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On some topological games involving networks

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ABSTRACT

In these notes we introduce and investigate two new games called R-nw-selective game and the M-nw-selective game. These games naturally arise from the corresponding selection principles involving networks introduced in [5].

1. Introduction

Throughout the paper we mean by "space" a topological Hausdorff space.

A family $\mathcal{N}$ of sets is called a network for X if for every $x \in X$ and for every open neighborhood U of x there exists an element N of $\mathcal{N}$ such that $x \in N \subseteq U$; $nw(X) = min\{|\mathcal{N}| : \mathcal{N}$ is a network for $X\}$ is the network weight of X. A family $\mathcal{P}$ of open sets is called a π-base for X if every nonempty open set in X contains a nonempty element of $\mathcal{P}$; $\pi w(X) = min\{|\mathcal{P}| : \mathcal{P}$ is a π-base for $X\}$ is the π-weight of X. It is known that $nw(X) \leq w(X)$, where $w(X)$ denotes the weight of the space X, and in the class of compact Hausdorff spaces $nw(X) = w(X)$ (see [9]); $\delta(X) = \sup\{d(Y) : Y$ is a dense subset of $X\}$, where $d(X)$ denotes the density of the space X.

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Recall that for $f, g \in \omega^\omega$, $f \leq^* g$ means that $f(n) \leq g(n)$ for all but finitely many n (and $f \leq g$ means that $f(n) \leq g(n)$ for all $n \in \omega$). A subset $D \subseteq \omega^\omega$ is dominating if for each $g \in \omega^\omega$ there is $f \in D$ such that $g \leq^* f$. The minimal cardinality of a dominating subset of ω^ω is denoted by $\mathfrak{d}$. The value of $\mathfrak{d}$ does not change if one considers the relation $\leq$ instead of $\leq^*$ [7, Theorem 3.6]. The symbol $\mathcal{M}$ denotes the family of all meager subsets of $\mathbb{R}$. $cov(\mathcal{M})$ is the minimum of the cardinalities of subfamilies $\mathcal{U} \subseteq \mathcal{M}$ such that $\bigcup \mathcal{U} = \mathbb{R}$. However, another description of the cardinal $cov(\mathcal{M})$ is the following. $cov(\mathcal{M})$ is the minimum cardinality of a family $F \subset \omega^\omega$ such that for every $g \in \omega^\omega$ there is $f \in F$ such that $f(n) \neq g(n)$ for all but finitely many $n \in \omega$ (see [2] and also [3, Theorem 2.4.1]). Thus if $F \subset \omega^\omega$ and $|F| < cov(\mathcal{M})$, then there is $g \in \omega^\omega$ such that for every $f \in F$, $f(n) = g(n)$ for infinitely many $n \in \omega$; it is often said that g *guesses* F. Also, if $\mathbb{P}$ is a countable poset and $\mathcal{D}$ is a family of dense sets of cardinality strictly less than $cov(\mathcal{M})$ then there exists a generic filter that meets all the dense sets of the family [3, Section 3].

In [11,13] a systematic approach was considered to describe selection principles. Given two collections $\mathcal{A}$ and $\mathcal{B}$ of some particular topological objects on a space X, Scheepers introduced the following notation:

$S_1(\mathcal{A}, \mathcal{B})$: For every sequence $(\mathcal{U}_n : n \in \omega)$ of elements of $\mathcal{A}$ there exists $U_n \in \mathcal{U}_n$, $n \in \omega$, such that $\{U_n : n \in \omega\}$ belongs to $\mathcal{B}$.

$S_{fin}(\mathcal{A}, \mathcal{B})$: For every sequence $(\mathcal{U}_n : n \in \omega)$ of elements of $\mathcal{A}$ there exists a finite subset $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ belongs to $\mathcal{B}$.

If one denotes by $\mathcal{O}$ the family of all open covers of a space X and by $\mathcal{D}$ the family of all dense subsets of a space X, a space is said to be Rothberger if it satisfies $S_1(\mathcal{O}, \mathcal{O})$, Menger if it satisfies $S_{fin}(\mathcal{O}, \mathcal{O})$, R-separable if it has the property $S_1(\mathcal{D}, \mathcal{D})$ and M-separable if it has the property $S_{fin}(\mathcal{D}, \mathcal{D})$.

It is shown in [4] that $\delta(X) = \omega$ for every M-separable space X; and if $\delta(X) = \omega$ and $\pi w(X) < \mathfrak{d}$, then X is M-separable (a stronger version of this fact is established in [13, Theorem 40]); moreover, if $\delta(X) = \omega$ and $\pi w(X) < \mathrm{cov}(\mathcal{M})$, then X is R-separable (a stronger version of this fact is established in [13, Theorem 29]).

Every space having a countable base is R-separable, therefore M-separable. However, not every space with countable network weight is M-separable. Hence in [5] the authors asked under which conditions a space of countable network weight must be M-separable and introduced and studied the following classes of spaces.

Definition 1.1. Let X be a space with $nw(X) = \omega$.

- X is M-nw-selective if for every sequence $(\mathcal{N}_n : n \in \omega)$ of countable networks for X one can select finite $\mathcal{F}_n \subset \mathcal{N}_n$, $n \in \omega$, such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a network for X.
- X is R-nw-selective if for every sequence $(\mathcal{N}_n : n \in \omega)$ of countable networks for X one can pick $F_n \in \mathcal{N}_n$, $n \in \omega$, such that $\{F_n : n \in \omega\}$ is a network for X.

In [5] it was proved that any R-nw-selective (M-nw-selective) space is both Rothberger and R-separable (Menger and M-separable). See also [6] for more details about these two properties.

Recall that topological games, introduced with a systematical structure in [11,13], are infinite games played by two different players, Alice and Bob, on a topological space X (see also [1]). We assume that the length (number of innings) of the games is ω and the two players pick in each inning some topological objects of a fixed space. The strategies of the two players are a priori defined; they are some functions that take care of the game history. At the end there is only one winner, so a draw is not allowed. Playing a game G on a space X gives rise to two properties: "Alice has a winning strategy in the game G on X"; "Bob has a winning strategy in the game G on X". Of course, since there is no draw, it is impossible for a space to have both these properties, but it can happen that the negation of both of them holds. In this case we say that the game G is indeterminate on the space X.

Given two families of topological objects $\mathcal{A}$ and $\mathcal{B}$, the followings are two games associated to selection principles.

$G_1(\mathcal{A}, \mathcal{B})$: is played according to the following rules.
- for every $n \in \omega$ Alice chooses $A_n \in \mathcal{A}$;
- Bob answers picking $b_n \in A_n$ for each $n \in \omega$;
- the winner is Bob if $\{b_n : n \in \omega\} \in \mathcal{B}$, otherwise Alice wins.

$G_{fin}(\mathcal{A}, \mathcal{B})$: is played according to the following rules.
- for every $n \in \omega$ Alice chooses $A_n \in \mathcal{A}$;
- Bob answers picking a finite subset $B_n \subseteq A_n$ for each $n \in \omega$;
- the winner is Bob if $\bigcup\{B_n : n \in \omega\} \in \mathcal{B}$, otherwise Alice wins.

The game $G_1(\mathcal{O}, \mathcal{O})$, called Rothberger game, is strictly related to the Rothberger property. In what follows we denote this game by Rothberger(X). The game $G_{fin}(\mathcal{O}, \mathcal{O})$, called Menger game, is strictly related to the Menger property. So everywhere in this paper we denote this game by Menger(X). The game $G_1(\mathcal{D}, \mathcal{D})$ is strictly related to the R-separability, from now on we denote this game by R-separable(X). Similarly, the game $G_{fin}(\mathcal{D}, \mathcal{D})$ is strictly related to the M-separability and in what follows we denote this game M-separable(X).

These games were largely studied and some important characterizations of "Alice does not have a winning strategy" and "Bob has a winning strategy" have been given [1,13,15,16]. Despite this some questions are still open. We denote by Bob $\uparrow G$ on X, the fact that "Bob has a winning strategy in the game G on X" and by Alice $\not\uparrow G$ on X, the fact that "Alice does not have a winning strategy in the game G on X".

Remark 1.2. In general the following implications hold.

1. Bob $\uparrow G_1(\mathcal{A}, \mathcal{B}) \implies$ Bob $\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$;
2. Alice $\not\uparrow G_1(\mathcal{A}, \mathcal{B}) \implies$ Alice $\not\uparrow G_{fin}(\mathcal{A}, \mathcal{B})$;
3. Bob $\uparrow G_1(\mathcal{A}, \mathcal{B}) \implies$ Alice $\not\uparrow G_1(\mathcal{A}, \mathcal{B}) \implies S_1(\mathcal{A}, \mathcal{B})$;
4. Bob $\uparrow G_{fin}(\mathcal{A}, \mathcal{B}) \implies$ Alice $\not\uparrow G_{fin}(\mathcal{A}, \mathcal{B}) \implies S_{fin}(\mathcal{A}, \mathcal{B})$.

For some properties the last two implications of points 3 and 4 are, in fact, characterizations, that is Alice $\not\uparrow G(\mathcal{A}, \mathcal{B}) \iff S(\mathcal{A}, \mathcal{B})$.

In [10] it is proved that a space X is Rothberger if, and only if, Alice $\not\uparrow$ Rothberger(X). In [15,8] it is proved that if X is a space in which each point is a G_δ, Bob $\uparrow$ Rothberger(X) if, and only if, X is countable. Similar arguments are valid for the Menger case: in [12,14] it is proved that a space X is Menger if, and only if, Alice$\not\uparrow$ Menger(X) and in [16] that if X is a metrizable space, Bob $\uparrow$ Menger(X) if, and only if, X is σ-compact. In [13] it is proved that Bob $\uparrow$ R-separable(X) if, and only if, $\pi w(X) = \omega$ and, under CH, it is given an example of a R-separable space X such that Alice $\uparrow$ R-separable(X).

Two topological games G and G' are called dual if both "Alice $\uparrow G \iff$ Bob $\uparrow G'$" and "Alice $\uparrow G' \iff$ Bob $\uparrow G$" hold. Sometimes this dual vision could be useful to apply different techniques in proofs. For instance, the Point-open game is the dual of the Rothberger game (see [8]), the Point-picking game is the dual of $G_1(\mathcal{D}, \mathcal{D})$ (see [13]), the Compact-open game is a possible dual of the Menger game (see [16]), but the question about the hypothesis to add to let them be dual is still open.

In Section 2 we study the R-nw-selective game. We present a characterization of the "Bob having a winning strategy" property and a sufficient condition for "Alice not having a winning" strategy property.

We also give a consistent characterization, in terms of games, of the R-nw-selective property in the class of spaces with countable netweight and weight strictly less than $cov(\mathcal{M})$. Moreover, we introduce the (Point, Open)-Set game and we prove that it is a promising candidate to be the dual of the R-nw-selective game.

In Section 3 we study the M-nw-selective game. We present, under some consistent hypothesis, a sufficient condition for "Alice not having a winning strategy" property and some necessary conditions for the "Bob having a winning strategy" property. We also give a consistent characterization, in terms of games, of the M-nw-selective property in the class of spaces with countable netweight and weight strictly less than $\mathfrak{d}$.

2. The R-nw-selective game

Definition 2.1. Let X be a space with $nw(X) = \omega$. The R-nw-selective game, denoted by R-nw-selective(X), is played according to the following rules. Alice chooses a countable network $\mathcal{N}_0$ and Bob answers picking an element $N_0 \in \mathcal{N}_0$. Then Alice chooses another countable network $\mathcal{N}_1$ and Bob answers in the same way and so on for countably many innings. At the end Bob wins if the set $\{N_n : n \in \omega\}$ of his selections is a network.

Simultaneously we consider the possible dual version of the R-nw-selective game.

Definition 2.2. The (Point, Open)-Set game on a space X, denoted by PO-set(X), is played according to the following rules. Alice chooses a point x_0 and an open set U_0 containing x_0. Then Bob picks N_0 a subset of X such that $x_0 \in N_0 \subseteq U_0$. The game goes ahead in this way for every $n \in \omega$ and Alice wins if the set $\{N_n : n \in \omega\}$ of Bob's choices is a network.

Theorem 2.3. *Let X be a space.* Bob $\uparrow$ *R-nw-selective*(X) *if, and only if, the space X is countable and second countable.*

Proof. Clearly, if X is a countable second countable space then it is easy to construct a winning strategy for Bob in the R-nw-selective game on X.

Let $\mathbb{M}$ be the collection of all countable networks of X. Let σ be a winning strategy for Bob.

First we prove that the space is countable.

Claim 1. $|\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}| \leq 1$.

Indeed, suppose that two distinct points, say x and y, belong to all the closure of the possible answers to $\mathcal{N}$, for any $\mathcal{N} \in \mathbb{M}$. Fix any countable network $\mathcal{N}$ and observe that $\mathcal{N}' = \{N \in \mathcal{N} : \{x, y\} \cap \overline{N} = \emptyset\} \cup \{\{x\}, \{y\}\}$ is also a network in X such that no element of $\mathcal{N}'$ contains the set $\{x, y\}$ in its closure. This gives a contradiction.

Claim 2. There exists a countable $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})}$.

Indeed, if $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \{x\}$ (it is the same if $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \emptyset$), the complements of all the closures form an open cover of $X \setminus \{x\}$ (or X) and then, since having countable network implies hereditary Lindelöfness, we can obtain a countable subcover of $X \setminus \{x\}$ (or of the all space X).

Claim 1. and Claim 2. hold for any inning $n \in \omega$, that is

Claim 1(n). $|\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N})}| \leq 1$.

Claim 2(n). There exists a countable $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N}_0, ..., \mathcal{N}_{n-2}, \mathcal{N})}$.

The proof of Claim 1(n). and Claim 2(n). is analogous to the one of Claim 1. and Claim 2., respectively.

Consider the following tree of possible evolution of the R-nw-selective game on X. By claim 2. there exists $(\mathcal{N}^n_\emptyset)_{n\in\omega}$, that is countably many possible choices of Alice in the first inning $\mathcal{N}^0_\emptyset, \mathcal{N}^1_\emptyset, \mathcal{N}^2_\emptyset, ...$, such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\sigma(\mathcal{N})} = \bigcap_{n\in\omega} \overline{\sigma(\mathcal{N}^n_\emptyset)}$.

Fix, for example, the branch with $\mathcal{N}^0_\emptyset$ then there exists a sequence $(\mathcal{N}^n_{<0>})_{n\in\omega}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N}^0_\emptyset,\mathcal{N})}$ $=\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^n_{<0>})}$.

Again consider, for example, $\mathcal{N}^1_{<0>}$, then there exists a sequence $(\mathcal{N}^n_{<0,1>})_{n\in\omega}$ such that $\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^1_{<0>},\mathcal{N}^n_{<0,1>})}=\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^1_{<0>},\mathcal{N})}$. By Claim 1, each intersection is empty or contains only one element. If the intersection $\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^n_\emptyset)}$ is non-empty we call this element $x^\emptyset$, otherwise we go on; if $\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^n_{<0>})}$ is not empty we call this element $x^{<0>}$; if the intersection $\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^1_{<0>},\mathcal{N}^n_{<0,1>})}$ is not empty we call this element $x^{<0,1>}$, and so on. We obtain a subset $X_0=\{x^s : s\in\omega^{<\omega}\}$ and now we want to prove that $X_0=X$. By contradiction, assume there exists $y\in X\setminus X_0$. Then $y\notin\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^n_\emptyset)}$; hence there exists an element of the sequence $\{\overline{\sigma(\mathcal{N}^n_\emptyset)} : n\in\omega\}$, say $\overline{\sigma(\mathcal{N}^{k_0}_\emptyset)}$, such that y does not belong to it. By hypothesis, $y\notin\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^{k_0}_\emptyset,\mathcal{N}^n_{<k_0>})}$; hence there exists an element of $\{\overline{\sigma(\mathcal{N}^n_{<k_0>})} : n\in\omega\}$, say $\sigma(\mathcal{N}^{k_1}_{<k_0>})$, such that y does not belong to it. Again, $y\notin\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^{k_0}_\emptyset,\mathcal{N}^{k_1}_{<k_0>},\mathcal{N}^n_{<k_0,k_1>})}$, there exists an element of $\{\sigma(\mathcal{N}^n_{<k_0,k_1>}) : n\in\omega\}$, say $\sigma(\mathcal{N}^{k_2}_{<k_0,k_1>})$, such that y does not belong to it. Proceeding in this way we obtain a branch consisting of elements that do not contain y; a contradiction, because such a branch is a network due to the fact that σ is a winning strategy for Bob. Then X is countable.

Now we prove that X is second countable.

Claim 3. If $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}=\{x\}$, there exists an open set V such that $x\in V\subset\bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}$.

Indeed, assume by contradiction that for every open set V such that $x\in V$ there exists $y_V\in V\setminus\overline{\sigma(\mathcal{N})}$, for every $\mathcal{N}\in\mathbb{M}$. Let $\mathcal{N}$ be a countable network and consider the family $\mathcal{N}'=(\mathcal{N}\setminus\mathcal{N}_x)\cup\{\{x,y_V\} : V\in\tau_x\}$, where τ_x denotes the family of all open sets containing x and $\mathcal{N}_x=\{N\in\mathcal{N} : x\in\overline{N}\}$. Since X is countable, $\mathcal{N}'$ is countable. Now we prove that $\mathcal{N}'$ is a network. Clearly, for construction $\mathcal{N}'$ is a network at x. Let $y\in X$, $y\neq x$, and let A be an open set such that $y\in A$. Since X is T_2, there exists an open set B such that $y\in B$ and $x\notin\overline{B}$. Then there exists $N\in\mathcal{N}$ such that $y\in N\subset A\cap B$. Therefore $N\in\mathcal{N}\setminus\mathcal{N}_x$.

Claim 4. If $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}=\{x\}$, there exists a countable $\mathbb{M}'\subset\mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'}\overline{\sigma(\mathcal{N})}=\{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'}\overline{\sigma(\mathcal{N})}=\bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}$.

Recall that, by Claim 2 there exists a countable subset $\mathbb{M}^*\subset\mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}=\bigcap_{\mathcal{N}\in\mathbb{M}^*}\overline{\sigma(\mathcal{N})}$; further, since X is countable, $\bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}$ is countable and then there exists a countable subset $\mathbb{M}^{**}\subset\mathbb{M}$ such that $\bigcup_{\mathcal{N}\in\mathbb{M}^{**}}\overline{\sigma(\mathcal{N})}=\bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N})}$. Then $\mathbb{M}'=\mathbb{M}^*\cup\mathbb{M}^{**}$.

Even Claim 3. and Claim 4. can be given for any inning $n\in\omega$, that is

Claim 3(n). If $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N}_0,...,\mathcal{N}_{n-2},\mathcal{N}))}=\{x\}$, there exists an open set V such that $x\in V\subset\bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N}_0,...,\mathcal{N}_{n-2},\mathcal{N}))}$.

Claim 4(n). If $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N}_0,...,\mathcal{N}_{n-2},\mathcal{N}))}=\{x\}$, there exists a countable $\mathbb{M}'\subset\mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'}\overline{\sigma(\mathcal{N}_0,...,\mathcal{N}_{n-2},\mathcal{N}))}=\{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'}\overline{\sigma(\mathcal{N}_0,...,\mathcal{N}_{n-2},\mathcal{N}))}=\bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\sigma(\mathcal{N}_0,...,\mathcal{N}_{n-2},\mathcal{N}))}$.

The proof of Claim 3(n). and Claim 4(n). is analogous to the one of Claim 3. and Claim 4., respectively.

Consider the construction of the tree in the previous part of the proof. We know that $|\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^n_\emptyset)}|\leq 1$. If $\bigcap_{n\in\omega}\overline{\sigma(\mathcal{N}^n_\emptyset)}\neq\emptyset$, fix $V_\emptyset$ as in Claim 3. If $\bigcap_{k\in\omega}\overline{\sigma(\mathcal{N}^n_\emptyset,\mathcal{N}^k_{<n>})}\neq\emptyset$, fix $V_{<n>}$ as in Claim 3 and so on. Now we prove that $\{V_s : s\in\omega^{<\omega}\}$ is a base. If it is not true, then there exist $x\in X$ and an open set A with $x\in A$ such that for every $s\in\omega^{<\omega}$ such that $x\in V_s$, V_s is not contained in A. In the first inning, we have a family $\mathbb{M}'$ of countably many networks obtained as in Claim 4. Consider the intersection $\bigcap_{N\in\mathbb{M}'}\overline{\sigma(\mathcal{N})}$. If $\bigcap_{N\in\mathbb{M}'}\overline{\sigma(\mathcal{N})}=\emptyset$, we can pick an $\mathcal{N}\in\mathbb{M}'$, such that $x\notin\sigma(\mathcal{N})$. If $\bigcap_{N\in\mathbb{M}'}\overline{\sigma(\mathcal{N})}=\{y\}$ we have two cases: if $y\neq x$, we can pick an $\mathcal{N}\in\mathbb{M}'$, such that $x\notin\sigma(\mathcal{N})$; if $y=x$, then we can pick, if there exists an $\mathcal{N}\in\mathbb{M}'$, such that $x\notin\sigma(\mathcal{N})$, otherwise by hypothesis and Claim 3 we can pick an $\mathcal{N}\in\mathbb{M}'$, such that $\overline{\sigma(\mathcal{N})}$ is not contained in A. Then, proceeding in this way for each inning, we find a branch of the tree, i.e., our construction provides a winning strategy for Alice in the R-nw-selective game on X which is a contradiction. □

The following proposition shows that the (Point, Open)-set game is a good candidate to be the dual of the R-nw-selective game.

Proposition 2.4. *Let X be a space. The following implications hold.*

1. Alice $\uparrow$ *PO-set*$(X) \implies$ Bob $\uparrow$ *R-nw-selective*(X).
2. Alice $\uparrow$ *R-nw-selective*$(X) \implies$ Bob $\uparrow$ *PO-set*(X).
3. Bob $\uparrow$ *R-nw-selective*$(X) \implies$ Alice $\uparrow$ *PO-set*(X).

Proof. The proof of Items 1. and 2. is trivial and Item 3. is an easy consequence of Theorem 2.3. □

Question 2.5. Does Bob $\uparrow$ PO-set(X) imply Alice $\uparrow$ R-nw-selective(X)?

Now we study the determinacy of the R-nw-selective game.

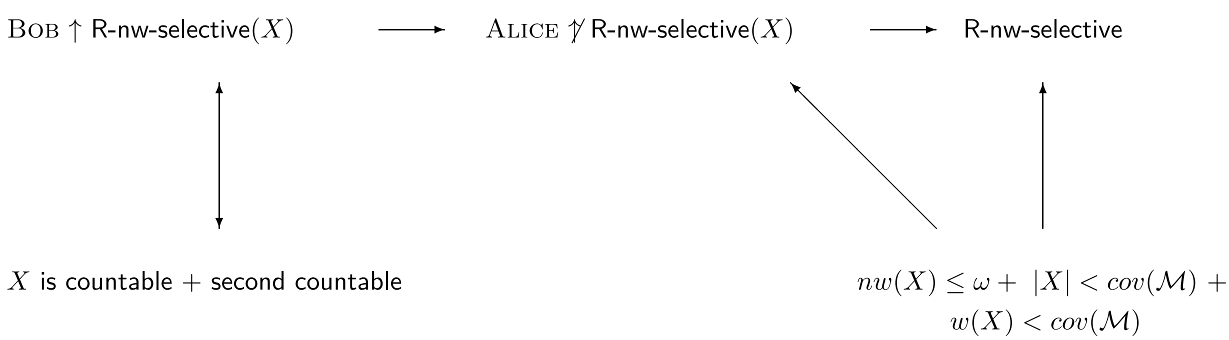
Proposition 2.6. *Let X be a space with $nw(X) = \omega$. If $|X| < cov(\mathcal{M})$ and $w(X) < cov(\mathcal{M})$, then* Alice $\not\uparrow$ *R-nw-selective*(X).

Proof. Suppose, by contradiction, that σ is a winning strategy for Alice in the R-nw-selective(X) and fix a base $\mathcal{B}$ of cardinality $w(X)$. Construct a countable tree using the strategy σ in such a way that $\sigma(\langle\rangle) = \mathcal{N}_0$; for each $N_0 \in \mathcal{N}_0$ apply the strategy and so on. Look at this tree as the poset of all finite branches ordered with the inverse natural extension. The nodes in this tree are the countable networks that are images under the function σ. Fix $x \in X$ and $B \in \mathcal{B}$ containing x. The set $D_{(x,B)}$ of all the finite sequences of the tree such that there exists an element of the sequence that is a $\sigma(\langle ..., N\rangle)$ with $x \in N \subset B$, is dense in the poset. Since the cardinality of the family $\{D_{(x,B)} : x \in X \text{ and } B \in \mathcal{B}\}$ is less than $cov(\mathcal{M})$ there exists a generic filter whose union is a branch of the tree that intersects all the dense sets of the family. This gives us a contradiction because this branch witnesses that there is a play in the R-nw-selective game on X in which Alice applies her strategy but Bob wins. □

Example 2.7. ($\omega_1 < cov(\mathcal{M})$) Consider a subspace $X \subset \mathbb{R}$ of cardinality ω_1. By Theorem 2.3 and Proposition 2.6 the R-nw-selective game on X is indeterminate.

Question 2.8. Is there any ZFC example of a space in which the R-nw-selective game turns out to be indeterminate?

The following diagram shows all the relations found above.



Question 2.9. Does R-nw-selectivity of a space X imply Alice $\not\uparrow$ R-nw-selective(X)?

Recall the following result.

Proposition 2.10. *[6] Let X be a space with $nw(X) = \omega$ and $w(X) < cov(\mathcal{M})$. Then the following are equivalent.*

1. $|X| < cov(\mathcal{M})$;
2. *X is R-nw-selective.*

Then it is possible to give a partial answer to Question 2.9.

Proposition 2.11. *Let X be a space with $nw(X) = \omega$ and $w(X) < cov(\mathcal{M})$. Then the following are equivalent.*

1. $|X| < cov(\mathcal{M})$;
2. Alice $\not\uparrow$ *R-nw-selective*(X);
3. *X is R-nw-selective.*

Now we will show that if Bob is forced to select a fixed number of elements from each network, then the respective game is equivalent to the R-nw-selective game for Bob. Let Nw denote the class of all countable networks of a fixed space X. Let $k \in \omega$ and $G_k(\mathrm{Nw}, \mathrm{Nw})$ on X be the game played in the following way: Alice chooses a countable network $\mathcal{N}_0$ and Bob answers picking a subset $\mathcal{F}_0 \subset \mathcal{N}_0$ such that $|\mathcal{F}_0| = k$. Then Alice chooses another countable network $\mathcal{N}_1$ and Bob answers picking a subset $\mathcal{F}_1 \subset \mathcal{N}_1$ such that $|\mathcal{F}_1| = k$ and so on for countably many innings. At the end Bob wins if the set $\bigcup\{\mathcal{F}_n : n \in \omega\}$ of his selections is a network.

It is straightforward to prove the following result.

Proposition 2.12. Alice $\uparrow G_k(\mathrm{Nw}, \mathrm{Nw})$ *on X implies that* Alice $\uparrow$ *R-nw-selective(X).*

Question 2.13. Is it true that if Alice $\uparrow$ R-nw-selective(X) then Alice $\uparrow G_k(\mathrm{Nw}, \mathrm{Nw})$ on X?

Proposition 2.14. Bob $\uparrow$ *R-nw-selective(X) if, and only if,* Bob $\uparrow G_k(\mathrm{Nw}, \mathrm{Nw})$ *on X.*

Proof. It suffices to prove that Bob $\uparrow G_k(\mathrm{Nw}, \mathrm{Nw})$ on X implies that the space X is countable and second countable. In fact the proof is similar to the one of Theorem 2.3. Let σ be a winning strategy for Bob in the $G_k(\mathrm{Nw}, \mathrm{Nw})$ on X and let $\mathbb{M}$ be the collection of all countable networks of the space X. We just need to prove the following claims.

Claim 1. $|\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}| \leq k$.

Assume that $x_0, ..., x_k$ are $k+1$ distinct points of X. Take any countable network $\mathcal{N}$ in the space X and observe that the family $\mathcal{N}' = \{N \in \mathcal{N} : x_i \notin \overline{N} \text{for every} i = 0, ..., k\} \cup \{\{x_0\}, ..., \{x_k\}\}$ is still a network in X and no element of $\mathcal{N}'$ contains more than one point of the set $\{x_1, ..., x_k\}$ in its closure. Now our claim easily follows.

Claim 2. There exists $\mathbb{M}' \subset \mathbb{M}$ countable such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.

Claim 3. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$, there exists an open set V such that $x \in V \subset \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.

Claim 4. If $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$, there exists $\mathbb{M}' \subset \mathbb{M}$ countable such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcup_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.

The proof of Claims 2., 3. and 4. are similar to the ones in Theorem 2.3. □

3. M-nw-selective game

Definition 3.1. Let X be a space with $nw(X) = \omega$. The M-nw-selective game, denoted by M-nw-selective(X), is played according to then following rules. ALICE chooses a countable network $\mathcal{N}_0$ and BOB answers picking a finite subset $\mathcal{F}_0 \subset \mathcal{N}_0$. Then ALICE chooses another countable network $\mathcal{N}_1$ and BOB answers in the same way and so on for countably many innings. At the end BOB wins if the set $\bigcup\{\mathcal{F}_n : n \in \omega\}$ of his selections is a network.

Proposition 3.2. ($MA[\mathfrak{d}]$) *Let X be a space with $nw(X) = \omega$. If $|X| < \mathfrak{d}$ and $w(X) < \mathfrak{d}$, then* ALICE $\not\uparrow$ *M-nw-selective*(X).

Proof. Similar to the proof of Proposition 2.6. □

Recall the following result.

Proposition 3.3. *[6] Let X be a space with $nw(X) = \omega$, $w(X) < \mathfrak{d}$. Then the following conditions are equivalent:*

1. $|X| < \mathfrak{d}$;
2. *X is M-nw-selective.*

Then it is possible to give the following equivalences.

Proposition 3.4. ($MA[\mathfrak{d}]$) *Let X be a space with $nw(X) = \omega$ and $w(X) < \mathfrak{d}$. The following are equivalent:*

1. $|X| < \mathfrak{d}$;
2. ALICE $\not\uparrow$ *M-nw-selective*(X);
3. *X is M-nw-selective.*

However, it is worthwhile to pose the following question.

Question 3.5. Does the M-nw-selectivity of a space X imply that ALICE $\not\uparrow$ M-nw-selective(X)?

Theorem 3.6. *Let X be a regular space such that* BOB $\uparrow$ *M-nw-selective(X). Then X is σ-compact.*

Proof. Let $\mathbb{M}$ be the collection of all countable networks of X and σ a winning strategy for BOB in M-nw-selective(X).

Claim 1: $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$ is compact.

Indeed, put $K = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$, let $\mathcal{U}$ be a cover made by open sets of X and $\mathcal{N} \in \mathbb{M}$. Consider the network $\mathcal{N}' = \{N \in \mathcal{N} : \overline{N} \subset U$ for some $U \in \mathcal{U}\} \cup \{N \in \mathcal{N} : \overline{N} \cap K = \emptyset\}$. Then $K \subset \sigma(\langle \mathcal{N}' \rangle)$ and considering the corresponding open sets we extract from $\mathcal{U}$ a finite subcover of K.

Claim 2: There exists a countable subset $\mathbb{M}' \subset \mathbb{M}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}'} \overline{\bigcup \sigma(\mathcal{N})} = \bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})}$.

The proof is similar to the one of Claim 2. in Theorem 2.3 and, as in there, these claims are true also for all the other innings.

There exists $(\mathcal{N}_\emptyset^n)_{n\in\omega}$, that is countably many possible first innings $\mathcal{N}_\emptyset^0, \mathcal{N}_\emptyset^1, \mathcal{N}_\emptyset^2, ...,$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N})} = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}_\emptyset^n)}$.

If Alice chooses $\mathcal{N}_\emptyset^0$, we can find $(\mathcal{N}_{<0>}^n)_{n\in\omega}$ such that $\bigcap_{\mathcal{N}\in\mathbb{M}} \overline{\bigcup \sigma(\mathcal{N}_\emptyset^0, \mathcal{N})} = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\mathcal{N}_\emptyset^0, \mathcal{N}_{<0>}^n)}$.

If then Alice chooses $\mathcal{N}^1_{<0>}$, we can find $(\mathcal{N}^n_{<0,1>})_{n\in\omega}$ such that $\bigcap_{n\in\omega}\overline{\bigcup\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^1_{<0>},\mathcal{N}^n_{<0,1>})} = \bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\bigcup\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^1_{<0>},\mathcal{N})}$. By Claims 1 and 2, each intersection, if it is not empty, is a compact subset. If the intersection is empty, we do not do anything and if $\bigcap_{n\in\omega}\overline{\bigcup\sigma(\mathcal{N}^n_\emptyset)}$ is a compact, we call this subset $K^\emptyset$. If $\bigcap_{n\in\omega}\overline{\bigcup\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^n_{<0>})}$ is a compact subset, we call it $K^{<0>}$. If $\bigcap_{n\in\omega}\overline{\bigcup\sigma(\mathcal{N}^0_\emptyset,\mathcal{N}^1_{<0>},\mathcal{N}^n_{<0,1>})}$ is a compact subset, we call this element $K^{<0,1>}$, and so on. Consider the set $X_0 = \bigcup\{K^s : s \in \omega^{<\omega}\}$. Now we prove that $X_0 = X$. By contradiction, assume there exists $y \in X \setminus X_0$. Then $y \notin \bigcap_{n\in\omega}\overline{\bigcup\sigma(\mathcal{N}^n_\emptyset)}$; hence there exists $n_0 \in \omega$ such that $y \notin \overline{\bigcup\sigma(\mathcal{N}^{n_0}_\emptyset)}$. Again, $y \notin \bigcap_{n\in\omega}\overline{\bigcup\sigma(\mathcal{N}^{n_0}_\emptyset,\mathcal{N}^n_{\langle n_0\rangle})}$; hence there exists $n_1 \in \omega$ such that $y \notin \overline{\bigcup\sigma(\mathcal{N}^{n_0}_\emptyset,\mathcal{N}^{n_1}_{\langle n_0\rangle})}$. Proceeding in this way we obtain a branch (or an evolution of the M-nw-selective(X)) in which Bob does not win, a contradiction, because σ is a winning strategy. Then X is σ-compact. □

Recall that a space is called σ-(metrizable compact) if it is union of countably many metrizable compact spaces. Then it is possible to obtain the following corollary.

Corollary 3.7. *Let X be a regular space in which* Bob $\uparrow$ *M-nw-selective*(X). *Then X is σ-(metrizable compact).*

Proof. By the previous theorem, X is σ-compact. Put $X = \bigcup_{n\in\omega} X_n$, where each X_n is compact. Since the space X has countable netweight, then each $nw(X_n) = \omega$ for every $n \in \omega$. By compactness of every X_n, each X_n is second countable. Therefore each X_n is metrizable. □

The following is a consistent example showing that the M-nw-selective game can be indeterminate.

Example 3.8. ($MA[\mathfrak{d}] + \omega_1 < \mathfrak{d}$) Consider a subset X of the irrational numbers having cardinality ω_1. By Proposition 3.2, Alice $\not\uparrow$ M-nw-selective(X). Since X is not σ-compact, by Theorem 3.6 we have that Bob $\not\uparrow$ M-nw-selective(X).

We prove the following result.

Proposition 3.9. *If X is a countable space in which* Bob $\uparrow$ *M-nw-selective*(X). *Then X is second countable.*

Proof. Similar to the proof of Theorem 2.3 replacing Claims 3. and 4. with the following.

Claim 3′. If $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\bigcup\sigma(\mathcal{N})} = \{x\}$, there exists an open set V such that $x \in V \subset \bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\bigcup\sigma(\mathcal{N})}$.

Claim 4′. If $\bigcap_{\mathcal{N}\in\mathbb{M}}\overline{\bigcup\sigma(\mathcal{N})} = \{x\}$, there exists $\mathbb{M}' \subset \mathbb{M}$ countable such that $\bigcap_{\mathcal{N}\in\mathbb{M}'}\overline{\bigcup\sigma(\mathcal{N})} = \{x\}$ and also such that $\bigcup_{\mathcal{N}\in\mathbb{M}'}\overline{\bigcup\sigma(\mathcal{N})} = \bigcup_{\mathcal{N}\in\mathbb{M}}\overline{\bigcup\sigma(\mathcal{N})}$. □

The next result uses the previous proposition to state that in the class of countable spaces the M-nw-selective and the R-nw-selective games are equivalent for Bob.

Corollary 3.10. *Let X be a countable space. The following are equivalent.*

1. Bob $\uparrow$ *R-nw-selective*(X);
2. Bob $\uparrow$ *M-nw-selective*(X);
3. *X is second countable.*

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