DOI: 10.1478/AAPP.1002A8

AAPP | Atti della Accademia Peloritana dei Pericolanti
Classe di Scienze Fisiche, Matematiche e Naturali
ISSN 1825-1242

Vol. 100, No. 2, A8 (2022)

TRANSIENT PROBLEMS IN COMPRESSIBLE FLUIDS

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ABSTRACT. The aim of this paper is the presentation of two simple transient phenomena in compressible fluids. We show two different time-dependent solutions based on Fourier analysis and obtained for an initial temperature perturbation suddenly applied to a gas, in bounded and unbounded 1D domains. In both cases, the linearized field equations yield a non vanishing velocity field which seems to equilibrate the temperature gradient and it disappears when the temperature reaches the equilibrium state.

1. Introduction

Analytical and numerical solutions of different transient problems are often present in literature (Whitham 1927; Lamb 1945; Thompson 1972; Lighthill 1980; Landau and Lifshitz 1987; Dennery and Krzywicki 1996; Feireisl 2006; Bargmann *et al.* 2008; Straughan 2011; Dafermos 2016), but most of them are related to incompressible fluids and isothermal flows, while compressible fluids are less studied because the determination of analytical solutions is quite more complex.

The aim of this paper is the study of two simple non-stationary problems in classical, ideal, monatomic gases where a little perturbation of the temperature at the initial time is applied. In order to derive more easily the solution, in Section 2 the field equations of classical thermodynamics are linearized around a constant solution. It is easily found that, contrary to the stationary case, the velocity field cannot vanish and instead it is a consequence of the heat flux. The Navier-Stokes-Fourier field equations are integrated in Sections 3 and 4 for two different problems. In Section 3 the case of the gas between two adiabatic infinite parallel plates is studied and an analytical solution of the field equations is derived by use of the classical method of separation of variables. In Section 4 the transfer problem of a gas in an unbounded 1D domain is considered. Here the solution is derived by use of the Fourier Transform which implies an analytical solution except for the numerical integration of the last integral. Finally in Section 5, for completeness, the particular case of a travelling wave is presented.

2. Field equations

In this paper, we use the field equations of classical thermodynamics (de Groot and Mazur 1984; I. Müller and W. H. Müller 2009) in order to describe some transient problems in compressible fluids. The field variables are the mass density $\rho(t,x_i)$, the velocity $v_j(t,x_i)$ and the temperature $T(t,x_i)$, while the field equations are based on the conservation laws of mass, momentum and energy:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial \rho v_k}{\partial x_k} = 0 \\ \dfrac{\partial \rho v_i}{\partial t} + \dfrac{\partial(\rho v_i v_k - t_{ik})}{\partial x_k} = 0, \\ \dfrac{\partial \rho u}{\partial t} + \dfrac{\partial(\rho u v_k + q_k)}{\partial x_k} = t_{ji}\dfrac{\partial v_j}{\partial x_i}. \end{cases} \tag{1}$$

These equations are closed by the Navier-Stokes and Fourier constitutive relations, which relate the stress tensor t_{ij} and the heat flux q_i to the gradients of the velocity and the temperature fields, so that

$$t_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_n}{\partial x_n}\delta_{ij}\right), \tag{2}$$

$$q_i = -K\frac{\partial T}{\partial x_i}. \tag{3}$$

The pressure $p(\rho,T)$ and the specific internal energy $u(\rho,T)$ are expressed respectively through the caloric and thermal equations of state and, in the particular case of a classical ideal monoatomic gas herein considered, they are given by

$$p = \frac{k_B}{m}\rho T, \quad u = \frac{3}{2}\frac{k_B}{m}T, \tag{4}$$

with k_B the Boltzmann constant and m the molecular mass. Furthermore, under the assumption of Maxwellian molecules, the shear viscosity η and the thermal conductivity K are linear functions of the temperature T of the form

$$\eta = \frac{k_B}{m}\frac{T}{a}, \quad K = \frac{15}{4}\left(\frac{k_B}{m}\right)^2\frac{T}{a}, \tag{5}$$

where a is a constant depending on the gas under consideration. Insertion of the constitutive equations (2) and (5) into the balance equations (1) provides a closed set of five PDEs, in the five fields $\rho(t,x_i), v_j(t,x_i)$ and $T(t,x_i)$.

In this paper, we assume that the gas lies between two infinite parallel plates, so that we can neglect the dependence of the fields on the x_2 and x_3 space variables. We consider the gas subject, at $t=0$, to a small initial perturbation from an equilibrium state and we study the evolution of the perturbation for $t>0$. Hence, we can linearize the field equations around the constant equilibrium state, characterized by ρ_0, T_0 and $v=0$, and write the corresponding equations in terms of the following dimensionless quantities and parameters:

$$x^\star = \frac{x}{L}, \quad t^\star = \frac{L}{c_0}t, \quad \rho^\star = \frac{\rho}{\rho_0}, \quad T^\star = \frac{T}{T_0}, \quad v^\star = \frac{v}{c_0}, \quad \tau = \frac{\sqrt{c_0}}{aL\rho_0}, \tag{6}$$

where L is a suitable length and $c_0 = \sqrt{k_B/mT_0}$ has the dimension of a velocity. In particular, $\sqrt{5/3}c_0$ represents the speed of the sound propagating in the gas. Finally, after dropping the star in the variables for notational simplicity, the linearized dimensionless field equations are given by

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial \rho}{\partial x} + \frac{\partial T}{\partial x} - \mu^\star \dfrac{\partial^2 v}{\partial x^2} = 0, \\ \dfrac{\partial T}{\partial t} - \dfrac{2}{3}K^\star \dfrac{\partial^2 T}{\partial x^2} + \dfrac{2}{3}\frac{\partial v}{\partial x} = 0, \end{cases} \tag{7}$$

with

$$\mu^\star = \frac{8}{9}\tau, \qquad K^\star = \frac{5}{2}\tau. \tag{8}$$

The resulting system forms a set of three partial differential equations for the variables $\rho(t,x)$, $v(t,x)$ and $T(t,x)$ that we are going to integrate for different initial and boundary conditions. In these equations all the quantities are explicit except for τ, the only parameter of the problem, which depends on the degree of rarefaction of the gas.

We emphasize that, in the case of a stationary problem, the field equations admit the solution with $v(x) = 0$. For time-dependent processes, that are processes in which the static solution it is not compatible with the initial or boundary conditions, the assumption $v(x,t) = 0$ leads to a contradiction and, therefore, it is not admissible. This can be easily seen as $v(x,t)$ in Eq. (7). The aim of this paper is to look for a solution of the field equations (7)-(8), which shows the behaviour of the temperature and the velocity fields as a function of time and space, that is driven by an initial temperature disturbance.

3. Solutions in a bounded domain

We start now considering our gas filling the space between two infinite parallel adiabatic plates, at a distance L, so that the dimensionless variable x takes values in $[0,1]$. We also assume that at the initial time the temperature T is perturbed from the equilibrium state and it is the known function

$$T(0,x) = T_0(x). \tag{9}$$

In order to look for solutions of this problem, we use Eqs. (7)-(8) and we apply the method of separation of variables. Therefore, we suppose that the three fields are given by the products

$$\rho(t,x) = \overline{R}(t)\hat{R}(x), \quad T(t,x) = \overline{T}(t)\hat{T}(x), \quad v(t,x) = \overline{v}(t)\hat{v}(x) \tag{10}$$

and substituting these expressions in equation $(7)_1$, we obtain

$$\bar{R}'(t) = H\,\bar{v}(t), \quad \hat{v}'(x) = -H\hat{R}(x), \tag{11}$$

while equation $(7)_3$ implies

$$\begin{aligned} &\frac{\bar{T}'(t)}{\bar{T}(t)} = N < 0, && \frac{\bar{v}(t)}{\bar{T}(t)} = M, \\ &-\frac{2}{3}\frac{\hat{v}'(x)}{\hat{T}(x)} = S, && -\frac{2}{3}K^\star \frac{\hat{T}''(x)}{\hat{T}(x)} = -N + SM \end{aligned} \tag{12}$$

with H, N, M and S unknown constants. We set $N < 0$, since only in this case a physically admissible solution can be obtained. The second equation furnishes

$$\begin{aligned} &\frac{\bar{v}'(t)}{\bar{T}(t)} = A, && \frac{\bar{R}(t) + \mu^\star H \bar{v}(t)}{\bar{T}(t)} = C, \\ &\frac{\hat{R}'(x)}{\hat{v}(x)} = D, && \frac{\hat{T}'(x)}{\hat{v}(x)} = B \end{aligned} \tag{13}$$

with other four integration constants A, B, C and D, satisfying

$$A + CD + B = 0. \tag{14}$$

Hence, integration of $(12)_{1,2}$ and $(13)_{1,2}$, implies

$$\begin{aligned} &\bar{T}(t) = \sigma e^{Nt}, \quad \bar{v}(t) = \sigma M e^{Nt}, \quad \bar{R}(t) = \sigma\left[C - \mu H M\right] e^{Nt}, \\ &A = MN\,, \quad HM(\mu^\star N + 1) = CN, \end{aligned} \tag{15}$$

while from the remaining equations, we have

$$\begin{aligned} \hat{T}(x) &= \alpha \cos(\lambda x) + \beta \sin(\lambda x), \\ \hat{v}(x) &= \frac{\lambda}{B}\left[-\alpha \sin(\lambda x) + \beta \cos(\lambda x)\right], \\ \hat{R}(x) &= \frac{\lambda^2}{HB}\left[\alpha \cos(\lambda x) + \beta \sin(\lambda x)\right], \end{aligned} \tag{16}$$

with

$$\lambda^2 = \frac{3}{2}\frac{SM - N}{K} = DH > 0, \quad S = \frac{2}{3}\frac{\lambda^2}{B}. \tag{17}$$

Therefore, the solutions of equations (7) and (8) are given by

$$\begin{aligned} \rho(t,x) &= \lambda^2 \frac{1}{N\tilde{B}} e^{Nt}\left[\tilde{\alpha}\cos(\lambda x) + \tilde{\beta}\sin(\lambda x)\right], \\ v(t,x) &= \lambda \frac{1}{\tilde{B}} e^{Nt}\left[-\tilde{\alpha}\sin(\lambda x) + \tilde{\beta}\cos(\lambda x)\right], \\ T(t,x) &= e^{Nt}\left[\tilde{\alpha}\cos(\lambda x) + \tilde{\beta}\sin(\lambda x)\right], \end{aligned} \tag{18}$$

where we have set

$$\tilde{\alpha} = \alpha\sigma, \quad \tilde{\beta} = \beta\sigma, \quad \tilde{B} = \frac{B}{H}. \tag{19}$$

In this way, the integration constants reduce to the five quantities $\tilde{B}, N, \lambda, \tilde{\alpha}, \tilde{\beta}$. Moreover, combining relations (14), $(15)_{4,5}$ and (17), we get

$$\tilde{B} = -\frac{N^2 + (1 + \mu^\star N)\lambda^2}{N} \tag{20}$$

and the algebraic relation

$$\frac{3}{2}N^3+\lambda^2\left[\frac{3}{2}\mu^\star+K^\star\right]N^2+\lambda^2\left[\frac{5}{2}+K^\star\mu^\star\lambda^2\right]N+K^\star\lambda^4=0, \tag{21}$$

which yield the values of N and $\tilde{B}$ in terms of λ and τ. So, in order to have the explicit analytical solution, we have to find the last three integration constants $\tilde{\alpha}$, $\tilde{\beta}$ and λ, that can be obtained expanding the initial temperature (9) in Fourier series. Assuming that the temperature field vanishes at the two boundaries $x=0$ and $x=1$ (this is a consequence of the adiabatic assumption for the two boundaries) we get

$$\begin{aligned}&\tilde{\alpha}_n=2\int_0^1 T_0(\xi)\cos(n\pi\xi)d\xi,\\&\lambda_n=n\pi,\quad \tilde{\beta}_n=0.\end{aligned} \tag{22}$$

In this way, the analytical solution becomes

$$\begin{aligned}\rho\,(t,x)&=\sum_{n=1}^{\infty}\frac{\tilde{\alpha}_n\lambda_n^2}{\tilde{B}_nN_n}e^{N_nt}\cos\left(\lambda_nx\right),\\ v\,(t,x)&=\sum_{n=1}^{\infty}\frac{\tilde{\alpha}_n\lambda_n}{\tilde{B}_n}e^{N_nt}\sin\left(\lambda_nx\right),\\ T\,(t,x)&=\frac{\tilde{\alpha}_0}{2}+\sum_{n=1}^{\infty}\tilde{\alpha}_ne^{N_nt}\cos\left(\lambda_nx\right),\end{aligned} \tag{23}$$

where N_n and $\tilde{B}_n$ are the roots of Eqs. (20) and (21) with $\lambda=\lambda_n$.

In Fig. 1, we show the values of N in terms of λ, derived from Eq. (21). As it can be easily seen, this equation admits one real negative root and two complex ones with negative real parts for $\lambda\tau<1.896$, while three negative solutions are present for $\lambda\tau>1.896$. In order to write the analytical solution $(23)_3$, we pick the real root of (21) and we consider the value $\tau=10^{-1}$ for the parameter of the problem and the function $T_0(x)=\exp\left(-[10(x-0.5)]^2\right)$ as initial temperature. In Fig. 2, we have shown in the analytical solution (23) in terms of these quantities. In particular, Fig. 2a shows the evolution of the temperature field, which follows from the initial disturbance. From the mathematical point of view, it could be possible using the other two roots for N, that would lead to an oscillatory and non physical behaviour since the remaining roots are complex. Figure 2b illustrates the velocity field generated from this disturbance. It can be observed that the velocity field is proportional to the heat flux and, as expected, it disappears when the temperature evolves to the constant equilibrium state. From the physical point of view the presence of an initial disturbance for the temperature implies also for the linear equations a non uniform velocity field which appears to redistribute the temperature gradient as $t>0$. We have to say that it is not possible to impose that the gas is at rest at $t=0$ in the analytical solution (23).

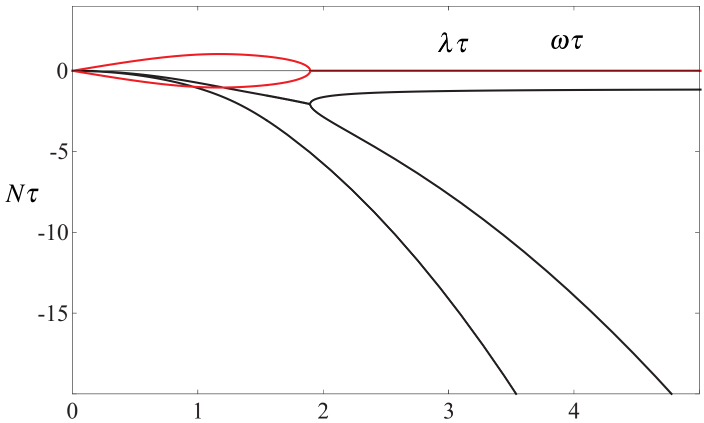
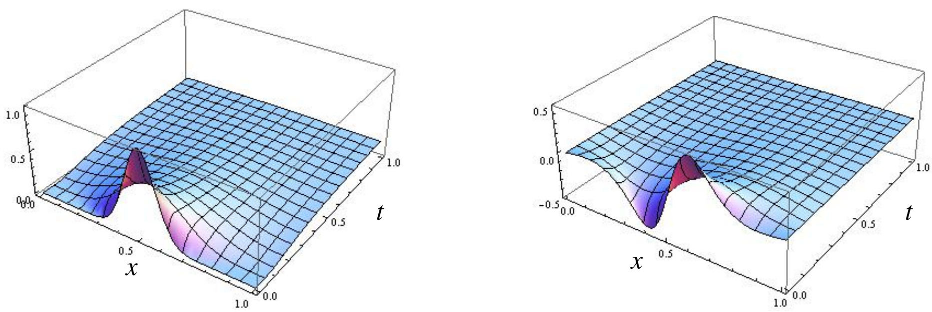


FIGURE 1. Plot of $N\tau$ in terms of $\lambda\tau$ obtained as solution of the algebraic relation (21), which coincide, as explained in the text, with the solution of $N\tau$ in terms of $\omega\tau$ obtained from (28). The black lines refer to the real parts of these solutions, while the red lines represent the imaginary parts. As it can be easily seen for $\lambda\tau$ or $\omega\tau < 1.896$, the algebraic relations admit one real root and two imaginary ones, while for $\lambda\tau$ or $\omega\tau > 1.896$ three real solutions are present.



2a) $T(t,x)$ 2b) $v(t,x)$

FIGURE 2. Solutions of system (7)-(8) for the temperature $T(t,x)$ and the velocity $v(t,x)$ in a bounded domain obtained through the method of separation of variables. Their analytical expression is given in (23).

4. Transient problem in an unbounded domain

We consider now an initial temperature perturbation in a gas filling an unbounded 1D-domain. In order to obtain the solution for this problem, we use the Fourier Transform as defined by Barozzi (2004):

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega x} f(x)dx, \tag{24}$$

where $f \in \mathrm{L}^1(\mathbb{R})$ and $\omega \in \mathbb{R}$ is the pulsation. Applying the Fourier Transform to equations (7)-(8), we get:

$$\begin{aligned} &\frac{\partial \hat{R}}{\partial t} + i\omega \hat{V} = 0, \\ &\frac{\partial \hat{V}}{\partial t} + i\omega \hat{R} + i\omega \hat{T} + \mu^\star \hat{V} = 0, \\ &\frac{\partial \hat{T}}{\partial t} + \frac{2}{3}K^\star \hat{T} + \frac{2}{3} i\omega \hat{V} = 0, \end{aligned} \tag{25}$$

where $\hat{R}(t,\omega), \hat{V}(t,\omega)$ and $\hat{T}(t,\omega)$ are the Fourier Transform of the density $\rho(t,x)$, the velocity $v(t,x)$ and the temperature $T(t,x)$, respectively.
System (25) is equivalent to the following third order linear PDE in the field $\hat{T}(t,\omega)$ with coefficients depending only on ω and τ:

$$\frac{3}{2}\frac{\partial^3 \hat{T}}{\partial t^3} + \left(\frac{3}{2}\mu^\star + K^\star\right)\omega^2 \frac{\partial^2 \hat{T}}{\partial t^2} + \left(\frac{5}{2} + K^\star \mu^\star \omega^2\right)\omega^2 \frac{\partial \hat{T}}{\partial t} + K^\star \omega^4 \hat{T} = 0, \tag{26}$$

that admits the analytical solution of the form

$$\hat{T}(t,\omega) = B(\omega)e^{Nt}. \tag{27}$$

Therefore, substituting (27) into equation (26), we get the algebraic relation for N:

$$\frac{3}{2}N^3 + \lambda^2\left[K^\star + \frac{3}{2}\mu^\star\right]N^2 + \omega^2\left[K^\star \mu^\star \omega^2 + \frac{5}{2}\right]N + K^\star \omega^4 = 0, \tag{28}$$

which implies the coefficient N in terms of the pulsation ω.

As expected, equation (28) has the same form of (21), with ω instead of λ. Taking into account the initial temperature (9) and the solution (27), the Fourier Transform $\hat{T}(t,\omega)$ becomes:

$$\hat{T}(t,\omega) = \hat{T}_0(\omega)e^{N(\omega)t}, \tag{29}$$

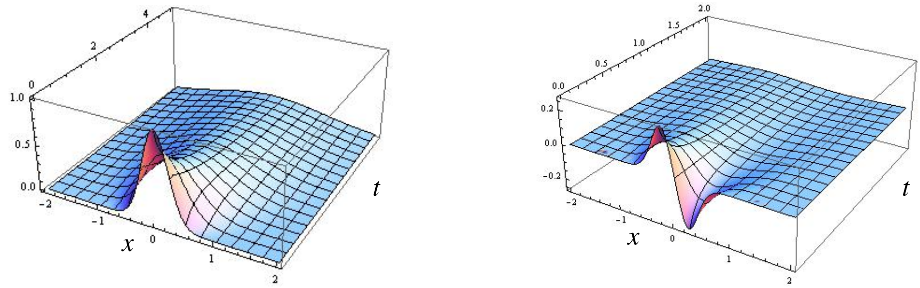
where $\hat{T}_0(\omega)$ is the Fourier Transform of $T_0(x)$. In this way, the temperature $T(t,x)$ is obtained applying the inverse Fourier Transform to solution (29), given by Barozzi (2004):

$$T(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{T}_0(\omega)e^{N(\omega)t + i\omega x} d\omega. \tag{30}$$

Due to the complexity of the calculation, we perform the integration of (30) numerically.

Figure 1 shows also the values of $N\tau$ in terms of $\omega\tau$, which coincide with the solution obtained in (21) changing $\lambda\tau$ with $\omega\tau$. As before, we choose the real root of Eq. (28) for the solution. Furthermore, we set the parameter $\tau = 10^{-1}$ and we consider the gas subject to the initial temperature $T_0(x) = \exp(-10x^2)$.

In Fig. 3, the solutions for the temperature and the velocity fields are depicted. Their behaviour is similar to the previous case, but here the perturbation is generated in an unbounded domain. After a time, the perturbation of the temperature and the velocity fields evolve to an equilibrium state with zero velocity and a constant temperature.



3a) $T(t,x)$ 3b) $v(t,x)$

FIGURE 3. Solutions of system (7)-(8) for the temperature $T(t,x)$ and the velocity $v(t,x)$ in a unbounded domain obtained through the Fourier Transform. The plots are obtained integrating numerically Eq. (30).

5. Travelling wave solution

Another possible way to integrate system (7) in an unbounded domain is to look for solutions, that are functions of the form $U = U(z)$, being $z = x - Vt$ the travelling wave coordinate with a constant wave speed $V > 0$. Therefore, considering $\rho(z), v(z)$ and $T(z)$, the PDEs (7) become the following system of three ODEs.

$$\begin{cases} -V\dfrac{d\rho}{dz} + \dfrac{dv}{dz} = 0, \\ -V\dfrac{dv}{dz} + \dfrac{d\rho}{dz} + \dfrac{dT}{dz} - \mu^\star \dfrac{d^2 v}{dz^2} = 0, \\ -V\dfrac{dT}{dz} - K^\star \dfrac{d^2 T}{dz^2} + \dfrac{2}{3}\dfrac{dv}{dz} = 0, \end{cases} \tag{31}$$

which can be integrated more easily. Moreover, assuming that the solution has the form

$$U = U^\star + \hat{U} e^{\xi z}, \tag{32}$$

where $U^\star$ represents the equilibrium state and ξ is a real parameter, we get from (31) a system of three algebraic equations in $\hat{\rho}$, $\hat{v}$ and $\hat{T}$, which admits a non-trivial solution iff the following condition holds:

$$\det \begin{bmatrix} -V & 1 & 0 \\ 1 & -V - \mu^\star \xi & 1 \\ 0 & \frac{2}{3} & -V - \dfrac{2}{3} K^\star \xi \end{bmatrix} = 0. \tag{33}$$

This furnishes the characteristic equation for ξ in terms of velocity V and the parameters of the gas:

$$V K^\star \mu^\star \xi^2 + \left[V^2 \left(K^\star + \frac{3}{2}\mu^\star \right) - K^\star \right] \xi + V \left(\frac{3}{2} V^2 - \frac{5}{2} \right) = 0. \tag{34}$$

Also this equation has the same form of (21) and (28) and, in particular, it can be obtained from (28) with the substitution $N = -V\xi$ and $\lambda = i\xi$. Here, contrary to the previous cases, we have to fix the velocity V and solve the equation in terms of the parameter ξ.

By inserting constitutive relations (8) into Eq. (34), it is easy to see that we have always two real solutions since the discriminant $\Delta \geq 0$ holds. This solutions are of opposite signs for $V < \sqrt{5/3}$, while they are both negative for $V > \sqrt{5/3}$. Then, in order to compute an explicit solution, we fix the value $V = 1.3$. Figure 4 shows the behaviour of the solution of system (31) for $\tau = 10^{-1}$. The first graph represents the temperature field connecting the initial perturbed value $T = 1$ to the equilibrium value 0.73. The second graph represents the velocity field which connects $V = 0.5$ with the state $V = 0$. In Fig. 4, the waves can be interpreted as the evolution of an initial perturbation in a position $\tilde{x}$ to the equilibrium value as time grows. The curves can also be seen a section of the 2-D surfaces in Figs. 2 and 3.

Christov *et al.* (2016) presented an interesting study about travelling waves in a similar case. The authors considered a non-isentropic fluid and an initial perturbation in the velocity field, so the evolution of an acoustic shock created at $t = 0$ is studied. In that case, an exact solution was obtained by Zverev (1950) and Morrison (1956) or alternatively a solution based on Fourier sine and Laplace Transform (Bargmann *et al.* 2008) was provided and the results are generalized to the non-linear case.

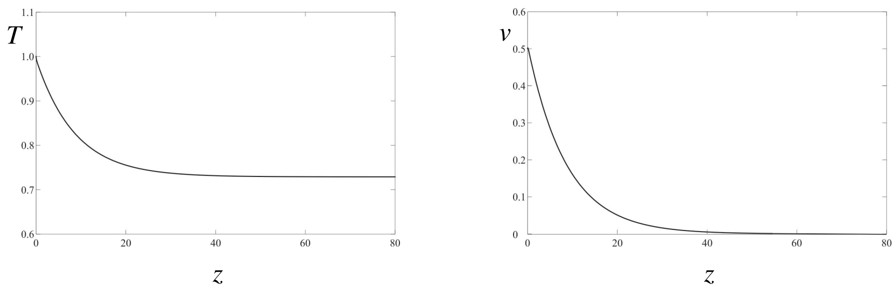


FIGURE 4. Solutions of system (31) for the temperature $T(z)$ and the velocity $v(z)$.

6. Conclusions and final remarks

In this paper, the linearized field equations of classical thermodynamics are integrated in an unbounded and bounded 1D domain. An initial perturbation in the temperature field is created and the transition from this disturbance to the equilibrium state is studied. The linearized field equations predict a non vanishing velocity field, which appears to equilibrate the temperature gradient. The Fourier Series and the Fourier Transform are used to find the analytical or numerical solution for both cases. A possible new approach to derive solutions to some differential problems can be obtained using the Laplace Transform. This subject is already under examination and it seems to give new interesting results. Furthermore, the numerical solution of traveling wave-like fields was determined. An interesting subject for future studies could be the analytical solution, as obtained for a different problem by Zverev (1950), Morrison (1956), Bargmann *et al.* (2008), and Christov *et al.* (2016), and the study of the weakly non-linear and fully non-linear theory as made by Christov *et al.* (2016).

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Communicated 28 May 2021; manuscript received 1 July 2022; published online 9 November 2022