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MONOMIAL *s*-SEQUENCES ARISING FROM GRAPH IDEALS

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(communicated by Liliana Restuccia)

ABSTRACT. Ideals arising from graphs are investigated via *s*-sequence theory. In particular, the notion of *s*-sequence for the generators of the edge ideal $I(G)$ of an acyclic graph G is considered for describing the Groebner basis of the relation ideal J of the symmetric algebra of $I(G)$. For ideals generated by a *s*-sequence, we are able to compute some standard algebraic invariants of their symmetric algebra in terms of the corresponding invariants of quotients of the polynomial ring related to such graphs. Because the initial ideal of J is well-determined with respect to a monomial order, it defines the edge ideals of supporting graphs to G, more suitable for instance in the management of sensitive data.

Introduction

In this paper the notion of *s*-sequence, first introduced by Herzog, Restuccia, and Tang (2001), is used to study the symmetric algebra of the edge ideals associated to certain graphs (see, for instance, Kühl 1982), as well as the Groebner basis of ideals which define such algebra.

Let G be a graph on n vertices, $R = K[X_1, \dots, X_n]$ be a polynomial ring over a field K. Let $I(G)$ be the ideal of R generated by the monomials $f_1, \dots, f_t$ representing edges of G. Let $J = (g_1, \dots, g_p)$ be the relation ideal of the symmetric algebra $Sym_R(I(G))$ in the polynomial ring $S = R[T_1, \dots, T_t]$, where $g_j = \sum_{i=1}^{t} a_{ij} T_i$, $j = 1, \dots, p$, and (a_{ij}) is the relation matrix of $I(G)$. We say that $f_1, \dots, f_t$ form a *s*-sequence if there exists an admissible monomial order $\prec$ on S with $T_1 \prec T_2 \prec \cdots \prec T_t$ such that $\mathrm{in}_\prec(J) = (\mathscr{I}_1 T_1, \cdots, \mathscr{I}_t T_t)$, with $\mathscr{I}_i = (f_1, \cdots, f_{i-1}) :_R f_i$. Our purpose is to explore interesting classes of acyclic graphs whose edge ideals are generated by *s*-sequences. Generally this happens when the linear forms that generate the relation ideal J form a Groebner basis. Hence, we may use the Groebner bases approach to give a description of the monomial initial ideal $\mathrm{in}_\prec(J)$. Under this situation, standard algebraic invariants are controllable passing from J to $\mathrm{in}_\prec(J)$, so that we can achieve formulas for these invariants in terms of the $\mathscr{I}_i$'s.

In detail, Section 1 is devoted to preliminary notions on graphs and main concepts on *s*-sequences of certain polynomials. Moreover, we deepen and improve results about acyclic finite simple graphs whose edge ideals are generated by a *s*-sequence. The main result shows that the generators of the edge ideal of a forest constitute a *s*-sequence.

In Section 2 we determine standard invariants, such as Krull dimension, multiplicity, etc. for the symmetric algebra of edge ideals of graphs in terms of their annihilator ideals. We apply the fact that the generators of the edge ideals associated to such graphs form a s-sequence.

In Section 3, using Groebner basis theory, we describe the initial ideal $\mathrm{in}_\prec(J)$ of the relation ideal of the symmetric algebra of the edge ideal of a star graph. With an appropriate change of variables, we note that such initial monomial ideal is associated with a bipartite graph, called Ferrers graph, whose edge ideal is isomorphic to $\mathrm{in}_\prec(J)$. This allows us among others to carry out for instance data transmissions in an easier and above all in a safe way.

1. Graph ideals generated by s-sequences

Let G be a graph, $V(G)$ and $E(G)$ be the sets of its vertices and edges respectively. G is said to be *simple* if, for all $\{v_i, v_j\} \in E(G)$, $i \neq j$, it holds that $v_i \neq v_j$. G is *connected* if has no isolated subgraph. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph. If $V(G) = \{v_1, \dots, v_n\}$ and $R = K[X_1, \dots, X_n]$ is the polynomial ring over a field K such that variables X_i correspond to vertices v_i, the *edge ideal* associated with G is the ideal $I(G) = \big(X_iX_j \,|\, \{v_i, v_j\} \in E(G)\big) \subset R$ (see Villarreal 2015). We investigate the symmetric algebra of classes of monomial ideals of the polynomial ring $R = K[X_1, \dots, X_n]$ that arise from graphs, using the theory of s-sequences (Imbesi and La Barbiera 2012; Imbesi, La Barbiera, and Tang 2015a,b; Barbera, Imbesi, and La Barbiera 2018).

Let $f_1, \dots, f_t$ be the minimal system of generators of $I(G)$. Let (a_{ij}), for $i = 1, \dots, t$, $j = 1, \dots, p$, be the relation matrix of $I(G)$. It is known that the symmetric algebra $Sym_R(I(G))$ has a presentation $R[T_1, \dots, T_t]/J$, where $R[T_1, \dots, T_t]$ is a polynomial ring in the variables $T_1, \dots, T_t$ and $J = (g_1, \dots, g_p)$ with $g_j = \sum_{i=1}^{t} a_{ij}T_i$, for $j = 1, \dots, p$. If we assign degree 1 to each variable T_i and degree 0 to the elements of R, then J is a graded ideal and $Sym_R(I(G))$ is a graded algebra over R. Set $S = R[T_1, \dots, T_t]$ and let $\prec$ be a monomial order on the monomials of S in the variables T_i such that $T_1 \prec T_2 \prec \cdots \prec T_t$. With respect to this term order, if $f = \sum a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha = T_1^{\alpha_1} \cdots T_t^{\alpha_t}$, we put $\mathrm{in}_\prec(f) = a_\alpha \underline{T}^\alpha$, where $\underline{T}^\alpha$ is the largest monomial in f such that $a_\alpha \neq 0$. So we can define the monomial ideal $\mathrm{in}_\prec(J) = (\mathrm{in}_\prec(f) \mid f \in J)$. For every $i = 1, \dots, t$, we set $I(G)_{i-1} = Rf_1 + \cdots + Rf_{i-1}$ and $\mathscr{I}_i = I(G)_{i-1} :_R f_i$. The ideals $\mathscr{I}_i$, called the annihilator ideals of the sequence $f_1, \dots, f_t$, depend on $f_1, \dots, f_t$ but not on the term order $\prec$. In general $(\mathscr{I}_1T_1, \mathscr{I}_2T_2, \dots, \mathscr{I}_tT_t) \subseteq \mathrm{in}_\prec(J)$, and the two ideals coincide in the linear case.

The sequence $f_1, \dots, f_t$ is called a *s-sequence* for $I(G)$ if $(\mathscr{I}_1T_1, \mathscr{I}_2T_2, \dots, \mathscr{I}_tT_t) = \mathrm{in}_\prec(J)$. When $\mathscr{I}_1 \subseteq \mathscr{I}_2 \subseteq \cdots \subseteq \mathscr{I}_t$, $f_1, \dots, f_t$ is said to be a *strong* s-sequence. We apply Groebner bases theory to compute $\mathrm{in}_\prec(J)$. Let $\prec$ be any term order on $K[X_1, \dots, X_n; T_1, \dots, T_t]$ with $T_1 \prec T_2 \prec \cdots \prec T_t$, $X_i \prec T_j$, for all i and j. Then for any Groebner basis B for $J \subset K[X_1, \dots, X_n, T_1, \dots, T_t]$ with respect to $\prec$, we have $\mathrm{in}_\prec(J) = (\mathrm{in}_\prec(f) \mid f \in B)$. If the elements of B are linear in the T_i, it follows that $f_1, \dots, f_t$ is a s-sequence for $I(G)$. Moreover, let $I(G) = (f_1, \dots, f_t)$. Set $f_{ij} = \dfrac{f_i}{[f_i, f_j]}$, for $i \neq j$, where $[f_i, f_j]$ is the greatest common divisor of the monomials f_i and f_j. J is generated by $g_{ij} = f_{ij}T_j - f_{ji}T_i$ for $1 \leq i < j \leq t$. The monomial sequence $f_1, \dots, f_t$ is a s-sequence if and only if g_{ij} for $1 \leq i < j \leq t$ is a

Groebner basis for J for any term order in $K[X_1,\dots,X_n;T_1,\dots,T_t]$ with $T_1 \prec T_2 \prec \cdots \prec T_t$, $X_i \prec T_j$, for all i,j.

Note that the annihilator ideals of the monomial sequence $f_1,\dots,f_t$ are the ideals $I_i = (f_{1i}, f_{2i},\dots,f_{i-1,i})$, for $i=1,\dots,t$ (see Herzog, Restuccia, and Tang 2001). Let us now examine the following classes of simple acyclic graphs and the edge ideals associated to them.

- P_{n-1}, the $(n-1)$-path graph: $I(P_{n-1}) = (X_1X_2, X_2X_3,\dots,X_{n-1}X_n)$;
- S_{n-1}, the $(n-1)$-star graph: $I(S_{n-1}) = (X_1X_n, X_2X_n,\dots,X_{n-1}X_n)$.

We observe that the generators of the edge ideals of each of them form a monomial M-sequence, consequently such generators form a s-sequence (for details, see Conca and De Negri 1999). Making use of the theory of Groebner bases, we present a direct approach to prove this.

Lemma 1.1. *Let $I(P_{n-1}) = (X_1X_2, X_2X_3,\dots,X_{n-1}X_n) \subset R = K[X_1,\dots,X_n]$ be the edge ideal of the path graph P_{n-1}. If $Sym_R(I(P_{n-1})) = R[T_1,\dots,T_{n-1}]/J$, then $J = (g_{ij}, 1 \le i < j \le n-1)$, where*

$$g_{ij} = \begin{cases} X_iT_j - X_{j+1}T_i & \text{if } j = i+1 \\ X_iX_{i+1}T_j - X_jX_{j+1}T_i & \text{if } j > i+1 \end{cases} .$$

Proof. The generators of $I(P_{n-1})$ are $f_1 = X_1X_2, f_2 = X_2X_3,\dots,f_{n-1} = X_{n-1}X_n$ and $f_{ij} = \dfrac{f_i}{[f_i,f_j]}$, for $i \neq j$, $i,j = 1,\dots,n-1$. From their computation we obtain $f_{12} = X_1, f_{13} = X_1X_2,\dots,f_{1,n-1} = X_1X_2, f_{23} = X_2, f_{24} = X_2X_3,\dots,f_{2,n-1} = X_2X_3,\dots,f_{n-2,n-1} = X_{n-2}$. In general, it is $f_{ij} = X_i$, for $j = i+1$ and $f_{ij} = X_iX_{i+1}$, for $j > i+1$, with $1 \le i < j \le n-1$ and $i < j$. In a similar way we have $f_{ji} = X_j$, for $j = 2,\dots,n-1$ and $i < j$. Then J is generated by the linear forms g_{ij}, with $g_{ij} = X_iT_j - X_{j+1}T_i$, if $j = i+1$, and $g_{ij} = X_iX_{i+1}T_j - X_jX_{j+1}T_i$, if $j > i+1$. ☐

Theorem 1.2. *The generators of the edge ideal $I(P_{n-1})$ form a s-sequence.*

Proof. Denoting with $f_1, f_2,\dots,f_{n-1}$ the generators of $I(P_{n-1})$, we observe that if $B = \{g_{ij} = f_{ij}T_j - f_{ji}T_i \mid 1 \le i < j \le n-1\}$ is a Groebner basis for J, then $f_1,\dots,f_{n-1}$ is a s-sequence. Hence we prove that $S(g_{ij}, g_{hl})$, with $i,j,h,l \in \{1,\dots,n-1\}$, has a standard expression with respect to B with remainder 0. Note that, to get a standard expression of $S(g_{ij}, g_{hl})$ is equivalent to find some $g_{st} \in B$ whose initial term divides the initial term of $S(g_{ij}, g_{hl})$ and substitute a multiple of g_{st} such that the remaindered polynomial has a smaller initial term and so on up to the remainder is 0. We have:

$$S(g_{ij}, g_{hl}) = \frac{f_{ij}f_{lh}}{[f_{ij}, f_{hl}]}T_jT_h - \frac{f_{hl}f_{ji}}{[f_{ij}, f_{hl}]}T_iT_l\,. \tag{1}$$

Let us find a standard expression of $S(g_{ij}, g_{hl})$, for all $i,j,h,l \in \{1,\dots,n-1\}$.
If $[\mathrm{in}_\prec(g_{ij}), \mathrm{in}_\prec(g_{hl})] = 1$, then $S(g_{ij}, g_{hl}) = f_{lh}g_{ij}T_h - f_{ji}g_{hl}T_i$.
If $[\mathrm{in}_\prec(g_{ij}), \mathrm{in}_\prec(g_{hl})] \neq 1$, we apply (1) in order to obtain a standard expression for the S-polynomials $S(g_{ij}, g_{hl})$. It results:

$S(g_{ij}, g_{il}) = -[f_{ji}, f_{li}]g_{jl}T_i$
$S(g_{ij}, g_{hj}) = [f_{ji}, f_{jh}]g_{ih}T_j$
$S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}](g_{ih}T_j - g_{jl}T_i)$ if $[f_{ji}, f_{hl}] = 1$ and $j < l$

$S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}](g_{ih}T_j - g_{lj}T_i)$ if $[f_{ji}, f_{hl}] = 1$ and $j > l$
$S(g_{ij}, g_{hl}) = [f_{ji}, f_{lh}](f_{lj}g_{ih}T_j - f_{hi}g_{jl}T_i)$ if $[f_{ji}, f_{hl}] \neq 1$.
Hence all the S-polynomials $S(g_{ij}, g_{hl})$ reduce to 0 with respect to B. □

Lemma 1.3. *Let $I(S_{n-1}) = (X_1X_n, X_2X_n, \dots, X_{n-1}X_n) \subset R = K[X_1, \dots, X_n]$ be the edge ideal of the star graph S_{n-1}. If $Sym_R(I(S_{n-1})) = R[T_1, \dots, T_{n-1}]/J$, then $J = (X_iT_j - X_jT_i, 1 \leq i < j \leq n-1)$.*

Proof. The generators of $I(S_{n-1})$ are $f_1 = X_1X_n, f_2 = X_2X_n, \dots, f_{n-1} = X_{n-1}X_n$ and $f_{ij} = \dfrac{f_i}{[f_i, f_j]}$, for $i < j$, $i, j = 1, \dots, n-1$. From their computation we obtain $f_{12} = X_1, f_{13} = X_1, \dots, f_{1,n-1} = X_1, f_{23} = X_2, \dots, f_{2,n-1} = X_2, \dots, f_{n-2,n-1} = X_{n-2}$. In general, it is $f_{ij} = X_i$, for $i = 1, \dots, n-2$ and $i < j$. In a similar way we have $f_{ji} = X_j$, for $j = 2, \dots, n-1$ and $i < j$. Because J is generated by the linear forms $g_{ij} = f_{ij}T_j - f_{ji}T_i$, for $1 \leq i < j \leq n-1$, then $J = (X_iT_j - X_jT_i, 1 \leq i < j \leq n-1)$. □

Remark 1.4. Lemma **1.3** can also be shown noting that $(X_1X_n, X_2X_n, \dots, X_{n-1}X_n)$ is isomorphic to $(X_1, \dots, X_{n-1})$ as R-module, so their symmetric algebras are isomorphic, then the results about the symmetric algebras of $(X_1X_n, X_2X_n, \dots, X_{n-1}X_n)$ follow.

Theorem 1.5. *The generators of the edge ideal $I(S_{n-1})$ form a s-sequence.*

Proof. Denoting with $f_1, f_2, \dots, f_{n-1}$ the generators of $I(S_{n-1})$, from Lemma **1.3** it follows that $f_{ij} = X_i$, $1 \leq i < j \leq n-1$. Thus $f_{ij} \neq f_{hl}$ when $i \neq h$ and $j \neq l$. Hence $[f_{ij}, f_{hl}] = 1$ for $i < j$, $h < l$, $i \neq h$, $j \neq l$ with $i, j, h, l \in \{1, \dots, n-1\}$. From Herzog, Restuccia, and Tang (2001, Proposition 1.7), it descends that $I(S_{n-1})$ is generated by a s-sequence. □

Imbesi and La Barbiera (2012) built a remarkable class of connected acyclic graphs by completing a star graph with path graphs connected to all the vertices of the star distinct from its hub, the so-called *generalized star* graphs. In particular, the edge ideal of a generic graph G on n vertices and $n-1$ edges that belongs to such a class is the following ideal of $R = K[X_1, \dots, X_n]$,

$$I(G) = (X_1X_r, X_2X_r, \dots, X_{r-1}X_r, X_1X_{r+1}, X_{r+1}X_{r+2}, \dots, X_{r+s_1-1}X_{r+s_1}, X_2X_{r+s_1+1}, \dots, X_{r-1}X_{r+s_1+\dots+s_{r-2}+1}, \dots, X_{n-1}X_n), \quad \text{where } n = r + s_1 + \dots, s_{r-1}\,.$$

Proposition 1.6. *Let $I(G)$ be as above. If $Sym_R(I(G)) = R[T_1, \dots, T_{n-1}]/J$, let $J = (\{g_{ij}, 1 \leqslant i < j \leqslant n-1\})$, with*

$$g_{ij} = \begin{cases} X_iT_j - X_jT_i & \text{if } 1 \leqslant i < j \leqslant r-1 \\ X_rT_j - X_{j+1}T_i & \text{if } i=1; j=r \text{ or } i=2,\dots,r-1; j=r+s_1+\dots+s_{i-1} \\ X_iT_{j+1} - X_{j+2}T_j & \text{if } i=1; j=r \text{ or } i=2,\dots,r-1; j=r+s_1+\dots+s_{i-1} \\ X_iT_j - X_{j+1}T_i & \text{if } j=i+1; i=r+1,\dots,r+s_1-2 \text{ or } j=i+1; i=r+ \\ & +s_1+\dots+s_{h-1}+k, h=2,\dots,r-1, k=1,\dots,s_h-2 \\ f_iT_j - f_jT_i & \text{otherwise}\,. \end{cases}$$

Then the generators of $I(G)$ form a s-sequence.

Proof. Look at Imbesi and La Barbiera (2012, Theorem 3.1). □

We can extend to a tree the assertion of Proposition **1.6**.

Theorem 1.7. *Let G be a tree on n vertices. The edge ideal $I(G) \subset R = K[X_1, \dots, X_n]$ is generated by a s-sequence.*

Proof. Look at Imbesi and La Barbiera (2012, Theorem 3.2) . □

As main result, we prove that the generators of the edge ideal of a forest form a s-sequence.

Theorem 1.8. *Let H be a forest on n vertices. The edge ideal $I(H) \subset R = K[X_1, \dots, X_n]$ is generated by a s-sequence.*

Proof. A forest H may be viewed as a set of disjoint trees, that means a composition of disjoint connected acyclic finite simple graphs. Each of these trees refers to extensions or restrictions of generalized star graphs, in particular isolated vertices, edges, path graphs, star graphs, etc. that can contain further vertices of degree > 2. In this way, for all such vertices, we can take in account the statements in the previous results.

Let $f_1, \dots, f_t$ denote the generators of the edge ideal $I(H)$ of the forest H. Following procedures similar to those of Lemma **1.1** and Imbesi and La Barbiera 2012, Propositions 4.1, 4.2, we are able to determine the generators $g_{ij} = f_{ij}T_j - f_{ji}T_i$, $1 \leqslant i < j \leqslant t$, of the relation ideal J of the symmetric algebra of $I(H)$. For showing that $f_1, \dots, f_t$ form a s-sequence, it is enough to see that the elements g_{ij} form a Groebner basis for J. In other words, we need to show that the S-polynomials $S(g_{ij}, g_{hl})$ such that $i, j, h, l \in \{1, \dots, t\}$, $i<j$, $i<h<l$, have a standard expression with respect to $\{g_{ij}\}$ with remainder 0.

Thinking to a generalization of the arguments in Theorem **1.2** and in Theorem 4.1 by Imbesi and La Barbiera (2012), also iterating the calculation to get standard expressions of the S-polynomials in every vertex of degree > 2 of H, we may conclude that all the S-polynomials reduce to 0 with respect to $\{g_{ij}\}$. □

2. Standard invariants associated to graph ideals

In this section we use the theory of s-sequences in order to compute standard algebraic invariants, such as *Krull dimension*, *multiplicity*, *Castelnuovo-Mumford regularity*, etc. of the symmetric algebra of some edge ideals of graphs, descending from the graph G examined in the final part of the previous section, in terms of their annihilator ideals. We know that the generators of the edge ideals associated to such graphs form a s-sequence.
First we analyze standard algebraic invariants of the symmetric algebra of the edge ideals of the $(n-1)$-path graph and the $(n-1)$-star graph.

Theorem 2.1. *Let P_{n-1} be the path graph and $I(P_{n-1}) = (X_1X_2, X_2X_3, \dots, X_{n-1}X_n) \subset R = K[X_1, \dots, X_n]$. For the symmetric algebra of $I(P_{n-1})$, it holds:*
a) $\dim(Sym_R(I(P_{n-1}))) = n+1$,
b) $\mathrm{e}(Sym_R(I(P_{n-1}))) = \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \dots$

Proof. Following Imbesi and La Barbiera (2012, Theorem 5.2), keeping in mind that the annihilator ideals of the generators of $I(P_{n-1})$ are $I_1 = (0), I_2 = (X_1), I_3 = (X_1X_2, X_2), I_i = (X_1X_2, X_2X_3, \dots, X_{i-3}X_{i-2}, X_{i-1})$, for $i = 4, \dots, n-1$. □

Theorem 2.2. *Let S_{n-1} be the star graph and $I(S_{n-1}) = (X_1X_n, X_2X_n, \dots, X_{n-1}X_n) \subset R = K[X_1, \dots, X_n]$. For the symmetric algebra $R[T_1, \dots, T_{n-1}]/J$ of $I(S_{n-1})$, it holds:*

a) $\dim(Sym_R(I(S_{n-1}))) = n+1$,
b) $\mathrm{e}(Sym_R(I(S_{n-1})) = n-1$,
c) $\mathrm{reg}(Sym_R(I(S_{n-1})) = 1$.

Proof. First, bear in mind that the annihilator ideals of the generators of $I(S_{n-1})$ are $I_1 = (0), I_i = (X_1,\dots,X_{i-1})$, for $i = 2,\dots,n-1$, so that the generators of $I(S_{n-1})$ form a strong s-sequence.

a) $\dim(Sym_R(I(G))) = \sup\{n+1, n-1\} = n+1$, where $n-1$ is the number of the edges of G.

b) From Herzog, Restuccia, and Tang (2001, Proposition 2.4), it follows that $\mathrm{e}(Sym_R(I(G))) = \sum_{i=1}^{n-1} \mathrm{e}(R/\mathscr{I}_i)$. The annihilator ideals I_i are generated by a regular sequence, then following Tang (2004, Theorem 4.8), $\mathrm{e}(R/\mathscr{I}_i) = 1$, for $i = 2,\dots,n-1$ and $\mathrm{e}(R/(0)) = 1$. Hence $\mathrm{e}(Sym_R(I(G))) = \sum_{i=1}^{n-1} e(R/\mathscr{I}_i) = n-1$.

c) Following Tang (2004, Theorem 4.8), $\mathrm{reg}(Sym_R(I(G))) = \mathrm{reg}(R[T_1,\dots,T_{n-1}]/J)$ $\leqslant \mathrm{reg}(R[T_1,\dots,T_{n-1}]/\operatorname{in}_\prec(J)) \leqslant \max_{2\leqslant j\leqslant n-1}\{\sum_{i=1}^{j-1}\deg(f_{ij}) - (j-2)\}$.

Then it results:

$$\mathrm{reg}(Sym_R(I(G))) \leqslant \max_{2\leqslant j\leqslant n-1}\{\sum_{i=1}^{j-1}\deg(X_i)-(j-2)\} = (j-1)-(j-2) = 1.$$

Moreover, J is generated by the linear forms of degree two $X_iT_j - X_jT_i$, for $i, j = 1,\dots,n-1$. Then $\mathrm{reg}(Sym_R(I(G))) = \mathrm{reg}(R[T_1,\dots,T_{n-1}]/J) \geq 1$, so $\mathrm{reg}(Sym_R(I(G))) = 1$. □

The simplest generalization of the graph considered in Theorem **2.2** can be obtained by adding an edge to that graph in a vertex of degree 1. The computation of standard algebraic invariants it is given by the following

Theorem 2.3. *Let G be the connected acyclic graph whose edge ideal is* $I(G) = (X_1X_n, X_2X_n,\dots,X_{n-1}X_n, X_\ell X_{n+1})$, $1 \leq \ell \leq n-1$, $I(G) \subset R = K[X_1,\dots,X_{n+1}]$. *For the symmetric algebra of* $I(G)$*, it holds:*
a) $\dim(Sym_R(I(G))) = n+1$,
b) $\mathrm{e}(Sym_R(I(G)) = 2(n-1)$.

Proof. With easy computations, the thesis descends from that of the previous theorem, knowing that the annihilator ideals of the generators of $I(G)$ are $I_1 = (0), I_i = (X_1,\dots,X_{i-1}), I_n = (X_n)$, for $i = 2,\dots,n-1$. □

More generally, the calculus of standard algebraic invariants of connected acyclic graphs formed by paths of different lengths with a common hub is not yet easy. But in some particular cases we were able to determine certain algebraic invariants (see Merlino 2017). Let G be composed of a star graph together with 2 further edges connected to distinct ends of the star. The following gives the annihilator ideals of the generators of the edge ideal of such a graph.

Proposition 2.4. *Let G be a graph on n vertices and edge ideal* $I(G) \subset R = K[X_1,\dots,X_n]$ *generated by* $(X_1X_{n-2}, X_2X_{n-2},\dots,X_{n-3}X_{n-2}, X_hX_{n-1}, X_kX_n)$, $1 \leq h \leq n-4$, $2 \leq k \leq n-3$, $h \neq k$. *The annihilator ideals of such generators are* $I_1 = (0)$, $I_2 = (X_1)$, $\dots$, $I_{n-3} = (X_1,\dots,X_{n-4})$, $I_{n-2} = (X_{n-2})$, $I_{n-1} = (X_{n-2}, X_hX_{n-1})$.

Proof. Let $I(G) = (f_1, f_2, \ldots, f_{n-2}, f_{n-1})$, where $f_1 = X_1X_{n-2}$, $f_2 = X_2X_{n-2}$, ..., $f_{n-3} = X_{n-3}X_{n-2}$, $f_{n-2} = X_hX_{n-1}$, $f_{n-1} = X_kX_n$, $h = 1, \ldots, n-4$, $k = 2, \ldots, n-3$, $h \neq k$. Set $f_{ij} = \frac{f_i}{[f_i, f_j]}$ for $i < j = 1, \ldots, n-1$. The annihilator ideals of the monomial sequence $f_1, \ldots, f_{n-1}$ are $I_\alpha = (f_{1\alpha}, f_{2\alpha}, \ldots, f_{\alpha-1,\alpha})$, for $\alpha = 1, \ldots, n-1$. So, it is

$$\begin{aligned}
I_1 &= (0), I_2 = (f_{12}) = (X_1), I_3 = (f_{13}, f_{23}) = (X_1, X_2), \ldots, \\
I_{n-3} &= (f_{1,n-3}, \ldots, f_{n-4,n-3}) = (X_1, X_2, \ldots, X_{n-4}), \\
I_{n-2} &= (f_{1,n-2}, \ldots, f_{n-3,n-2}) = \\
&= (X_1X_{n-2}, \ldots, X_{h-1}X_{n-2}, X_{n-2}, X_{h+1}X_{n-2}, \ldots, X_{n-3}X_{n-2}) = (X_{n-2}), \\
I_{n-1} &= (f_{1,n-1}, \ldots, f_{n-2,n-1}) = \\
&= (X_1X_{n-2}, \ldots, X_{k-1}X_{n-2}, X_{n-2}, X_{k+1}X_{n-2}, \ldots, X_{n-3}X_{n-2}, X_hX_{n-1}) = \\
&= (X_{n-2}, X_hX_{n-1}).
\end{aligned}$$

□

The dimension and the multiplicity of the symmetric algebra associated to the last graph are computed in the following

Theorem 2.5. *Let G be a graph on $n \geq 5$ vertices and edge ideal $I(G) = (X_1X_{n-2}, X_2X_{n-2}, \ldots, X_{n-3}X_{n-2}, X_hX_{n-1}, X_kX_n)$, $1 \leq h \leq n-4$, $2 \leq k \leq n-3$, $h \neq k$. For the symmetric algebra of $I(G) \subset R = K[X_1, \ldots, X_n]$ it holds*

a) $\dim(Sym_R(I(G))) = n+1$,

b) $$\mathrm{e}(Sym_R(I(G))) = \begin{cases} 7 & \textit{for } n = 5 \\ \dfrac{n(n-1)}{2} - 4 & \textit{for } n \geq 6 \ . \end{cases}$$

Proof. a) $\dim(Sym_R(I(G))) = \sup\{n+1, n-1\}$, where $n-1$ is the number of the edges of G (see Villarreal 2015).
b) Following Herzog, Restuccia, and Tang (2001, Proposition 2.4) ,

$$\mathrm{e}(Sym_R(I(G))) = \sum_{1 \leq i_1 < \cdots < i_r \leq n-1} \mathrm{e}(R/(I_{i_1}, \ldots, I_{i_r})) = d - r,$$

where $d = \dim(Sym_R(I(G))) = n+1$ and $1 \leq r \leq n-1$.
Let $d' = \dim(R/(I_{i_1}, \ldots, I_{i_r})) = n+1-r$.
The multiplicity $\mathrm{e}(Sym_R(I(G)))$ is the sum of the following multiplicities:

$r = 1$, $\mathrm{e}(R/I_1) = 1$ (because $d' = \dim(R/I_1) = n$),
$r = 2$, $\mathrm{e}(R/(I_1+I_2)) + \mathrm{e}(R/(I_1+I_3)) = 2$
(because $d' = \dim(R/(I_1+I_2)) = \dim(R/(I_1+I_3)) = n-1$),
$r = 3$, $\mathrm{e}(R/(I_1+I_2+I_3)) + \mathrm{e}(R/(I_1+I_2+I_4)) + \mathrm{e}(R/(I_1+I_2+I_{n-1})) = 3$
(because $d' = \dim(R/(I_1+I_2+I_3)) = \cdots = \dim(R/(I_1+I_2+I_{n-1})) = n-2$),

and so on till $r = n-1$.
For $n = 5$, the graph is the path graph P_4 and $\mathrm{e}_5 = \binom{4}{1} + \binom{3}{2} = 7$ (see Theorem **2.1**). For $n \geq 6$,

$$\mathrm{e}_n = (n-1) + \mathrm{e}_{n-1} = (n-1) + (n-2) + \mathrm{e}_{n-2} = (n-1) + (n-2) + (n-3) + \mathrm{e}_{n-3} = \\ = \ldots = (n-1) + \ldots + 6 + \mathrm{e}_6 = \frac{n(n-1)}{2} - 15 + \mathrm{e}_6 = \frac{n(n-1)}{2} - 4 . \qquad \square$$

Let now G be constituted by a star graph and a path graph which are connected in a vertex of degree 1 of them. In the following we compute the multiplicity of the symmetric algebra of the edge ideal of such a graph.

Theorem 2.6. *Let G be a connected graph on n vertices constituted by a star graph on m vertices, with $m \leq n$, and a path graph, with $n-m$ edges, $n-m \leq 4$. The multiplicity* e *of the symmetric algebra of $I(G) = (X_1X_m, \ldots, X_{m-1}X_m, X_\ell X_{m+1}, X_{m+1}X_{m+2}, \ldots, X_{n-1}X_n) \subset R = K[X_1, \ldots, X_n], 1 \leq \ell \leq m-1$, is:*

- *when $n = m$,* $\mathrm{e} = 1(n-3) + 2$ *(in compliance with Theorem* **2.2***),*
- *when $n-m = 1$,* $\mathrm{e} = 2(n-4) + 4$ *(in compliance with Theorem* **2.3***),*
- *when $n-m = 2$,* $\mathrm{e} = 4(n-5) + 7$,
- *when $n-m = 3$,* $\mathrm{e} = 7(n-6) + 12$,
- *when $n-m = 4$,* $\mathrm{e} = 11(n-7) + 20$.

Proof. Similar to that one in item b) of Theorem **2.5** . □

Furthermore, we could get a general formula for the above multiplicity.

Conjecture 2.7. *Let G be the graph as in Theorem* **2.6** *with $n-m = r$ any positive integer. The multiplicity of the symmetric algebra of $I(G)$ is*

$$\mathrm{e} = \frac{r^2+r+2}{2}(n-(3+r)) + \binom{r+2}{1} + \binom{r+1}{2} + \binom{r}{3} + \ldots \,. \qquad \square$$

For such graph, let us calculate the dimension and the multiplicity of the symmetric algebra of its edge ideal in the case $n = 6$ and $n-m = 2$.

Example 2.8. Let $R = K[X_1, \ldots, X_6]$ and $I(G) = (X_1X_4, X_2X_4, X_2X_5, X_3X_4, X_5X_6)$. Then

$$d = \dim(Sym_R(I(G))) = \max_{\substack{0 \leq r \leq 5 \\ 1 \leq i_1 \leq \cdots \leq i_r \leq 5}} \{\dim(R/(I_{i_1} + \cdots + I_{i_r})) + r\} .$$

The annihilator ideals of $f_1 = X_1X_4, f_2 = X_2X_4, f_3 = X_2X_5, f_4 = X_3X_4, f_5 = X_5X_6$ are:

$$I_1 = (0) : (f_1) = (0)$$
$$I_2 = (f_1) : (f_2) = (X_1X_4) : (X_2X_4) = (X_1)$$
$$I_3 = (f_1, f_2) : (f_3) = (X_1X_4, X_2X_4) : (X_2X_5) = (X_1X_4, X_4) = (X_4)$$
$$I_4 = (f_1, f_2, f_3) : (f_4) = (X_1X_4, X_2X_4, X_2X_5) : (X_3X_4) = (X_1, X_2, X_2X_5) = (X_1, X_2)$$
$$I_5 = (f_1, f_2, f_3, f_4) : (f_5) = (X_1X_4, X_2X_4, X_2X_5, X_3X_4) : (X_5X_6) = \\ = (X_1X_4, X_2X_4, X_2, X_3X_4) = (X_2, X_4)$$

For $r = 0$ it is $\dim(R) = 6$

For $r = 1$ it is

$$\begin{aligned}
&\dim(R/I_1) = \dim(R/(0)) = \dim(R) = 6\,.\\
&\dim(R/I_2) = \dim(R/(X_1)) = \dim(R) - 1 = 5\\
&\dim(R/I_3) = \dim(R/(X_4)) = \dim(R) - 1 = 5\\
&\dim(R/I_4) = \dim(R/(X_1,X_2)) = \dim(R) - 2 = 4\\
&\dim(R/I_5) = \dim(R/(X_2,X_4)) = \dim(R) - 2 = 4\,.
\end{aligned}$$

For $r = 2$ it is

$$\begin{aligned}
&\dim(R/(I_1+I_2)) = \dim(R/(X_1)) = \dim(R) - 1 = 5\\
&\dim(R/(I_1+I_3)) = \dim(R/(X_4)) = \dim(R) - 1 = 5\\
&\dim(R/(I_1+I_4)) = \dim(R/(X_1,X_2)) = \dim(R) - 2 = 4\\
&\dim(R/(I_1+I_5)) = \dim(R/(X_2,X_4)) = \dim(R) - 2 = 4\\
&\dim(R/(I_2+I_3)) = \dim(R/(X_1,X_4)) = \dim(R) - 2 = 4\\
&\dim(R/(I_2+I_4)) = \dim(R/(X_1,X_2)) = \dim(R) - 2 = 4\\
&\dim(R/(I_2+I_4)) = \dim(R/(X_1,X_2)) = \dim(R) - 2 = 4\\
&\dim(R/(I_2+I_5)) = \dim(R/(X_1,X_2,X_4)) = \dim(R) - 3 = 3\\
&\dim(R/(I_3+I_4)) = \dim(R/(X_1,X_2,X_4)) = \dim(R) - 3 = 3\\
&\dim(R/(I_3+I_5)) = \dim(R/(X_2,X_4)) = \dim(R) - 2 = 4\\
&\dim(R/(I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = \dim(R) - 3 = 3\,.
\end{aligned}$$

For $r = 3$ it is

$$\begin{aligned}
&\dim(R/(I_1+I_2+I_3)) = \dim(R/(X_1,X_4)) = \dim(R) - 2 = 4\\
&\dim(R/(I_1+I_2+I_4)) = \dim(R/(X_1,X_2)) = \dim(R) - 2 = 4\\
&\dim(R/(I_1+I_2+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\\
&\dim(R/(I_1+I_3+I_4)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\\
&\dim(R/(I_1+I_3+I_5)) = \dim(R/(X_2,X_4)) = \dim(R) - 2 = 4\\
&\dim(R/(I_1+I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\\
&\dim(R/(I_2+I_3+I_4)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\\
&\dim(R/(I_2+I_3+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\\
&\dim(R/(I_2+I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\\
&\dim(R/(I_3+I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6 - 3 = 3\,.
\end{aligned}$$

For $r = 4$ it is

$$\begin{aligned}
&\dim(R/(I_1+I_2+I_3+I_4)) = \dim(R/(X_1,X_2,X_4)) = 6-3 = 3\\
&\dim(R/(I_1+I_2+I_3+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6-3 = 3\\
&\dim(R/(I_1+I_2+I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6-3 = 3\\
&\dim(R/(I_1+I_3+I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6-3 = 3\\
&\dim(R/(I_2+I_3+I_4+I_5)) = \dim(R/(X_1,X_2,X_4)) = 6-3 = 3\,.
\end{aligned}$$

For $r = 5$ it is $\dim(R/(I_1+I_2+I_3+I_4+I_5)) = 3\,.$

Consequently,

$$d = \max_{0\le r\le 5}\{\dim(R/(I_{i_1}+\cdots+I_{i_r})) + r\} \overset{\text{for } r=1}{=} 6+1 = 7\,.$$

$$e(Sym_R(I(G))) = \sum_{\substack{0 \le r \le 5 \\ 1 \le i_1 \le \cdots \le i_r \le 5}} e(R/(I_{i_1} + \cdots + I_{i_r}))$$

and $d' = \dim(R/(I_{i_1} + \cdots + I_{i_r})) = d - r = 7 - r$.
For $r = 0$, $d' = 7$, but no quotient ring $(R/(I_{i_1} + \cdots + I_{i_r}))$ has dimension $d' = 7$.
For $r = 1$, $d' = 6$, in particular $R/I_1 = R/(0)$ has dimension $d' = 6$. Then $e(R/(0)) = 1$.
For $r = 2$, $d' = 5$, in particular $R/(I_1 + I_2)$ and $R/(I_1 + I_3)$ have dimension $d' = 5$. Then:

$$e(R/(I_1 + I_2)) + e(R/(I_1 + I_3)) = 1 + 1 = 2\,.$$

For $r = 3$, $d' = 4$, in particular $R/(I_1 + I_2 + I_3)$, $R/(I_1 + I_2 + I_4)$ and $R/(I_1 + I_3 + I_5)$ have dimension $d' = 4$. Then:

$$e(R/(I_1 + I_2 + I_3)) + e(R/(I_1 + I_2 + I_4)) + e(R/(I_1 + I_3 + I_5)) = 1 + 1 + 1 = 3\,.$$

For $r = 4$, $d' = 3$, in particular $R/(I_1+I_2+I_3+I_4)$, $R/(I_1+I_2+I_3+I_5)$, $R/(I_1+I_2+I_4+I_5)$, $R/(I_1 + I_3 + I_4 + I_5)$ and $R/(I_2 + I_3 + I_4 + I_5)$ have dimension $d' = 3$. Then:

$$e(R/(I_1 + I_2 + I_3 + I_4)) + e(R/(I_1 + I_2 + I_3 + I_5)) + e(R/(I_1 + I_2 + I_4 + I_5)) +$$

$$+ e(R/(I_1 + I_3 + I_4 + I_5)) + e(R/(I_2 + I_3 + I_4 + I_5)) = 1 + 1 + 1 + 1 + 1 = 5\,.$$

For $r = 5$, $d' = 2$, but no quotient ring $(R/(I_{i_1} + \cdots + I_{i_r}))$ has dimension $d' = 2$.
In conclusion, it is $e(Sym_A(I(G)) = 1 + 2 + 3 + 5 = 11$.

3. Initial ideals and Ferrers graphs

Let us study connected acyclic graphs G whose edge ideal is generated by a strong s-sequence. In particular, we examine the star graph S_{n-1} whose edge ideal is $I(S_{n-1}) = (X_1X_n, X_2X_n, \ldots, X_{n-1}X_n) \subset R = K[X_1, \ldots, X_n]$. We may associate S_{n-1} to any graph F whose edge ideal is defined starting from $\mathrm{in}_\prec(J)$, where $J = (g_1, \ldots, g_p)$ is the relation ideal of the symmetric algebra $Sym_R(I(S_{n-1}))$. In general, by definition, $\mathrm{in}_\prec(J) = (\mathrm{in}_\prec(f) \mid f \in J)$, but if $g_1, \ldots, g_p$ form a Groebner basis for J, then it is known that $\mathrm{in}_\prec(J) = (\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_p))$.

Proposition 3.1. *Let S_{n-1} be the $(n-1)$-star graph, $I(S_{n-1}) = (X_1X_n, X_2X_n, \ldots, X_{n-1}X_n) \subset R = K[X_1, \ldots, X_n]$ be its edge ideal. Then*

$$\mathrm{in}_\prec(J) = ((X_1)\,T_2, (X_1, X_2)\,T_3, \ldots, (X_1, X_2, \ldots, X_{n-2})\,T_{n-1})\,.$$

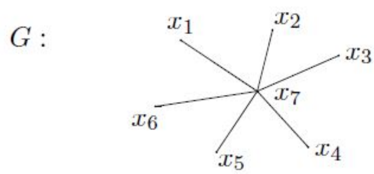
Proof. Let $f_1 = X_1X_n, \ldots, f_{n-1} = X_{n-1}X_n$. Since $I(S_{n-1}) = (f_1, \ldots, f_{n-1})$ is generated by a s-sequence, then $g_{ij} = f_{ij}T_j - f_{ji}T_i$, for $1 \le i < j \le n-1$, form a Groebner basis of J. Hence by Lemma **1.3** $\mathrm{in}_\prec(J) = (f_{ij}T_j \,|\, 1 \le i < j \le n-1)$, where $f_{ij} = X_i$ for $2 < j \le n-1$. □

Remark 3.2. Since the generators of $I(S_{n-1})$ also form a M-sequence, the Groebner basis of J coincides with the Groebner basis of the ideal of presentation of the Rees algebra $\mathfrak{R}(I(S_{n-1}))$. Hence $I(S_{n-1})$ is of linear type (see Conca and De Negri 1999).

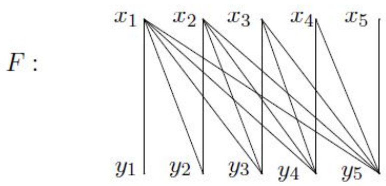
Let us now consider $\mathrm{in}_\prec(J) = ((X_1)\,T_2, (X_1, X_2)\,T_3, \ldots, (X_1, X_2, \ldots, X_{n-2})\,T_{n-1})$.

If we replace the set of variables $\{T_2, \ldots, T_{n-1}\}$ with $\{Y_1, \ldots, Y_{n-2}\}$, then $\mathrm{in}_\prec(J) = ((X_1)\,Y_1, (X_1, X_2)\,Y_2, \ldots, (X_1, \ldots, X_{n-2})\,Y_{n-2})$ is a monomial ideal of $R = K[X_1, \ldots, X_{n-2}; Y_1, \ldots, Y_{n-2}]$ associated with a bipartite graph F on distinct vertex sets $\{x_1, \ldots, x_{n-2}\}$ and $\{y_1, \ldots, y_{n-2}\}$ that correspond to the sets of variables $\{X_1, \ldots, X_{n-2}\}$ and $\{Y_1, \ldots, Y_{n-2}\}$ respectively. In particular, let us introduce the following

Example 3.3. Let $R = K[X_1,\dots,X_7]$, G be the star graph on vertex set $\{x_1,...,x_7\}$ whose edge ideal is $I(G) = (X_1X_7, X_2X_7,\dots,X_6X_7)$ and set $f_1 = X_1X_7$, $f_2 = X_2X_7$, ..., $f_6 = X_6X_7$. By Proposition **3.1**, $\text{in}_\prec(J) = ((X_1)T_2,(X_1,X_2)T_3,(X_1,X_2,X_3)T_4,(X_1, X_2,X_3,X_4)T_5,$ $(X_1,X_2,X_3,X_4,X_5)T_6)$.



If we replace $\{T_2,T_3,T_4,T_5,T_6\}$ with $\{Y_1,Y_2,Y_3,Y_4,Y_5\}$, the bipartite graph



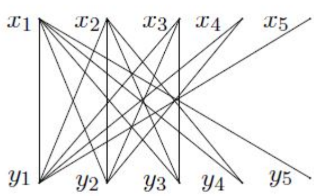
has edge ideal $I(F) = \text{in}_\prec(J) = (X_1Y_1,X_1Y_2,X_1Y_3,X_1Y_4,X_1Y_5,X_2Y_2,X_2Y_3,X_2Y_4,X_2Y_5,X_3Y_3,$ $X_3Y_4,X_3Y_5,X_4Y_4,X_4Y_5,X_5Y_5)$.

When we replace the set of variables $\{T_2,T_3,\dots,T_{n-1}\}$ with the set $\{Y_{n-2},Y_{n-3},\dots,Y_1\}$, then

$$\text{in}_\prec(J) = ((X_1)Y_{n-2},(X_1,X_2)Y_{n-3},\dots,(X_1,X_2,\dots,X_{n-2})Y_1)$$

is a monomial ideal of $R = K[X_1,\dots,X_{n-2};Y_1,\dots,Y_{n-2}]$ associated with a special graph F on distinct vertex sets $\{x_1,\dots,x_{n-2}\}$ and $\{y_1,\dots,y_{n-2}\}$. More precisely, F is a *Ferrers graph* (see Corso and Nagel 2009), namely a bipartite graph on vertex sets $\{x_1,\dots,x_{n-2}\}$ and $\{y_1,\dots,y_{n-2}\}$ such that whenever $\{x_i,y_j\}$ is an edge of F, then so is $\{x_r,y_s\}$ for $1 \leqslant r \leqslant i$ and $1 \leqslant s \leqslant j$, and $\{x_1,y_{n-2}\}$, $\{x_{n-2},y_1\}$ are edges of F.

In the Example **3.3**, the graph F is equivalent to the following Ferrers graph:



The previous description lends itself to being applied to the transmission of confidential data. For example, it must be communicated the position of submarines with nuclear warheads and those with conventional armaments of a superpower fleet. Such positions can be represented by the vertex set of a graph G, $\{x_1,\dots,x_n\}$. The equipments of submarines can be classified through the vertex set $\{y_1,\dots,y_{n-2}\}$ of an unknown graph F. The message to be sent is the drawing of G. It is elaborated via Groebner bases finding the initial ideal $\mathrm{in}_{\prec}(J)$. So it can be built the bipartite graph F on disjoint vertex sets $\{x_1,\dots,x_{n-2}\}$, $\{y_1,\dots,y_{n-2}\}$ that is the graph associated with $\mathrm{in}_{\prec}(J)$. F contains the real meaning of the message because it gives the connection among the location of submarines and their arming. So the receiver safely obtains the desired information.

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