1^*, t_{12}^*) solves the following problem: for given $\mathcal{E} \in \mathbb{R}^5_{++}$, $t_{21}^* \in [0, e_2 + k]$, $x_2^* \in \mathbb{R}_+$, $t_{12}^e \in [0, e_1 + k]$,

$$\max_{(x_1,t_{12})\in\mathbb{R}^2} \quad \log x_1 + \beta_1 \log (x_2^* - t_{12}^e + t_{12}) \quad \text{s.t.} \quad \begin{array}{ll} -x_1 - t_{12} + e_1 + t_{21}^* & \geq 0 \\ t_{12} & \geq 0, \\ e_1 + k - t_{12} & \geq 0 \\ x_1 & > 0 \\ x_2^* - t_{12}^e + t_{12} & > 0 \end{array}$$

and similar condition holds for (x_2^*, t_{21}^*), and

2. $x_1^* + x_2^* = e_1 + e_2$,
3. $t^e = t^*$.

As an easy application of Kuhn-Tucker theorems, it is possible to prove the following result.

Proposition 6.2. *A vector $(x^*, t^*, t^e) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ is an allocation-transfer equilibrium associated with the economy $(e_1, e_2, \beta_1, \beta_2, k)$ iff there exists a vector of Kuhn-Tucker multipliers $(\lambda^*, \gamma^*, \delta^*) := (\lambda_1^*, \lambda_2^*, \gamma_{12}^*, \gamma_{21}^*, \delta_{12}^*, \delta_{21}^*) \in \mathbb{R}^6$ such that*

$(x^*, t^*, \lambda^*, \gamma^*, \delta^*)$ *is a solution to the following system.*

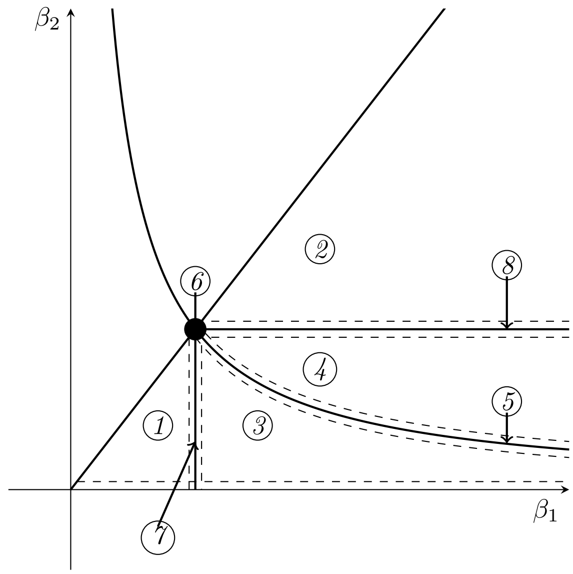
$$\begin{aligned}
&\tfrac{1}{x_1} - \lambda_1 &&= 0\\
&\beta_1 \tfrac{1}{x_2} - \lambda_1 + \gamma_{12} - \delta_{12} &&= 0\\
&-x_1 - t_{12} + e_1 + t_{21} &&= 0\\
&\min\{\gamma_{12}, t_{12}\} &&= 0\\
&\min\{\delta_{12}, e_1 + k - t_{12}\} &&= 0\\
&x_2 &&> 0\\
\\
&\tfrac{1}{x_2} - \lambda_2 &&= 0\\
&\beta_2 \tfrac{1}{x_1} - \lambda_2 + \gamma_{21} - \delta_{21} &&= 0\\
&-x_2 - t_{21} + e_2 + t_{12} &&= 0\\
&\min\{\gamma_{21}, t_{21}\} &&= 0\\
&\min\{\delta_{21}, e_2 + k - t_{21}\} &&= 0\\
&x_1 &&> 0\\
\\
&\textstyle\sum_{h \in \mathcal{H}} (x_h - e_h) &&= 0
\end{aligned} \tag{28}$$

Proposition 6.3. *Assume that $e_1 = e_2 = 1$. The allocation-transfer equilibrium associated with $(\beta_1, \beta_2, k) \in \mathbb{R}^3_{++}$ are those described in the Table below. In each of the eight cases, we write a subtable in which the first column lists the equilibrium values for household 1 and the second column the equilibrium values of household 2, i.e., $(x_1, t_{12}, \gamma_{12}, \delta_{12})$ and $(x_2, t_{21}, \gamma_{21}, \delta_{21})$, respectively.*

1		***2***		***3***		***4***	
$\beta_1\beta_2 < 1$		$\beta_1\beta_2 > 1$		$\beta_1\beta_2 < 1$		$\beta_1\beta_2 > 1$	
$\beta_1 < 1,\ \beta_2 < 1$		$\beta_1 > 1,\ \beta_2 > 1$		$\beta_1 > 1,\ \beta_2 < 1$		$\beta_1 > 1,\ \beta_2 < 1$	
1	1	1	1	$\frac{2}{1+\beta_1}$	$\frac{2\beta_1}{1+\beta_1}$	$\frac{2\beta_2}{1+\beta_2}$	$\frac{2}{1+\beta_2}$
0	0	$1+k$	$1+k$	$\frac{\beta_1-1}{1+\beta_1}$	0	$1+k$	$1+k-\frac{1-\beta_2}{1+\beta_2}$
$1-\beta_1$	$1-\beta_2$	0	0	0	$\frac{(\beta_1+1)(1-\beta_1\beta_2)}{2\beta_1}$	0	0
0	0	β_1-1	β_2-1	0	0	$\frac{(1+\beta_2)(\beta_1\beta_2-1)}{2\beta_2}$	0

5		***6***		***7***		***8***	
$\beta_1\beta_2 = 1$		$\beta_1 = \beta_2 = 1$		$\beta_1\beta_2 < 1$		$\beta_1\beta_2 > 1$	
$\beta_1 > 1,\ \beta_2 < 1$				$\beta_1 = 1,\ \beta_2 < 1$		$\beta_1 > 1,\ \beta_2 = 1$	
$\frac{2}{1+\beta_1}$	$\frac{2}{1+\beta_2}$	1	1	1	1	1	1
$t_{21} + \frac{\beta_1-1}{1+\beta_1}$	$\in \left[0,\ 1-\frac{\beta_1-1}{1+\beta_1}\right]$	t_{21}	$\in [0,\ 1+k]$	0	0	$1+k$	$1+k$
0	0	0	0	0	$1-\beta_2$	0	0
0	0	0	0	0	0	β_1-1	0

TABLE 1. Summary of cases based on values of β_1 and β_2.



Remark 6.4. In the proposition, we consider the case $\beta_1 \geq \beta_2$, as the case $\beta_2 \geq \beta_1$ is entirely symmetric. An infinite number of equilibria arises only in Cases 5 and 6, where $\beta_1\beta_2 = 1$; in all other cases, the equilibrium is unique. Even when multiple equilibria exist, the allocations—and therefore the utility levels—of both households remain constant across equilibria. It is worth noting that when $\beta_1\beta_2 > 1$, at least one household chooses a transfer equal to its upper bound. Without this artificial constraint, an equilibrium would not exist.

Proof. **of Proposition 6.3**

The main idea to prove the desired results is to proceed as follows:

1. Determine the best response function of household 1 in the cases $\beta_1 < 1$, $\beta_1 = 1$ and $\beta_1 > 1$. In each case, formulate a conjecture based on the observation that $\beta_1 \geq 1$ implies $t_{12} = 1 + k$ and $\beta_1 < 1$ implies $t_{12} = 0$. By symmetry, construct the reaction function for household 2.

2. Use the best response functions to determine the equilibrium values of (t_{12}, t_{21}).

3. Compute the remaining equilibrium variables based on the equilibrium values of t_{12} and t_{21} obtained in step 2, as illustrated below.

$$\begin{array}{ll} x_1 = 1 - t_{12} + t_{21} & x_2 = 1 - t_{21} + t_{12} \\ \gamma_{12} = \delta_{12} - \frac{\beta_1}{x_2} + \frac{1}{x_1} & \gamma_{21} = \delta_{21} - \frac{\beta_2}{x_1} + \frac{1}{x_2} \\ \delta_{12} = \gamma_{12} + \frac{\beta_1}{x_2} - \frac{1}{x_1} & \delta_{21} = \gamma_{21} + \frac{\beta_2}{x_1} - \frac{1}{x_2} \end{array} \tag{29}$$

□

The following picture describes the above results.

7. **Conclusion.** In this paper, we have analyzed the existence problem in a model with other-regarding, consumption-based preferences and the possibility of transfers among households, using a generalized game approach. Our analysis highlights several issues in the framework proposed by Mercier Ythier in [12]. Under standard continuity and concavity assumptions, and by imposing an ad hoc upper bound

on transfers, we establish the existence of equilibria. This existence result is a crucial first step in analyzing the properties of equilibria. A natural next step is to demonstrate a so-called generic regularity result - that is, to show that, typically, in the space of economies, the number of equilibria is finite and they vary smoothly with changes in exogenous variables. Finally, exploring the efficiency properties of equilibria and assessing the effectiveness of policy interventions represent compelling directions for future research, both analytically and economically.

8. **Appendix A: Proof of equivalence results.**

Proof of Proposition 3.2. 1. As a preliminary observation, notice that, by assumption for any $h \in \mathcal{H}$,

$$z_h^* = x_h^* - (e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}) \quad \text{and} \quad x_h^* = z_h^* + (e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}).$$

First of all, we have to show that market clearing is satisfied, i.e., $\sum_{h\in\mathcal{H}} z_h^* = 0$. Indeed,

$$\sum_{h\in\mathcal{H}} z_h^* = \sum_{h\in\mathcal{H}} (x_h^* - e_h - t^*_{\longrightarrow h} + t^*_{h\longrightarrow}) = \sum_{h\in\mathcal{H}} (x_h^* - e_h) - \sum_{h\in\mathcal{H}} (t^*_{h\rightarrow} - t^*_{\rightarrow h}) = 0,$$

since, by assumption $\sum_{h\in\mathcal{H}} (x_h^* - e_h) = 0$ and $\sum_{h\in\mathcal{H}} (t^*_{h\rightarrow} - t^*_{\rightarrow h}) = 0$ from Proposition 2.6.

Observe that by assumption, $t^e = t^*$.

We now want to show that

a. $(z_h^*, t_h^*) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h'\in\mathcal{B}_h})$ and

b. it is a maximizer for problem (4).

Indeed,

a. $-p^* z_h^* = -p^* (x_h^* - e_h - t^*_{\longrightarrow h} + t^*_{h\longrightarrow}) \geq 0$;

$z_h^* + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow} = x_h^* - e_h - t^*_{\longrightarrow h} + t^*_{h\longrightarrow} + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow} = x_h^* \geq 0$;

for any $h' \in \mathcal{B}_h$, $z^*_{h'} + e_{h'} + t^*_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow} = x_h^* - (e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}) + e_{h'} + t^*_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow} = x^*_{h'} \geq 0$;

and the other two constraints ($0 \leq t_h \leq k_h$) are clearly satisfied because they are the same in both definitions of equilibria.

b. Suppose otherwise, i.e., there exists $(\widehat{z}_h, \widehat{t}_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ such that

$$(\widehat{z}_h, \widehat{t}_h) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h'\in\mathcal{B}_h}) \tag{30}$$

and

$$\begin{aligned} &u_h(\widehat{z}_h + e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}, (z^*_{h'} + e_{h'} + \widehat{t}_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow})_{h'\in\mathcal{B}_h}) > \\ &u_h(z_h^* + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}, (z^*_{h'} + e_{h'} + t^*_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow})_{h'\in\mathcal{B}_h}) \end{aligned} \tag{31}$$

Define

$$\widehat{x}_h = \widehat{z}_h + e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}. \tag{32}$$

We want to show that A. $(\widehat{x}_h, \widehat{t}_h) \in \Gamma_h(p^*, t^*_{\setminus h}, x^*_{\mathcal{B}_h}, t^e_h)$, and B. it gives a higher utility then (x_h^*, t_h^*), which is the desired contradiction.

A. $((\widehat{x}_h, \widehat{t}_h) \in \Gamma_h(p^*, t^*_{\setminus h}, x^*_{\mathcal{B}_h}, t^e_h)$:

$$\begin{aligned} &-p^* \left(\widehat{x}_h - e_h - t^*_{\longrightarrow h} + \widehat{t}_{h\longrightarrow}\right) \overset{(32)}{=} -p^* \widehat{z}_h \overset{(30)}{\geq} 0; \\ &\widehat{x}_h \overset{(32)}{=} \widehat{z}_h + \left(e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}\right) \overset{(30)}{\geq} 0. \end{aligned}$$

B.

$$u_h\left(\widehat{x}_h, (x^*_{h'} + \widehat{t}_{hh'} - t^e_{hh'})_{h'\in\mathcal{B}_h}\right) \overset{(32),\text{ condition 3. in Def. 2.4}}{=}$$
$$u_h\left(\widehat{z}_h + e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}, (z^*_{h'} + (e_{h'} + t^*_{\longrightarrow h'} - t^*_{h'\longrightarrow}) + \widehat{t}_{hh'} - t^*_{hh'})_{h'\in\mathcal{B}_h}\right) =$$
$$u_h\left(\widehat{z}_h + e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}, (z^*_{h'} + e_{h'} + \widehat{t}_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow})_{h'\in\mathcal{B}_h}\right) \overset{(31)}{>}$$
$$u_h\left(z^*_h + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}, (z^*_{h'} + e_{h'} + t^*_{hh'} + t^*_{\rightarrow h'\neg h} - t^*_{h'\rightarrow})_{h'\in\mathcal{B}_h}\right) =$$
$$u_h\left(x^*_h, (x^*_{h'})_{h'\in\mathcal{B}_h}\right)$$

2.
We now want to show that

$$\begin{array}{l}((z^*_h + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow})_{h\in\mathcal{H}}, t^*, t^*, p^*) \in \mathbb{R}^{CH} \times \mathbb{R}^{CB} \times \mathbb{R}^{CB} \times \Delta \\ \text{is an } \textbf{allocation}\text{-transfer-price equilibrium for the economy } \mathcal{E} \in \mathbb{E}.\end{array} \tag{33}$$

Similarly to what done above, observe that by assumption for any $h \in \mathcal{H}$,

$$x^*_h = z^*_h + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow} \text{ and } z^*_h = x^*_h - e_h - t^*_{\longrightarrow h} + t^*_{h\longrightarrow}.$$

Firstly, we prove that market clearing is satisfied:

$$\begin{aligned}&\textstyle\sum_{h\in\mathcal{H}}(x^*_h - e_h) = \sum_{h\in\mathcal{H}}\left(z^*_{h'} + \sum_{h'\in\mathcal{B}_h} t^*_{h'h} - \sum_{h'\in\mathcal{B}_{\rightarrow h}} t^*_{hh'}\right) = \\ &= \textstyle\sum_{h\in\mathcal{H}} z^*_{h'} - \sum_{h\in\mathcal{H}}(t^*_{h\rightarrow} - t^*_{\rightarrow h}).\end{aligned}$$

By assumption $\sum_{h\in\mathcal{H}} z^*_h = 0$. Then, from Proposition 2.6, we get the desired result.

Condition 3 in the definition of equilibrium is satisfied by (33), in which we require

$$t^e = t^*. \tag{34}$$

We want to show that a. $(x^*_h, t^*_h) \in \Gamma_h(p^*, t^*_{\setminus h}, x^*_{\mathcal{B}_h}, t^e_h)$ and b. it is a maximizer.

Indeed,

a. $-p^*(x^*_h - (e_h + t^*_{\rightarrow h} - t^*_{h\rightarrow})) = -p^*(z^*_h + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow} - e_h - t^*_{\longrightarrow h} + t^*_{h\longrightarrow}) = -p^* z^*_h \overset{\text{Assu.}}{\geq} 0$;

$x^*_h = z^*_h + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow} \overset{\text{Assu.}}{\geq} 0$; for any $h' \in \mathcal{B}_h$, $x^*_{h'} - t^e_{hh'} + t^*_{hh'} = z^*_h + (e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}) - t^*_{hh'} + t^*_{hh'} \geq 0$; and the other two constraints are clearly satisfied because they are the same in both definitions of equilibria.

b. Suppose otherwise, i.e., there exists $(\widehat{x}_h, \widehat{t}_h) \in \mathbb{R}^C \times \mathbb{R}^{CB_h}$ such that

$$(\widehat{x}_h, \widehat{t}_h) \in \Gamma_h(p^*, t^*_{\setminus h}, x^*_{\mathcal{B}_h}, t^e_h) \tag{35}$$

and

$$\begin{aligned}&u_h(\widehat{x}_h, (x^*_{h'} - t^e_{hh'} + \widehat{t}_{hh'}))_{h'\in\mathcal{B}_h}) \overset{(34)}{=} \\ &u_h(\widehat{x}_h, (x^*_{h'} - t^*_{hh'} + \widehat{t}_{hh'}))_{h'\in\mathcal{B}_h}) > u_h(x^*_h, (x^*_{h'}))_{h'\in\mathcal{B}_h}).\end{aligned} \tag{36}$$

Define

$$\widehat{z}_h = \widehat{x}_h - e_h - t^*_{\longrightarrow h} + \widehat{t}_{h\longrightarrow}. \tag{37}$$

We want to show that A. $(\widehat{z}_h, \widehat{t}_h) \in \Psi_h(p^*, t^*_{\setminus h}, (z^*_{h'})_{h'\in\mathcal{B}_h})$ and B. it gives a higher utility than (z^*_h, t^*_h), which is the desired contradiction.

A. $(\widehat{z}_h, \widehat{t}_h) \in \Psi_h(p^*, t^*_{\backslash h}, (z^*_{h'})_{h' \in \mathcal{B}_h})$:

$$-p^* \widehat{z}_h \overset{(37)}{=} -p^* \left(\widehat{x}_h - e_h - t^*_{\longrightarrow h} + \widehat{t}_{h\longrightarrow}\right) \overset{(35)}{\geq} 0;$$
$$\widehat{z}_h + \left(e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}\right) \overset{(37)}{=} \widehat{x}_h \overset{(35)}{\geq} 0.$$

Moreover, taking into account that $(\widehat{x}_h, \widehat{t}_h) \in \Gamma_h(p^*, t^*_{\backslash h}, x^*_{\mathcal{B}_h}, t^e_h)$, for any $h' \in \mathcal{B}_h$, we get

$$z^*_{h'} + e_{h'} + \widehat{t}_{hh'} + t^*_{\rightarrow h' \neg h} - t^*_{h' \rightarrow} =$$

$$x^*_{h'} - e_{h'} - \sum_{h'' \in \mathcal{B}_{\rightarrow h'}} t^*_{h''h'} + t^*_{h'\rightarrow} + e_{h'} + \widehat{t}_{hh'} + t^*_{\rightarrow h' \neg h} - t^*_{h'\rightarrow} =$$

$$x^*_{h'} - t^*_{hh'} + \widehat{t}_{hh'} = x^*_{h'} - t^e_{hh'} + \widehat{t}_{hh'} \geq 0.$$

B.

$$u_h \left(\widehat{z}_h + e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}, (z^*_{h'} + e_{h'} + \widehat{t}_{hh'} + t^*_{\rightarrow h' \neg h} - t^*_{h'\rightarrow})_{h' \in \mathcal{B}_h}\right)$$
$$= u_h \left(\widehat{z}_h + e_h + t^*_{\longrightarrow h} - \widehat{t}_{h\longrightarrow}, (z^*_{h'} + (e_{h'} + t^*_{\longrightarrow h'} - t^*_{h'\longrightarrow}) + \widehat{t}_{hh'} - t^*_{hh'})_{h' \in \mathcal{B}_h}\right)$$
$$\overset{(37),(34)}{=} u_h \left(\widehat{x}_h, (x^*_{h'} + \widehat{t}_{hh'} - t^e_{hh'})_{h' \in \mathcal{B}_h}\right) \overset{(36)}{>} u_h \left(x^*_h, (x^*_{h'})_{h' \in \mathcal{B}_h}\right)$$
$$= u_h \left(z^*_h + e_h + t^*_{\longrightarrow h} - t^*_{h\longrightarrow}, (z^*_{h'} + e_{h'} + t^*_{hh'} + t^*_{\rightarrow h' \neg h} - t^*_{h'\rightarrow})_{h' \in \mathcal{B}_h}\right).$$

□

9. Appendix B: Mercier Ythier 's maintained assumptions.

In this appendix we compare our assumptions with those adopted in [12].

In our paper and in [12], the same so called "survivor assumptions" is made, i.e., for any $h \in \mathcal{H}$, $e_h \in \mathbb{R}^C_{++}$.

We assume that each household utility function is Lipschitz continuous, strictly increasing and concave; we impose that transfers have to satisfy an ad hoc upper bound requirement. Below, we list and briefly discuss Mercier Ythier assumptions on households preferences - using our own notation.

On page 47 in [12], the author defines the following utility functions the selfish utility function for household h as

$$\upsilon_h : \mathbb{R}^C \longrightarrow \mathbb{R},\ x_h \mapsto \upsilon_h(x_h), \tag{38}$$

where x_h is the consumption vector by household h; the vector utility function as

$$\upsilon : \mathbb{R}^{CH} \longrightarrow \mathbb{R}^H,\ x = (x_h)_{h \in \mathcal{H}} \mapsto (\upsilon_h(x_h))_{h \in \mathcal{H}};$$

a utility function describing preferences on other households utility vector $\theta \in \mathbb{R}^H$ as

$$\omega_h : \mathbb{R}^H \longrightarrow \mathbb{R}, \qquad \theta \mapsto \omega_h(\theta),$$

and, finally, an overall utility function as

$$\Upsilon_h := \omega_h \circ \upsilon : \mathbb{R}^{CH} \longrightarrow \mathbb{R}, \quad x = (x_{h'})_{h' \in \mathcal{H}} \mapsto \omega_h\left((\upsilon_{h'}(x_h))_{h' \in \mathcal{H}}\right).$$

Below, we list the assumptions on the above defined utility functions assumed in [12].

Assumption 0. $\upsilon_h(0) = 0$.

Assumption 1. (i) a. υ_h is continuous on $\mathbb{R}^C_+$ and differentiable on $\mathbb{R}^C_{++}$; b. υ_h is strictly increasing on $\mathbb{R}^C_{++}$; c. for any $x_h \in \mathbb{R}^C$ such that $\upsilon_h(x_h) > \upsilon_h(0)$, $x_h \in \mathbb{R}^C_{++}$;

(ii) a. ω_h is continuous on $\mathbb{R}^H_+$ and for any $h' \in \mathcal{H}$ and for any $\theta \in \mathbb{R}^H_+ \setminus \{0\}$, ω_h admits partial derivatives; b. ω_h is increasing;

(iii) a. Υ_h is strictly quasi-concave; b. for any $\theta_{\setminus h} \in \mathbb{R}^{H-1}$, $\omega_h \left(0, \theta_{\setminus h}\right) = 0$.

Then, a sort of "Walrasian indirect utility function" is introduced as follows. Given $(p, \theta_h) \in \mathbb{R}^C_{++} \times \mathbb{R}_{++}$, consider the problem

$$\max_{x_h \in \mathbb{R}^C} \quad \upsilon_h(x_h) \qquad \text{s.t..} \qquad \begin{aligned} &p x_h \leq \theta_h \\ &x_h \geq 0. \end{aligned} \tag{39}$$

Under the above assumptions, we can the define the associated indirect utility function[7] as

$$V_h : \mathbb{R}^C_{++} \times \mathbb{R}_{++} \to \mathbb{R}, \qquad (p, \theta_h) \longmapsto \max (39).$$

For any $h \in \mathcal{H}$, $h' \in \mathcal{H} \setminus \{h\}$, for any $(p, \theta) \in \mathbb{R}^C_{++} \times \mathbb{R}^H_{++}$, the author defines $V_{hh',(p,\theta)} : \mathbb{R}_+ \longrightarrow \mathbb{R}$ such that

$$\tau \mapsto \omega_h \Big(V_h (p, \theta_h - \tau), V_{h'} (p, \theta_{h'} + \tau), (V_{h''} (p, \theta_{h''}))_{h'' \in \mathcal{H} \setminus \{h, h'\}} \Big),$$

which is a function which describes the total effect on household h 's utility as a consequence of a transfer of wealth to household $h' \in \mathcal{H} \setminus \{h\}$. In [12] the following assumption on that function is assumed.

Assumption 2. For any $h \in \mathcal{H}$, (i) V_h is differentiable; (ii) for any $h' \in \mathcal{H} \setminus \{h\}$, for any $p \in \mathbb{R}^C_{++}$, for any $\theta \in \mathbb{R}^H$ such that $\theta_h \leq \theta_{h'}$, $V_{hh',(p,\theta)}$ is decreasing.

Below, we discuss the assumptions underlying the model.

The assumption that the consumption set is the entire Euclidean space $\mathbb{R}^C$, as implied by the definition of the utility function in (38), is difficult to justify: there is no reasonable interpretation for the consumption of a negative quantity of a good. If, instead, we restrict the consumption set to $\mathbb{R}^C_+$, as suggested by Assumption 1, then the emptiness problem presented at the beginning of Section 4 arises. Nevertheless, this issue is not addressed in [12].

Assumption 0, together with monotonicity, implies that υ_h is bounded below, but this comes at the cost of some loss of generality.[8]

Assumption 1 effectively requires a sufficient level of differentiability to apply Kuhn-Tucker conditions.

Assumption 2 concerns an endogenous object: the indirect utility function. This introduces a methodological issue. An economic model is an abstract construction in which assumptions are made about exogenous objects, and conclusions are derived about endogenous ones. Making assumptions about endogenous variables conflicts with the very notion of endogeneity. Moreover, Assumption 2, roughly speaking, posits that an individual's utility decreases if they transfer wealth to households richer than themselves. While this may be a reasonable assumption in many contexts, it is not universally valid. For example, it is violated in the case of political donations to the campaign of a wealthier candidate.

[7] See any textbook in graduate microeconomics as, for example, [9], Proposition 3.D.3, page 56.
[8] Our assumptions do imply boundedness below as well.

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