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NIL SETS DEFINED THROUGH ADDITIVE MAPS IN PRIME AND SEMIPRIME RINGS

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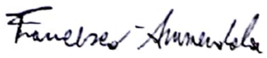

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"A mathematician is a person who can find analogies between theorems; a better mathematician is one who can see analogies between proofs and the best mathematician can notice analogies between theories. One can imagine that the ultimate mathematician is one who can see analogies between analogies."

Stefan Banach

UNIVERSITY OF MESSINA

Abstract

Department of Mathematical and Computer Sciences,
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Doctor of Philosophy

NIL SETS DEFINED THROUGH ADDITIVE MAPS IN PRIME AND SEMIPRIME RINGS

by FRANCESCO AMMENDOLIA

In this dissertation, we will focus on analyzing the structure of prime and semiprime rings equipped with additive maps that have nilpotent evaluations on appropriate subsets of the ring in which they are defined. In particular the involved maps on R will be derivations, generalized derivations and generalized skew derivations. We remark that the results in literature show that there is a strong relationship between the behaviour of derivations, generalized derivations and generalized skew derivations in prime and semiprime rings and the structure of the rings.

Research on how nilpotent maps behave in these contexts has led to important results, such as the characterization of certain classes of rings and the understanding of their structural properties. In particular there is a close connection between the existence of nilpotent maps in a ring and the potential commutativity of the ring itself. Here is why the idea of studying the potential existence of nilpotent subsets of a prime ring naturally arises.

Without any doubt, the two milestones of this line of research reside in the articles by S. Montgomery (1974) and I.N. Herstein (1979), in which, respectively, nilpotent traces and nilpotent derivations in prime rings are analyzed.

Many researchers have followed up on the work of Montgomery and Herstein over the years, from various points of view on operator algebras, leading to their generalizations: the concept of generalized derivations and generalized skew derivations having nilpotent values.

In all that we will examine and study in the continuation of the thesis, R will be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid. We will characterize the structure of R and any possible form of some additive maps, which are denfined of R, in the case appropriate subsets of R have nilpotent values.

More precisely, we firstly study nil sets defined through commuting iterates of generalized derivations F and G of R. We prove that, if L is a non-central Lie ideal of R and $n \geq 1$ is a fixed integer such that

$$\left\{F^2(x)x - xG^2(x)\right\}^n = 0$$

for all $x \in L$, then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one of the following holds:

(1) $F(x) = xa$ and $G(x) = xc$ for all $x \in R$ with $a^2 = c^2 \in C$;

(2) $F(x) = xa$ and $G(x) = cx$ for all $x \in R$ with $a^2 = c^2$;

(3) $F(x) = ax$ and $G(x) = xc$ for all $x \in R$ with $a^2 = c^2 \in C$;

(4) $F(x) = ax$ and $G(x) = cx$ for all $x \in R$ with $a^2 = c^2 \in C$.

Moreover we will also analyze nil sets defined through generalized derivations having homomorphism-like behavior. In this case we show that, if F and G are two generalized derivations of R, L is a non-central Lie ideal of R and $n \geq 1$ is a fixed integer such that

$$\big(F(xy) - G(x)G(y)\big)^n = 0$$

for any $x, y \in L$, then there exists $\lambda \in C$ such that $F(x) = \lambda^2 x$ and $G(x) = \lambda x$, for any $x \in R$.

Finally, we will study nil sets defined through generalized skew derivations F and G of R that emulate the behavior of Jordan derivations, and prove that if

$$\Big\{F(x^2) - G(x)x - xG(x)\Big\}^n = 0$$

for all $x \in R$, then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one of the following holds:

(1) $F(x) = G(x) = [x, p]$, for all $x \in R$.

(2) $\exists \eta \in C$, $\eta \neq 0$ such that $F(x) = [x, p] + 2\eta x$ and $G(x) = [x, p] + \eta x$, for all $x \in R$.

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Contents

Chapter 1

Introduction

The present thesis is devoted to the study of nil sets defined through appropriate functional identities in prime and semiprime rings.
Let $N \subset R$ be a subset of the ring R. N is said to be a nil set if every element of N is nilpotent, that is, for any $x \in N$ there exists an integer $m \geq 1$ such that $x^m = 0$. In particular, a nil set of bounded degree in a ring is a nil set where the index of nilpotence is bounded by a constant, regardless of the specific element in the ideal. In other words, there is a fixed integer $m \geq 1$, such that $x^m = 0$, for any $x \in N$.
In 1974, S. Montgomery begin the study of subsets of rings whose elements are nilpotent, presenting a result that led to a series of developments forming the foundation of the theory we have chosen to focus on in this thesis. More precisely, in [50] she proves the following:

Theorem 1. Let R be a prime ring with involution $*$ (an antiautomorphism of period 2), $T = \{x + x^* | x \in R\}$, the set of traces.
If every element of T is nilpotent, then R is commutative, $2R = 0$, and $*$ is the identity.

Let's now highlight some important findings in the literature regarding additive maps having nilpotent values. The first such result is due to Herstein [28]. He proves that if R is a prime ring and d an inner derivation of R satisfying $d(x)^n = 0$, for all $x \in R$ and n a fixed integer, then $d = 0$. We recall that an additive map $d : R \to R$ is called derivation of R if $d(xy) = d(x)y + xd(y)$, for any $x, y \in R$. The Herstein's result is extended to arbitrary derivations by A. Giambruno and I.N. Herstein in 1981 in [26]. They prove that:

Theorem 2. Let R be a semiprime ring, $n \geq 1$ a fixed integer, $d : R \to R$ a derivation of R. If $d(x)^n = 0$, for any $x \in R$, then $d = 0$.

A generalization for semiprime rings and derivations acting on two-sided ideals is provided by B. Felzenswalb and C. Lanski in 1983 in [25] and proved the following result:

Theorem 3. Let R be a semiprime ring containing no non-zero nil right ideal, $d : R \to R$ a derivation of R, I an ideal of R. If, for any $x \in I$, there is $n = n(x) \geq 1$ such that $d(x)^n = 0$, then $d(I) = 0$. In particular, if R is prime, then $d = 0$

Many authors extend this result to arbitrary derivations which act either on Lie ideals or on multilinear polynomials in prime and semiprime rings.
In 1985, L. Carini and A. Giambruno in [6] prove the following result:

Theorem 4. Let R be a prime ring containing no non-zero nil right ideal, $d : R \to R$ a non-zero derivation of R, L a Lie ideal of R. If, for any $x \in L$, there is $n = n(x) \geq 1$ such that $d(x)^n = 0$, then $d(L) = 0$.

The previous theorem is extended in 1990 by C. Lanski in [41], as follows:

Theorem 5. Let R be a prime ring containing no non-zero nil right ideal, $d : R \to R$ a non-zero derivation of R, L a Lie ideal of R. If, for any $x \in L$, there is $n = n(x) \geq 1$ such that $d(x)^n = 0$, then either L is commutative or $d(I) = 0$, where I is the ideal of R generated by $[L, L]$.

In 1996 T.L. Wong in [58] generalises these results to the case of derivations acting on multilinear polynomials, more precisely he prove the following result:

Theorem 6. Let R be a prime ring containing no non-zero nil right ideal, $d : R \to R$ a non-zero derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, $S = \{f(r_1, \ldots, r_n) \,|r_1, \ldots, r_n \in R\}$. If, for any $x \in S$, there is $n = n(x) \geq 1$ such that $d(x)^n = 0$, then $S \subset Z(R)$ and $d\Big(Z(R)\Big) = 0$.

Later, analogous results are obtained for other kinds of additive maps generalizing the concept of derivation. For example, in 2006, J.S. Lin and C.K. Liu [48] extend Wong's theorem for generalized derivations.

Definition 1. We recall that an additive map $F : R \to R$ is a generalized derivation of R if $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$, where d is an ordinary derivation of R.

J.S. Lin and C.K. Liu [48] prove the following result:

Theorem 7. Let R be a prime ring containing no non-zero nil right ideal, $G : R \to R$ a non-zero generalized derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, $S = \{f(r_1, \ldots, r_n) \,|r_1, \ldots, r_n \in R\}$. If, for any $x \in S$, there is $n = n(x) \geq 1$ such that $G(x)^n = 0$, then $S \subset Z(R)$.

In the same paper, J.S. Lin and C.K. Liu provide a semiprime version of their result, which is the following:

Theorem 8. Let R be a semiprime ring, $G : R \to R$ a non-zero generalized derivation of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over C, $n \geq 1$ a fixed integer and $S = \{f(r_1, \ldots, r_n) \,|r_1, \ldots, r_n \in R\}$. If $G(x)^n = 0$, for any $x \in S$, then $[S, R]G(R) = 0$.

Starting from the previous cited results, many researchers over the years investigate from various points of view, other generalizations of derivations, as skew derivations and generalized skew derivations.

Definition 2. Suppose α is an automorphism R. An additive map $\delta : R \to R$ is said to be a skew derivation of R, associated with α, if $\delta(xy) = \delta(x)y + \alpha(x)\delta(y)$, for all $x, y \in R$.

Definition 3. Suppose d is a skew derivation of R, associated with the automorphism α of R. An additive map $F : R \to R$ is said to be a generalized skew derivation of R, associated with d and α, if $F(xy) = F(x)y + \alpha(x)d(y)$, for all $x, y \in R$.

In 2010, J.C. Chang in [7] prove the following:

Theorem 9. Let R be a prime ring, $F : R \to R$ a non-zero generalized skew derivation of R, L a Lie ideal of R and $n \geq 1$ a fixed integer. If $F(x)^n = 0$, for any $x \in L$, then L is commutative.

At this point, it becomes clear that there is a close connection between the existence of nilpotent maps in a ring and the potential commutativity of the ring itself. Here's why the idea of studying the potential existence of nilpotent subsets naturally arises.

The tools we will use for our arguments are those related to the theory of functional identities. By a functional identity on a ring R we mean, roughly speaking, an identical realtion satisfied by elements in R, which involves some maps of R. The usual goal when treating a functional identity is to either describe the form of the maps appearing in the identity or, when this is not possible, to determine the structure of the ring admitting this identity.
In particular we will focus our attention to the cases when the involved maps on R will be derivations, generalized derivations and generalized skew derivations. We remark that the results in literature show that there is a strong relationship between the behaviour of derivations, generalized derivations and generalized skew derivations in prime and semiprime rings and the structure of the rings.
Our main idea can be described as follows: under suitable assumptions on the ring R, we'll find all possible solutions of appropriate functional identities that describe the nilpotency of elements of some appropriately defined subsets in a prime (or semiprime) ring, and extend some results in literature.
The thesis is organized as follows. In Chapter 2 we briefly survey basic concepts and some results on generalized polynomial identities and differential polynomial identities and provide necessary tools for proving our theorems in the sequel. The main reason for including Chapter 2 is to make the thesis readable and as self-contained as possible. The body of the thesis are Chapters 3,4 and 5.
In Chapter 3 we initiate the study of an algebraic condition, involving generalized derivations, that generalizes the concept of a commuting map. We recall that an additive map g on R is said to be commuting on a subset S of R if $[g(x), x] = 0$ for all $x \in S$, and is said to be centralizing on S if $[g(x), x] \in Z(R)$, the center of R, for all $x \in S$.
The first important result on maps which satisfy commuting conditions is Pošner's theorem in [**3**]. In that theorem, Pošner proves that the existence of a nonzero derivation d on a prime ring R such that $[d(x), x] \in Z(R)$, for all $x \in R$, forces R to be commutative.
This was the starting point for the research of several authors: the main topics arising directly from Pošner's theorem are the study of commuting derivations, commuting additive maps, commuting traces of multiadditive maps, and various generalizations of the notion of a commuting map, with many applications to different areas, in particular to Lie theory (for a complete and detailed discussion about these topics, we refer the reader to Bresar's survey on commuting maps [5]).
As a generalization of the concept of commuting maps, recently in [2] Argaç and De Filippis study the case $H(x)x - xG(x) = 0$, where F and G are two generalized derivations of R acting on a multilinear polynomial. More precisely, they prove the following:

Theorem 10. Let R be a non-commutative prime ring, U the Utumi ring of quotients of R, $C = Z(U)$ the extended centroid of R, I a non-zero two-sided ideal of R, H and G non-zero generalized derivations of R and $f(x_1, \ldots, x_n)$ is a non-central multilinear polynomial over C. Consider the following subset of R:

$$S = \{H(f(X))f(X) - f(X)G(f(X)) \ |X = (x_1, \ldots, x_n) \in I^n\}.$$

If $S = 0$, then one of the following holds:

1. there exists $a \in U$ such that, $H(x) = xa$ and $G(x) = ax$ for all $x \in R$;
2. $f(x_1, \ldots, x_n)^2$ is central valued on R and there exist $a, b \in U$ such that $H(x) = ax + xb$, $G(x) = bx + xa$, for all $x \in R$;
3. $char(R) = 2$ and R satisfies s_4, the standard identity of degree 4.

The natural question could be whether the results obtained for commuting generalized derivations can be extended to generalized derivations which satisfy a mixed condition of nilpotency and commutation on the ring R. In this sense, in Chapter 3, we will prove that, if the set

$$S = \{H^2(x)x - xG^2(x) \ \ |x \in L\}$$

is a nil set of R of bounded degree, where H and G are generalized derivations and L is a Lie ideal of R, then such generalized derivations are right or left centralizers of R, unless in the case R satisfies $s_4(x_1, \ldots, x_4)$, the standard identity of degree 4 . Following this line of investigation in Chapter 4 we will study the subset

$$T = \{F(xy) - G(x)G(y) \ \ |x \in L\}$$

where F and G are generalized derivations and L is once again a Lie ideal of R. We will prove that, if T is nil of bounded degree, then one may describe precisely the forms of both F and G. We will also provide a generalization for semiprime rings. The last Chapter is dedicated to generalized skew derivations. We will analyse the subset

$$W = \{F(x^2) - G(x)x - xG(x) \ \ |x \in R\}$$

where F and G are generalized skew derivations and R is a prime ring having characteristic different from 2 and 3. In case W is nil of bounded degree, we will conclude that $R \subseteq M_2(K)$, the 2×2 matrix ring over the field K, unless when F and G are both generalized derivations whose form is precisely described.

Chapter 2

Preliminaries

2.1 Maximal right rings of quotients

The rings of quotients play a crucial role in the study of generalized identities in prime and semiprime rings. More precisely, not only it is possible to extend generalized identities from a ring R to its rings of quotients but if one wants to define a generalized identity it is necessary the presence of a suitable ring of quotients.
In this chapter with R will be denoted a semiprime ring.

Definition 4. *Let S a subset of R. Denote by $l(S)$ the left annihilator of S on R; it is defined*

$$l(S) = \{x \in R \,:\, xS = (0)\}.$$

Similarly, we define the right annihilator $r(S)$.

Definition 5. *A right ideal J of R is dense if*

$$\forall x \neq 0, y \in R, \exists r \in R \,:\, xr \neq 0 \text{ and } yr \in J$$

Analogously, it is possible to define a dense left ideal. The collection of all dense right ideals of R will be denoted by $D = D(R)$.

Definition 6. *Let M be an R-module, for any submodule J of M and subset S of M, we define the set*

$$(S : J)_R = \{x \in R \,:\, Sx \subseteq J\}.$$

In order to define the rings of quotients we consider the following results (for more details see chapter 2 in [35]):

Proposition 1. *Let $I, J, S \in D(R)$ and let $f : I \longrightarrow R$ a homomorphism of right R-modules. Then:*

1. $f^{-1}(J) = \{a \in I \,:\, f(a) \in J\} \in D(R)$;
2. $(a : J) \in D(R)$, *for all* $a \in R$;
3. $I \cap J \in D(R)$;
4. *If K is a right ideal of R and $I \subseteq K$, then $K \in D(R)$;*
5. $l(I) = 0 = r(I)$;
6. *If K is a right ideal of R and $(a : K) \in D(R)$, for all $a \in I$, then $K \in D(R)$;*
7. *If L is a right ideal of R and $g : L \longrightarrow R$ a homomorphism of right S-modules, then g is a homomorphism of right R-modules;*

8. $IJ \in D(R)$.

Corollary 1. *Let J be a right ideal of R. Then $J \in D$ if and only if $l((a : J)) = 0$, for all $a \in R$.*

Remark 1. *Let I be a two-sided ideal of R. Then the following conditions are equivalent:*

1. $l(I) = 0$;
2. *I is a dense right ideal;*
3. *for any non-zero ideal J, $I \cap J \neq 0$.*

In case R is prime ring then any non-zero two-sided ideal of R is dense.

Remark 2. *Let I be a two-sided ideal of R. Then:*

1. $l(I) = r(I)$;
2. $l(I) \cap I = 0$;
3. *$I + l(I)$ is a dense right ideal of R.*

Proposition 2. *Let J a right ideal of R and let $f : J \longrightarrow R$ be a right R-module homomorphism. Then*

1. *$a \in R$ and $r(a) \in D(R)$ implies $a = 0$;*
2. *$ker(f) \in D(R)$ implies $f = 0$.*

Definition 7. *Consider the following set*

$$H = \{(f; J) \ : \ J \in D(R) \text{ and } f : J_R \longrightarrow R_R\}.$$

We define $(f; J) \equiv (g; K)$ if there exists $L \subseteq J \cap K$ such that $L \in D(R)$ and $f = g$ in L. Notice that $\equiv$ is an equivalence relation and $[f; J]$ denotes the equivalence class determined by $(f; J) \in H$. Moreover, we define addition and multiplication of equivalence classes as follows:

$$[f; J] + [g; K] = [f + g; J \cap K]$$
$$[f; J][g; K] = [fg; g^{-1}(J)].$$

It is very easy to prove that they are well defined. We will denote by $Q_{mr} = Q_{mr}(R)$ the ring of all equivalence classes $[f; J]$ with operations defined above and we will call it the maximal right ring of quotients of R. Similarly, it is possible define $Q_{ml}(R)$, considering J as dense left ideal.

Notice that there exists a ring injection $\beta : R \longrightarrow Q_{mr}$ given by $\beta(a) = [l_a; R]$, where l_a is the left multiplication determined by a. Analogously, there exists the injection $\alpha : R \longrightarrow Q_{ml}$ given by $\alpha(a) = [R; r_a]$, where r_a is the right multiplication by a. In any situation we will work with one of them; for this reason we will use the right ring of quotients. Moreover, to simplify the notation we identify R with its isomorphic image $\beta(R)$; more precisely, now R is contained in Q_{mr}.

Proposition 3. *Q_{mr} satisfies:*

1. *R is a subring of Q_{mr};*
2. *$\forall q \in Q_{mr}, \exists J \in D$ such that $qJ \subseteq R$;*

3. *$\forall q \in Q_{mr}, J \in D$, $qJ = 0$ if and only if $q = 0$;*

4. *$\forall J \in D$ and $f : J \longrightarrow R$, $\exists q \in Q_{mr}$ such that $f(x) = q(x)$, $\forall x \in J$.*

Lemma 1. *Let K be a dense right ideal of a semiprime ring R and S a subring of Q_{mr} such that $K \subseteq S$. Then :*

1. *S is a semiprime ring;*

2. *a right ideal J of S is dense if and only if $(J \cap R)K \in D$ (in particular $IS \in D(S)$ if $I \in D$).*

Proposition 4. *Let K be a dense right ideal of a semiprime ring R and S a subring of Q_{mr} such that $K \subseteq S$. Then $Q_{mr}(S) = Q_{mr}$.*

The following results are immediate consequences of the previuos Proposition:

Theorem 11. *Let $U = Q_{mr}$ the right ring of quotients of a semiprime ring R. Then $Q_{mr}(U) = U$.*

Corollary 2. *Let I be an ideal of R and $J = l_R(I)$. Then $Q_{mr} = Q_{mr}(I) \oplus Q_{mr}(J)$.*

Throughout this thesis the maximal right ring of quotients $Q_{mr}(R)$ of R will be called Utumi ring of quotients and will be denoted by U.

2.2 Martindale quotients ring and symmetric ring of quotients

The notion and the construction of the two-sided ring of quotients introduced by Martindale in [49] are simpler than of the maximal ring of quotients. Since the annihilator of any non-zero two-sided ideal of a prime ring is equal to zero, any non-zero two-sided ideal of a prime ring is dense. Moreover, it does not hold for semiprime rings, we will describe the construction of the Martindale quotients ring for semiprime rings.

Definition 8. *Let R be a semiprime ring and consider the following sets*

$$\Gamma = \{I \ : \ I \text{ is an ideal of } R \text{ and } l(I) = 0\}$$

$$T = \{(f;J) \ : \ J \in \Gamma,\ f : J_R \longrightarrow R_R\}.$$

Moreover, we define the following equivalence relation:

$$(f;J) \equiv (g;K) \text{ if } \exists L \subseteq J \cap K \ : \ L \in \Gamma \text{ and } f = g \text{ in } L$$

denoting by $\{f;J\}$ the equivalence class determined by $(f,J) \in T$.
We now define addition and multiplication of equivalence classes as follows:

$$\{f;J\} + \{g;K\} = \{f+g;KJ\}$$

$$\{f;J\}\{g;K\} = \{fg;KJ\}.$$

So that, we get $Q_r(R) = Q_r$, called Martindale quotients ring (on the right) of R.

The ring R can be embeded in $Q_r(R)$ thank to the monomorphism $\gamma : R \longrightarrow Q_r$, with $\gamma(x) = \{l_a;R\}$, where l_a is the left multiplication by a.

Proposition 5. *Let R be a semiprime ring. Then $Q_r(R)$ satisfies:*

1. *R is a subring of Q_r;*
2. *$\forall q \in Q_r \ \exists J \in \Gamma$ such that $qJ \subseteq R$;*
3. *$\forall q \in Q_r$ and $J \in \Gamma$, $qJ = 0$ if and only if $q = 0$;*
4. *$\forall J \in \Gamma$ and $f : J \longrightarrow R \ \exists q \in Q_r$ such that $f(x) = qx, \forall x \in J$.*

Proposition 6. *Let R be a semiprime ring. Then there exists a unique ring monomorphism $\sigma : Q_r(R) \longrightarrow U$ such that $\sigma(r) = r, \forall r \in R$. Moreover*

$$Im(\sigma) = \{q \in U \ : \ qJ \subseteq R, \ \text{for some} \ \ J \in \Gamma\}.$$

Definition 9. *We call symmetric ring of quotients of R (on the right) the following:*

$$Q_s = \{q \in U \ : \ qJ \cup Jq \subseteq R \text{ for some } J \in \Gamma\}.$$

One can easily check that Q_s is a subring of Q_r. Q_s is characterized by four properties analogous to those which characterize U:

Proposition 7. *Let R be a semiprime ring. Then $Q_s(R)$ satisfies:*

1. *R is a subring of Q_s;*
2. *$\forall q \in Q_s \ \exists J \in \Gamma$ such that $qJ \cup Jq \subseteq R$;*
3. *$\forall q \in Q_s$ and $J \in \Gamma$, $qJ = 0$ if and only if $q = 0$;*
4. *$\forall J \in \Gamma$ and $f : J_R \longrightarrow R_R$ homomorphism of right R-modules, $g :_R J \longrightarrow_R R$ homomorphism of left R-modules, such that $xf(y) = g(x)y$, for all $x, y \in J$, $\exists q \in Q_s$ such that $f(v) = qv$ and $g(v) = vq$, for all $v \in J$.*

Definition 10. *The center of the Martindale quotients ring $C = Z(Q_r)$ of a ring R is called extended centroid of R. The subring RC of U is said central closure of R. Further, R is called centrally closed if $R = RC$.*

Remark 3. *If R is a semiprime ring then*

$$Z(Q_r) = Z(Q_s) = C = Z(U) = \{q \in U \ : \ qr = rq \ \forall r \in R\}.$$

Remark 4. *In case R is a prime ring, all that we need here about these objects is that:*

- *$R \subseteq Q_r \subseteq U$;*
- *U and Q_r are prime rings;*
- *for all $q \in Q_r$ there exists a dense left ideal M of R such that $Mq \subseteq R$, moreover if $Mq = 0$, for some dense left ideal M of R, then $q = 0$.*
- *C is a field.*

2.3 Polynomial identity

In this paragraph R will always denote an algebra. Although many results hold for algebras over an arbitrary commutative base ring, or at least over a Noetherian ring. It is convenient in this paragraph to consider only algebras over a field F. We also denote by $Z(R)$ the center of R. An identity of an algebra R is an element f of the free associative algebra $F\{X\}$, all of whose substitutions in R are 0. In this case we also say that R satisfies the identity f. For example, any commutative algebra R satisfies the identity $xy - yx$. R is a PI- algebra if R satisfies some identity, at least one of whose coefficients is 1.
It is easy to see that any algebra R spanned by a finite number of elements $b_1, \ldots, b_m$ satisfies the standard identity

$$S_k(x_1, \ldots, x_k) = \sum_{\pi \in Sym(k)} sgn(\pi) x_{\pi_1} \ldots x_{\pi_k}$$

for any $k > m$. More generally, one can define the *Capelli polynomial*

$$C_k(x_1, \ldots, x_k, y_0, \ldots, y_k) = \sum_{\pi \in Sym(k)} sgn(\pi) y_0 x_{\pi_1} y_1 \ldots x_{\pi_k} y_k$$

which is also an identity of R for any $k > m$. In particular, C_{n^2+1} (and thus s_{n^2+1}) is an identity of the $n \times n$ matrix algebra $M_n(F)$. In fact, Amitsur and Levitzki (1950) proved that S_{2n} is an identity of $M_n(F)$ and has the lowest degree (as a polynomial) of all identities of $M_n(F)$.
Although PI-theory was initiated by Dehn (1922), the groundwork for the structure theory was laid by Levitzki, Jacobson, Kaplansky and Amitsur. Nevertheless, the crucial connection with commutativity came only with the discovery of central polynomials for $M_n(F)$, independently by Formanek and Razmyslov (1972). They proved that these are not identities of R which take on only scalar values.
An arbitrary PI-algebra R, with center $Z(R)$, has the following structure:

1. (Kaplansky's Theorem) All primitive rings are simple of finite dimension over their centers;

2. (Pošner - Formanek - Razmyslov - Rowen) If R is semiprime, then R has a central-polynomial and any non-zero ideal intersects $Z(R)$ non-trivially. In particular, if R is prime then localizing at $Z(R) - \{0\}$ produces a simple PI-algebra. In general, any semiprime PI-algebra R can be embedded in to a matrix algebra over a commutative algebra.

We would like to remark that, in case R is prime, the result stated in (2) is usually called " The Classical Pošner's Theorem " (see [[30], Theorem 1.4.3 p. 40]).
Also Regev (1972) proved that the tensor product of PI-algebras satisfies a PI. These results show that PI's are closely related to finite dimensional representation of algebras; an irreducible representation of dimension n is just a simple homomorphic ring satisfying S_{2n}.

2.4 Generalized polynomial identity

Let R be a semiprime ring, U the Utumi ring of quotients (on the right) of R, C its extended centroid of R. Let $X = \{x_1, \ldots, x_n, \ldots\}$ a countable set of indeterminates $x_1, \ldots, x_n, \ldots$ and $C\{X\}$ the free algebra on X with coefficients in C. Denote by

$T = U * C\{X\}$ the free product on C of U and $C\{X\}$.
The elements of T are called generalized polynomial. Every $m \in T$ of the form $q_0 y_1 q_1 y_2 \ldots y_n q_n$, where $q_i \in U$ and $y_i \in X$, is said monomial and $q_1, \ldots, q_n$ are its coefficients. Every $f \in T$ can be represented as a finite sum of monomials and this representation is not unique. Consider $V \subseteq U$, then $f \in T$ is said V-generalized polynomial if and only if it has a representation that has all coefficients in V. Obviously, every V-generalized polynomial is a U-generalized polynomial.
Let B a set containing C-independent vectors of U; a B-monomial is a monomial on the form $q_0 y_1 q_1 y_2 \ldots y_n q_n$, where $q_i \in B$ and $y_i \in X$. Let $V = BC$ the C-subspace generated by B; then every V-generalized polynomial f can be written $f = \sum_i \alpha_i m_i$, where $\alpha_i \in C$ and m_i are B-monomials. In order to express in this form f we fix a representation of f with all coefficients in V; after that, we express every coefficient as a linear combination of elements of B, so that we get the required form. Now, this representation is unique and it follows that the B-monomials are a basis for the vector space over C that contains all the BC-generalized monomials. If B is a basis of U over C then the B-monomials generate the whole T.

Remark 5. *In [13] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if any α_i is zero. As a consequence, let $a_1, \ldots, a_k \in U$ be linearly independent over C and $a_1 g_1(x_1, \ldots, x_n) + \ldots + a_k g_k(x_1, \ldots, x_n) = 0 \in T$, for some $g_1, \ldots, g_k \in T$. If, for any i, $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j h_j(x_1, \ldots, x_n)$ and $h_j(x_1, \ldots, x_n) \in T$, then $g_1(x_1, \ldots, x_n)$,..., $g_k(x_1, \ldots, x_n)$ are the zero element of T. The same conclusion holds if $g_1(x_1, \ldots, x_n)a_1 + \ldots + g_k(x_1, \ldots, x_n)a_k = 0 \in T$, and $g_i(x_1, \ldots, x_n) = \sum_{j=1}^{n} h_j(x_1, \ldots, x_n)x_j$ for some $h_j(x_1, \ldots, x_n) \in T$.*

Definition 11. *Let $f(x_1, \ldots, x_n) \in T$ be a non-trivial generalized polynomial; it is a non-trivial generalized polynomial identity for a semiprime ring R if and only if $f(r_1, \ldots, r_n) = 0$, for all $r_1, \ldots, r_n \in R$. In this case, R is said satisfying a non-trivial generalized polynomial identity with coefficients in U.*

Theorem 12. *([13]) Let R be a prime ring. If I is a nonzero ideal of R, then R, I and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r.*

Theorem 13. *([49]) Let R be a prime ring and $S = RC$ its central closure. Then R satisfies a non-trivial GPI with coefficients in R if and only if S is a primitive ring containing a minimal right ideal eS, with $e = e^2$, such that eSe is a division ring of finite dimension over C.*

Corollary 3. *Let R be a simple ring with unit, then R satisfies a GPI if and only if R is a simple algebra of finite dimension over its center. Therefore, $R \cong M_n(D)$ and $[D : Z(D)]$ is finite.*

2.5 Differential polynomial identity

The first author that studied some differential identity in associative rings was V.K. Kharchenko, in [37] and [38], respectively, in 1978 and 1979.
Also C.L. Chuang [10], C. Lanski [42] and T.K. Lee [46] considered particular differential identities. In this paragraph we will give some definitions and some results useful for the drawing up of this thesis.
An additive map $d : R \to R$, for any ring R, is called *derivation* of R if $d(xy) = d(x)y + xd(y)$, for any $x, y \in R$.

Lemma 2. *Let R be a semiprime ring, U its Utumi ring of quotients and $d : R \longrightarrow U$ an additive map satisfying the Leibnitz rule $d(xy) = d(x)y + xd(y)$. Then d can be uniquely extended to a derivation in U.*

Proof. Let $q \in U$ and consider a dense right ideal I of R such that $qI \subseteq R$. Notice that IU is a dense right ideal of U. Define the map $\varphi : IU \longrightarrow U$ as follows: if $z = \sum_i x_i y_i \in IU$, where $x_i \in I$ and $y_i \in U$, then $\varphi(z) = \sum_i (d(qx_i) - qd(x_i))y_i$.
It is possible to show that φ is well defined and it is a map between right U-modules. Moreover, since maximal right ring of quotients of U is U, then φ defines an element of U and this element is $d(q)$. Therefore, d can be extended to a map $\varphi : U \longrightarrow U$, but, for brevity, we will denote it by d. □

Denote by $Der(U)$ the set of all derivations on U. We define as " derivations word " each additive map $\Delta \in End(U)$ such that it is the composition of a finite number of derivations, that is $\Delta = d_1 d_2 \dots d_n$, where $d_i \in Der(U)$, for all i.
A differential polynomial is a generalized polynomial on the form $p(\Delta_i(x_j))$, where the words Δ_i act on non-commutative variables x_j, as unary operation, and its coefficients are in U.
The polynomial $p(\Delta_i(x_j))$ is called differential identity on a subset T of U if it assume the value 0 for all substitutions of the indeterminates x_j with elements of T.
Let $d \in Der(U)$, $\alpha \in C$; we define $(\alpha d) \in Der(U)$ in the following way:

$$(\alpha d)(u) = \alpha(d(u))$$

for all $u \in U$. Moreover, consider $d, \delta \in Der(U)$, then the commutator $[d, \delta] \in Der(U)$ acts as follows:

$$[d, \delta](u) = d(\delta(u)) - \delta(d(u))$$

for all $u \in U$. From the definition of derivation and above arguments, recall the following basic differential identities:

(B1) $d(xy) - d(x)y - xd(y)$, where $d \in Der(U)$.

(B2) $d(x + y) - (d(x) + d(y))$, where $d \in Der(U)$.

(B3) $d(x) - (ax - xa)$, where d is an inner derivation induced by $a \in U$.

(B4) $\underbrace{d(d(\dots d(x)))}_{p-times} - d^p(x)$, where $d \in Der(U)$, R has characteristic p, and either p is a prime number or $p = 0$. In this latter case, our identity assume the form $x = x$.

(B5) $[d, \delta](x) - (\delta(d(x))) - d(\delta(x))$, where $d, \delta \in Der(U)$.

(B6) $(\alpha d + \beta\delta)(x) - (\alpha d(x)) - \beta\delta(x)$, where $d, \delta \in Der(U)$, $\alpha, \beta \in C$.

Assume that R is a prime ring; in this case C is a field, then $Der(U)$ is a vectorial space over C. Let $D_{int}(U)$ be the subspace of $Der(U)$ that contains all inner derivations of U. Choose a fixed basis M_0 of $D_{int}(U)$ and augment it to a basis M for

$Der(U)$. Fix a total order in M such that $d_0 \succ d$, where $d_0 \in M_0$ and $d \in M - M_0$, and extend this order to the set of all derivation words by assuming that for all $d, \delta \in M$

- if d is longer than δ then $d \succ \delta$;
- if d and δ have the same length then are ordered lexicographically.

Moreover, by a " regular word " we mean a derivation word of the form $\Delta = d_1^{s_1} d_2^{s_2} \ldots d_m^{s_m}$ possessing the following properties:

(**W1**) $d^{s_i} \in M - M_0$, for $i = 1, \ldots, m$;

(**W2**) $d_1 \prec d_2 \prec \ldots \prec d_m$;

(**W3**) $s_i < p$ for $i = 1, \ldots, m$, if $char(R) = p > 0$.

Notice that, by $(B1) - (B6)$, each differential identity can be transformed into a form $p(\Delta_i(x_j))$, where

(**R1**) $p(z_{ij})$ is a generalized polynomial over U in non-commuting variables z_{ij};

(**R2**) the Δ_i are distinct regular words over M.

A differential polynomial is called " reduced " if it assumes the form $p(\Delta_i(x_j))$ satisfying $(R1)$ and $(R2)$. If it is a differential identity, then we call it reduced differential identity.
The revised form of the Kharchenko's Theorem, that Lee gave in [46, Theorem 1], is the following:

Theorem 14. *Let R be a prime ring, U its maximal right ring of quotients and I_R a dense R-submodule of U. Assume that $p(\Delta_i(x_j))$ is a reduced differential identity for I_R. Then $p(z_{ij})$ is a generalized polynomial identity for U, where the z_{ij} are distinct variables.*

In Theorem 2 of the same paper, Lee also proved:

Theorem 15. *Let R be a prime ring, U its maximal right ring of quotients and I_R a dense R-submodule of U. Then I_R, R, Q_r and U satisfy the same differential identities.*

This last result was generalized for semiprime rings by T.K. Lee in [46, Theorem 3]:

Theorem 16. *Let R be a semiprime ring and I_R a dense R-submodule of Q_r. Then I_R, R and Q_r and U satisfy the same differential identities.*

Finally Lee proved (Theorem 4 in [46]) also the following result:

Theorem 17. *Let R be a semiprime ring, U its maximal right ring of quotients and I a non-zero two-sided ideal of R. If $f(\Delta_i(x_j)) = 0$, for all $x_j \in I$, then $f(\Delta_i(x_j))y = 0$, for all $x_j \in U$, $y \in I$. In particular, if R is prime, then $f(\Delta_i(x_j)) = 0$, for all $x_j \in U$.*

The simplest case is when R is a prime ring and the derivations words Δ_i are all equal to $0 \neq d \in Der(U)$; in this case, either d is an inner derivation of U, that is d belongs to M_0, or d may augment the basis M_0 of D_{int} to the basis M for $Der(U)$; more precisely, we may assume that $d \in M - M_0$. In this last case the differential polynomial

$$f(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$$

is a reduced differential polynomial; hence, it is a differential polynomial if and only if the generalized polynomial

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n)$$

is a generalized polynomial identity for U.
In the first case, when d is inner, the differential polynomial is so written:

$$f(x_1, \ldots, x_n, [q, x_1], \ldots, [q, x_n])$$

and it is a generalized polynomial for T. Therefore, U, or equivalently a dense R-submodule of U, satisfies the differential identity $f(x_1, \ldots, x_n, d(x_1), \ldots, d(x_n))$ if and only if U satisfies the generalized polynomial identity $f(x_1, \ldots, x_n, [q, x_1], \ldots, [q, x_n])$, that is not trivial if it is different from zero in T.
In this context we also would like to describe the case when the derivations words are composition of two different derivations $d, \delta \in Der(U)$. Under this assumption, Theorem 14 reduces to the following:

Theorem 18. *Let R be a prime ring, $d, \delta \in Der(U)$ and $p(\Delta_i(x_j))$ is a differential identity for R, where Δ_i are derivations words of the following form $d, \delta, d^2, d\delta, \delta^2$. Then one of the following holds:*

1. *either there exist $\alpha, \beta \in C$ and $g \in D_{int}$ such that $\alpha d(x) + \beta\delta(x) = g(x)$, for all $x \in U$;*
2. *or $p(z_{ij})$ is a generalized polynomial identity for U, where the z_{ij} are distinct indeterminates.*

2.6 Generalized derivations and generalized skew derivations

More recently, several authors have considered the study of prime and semiprime rings which satisfy generalized polynomial identity involving other different kinds of additive maps. In this context we will focus our attention on the analisys of the behaviour of some particular maps called respectively generalized derivations and generalized skew derivations (we refer the reader to Definitions 1 and 3). We would like to point out that basic examples of generalized derivations are the usual derivations on R and left R-module mappings from R into itself. An important example is a map of the form $G(x) = ax + xb$, for some $a, b \in R$; such generalized derivations are called inner.
Generalized derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [32] and [33]). In [44, Theorem 3], Lee proved the following theorem:

Theorem 19. *Let G be a generalized derivation on a dense right ideal of R. Then G can be uniquely extended to a generalized derivation of U and assume the form $G(x) = bx + d(x)$ for all $x \in U$, for some $b \in U$ and a derivation d of U.*

We also recall that basic examples of α-derivations, that is skew derivations associated with automorphism α, are the usual derivations and the map $D = \alpha - id$, where id denotes the identity map. Let $b \in Q_r$ be a fixed element. Then a map $D : R \longrightarrow R$ defined by $D(x) = bx - \alpha(x)b$, for all $x \in R$, is an α-derivation on R and it is called an inner α-derivation (an inner skew derivation) defined by b. If a skew derivation D is not inner, then it is called outer.
Let us also mention that an automorphism $\alpha : R \longrightarrow R$ is inner if there exists an invertible $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$. If an automorphism $\alpha \in Aut(R)$ is not inner, then it is called outer. For more details on skew derivations, generalized skew derivations and their associated automorphisms, see [45] and [46].
We denote the set of all skew-derivations on Q_r by $SDer(Q_r)$. By a skew-derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \dots d_m$, with each di $d_i \in SDer(Q_r)$. Then a skew-differential polynomial is a generalized polynomial, with coefficents in Q_r, of the form $p(\Delta_j(x_i))$ involving non-commutative indeterminates x_i on which the skew derivations words Δ_j act as unary operations. The skew-differential polynomial $p(\Delta_j(x_i))$ is said a skew-differential identity on a subset T of Q_r if it vanishes for any assignment of values from T to its indeterminates x_i.
We conclude this section with the following known result on generalized identities:

Fact 1. Let R be a prime ring and I a two-sided ideal of R. Then I, R, and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [13]). Furthermore, I, R, and Q_r satisfy the same generalized polynomial identities with automorphisms (Theorem 1 in [12]).

Chapter 3

Nil sets defined through commuting iterates of generalized derivations

In [20] Dhara and De Filippis studied the situation $F^2(x)x - xG^2(x) = 0$ for all $x \in \{f(x_1, \ldots, x_n) | x_1, \ldots, x_n \in I\}$, where F and G are two generalized derivations of R, $f(x_1, \ldots, x_n)$ is a multilinear polynomial over C and I is an ideal of R and proved that:

Theorem 20. *Let R be a prime ring of characteristic different from 2 with Utumi ring of quotients U and extended centroid C, F and G two nonzero generalized derivations of R, I an ideal of R and $f(x_1, \ldots, x_n)$ a multilinear polynomial over C which is not central valued on R. If*

$$F^2(f(x_1, \ldots, x_n))f(x_1, \ldots, x_n) - f(x_1, \ldots, x_n)G^2(f(x_1, \ldots, x_n)) = 0$$

for all $x_1, \ldots, x_n \in I$, then one of the following holds:

1. *$F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;*
2. *$F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;*
3. *$F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;*
4. *$F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2$ and $f(x_1, \ldots, x_n)^2$ is central valued on R;*
5. *$F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.*

As a special case of Theorem 20 we also have:

Theorem 21. *Let R be a prime ring of characteristic different from 2 with Utumi ring of quotient U and extended centroid C, F and G two nonzero generalized derivations of R, L a non-central Lie ideal of R. If*

$$F^2(u)u - uG^2(u) = 0$$

for all $u \in L$, then one of the following holds:

1. *$F(x) = xa$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;*
2. *$F(x) = xa$ and $G(x) = bx$ for all $x \in R$ with $a^2 = b^2$;*
3. *$F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2 \in C$;*
4. *$F(x) = ax$ and $G(x) = xb$ for all $x \in R$ with $a^2 = b^2$ and R satisfies the standard identity of degree 4;*

5. *$F(x) = ax$ and $G(x) = bx$ for all $x \in R$, with $a^2 = b^2 \in C$.*

Here our aim is to extend the previous cited result to the case that the first member of the equation is raised to the n-th power. In fact, the main result of the present chapter is the following:

Theorem 22. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, F and G two nonzero generalized derivations of R, L a non-central Lie ideal of R and $n \geq 1$ a fixed integer. If*

$$\left\{F^2(x)x - xG^2(x)\right\}^n = 0$$

for all $x \in L$, then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one ot the following holds:

1. *$F(x) = xa$ and $G(x) = xc$ for all $x \in R$ with $a^2 = c^2 \in C$;*
2. *$F(x) = xa$ and $G(x) = cx$ for all $x \in R$ with $a^2 = c^2$;*
3. *$F(x) = ax$ and $G(x) = xc$ for all $x \in R$ with $a^2 = c^2 \in C$;*
4. *$F(x) = ax$ and $G(x) = cx$ for all $x \in R$ with $a^2 = c^2 \in C$.*

Proof. In order to prove our result, we firstly fix the following well known fact:

Fact 2. If L is not central and $char(R) \neq 2$, then there exists a non-zero two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$ (see pp. 4 and 5 in [31] , Lemma 2 and Proposition 1 in [21], and Theorem 4 in [43]).

By hypothesis and Fact 2, we may assume that there exists a non-central ideal of R such that

$$\left\{F^2([x_1,x_2])[x_1,x_2] - [x_1,x_2]G^2([x_1,x_2])\right\}^n = 0 \tag{3.1}$$

is a generalized differential identity for I.
Moreover, in light of Theorem 19, there exist $a, c \in Q_r$ and d, δ derivations of R such that $F(x) = ax + d(x)$, $G(x) = cx + \delta(x)$ for all $x \in R$. Hence I satisfies

$$\begin{aligned}\Phi(x_1,x_2) = \Big\{&\Big(F(a)[x_1,x_2] + 2ad([x_1,x_2]) + d^2([x_1,x_2])\Big)[x_1,x_2] \\ &- [x_1,x_2]\Big(G(c)[x_1,x_2] + 2c\delta([x_1,x_2]) + \delta^2([x_1,x_2])\Big)\Big\}^n\end{aligned} \tag{3.2}$$

where $F(a) = a^2 + d(a)$ and $G(c) = c^2 + \delta(c)$. Since R, I and Q_r satisfy the same generalized differential identities, without loss of generality we may assume that Q_r satisfies (3.2). Moreover

$$d([x_1,x_2]) = [d(x_1),x_2] + [x_1,d(x_2)]$$
$$\delta([x_1,x_2]) = [\delta(x_1),x_2] + [x_1,\delta(x_2)]$$

and

$$d^2([x_1,x_2]) = [d^2(x_1),x_2] + [x_1,d^2(x_2)] + 2[d(x_1),d(x_2)]$$
$$\delta^2([x_1,x_2]) = [\delta^2(x_1),x_2] + [x_1,\delta^2(x_2)] + 2[\delta(x_1),\delta(x_2)].$$

Thus, from (3.2), we have that Q_r satisfies

$$\begin{aligned}&\Big\{\Big(F(a)[x_1,x_2]+2a\Big([d(x_1),x_2]+[x_1,d(x_2)]\Big)+[d^2(x_1),x_2])\\&+[x_1,d^2(x_2)]+2[d(x_1),d(x_2)]\Big)[x_1,x_2]\\&-[x_1,x_2]\Big(G(c)[x_1,x_2]+2c\Big([\delta(x_1),x_2]+[x_1,\delta(x_2)]\Big)\\&+[\delta^2(x_1),x_2]+[x_1,\delta^2(x_2)]+2[\delta(x_1),\delta(x_2)]\Big)\Big\}^n=0.\end{aligned} \tag{3.3}$$

3.1 F and G both are inner generalized derivations of R

In this section, we study the case when both the derivations F and G are inner generalized derivations, respectively defined as follows:

$$F(x) = ax + xb \quad \text{and} \quad G(x) = cx + xq$$

where a, b, c, q are fixed element of Q_r.
In what follows, we denote

$$\begin{aligned}\Phi(x_1,x_2) = &\Big\{\Big(a^2[x_1,x_2]+2a[x_1,x_2]b+[x_1,x_2]b^2\Big)[x_1,x_2]\\&-[x_1,x_2]\Big(c^2[x_1,x_2]+2c[x_1,x_2]q+[x_1,x_2]q^2\Big)\Big\}^n\end{aligned} \tag{3.4}$$

and assume that R satisfies the generalized identity $\Phi(x_1, x_2)$.
The aim of this section is to prove the following:

Proposition 8. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, $F(x) = ax + xb$ and $G(x) = cx + xq$ two nonzero inner generalized derivations of R where a, b, c, q are fixed element of Q_r. If R satisfies (3.4), then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one ot the following holds:*

1. *$a, c \in C$, $F(x) = x(a + b)$ and $G(x) = x(c + q)$ for all $x \in R$ with $(a + b)^2 = (c + q)^2 \in C$;*
2. *$a, q \in C$, $F(x) = x(a + b)$ and $G(x) = (c + q)x$ for all $x \in R$ with $(a + b)^2 = (c + q)^2$;*
3. *$b, c \in C$, $F(x) = (a + b)x$ and $G(x) = x(c + q)$ for all $x \in R$ with $(a + b)^2 = (c + q)^2 \in C$;*
4. *$b, q \in C$, $F(x) = (a + b)x$ and $G(x) = (c + q)x$ for all $x \in R$ with $(a + b)^2 = (c + q)^2 \in C$.*

The proof of Proposition 8 is a consequence of several Lemmas. We begin with:

Lemma 3. *Either (3.4) is a non trivial GPI for R or one of the conclusions in Proposition 8 must occur.*

Proof. Assume that $\Phi(x_1, x_2)$ is a trivial generalized polynomial identity for R. Let $T = Q_r *_C C\{X\}$ be the free product over C of the C-algebra Q_r and the free C-algebra $C\{X\}$, with X the set consisting of non-commuting indeterminates x_1, x_2.

By our hypothesis, $\Phi(x_1, x_2) = 0 \in Q_r *_C C\{X\}$.
We denote

$$\Psi(x_1, x_2) = \Big(a^2[x_1, x_2] + 2a[x_1, x_2]b + [x_1, x_2]b^2\Big)[x_1, x_2]$$
$$- [x_1, x_2]\Big(c^2[x_1, x_2] + 2c[x_1, x_2]q + [x_1, x_2]q^2\Big)$$

hence

$$\Phi(x_1, x_2) = \Psi(x_1, x_2)^n$$

whence R satisfies

$$\Psi(x_1, x_2)^{n-1} \cdot \Big\{\Big(a^2[x_1, x_2] + 2a[x_1, x_2]b + [x_1, x_2]b^2\Big)[x_1, x_2]$$
$$- [x_1, x_2]\Big(c^2[x_1, x_2] + 2c[x_1, x_2]q + [x_1, x_2]q^2\Big)\Big\} =$$
$$\Psi(x_1, x_2)^{n-1} \cdot \Big(a^2[x_1, x_2] + 2a[x_1, x_2]b + [x_1, x_2]b^2\Big)[x_1, x_2]$$
$$- \Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]\Big(c^2[x_1, x_2] + 2c[x_1, x_2]q + [x_1, x_2]q^2\Big).$$

If we set

$$g(x_1, x_2) = \Big(a^2[x_1, x_2] + 2a[x_1, x_2]b + [x_1, x_2]b^2\Big)[x_1, x_2] - [x_1, x_2]c^2[x_1, x_2]$$

then we have

$$\Psi(x_1, x_2)^{n-1} \cdot g(x_1, x_2)$$
$$- \Psi(x_1, x_2)^{n-1} \cdot 2[x_1, x_2]c[x_1, x_2]q$$
$$- \Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]^2 q^2 = 0 \in T.$$

This implies that $\{q^2, q, 1\}$ is linearly C-dependent. Let $\alpha q^2 + \beta q + \gamma = 0$. If $\alpha = 0$, then $\beta \neq 0$, and hence $q \in C$. If $\alpha \neq 0$, then $q^2 = \lambda q + \mu$ for some $\lambda, \mu \in C$. In this case our identity reduces to

$$\Psi(x_1, x_2)^{n-1} \cdot g(x_1, x_2)$$
$$- \Psi(x_1, x_2)^{n-1} \cdot 2[x_1, x_2]c[x_1, x_2]q$$
$$- \Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]^2(\lambda q + \mu) = 0 \in T.$$

If $q \notin C$, then

$$\Psi(x_1, x_2)^{n-1} \cdot 2[x_1, x_2]c[x_1, x_2]q + \Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]^2 \lambda q = 0 \in T$$

that is

$$\Psi(x_1, x_2)^{n-1}[x_1, x_2] \cdot (2c + \lambda)[x_1, x_2]q = 0 \in T.$$

This is implies $2c + \lambda = 0$. Since $char(R) \neq 2$, this implies $c \in C$. Thus we conclude that either $q \in C$ or $c \in C$. Similarly we can prove that either either $a \in C$ or $b \in C$. So we obtain the following four cases:

Assume $a, c \in C$

In this case $F(x) = x\overline{a}$ for $\overline{a} = a + b$ and $G(x) = x\overline{c}$ for $\overline{c} = c + q$. So

$$\Phi(x_1, x_2) = \Psi(x_1, x_2)^n = \left\{[x_1, x_2]\overline{a}^2[x_1, x_2] - [x_1, x_2]^2\overline{c}^2\right\}^n =$$
$$\Psi(x_1, x_2)^{n-1} \cdot \left\{[x_1, x_2]\overline{a}^2[x_1, x_2] - [x_1, x_2]^2\overline{c}^2\right\} =$$
$$\Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]\overline{a}^2[x_1, x_2] - \Psi(x_1, x_2)^{n-1}[x_1, x_2]^2\overline{c}^2 = 0 \in T. \tag{3.5}$$

If $\{1, \overline{c}^2\}$ were linearly C-independent then

$$\Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]^2\overline{c}^2 = 0 \in T$$

and again

$$\Psi(x_1, x_2)^{n-1} = 0 \in T.$$

Continuing similarly, we would have

$$\Psi(x_1, x_2) = 0 \in T$$

namely

$$[x_1, x_2]\overline{a}^2[x_1, x_2] - [x_1, x_2]^2\overline{c}^2 = 0 \in T.$$

This would lead to the contradiction $[x_1, x_2]^2\overline{c}^2 = 0 \in T$. We can then admit that $\overline{c}^2 \in C$. It follows from (5.6) that

$$\Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2](\overline{a}^2 - \overline{c}^2)[x_1, x_2] = 0 \in T.$$

Since $\Phi(x_1, x_2)$ is trivial then $\overline{a}^2 = \overline{c}^2 \in C$ (case (a) of Proposition 8).

Assume $a, q \in C$

In this case $F(x) = x\overline{a}$ for $\overline{a} = a + b$ and $G(x) = \overline{c}x$ for $\overline{c} = c + q$. So

$$\Phi(x_1, x_2) = \Psi(x_1, x_2)^n = \left\{[x_1, x_2]\overline{a}^2[x_1, x_2] - [x_1, x_2]\overline{c}^2[x_1, x_2]\right\}^n = 0 \in T$$

namely

$$\Psi(x_1, x_2)^{n-1} \cdot \left\{[x_1, x_2]\overline{a}^2[x_1, x_2] - [x_1, x_2]\overline{c}^2[x_1, x_2]\right\} =$$
$$\Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2](\overline{a}^2 - \overline{c}^2)[x_1, x_2] = 0 \in T.$$

Since $\Phi(x_1, x_2)$ is trivial then $\overline{a}^2 = \overline{c}^2$ (case (b) of Proposition 8).

Assume $b, c \in C$

In this case $F(x) = \overline{a}x$ for $\overline{a} = a + b$ and $G(x) = x\overline{c}$ for $\overline{c} = c + q$. So

$$\Phi(x_1, x_2) = \Psi(x_1, x_2)^n = \left\{\overline{a}^2[x_1, x_2]^2 - [x_1, x_2]^2\overline{c}^2\right\}^n = 0 \in T$$

namely

$$\Psi(x_1, x_2)^{n-1} \cdot \left\{\overline{a}^2[x_1, x_2]^2 - [x_1, x_2]^2\overline{c}^2\right\} =$$
$$\Psi(x_1, x_2)^{n-1} \cdot \overline{a}^2[x_1, x_2]^2 - \Psi(x_1, x_2)^{n-1} \cdot [x_1, x_2]^2\overline{c}^2 = 0 \in T. \tag{3.6}$$

If $\{1, \bar{a}^2\}$ were linearly C-independent then

$$\Psi(x_1, x_2)^{n-1} \cdot \bar{a}^2 [x_1, x_2]^2 = 0 \in T$$

and again

$$\Psi(x_1, x_2)^{n-1} = 0 \in T.$$

Continuing similarly, we would have

$$\Psi(x_1, x_2) = 0 \in T$$

namely

$$\bar{a}^2 [x_1, x_2]^2 - [x_1, x_2]^2 \bar{c}^2 = 0 \in T.$$

This would lead to the contradiction $\bar{a}^2 [x_1, x_2]^2 = 0 \in T$. We can then admit that $\bar{a}^2 \in C$. It follows from (5.4) that

$$\Psi(x_1, x_2)^{n-1} [x_1, x_2]^2 (\bar{a}^2 - \bar{c}^2) = 0 \in T.$$

Since $\Phi(x_1, x_2)$ is trivial then $\bar{c}^2 = \bar{a}^2 \in C$ (case (c) of Proposition 8).

Assume $b, q \in C$

In this case $F(x) = \bar{a}x$ for $\bar{a} = a + b$ and $G(x) = \bar{c}x$ for $\bar{c} = c + q$. So

$$\Phi(x_1, x_2) = \Psi(x_1, x_2)^n = \left\{ \bar{a}^2 [x_1, x_2] - [x_1, x_2]^2 \bar{c}^2 [x_1, x_2] \right\}^n =$$
$$\left\{ [x_1, x_2] \bar{a}^2 [x_1, x_2] - [x_1, x_2]^2 \bar{c}^2 \right\} \cdot \Psi(x_1, x_2)^{n-1} =$$
$$[x_1, x_2] \bar{a}^2 [x_1, x_2] \cdot \Psi(x_1, x_2)^{n-1} - [x_1, x_2]^2 \bar{c}^2 \cdot \Psi(x_1, x_2)^{n-1} = 0 \in T. \tag{3.7}$$

If $\{1, \bar{c}^2\}$ were linearly C-independent then

$$[x_1, x_2]^2 \bar{c}^2 \cdot \Psi(x_1, x_2)^{n-1} = 0 \in T.$$

and again

$$\Psi(x_1, x_2)^{n-1} = 0 \in T.$$

Continuing similarly, we would have

$$\Psi(x_1, x_2) = 0 \in T$$

namely

$$[x_1, x_2] \bar{a}^2 [x_1, x_2] - [x_1, x_2]^2 \bar{c}^2 = 0 \in T.$$

This would lead to the contradiction $[x_1, x_2]^2 \bar{c}^2 = 0 \in T$. We can then admit that $\bar{c}^2 \in C$. It follows from (3.7) that

$$[x_1, x_2] (\bar{a}^2 - \bar{c}^2) [x_1, x_2] \cdot \Psi(x_1, x_2)^{n-1} = 0 \in T.$$

Since $\Phi(x_1, x_2)$ is trivial then $\bar{a}^2 = \bar{c}^2 \in C$ (case (d) of Proposition 8). □

Remark 6. By Lemma 3, we may assume that R satisfies the non-trivial generalized polynomial identity (3.4). Since Q_r and R satisfy the same generalized polynomial identities with coeffcients in Q_r , $\Phi(x_1, x_2)$ is also a non-trivial generalized polynomial identity for Q_r. In case C is infinite, we have $\Phi(r_1, r_2) = 0$ for all

$r_1, r_2 \in Q_r \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q_r and $Q_r \otimes_C \overline{C}$ are centrally closed (Theorems 2.5 and 3.5 in [22]), we may replace R by Q_r or $Q_r \otimes_C \overline{C}$ according as C is finite or infinite. Thus, without loss of generality, we may consider the case when R is centrally closed over C which is either finite or algebraically closed and $\Phi(r_1, r_2) = 0$ for all $r_1, r_2 \in R$. By Martindale's theorem [49], R is a primitive ring having a nonzero socle with C as the associated division ring. In light of Jacobson's theorem (p. 75 in [34]), R is isomorphic to a dense ring of linear transformations on some vector space V over C.
In case $dim_C V = 2$, we are done.
Hence in all that follows we assume $dim_C V \geq 3$.

Lemma 4. *Let $a \in Q_r$ be such that*

$$\Phi'(x_1, x_2) = \Big\{[x_1, x_2]a[x_1, x_2]\Big\}^n$$

is a generalized polynomial identity for R. Then $a = 0$.

Proof. If $0 \neq a \in C$ then, by our hypothesis, R should satisfy the generalized polynomial identity $a^n[x_1, x_2]^{2n}$, which is a contradiction, since a is not a zero-divisor and $[x_1, x_2]^{2n}$ cannot be an identity for R (we remember that the nilpotence of the two-element commutator would imply commutativity of the ring, contrary to what we assumed). Let now $a \notin C$. Then there exists $v' \in V$ such that $\{v', av'\}$ is linearly C-independent. Since $\dim_C V \geq 3$ then there exists $w' \in V$ such that $\{v', av', w'\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 w' = 0, \quad r_2 w' = v', \quad r_1 v' = v' \quad \text{implying that} \quad [r_1, r_2]w' = v';$$

$$r_1 av' = v', \quad r_2 av' = 0, \quad r_2 v' = -w' \quad \text{implying that} \quad [r_1, r_2]av' = w'.$$

Now we get the following contradiction:

$$0 = \Phi'(r_1, r_2)w' = w' \neq 0.$$

□

Lemma 5. *Let $a, b \in Q_r$ be such that*

$$\Phi''(x_1, x_2) = \Big\{[x_1, x_2]a[x_1, x_2] - [x_1, x_2]^2 b\Big\}^n$$

is a generalized polynomial identity for R. Then $a = b \in C$.

Proof. We suppose that there exists a vector $v \in V$ such that $\{v, bv\}$ is linearly C-independent. Since $k \geq 3$ there exists a vector $w \in V$ such that $\{v, bv, w\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = 0, \quad r_2 v = 0 \quad \text{implying that} \quad [r_1, r_2]v = 0;$$

$$r_1 bv = 0, \quad r_2 bv = w, \quad r_1 w = w \quad \text{implying that} \quad [r_1, r_2]bv = w;$$

$$r_1 w = w, \quad r_1 v = 0, \quad r_2 w = -v \quad \text{implying that} \quad [r_1, r_2]w = v.$$

Therefore we get the following contradiction:

$$0 = \Phi''(r_1, r_2)v = (-1)^n v \neq 0.$$

Hence $\{v, bv\}$ is linearly C-dependent, for any $v \in V$, implying that $b \in C$ and R satisfies

$$\Phi''(r_1, r_2) = \Big\{[x_1, x_2](a - b)[x_1, x_2]\Big\}^n.$$

Therefore $a = b$ follows from Lemma 4. □

Lemma 6. *Let $a, b \in Q_r$ be such that*

$$\Phi'''(x_1, x_2) = \Big\{a[x_1, x_2]^2 - [x_1, x_2]^2 b\Big\}^n$$

is a generalized polynomial identity for R. Then $a = b \in C$.

Proof. If $b \notin C$ then there exists $v \in V$ such that $\{v, bv\}$ is linearly C-independent. Moreover, $dim_C V \geq 3$ implies that there exists $w \in V$ such that $\{v, bv, w\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = 0, \quad r_2 v = 0 \quad \text{implying that} \quad [r_1, r_2]v = 0;$$

$$r_1 bv = 0, \quad r_2 bv = w \quad \text{implying that} \quad [r_1, r_2]bv = w;$$

$$r_1 w = w, \quad r_2 w = -v \quad \text{implying that} \quad [r_1, r_2]w = v.$$

Now we get the following contradiction:

$$0 = \Phi'''(r_1, r_2)v = (-1)^n v \neq 0.$$

Thus we may assume that $b \in C$ and Q_r satisfies

$$\Big\{(a - b)[x_1, x_2]^2\Big\}^n = 0.$$

If $a - b \notin C$ then there exists $v' \in V$ such that $\{v', (a - b)v'\}$ is linearly C-independent. As above, there exists $w' \in V$ such that $\{v', (a - b)v', w'\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v' = v', \quad r_2 v' = -w' \quad \text{implying that} \quad [r_1, r_2]v' = w';$$

$$r_1(a - b)^2 v' = v', \quad r_2(a - b)^2 v' = 0 \quad \text{implying that} \quad [r_1, r_2](a - b)v' = w';$$

$$r_1 w' = 0, \quad r_2 w' = v' \quad \text{implying that} \quad [r_1, r_2]w' = v'.$$

Thus we get the following contradiction:

$$0 = \Big\{(a - b)[r_1, r_2]^2\Big\}^n v' = (a - b)v' \neq 0.$$

Hence $\{v', (a - b)v'\}$ is linearly C-dependent, for any $v' \in V$, implying that $a - b \in C$, so that $a = b$. □

Lemma 7. *Let $a, b \in Q_r$ be such that*

$$\Phi^{iv}(x_1, x_2) = \Big\{a[x_1, x_2]^2 - [x_1, x_2]b[x_1, x_2]\Big\}^n$$

is a generalized polynomial identity for R. Then $a = b \in C$.

Proof. If $b \notin C$ then there exists $v \in V$ such that $\{v, bv\}$ is linearly C-independent. As above, there exists $w \in V$ such that $\{v, bv, w\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = v, \quad r_2 v = 0 \quad \text{implying that} \quad [r_1, r_2]v = 0;$$

$$r_1 w = v, \quad r_2 w = w \quad \text{implying that} \quad [r_1, r_2]w = v;$$

$$r_1 bv = -w, \quad r_2 bv = 0 \quad \text{implying that} \quad [r_1, r_2]bv = w.$$

Now we get the following contradiction:

$$0 = \Phi^{iv}(r_1, r_2)w = (-1)^n w \neq 0.$$

Thus $\{v, bv\}$ is linearly C-dependent, for any $v \in V$, and Q_r satisfies

$$\left\{(a - b)[x_1, x_2]^2\right\}^n = 0.$$

From Lemma 6, it follows that $a = b \in C$. □

Lemma 8. *Let $a, b, v, w \in Q_r$ be such that*

$$\Phi^v(x_1, x_2) = \left\{v[x_1, x_2]^2 + a[x_1, x_2]b[x_1, x_2] + [x_1, x_2]w[x_1, x_2]\right\}^n \tag{3.8}$$

is a generalized polynomial identity for R. Then either $a \in C$ or $b \in C$.

Proof. Firstly we assume $\dim_C V = k$ is a finite integer, then $Q_r \simeq M_k(C)$ (with $k \geq 3$) and let e_{ij} be the unit matrix with 1 in (i, j) entry and 0 elsewhere. In (5.20) we set $[x_1, x_2] = [e_{ii}, e_{ij}] = e_{ij}$ for $i \neq j$. So left multiplying by e_{ij}, we obtain

$$e_{ij}\Phi^v(e_{ii}, e_{ij}) = e_{ij}(ae_{ij}be_{ij})^n = 0$$

with $a = \sum \alpha_{rs}e_{rs}$ and $b = \sum \beta_{rs}e_{rs}$. Since $(\alpha_{ji}\beta_{ji})^n = 0$ then $\alpha_{ji}\beta_{ji} = 0$. By Proposition 1 in [55] we have that either a or b are in C.
Now let $\dim_C V = \infty$ and $e^2 = e \in H = \mathrm{Soc}(RC)$. By hypothesis, eRe satisfies (5.20), that is R satisfies $\Phi(ex_1e, ex_2e)$:

$$\Phi^v(ex_1e, ex_2e) = \Big\{v[ex_1e, ex_2e]^2 + a[ex_1e, ex_2e]b[ex_1e, ex_2e] + [ex_1e, ex_2e]w[ex_1e, ex_2e]\Big\}^n = 0. \tag{3.9}$$

In particular, for all $r_1, r_2 \in R$

$$\Phi^v(er_1e, er_2e) = \Big\{eve[er_1e, er_2e]^2 + eae[er_1e, er_2e]ebe[er_1e, er_2e] + [er_1e, er_2e]ewe[er_1e, er_2e]\Big\}^n = 0.$$

Suppose that $a \notin C$ and $b \notin C$. Thus, $[a, H] \neq 0$ and $[b, H] \neq 0$. In other words, there exist $h_1, h_2 \in H$ such that $[a, h_1] \neq 0$ and $[b, h_2] \neq 0$.
Because of the infinite-dimensionality, H does not satisfy the polynomial $s_4(x_1, \ldots, x_4)$, that is, there exist $h_3, h_4, h_5, h_6 \in H$ such that $s_4(h_3, \ldots, h_6) \neq 0$. By Litoff's theorem in [23], there exists an $e^2 = e \in Q_r$ such that $h_1, \ldots, h_6, ah_1, h_1a, bh_2, h_2b \in eQ_re \simeq M_t(C)$. Moreover, eQ_re is a central simple algebra finite-dimensional over its center. Since $s_4(h_3, \ldots, h_6) \neq 0$, we have $eQ_re \simeq M_t(C)$ for $t \geq 3$. We know that (3.9) is a generalized polynomial identity for eQ_re, then, by the finite-dimensional case, we

have that eae or ebe are in $Z(eRe)$, thus one of the following is a contradiction:

$$ah_1 = eah_1 = eaeh_1 = h_1eae = h_1ae = h_1a,$$
$$bh_2 = ebh_2 = ebeh_2 = h_2ebe = h_2be = h_2b.$$

Therefore, either $a \in C$ or $b \in C$, as required. □

Lemma 9. *Let $a, b, v, w \in Q_r$ be such that*

$$\Phi^{vi}(x_1, x_2) = \left\{v[x_1, x_2]^2 + a[x_1, x_2]b[x_1, x_2] + [x_1, x_2]w[x_1, x_2] - [x_1, x_2]^2u\right\}^n \quad (3.10)$$

is a generalized polynomial identity for R. Then either $a \in C$ or $b \in C$.

Proof. If $u \notin C$ then there exists $w_0 \in V$ such that $\{w_0, uw_0\}$ is linearly C-independent. Since $\dim_C V \geq 3$ then there exists $w_1 \in V$ such that $\{w_0, uw_0, w_1\}$ is linearly C-independent. Then by the density of Q_r, there exist $r_1, r_2 \in Q_r$ such that

$$r_1w_0 = 0, \quad r_2w_0 = w_0 \quad \text{implying that} \quad [r_1, r_2]w_0 = 0;$$
$$r_1(uw_0) = 0, \quad r_2(uw_0) = w_1, \quad r_1w_1 = w_1 \quad \text{implying that} \quad [r_1, r_2](uw_0) = w_1;$$
$$r_2w_1 = w_0 \quad \text{implying that} \quad [r_1, r_2]w_1 = -w_0.$$

Then by (5.19) we get the following contradiction:

$$0 = \Phi^{vi}(r_1, r_2)w_0 = w_0 \neq 0.$$

Therefore $u \in C$ and we conclude by Lemma 8. □

Lemma 10. *Let $a_1, a_2, a_3, a_4, a_5, a_6, a_7 \in Q_r$ be such that*

$$\begin{aligned}\Phi^{vii}(x_1, x_2) = \Big\{&\Big(a_1[x_1, x_2] + a_2[x_1, x_2]a_3 + [x_1, x_2]a_4\Big)[x_1, x_2] \\ &- [x_1, x_2]\Big(a_5[x_1, x_2]a_6 + [x_1, x_2]a_7\Big)\Big\}^n\end{aligned} \quad (3.11)$$

is a generalized polynomial identity for R. If $a_6 \notin C$ then $a_5 \in C$.

Proof. As above we firstly assume $\dim_C V = k$ is a finite integer. Thus $Q_r \simeq M_k(C)$ with $k \geq 3$. So, for $[x_1, x_2] = e_{ij} = [e_{ii}, e_{ij}]$ in (3.4) and right multiplying by e_{ij}, we obtain

$$\Phi^{vii}(e_{ii}, e_{ij})e_{ij} = (-1)^n(e_{ij}a_5e_{ij}a_6)^ne_{ij} = 0,$$

then by Proposition 1 in [55] we have that $a_5 \in C$.
Let now $\dim_C V = \infty$. Since $a_6 \notin C$ then there exists $v \in V$ such that $\{v, a_6v\}$ is linearly C-independent.
If $\{a_7v, v, a_6v\}$ is linearly C-independent then there exist $w, w' \in V$ such that $\{a_7v, v, a_6v, w, w'\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1v = 0, \quad r_2v = v \quad \text{implying that} \quad [r_1, r_2]v = 0;$$
$$r_1a_6v = 0, \quad r_2a_6v = v \quad \text{implying that} \quad [r_1, r_2]a_6v = 0;$$
$$r_1a_7v = 0, \quad r_2a_7v = w', \quad r_1w' = w \quad \text{implying that} \quad [r_1, r_2]a_7v = w;$$
$$r_1w = v, \quad r_2w = 0 \quad \text{implying that} \quad [r_1, r_2]w = -v.$$

Thus by (3.11) we get the following contradiction:

$$0 = \Phi^{vii}(r_1, r_2)v = v \neq 0.$$

This contradiction implies that $a_7 v = \lambda_v v + \mu_v a_6 v$ for some $\lambda_v, \mu_v \in C$.
Next goal is to prove that $a_5 v \in \text{Span}(v, a_6 v)$. On the contrary we assume that $\{a_5 v, v, a_6 v\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = 0, \quad r_2 v = v \quad \text{implying that} \quad [r_1, r_2]v = 0;$$

$$r_1 a_6 v = -v, \quad r_2 a_6 v = 0 \quad \text{implying that} \quad [r_1, r_2]a_6 v = v$$

$$r_1 a_5 v = v, \quad r_2 a_5 v = 0 \quad \text{implying that} \quad [r_1, r_2]a_5 v = -v.$$

Hence by (3.11) we get the following contradiction:

$$0 = \Phi^{vii}(r_1, r_2)v = v \neq 0.$$

Thus there exist $\alpha_v, \beta_v \in C$ such that $a_5 v = \alpha_v v + \beta_v a_6 v$.
On the other hand, again by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = 0, \quad r_2 v = v \quad \text{implying that} \quad [r_1, r_2]v = 0;$$

$$r_1 a_6 v = -v, \quad r_2 a_6 v = 0 \quad \text{implying that} \quad [r_1, r_2]a_6 v = v.$$

By (3.11) we get $\Phi^{vii}(r_1, r_2)v = (-1)^n \beta_v^n v = 0$, hence $\beta_v = 0$ and $a_5 v \in \text{Span}(v)$, then $a_5 v = \alpha_v v$.
The final goal is to prove that $\{w, a_5 w\}$ is linearly C-dependent for all $w \in V$.
If $w = \theta v$ for some $\theta \in C$, then $a_5 w = \theta a_5 v = \theta \alpha_v v = \alpha_v \theta_v v = \alpha_v w$ as required.
Let $w \in V$ such that $\{w, v\}$ is linearly C-independent and assume that $\{w, a_5 w\}$ is linearly C-independent. We will prove that this leads to a contradiction.
By the previous argument, there exists $\lambda_w \in C$ such that $a_6 w = \lambda_w w$. So we consider the following linearly C-independent sets

$$\{v, w + v\}, \quad \{v, w - v\}.$$

If $\{w + v, a_6(w + v)\}$ and $\{w - v, a_6(w - v)\}$ were linearly C-independent, then as above there exist $\beta, \gamma \in C$ such that

$$\begin{cases} a_5(w + v) = \beta(w + v) \\ a_5(w - v) = \gamma(w - v) \end{cases}$$

$$\begin{cases} a_5 w + \alpha_v v = \beta w + \beta v \\ a_5 w - \alpha_v v = \gamma w - \gamma v. \end{cases}$$

If we subtract from each other, we obtain

$$2\alpha_v v = (\beta - \gamma)w + (\beta + \gamma)v.$$

Since $\{w, v\}$ is linearly C-independent, then $\beta = \gamma = \alpha_v$ hence $a_5 w = \beta w$, that is $\{w, a_5 w\}$ is linearly C-dependent, a contradiction.
Now we may assume without loss of generality, that there exists $\eta \in C$ such that $a_6(w + v) = \eta(w + v)$ then $\lambda_w w + a_6 v = \eta w + \eta v$. Since $\{a_6 v, v\}$ is linearly C-independent, then $\lambda_w \neq \eta$. Hence $w \in \text{Span}(v, a_6 v)$.
In conclusion, we can state that for all $w \in V$ either there exists $\alpha_w \in C$ such that

$a_5w = \alpha_w w$ or $w \in \mathrm{Span}(v, a_6v)$.
Let now $w_1, w_2 \in V$ be such that $\{w_1, w_2\}$ is linearly C-independent, $w_1 \notin \mathrm{Span}(v, a_6v)$ and $\{w_2, a_5w_2\}$ is linearly C-independent. The above argument implies that there is $\lambda_1 \in C$ such that $a_5w_1 = \lambda_1 w_1$ and also $w_2 \in \mathrm{Span}(v, a_6v)$.
If $w_1 + w_2 \in \mathrm{Span}(v, a_6v)$, then $w_1 \in \mathrm{Span}(v, a_6v)$, that is a contradiction. Hence there exists $\eta_1 \in C$ such that $a_5(w_1 + w_2) = \eta_1(w_1 + w_2)$, that is

$$\lambda_1 w_1 + a_5 w_2 = \eta_1 w_1 + \eta_1 w_2. \tag{3.12}$$

Analogously, $w_1 - w_2 \notin \mathrm{Span}(v, a_6v)$, then there exists $\eta_2 \in C$ such that $a_5(w_1 - w_2) = \eta_2(w_1 + w_2)$, that is

$$\lambda_1 w_1 - a_5 w_2 = \eta_2 w_1 - \eta_2 w_2. \tag{3.13}$$

the sum of relations (3.12) and (3.13) leads to

$$2\lambda_1 w_1 = (\eta_1 + \eta_2)w_1 + (\eta_1 - \eta_2)w_2.$$

Since $\{w_1, w_2\}$ is linearly C-independent, one has $eta_1 = eta_2 = \lambda_1$. In particular $a_5w_2 = \eta_1 w_2$ follows, which is the required contradiction.
Therefore, $\{w, a_5w\}$ is linearly C-dependent for all $w \in V$, implying that $a_5 \in C$. □

Proof of Proposition 8. We recall that

$$\Phi(x_1, x_2) = \Big\{ \Big(a^2[x_1, x_2] + 2a[x_1, x_2]b + [x_1, x_2]b^2\Big)[x_1, x_2] \\ - [x_1, x_2]\Big(c^2[x_1, x_2] + 2c[x_1, x_2]q + [x_1, x_2]q^2\Big) \Big\}^n$$

is a generalized polynomial identity for R. Application of Lemmas 8, 9 and 10 implies that one of the following cases must occur:

1. $a, c \in C$;
2. $a, q \in C$;
3. $b, c \in C$;
4. $b, q \in C$.

In the sequel, we analyze all cases in detail.

$a, c \in C$

In this case $F(x) = x\overline{a}$ for $\overline{a} = a + b$ and $G(x) = x\overline{c}$ for $\overline{c} = c + q$. So R satisfies

$$\Phi(x_1, x_2) = \Big\{ [x_1, x_2]\overline{a}^2[x_1, x_2] - [x_1, x_2]^2\overline{c}^2 \Big\}^n.$$

Then, by Lemma 5, we get $\overline{c}^2 = \overline{a}^2 \in C$.

$a, q \in C$

In this case $F(x) = x\overline{a}$ for $\overline{a} = a + b$ and $G(x) = \overline{c}x$ for $\overline{c} = c + q$. So

$$\Phi(x_1, x_2) = \Big\{ [x_1, x_2](\overline{a}^2 - \overline{c}^2)[x_1, x_2] \Big\}^n.$$

By Lemma 4, $\overline{a}^2 = \overline{c}^2$.

$b, c \in C$

In this case $F(x) = \overline{a}x$ for $\overline{a} = a + b$ and $G(x) = x\overline{c}$ for $\overline{c} = c + q$. So

$$\Phi(x_1, x_2) = \left\{ \overline{a}^2 [x_1, x_2]^2 - [x_1, x_2]^2 \overline{c}^2 \right\}^n$$

and, by Lemma 6, $\overline{a}^2 = \overline{c}^2 \in C$.

$b, q \in C$

In this case $F(x) = \overline{a}x$ for $\overline{a} = a + b$ and $G(x) = \overline{c}x$ for $\overline{c} = c + q$. So

$$\Phi(x_1, x_2) = \left\{ \overline{a}^2 [x_1, x_2]^2 - [x_1, x_2] \overline{c}^2 [x_1, x_2] \right\}^n$$

and $\overline{a}^2 = \overline{c}^2 \in C$ follows from Lemma 7. □

3.2 The proof of Theorem 22

In light of our previous results, we may assume that F and G are not simultaneously inner.
Under this assumption, we will prove that a number of contradictions follows.
We firstly assume that d and δ are C-dependent modulo inner derivations of Q_r, Thus there exist $\alpha, \beta \in C$ such that $\alpha d + \beta \delta = ad_p$, where $p \in Q_r$ and $ad_p(x) = [p, x]$ for all $x \in R$.

Assume $\alpha = 0$.

In this case $\delta(x) = \lambda ad_p(x) = \lambda[p, x]$ where $0 \neq \lambda = \beta^{-1} \in C$. Moreover d is not inner and

$$G(x) = cx + \lambda[p, x] = bx + xq \quad \forall x \in R$$

where $b = c + \lambda p$ and $q = -\lambda p$. Hence

$$G^2(x) = b^2 x + 2bxq + xq^2 \quad \forall x \in R.$$

From (3.1), we obtain that Q_r satisfies

$$\begin{aligned}
&\Big\{ \Big(F(a)[x_1, x_2] + 2a\big([d(x_1), x_2] + [x_1, d(x_2)]\big) \\
&+ [d^2(x_1), x_2] + [x_1, d^2(x_2)] + 2[d(x_1), d(x_2)] \Big) [x_1, x_2] \\
&- [x_1, x_2] \Big(G(c)[x_1, x_2] + 2c\lambda[p, [x_1, x_2]] + \lambda^2 [p, [p, [x_1, x_2]]] \Big) \Big\}^n = \\
&\Big\{ \Big(F(a)[x_1, x_2] + 2a\big([d(x_1), x_2] + [x_1, d(x_2)]\big) \\
&+ [d^2(x_1), x_2] + [x_1, d^2(x_2)] + 2[d(x_1), d(x_2)] \Big) [x_1, x_2] \\
&- [x_1, x_2] \Big(b^2 [x_1, x_2] + 2b[x_1, x_2]q + [x_1, x_2]q^2 \Big) \Big\}^n .
\end{aligned} \tag{3.14}$$

Then by Kharchenko's theorem (see Theorem 14), we have that Q_r satisfies

$$\Big\{\Big(F(a)[x_1,x_2]+2a\big([y_1,x_2]+[x_1,y_2]\big)+[t_1,x_2]+[x_1,t_2]+2[y_1,y_2]\Big)[x_1,x_2] \\ -[x_1,x_2]\Big(b^2[x_1,x_2]+2b[x_1,x_2]q+[x_1,x_2]q^2\Big)\Big\}^n. \tag{3.15}$$

In particular, when $y_1=y_2=t_1=t_2=0$ in (4.10), we have that

$$\Big\{F(a)[x_1,x_2]^2-[x_1,x_2]\Big(b^2[x_1,x_2]+2b[x_1,x_2]q+[x_1,x_2]q^2\Big)\Big\}^n \tag{3.16}$$

is a generalized identity for Q_r. Application of Lemma 10 implies that either $b\in C$ or $q\in C$.
If $b\in C$, then $G(x)=xc$, for any $x\in R$, and we may rewrite (4.10) as follows

$$\Big\{\Big(F(a)[x_1,x_2]+2a\big([y_1,x_2]+[x_1,y_2]\big)+[t_1,x_2]+[x_1,t_2]+2[y_1,y_2]\Big)[x_1,x_2]-[x_1,x_2]^2c^2\Big\}^n. \tag{3.17}$$

In particular Q_r satisfies

$$\Big\{F(a)[x_1,x_2]^2-[x_1,x_2]^2c^2\Big\}^n.$$

Hence, by Lemma 6, $F(a)=c^2\in C$, that is $F(a)=G(c)\in C$. Therefore, for $y_1=y_2=0$, $t_1=x_1$ and $t_2=x_2$ in (4.12), Q_r should satisfy $2^n[x_1,x_2]^{2n}$, that is Q_r should be commutative, a contradiction.
On the other hand, if $q\in C$, then $G(x)=cx$, for any $x\in R$, and we may rewrite (4.10) as follows

$$\Big\{\Big(F(a)[x_1,x_2]+2a\big([y_1,x_2]+[x_1,y_2]\big)+[t_1,x_2]+[x_1,t_2]+2[y_1,y_2]\Big)[x_1,x_2]-[x_1,x_2]c^2[x_1,x_2]\Big\}^n. \tag{3.18}$$

For $y_1=y_2=t_1=t_2=0$ in (3.18), it follows that

$$\Big\{\Big(F(a)[x_1,x_2]^2-[x_1,x_2]c^2[x_1,x_2]\Big\}^n$$

is a generalized identity for Q_r. By Lemma 7, $F(a)=c^2\in C$ and, by using a similar argument as above, we get a contradiction.

Assume $\beta=0$.

In this case $d(x)=\mu ad_p(x)=\mu[p,x]$ where $0\neq\mu=\alpha^{-1}\in C$. Moreover δ is not inner and

$$F(x)=ax+\mu[p,x]=b'x+xq' \quad \forall x\in R$$

where $b'=a+\mu p$ and $q'=-\mu p$.
By using the same above argument, and applying Lemmas 7, 4 (instead of Lemmas 10, 6, 7), we obtain the same above contradiction.
We omit the computations for brevity.

Assume $\alpha \neq 0$ and $\beta \neq 0$.

Say $d = \lambda\delta + ad_q$, where $\lambda = -\alpha^{-1}\beta \in C$, $q = \alpha^{-1}p \in Q_r$ and $ad_q(x) = [q, x]$, for all $x \in R$. Moreover,

$$\begin{aligned} d^2(x) &= \lambda\delta\big(\lambda\delta(x) + [q, x]\big) + \big[q, \lambda\delta(x) + [q, x]\big] = \\ &= \lambda d(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda[q, \delta(x)] + \lambda[\delta(q), x] + [q, [q, x]]. \end{aligned}$$

From (3.3), we obtain that Q_r satisfies

$$\begin{aligned} &\Big\{\Big(F(a)[x_1, x_2] + 2\lambda a\delta([x_1, x_2]) + 2a[q, [x_1, x_2]] \\ &+ \lambda\delta(\lambda)\delta([x_1, x_2]) + \lambda^2\delta^2([x_1, x_2]) + 2\lambda[q, \delta([x_1, x_2])] \\ &+ \lambda[\delta(q), [x_1, x_2]] + [q, [q, [x_1, x_2]]]\Big)[x_1, x_2] \\ &- [x_1, x_2]\Big(G(c)[x_1, x_2] + 2c\delta([x_1, x_2]) + \delta^2([x_1, x_2])\Big)\Big\}^n \end{aligned}$$

that is

$$\begin{aligned} &\Big\{\Big(F(a)[x_1, x_2] + (2\lambda a + \lambda\delta(\lambda))\Big([\delta(x_1), x_2] + [x_1, \delta(x_2)]\Big) \\ &+ 2a[q, [x_1, x_2]] + \lambda^2\Big([\delta^2(x_1), x_2] + [x_1, \delta^2(x_2)] + 2[\delta(x_1), \delta(x_2)]\Big) \\ &+ 2\lambda[q, [\delta(x_1), x_2] + [x_1, \delta(x_2)]] + \lambda[\delta(q), [x_1, x_2]] \\ &+ [q, [q, [x_1, x_2]]]\Big)[x_1, x_2] - [x_1, x_2]\Big(G(c)[x_1, x_2] + 2c\Big([\delta(x_1), x_2] + [x_1, \delta(x_2)]\Big) \\ &+ [\delta^2(x_1), x_2] + [x_1, \delta^2(x_2)] + 2[\delta(x_1), \delta(x_2)]\Big)\Big\}^n. \end{aligned}$$

Then by Kharchenko's theorem (see Theorem 14), we have that Q_r satisfies

$$\begin{aligned} &\Big\{\Big(F(a)[x_1, x_2] + (2\lambda a + \lambda\delta(\lambda))\Big([y_1, x_2] + [x_1, y_2]\Big) \\ &+ 2a[q, [x_1, x_2]] + \lambda^2\Big([t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big) \\ &+ 2\lambda[q, [y_1, x_2] + [x_1, y_2]] + \lambda[\delta(q), [x_1, x_2]] \\ &+ [q, [q, [x_1, x_2]]]\Big)[x_1, x_2] - [x_1, x_2]\Big(G(c)[x_1, x_2] + 2c\Big([y_1, x_2] + [x_1, y_2]\Big) \\ &+ [t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big)\Big\}^n. \end{aligned} \tag{3.19}$$

For $y_1 = y_2 = t_1 = t_2 = 0$ in (3.19), we have that

$$\begin{aligned} &\Big\{\Big(F(a)[x_1, x_2] + 2a[q, [x_1, x_2]] + \lambda[\delta(q), [x_1, x_2]] + [q, [q, [x_1, x_2]]]\Big)[x_1, x_2] \\ &- [x_1, x_2]G(c)[x_1, x_2]\Big\} \end{aligned} \tag{3.20}$$

is a generalized identity for Q_r. Rearranging the terms in relation (3.20) appropriately, we may state that Q_r satisfies

$$\Big\{a_1[x_1, x_2]^2 + [x_1, x_2]a_2[x_1, x_2] + a_3[x_1, x_2]a_4[x_1, x_2]\Big\}^n \tag{3.21}$$

where $a_1 = F(a) + 2aq + \lambda\delta(q) + q^2$, $a_2 = q^2 - \lambda\delta(q) - G(c)$, $a_3 = -2(a + q)$ and $a_4 = q$. Application of Lemma 8 to relation (3.21) implies that either $a_3 \in C$ or $a_4 \in C$.

If $a_3 \in C$ then $a+q \in C$ and $F(x) = \lambda\delta(x) + xa$, for any $x \in R$. Moreover, (3.21) reduces to

$$\Big\{a_1[x_1,x_2]^2 + [x_1,x_2](a_2+a_3a_4)[x_1,x_2]\Big\}^n$$

and, by Lemma 7, it follows $a_1 = -a_2 - a_3a_4 \in C$, that is

$$0 = a_1 + a_2 + a_3a_4 = F(a) - G(c).$$

Hence

$$\begin{aligned}&F^2(x) = \lambda\delta\Big(\lambda\delta(x) + xa\Big) + \Big(\lambda\delta(x) + xa\Big)a = \\ &\lambda\delta(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda\delta(x)a + \lambda x\delta(a) + xa^2 = \\ &\lambda\delta(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda\delta(x)a + x(a^2 + \lambda\delta(a)) = \\ &\lambda\delta(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda\delta(x)a + xG(c).\end{aligned}$$

Then, by (3.3), we have that Q_r satisfies

$$\begin{aligned}&\Big\{\Big(\lambda\delta(\lambda)\big([\delta(x_1),x_2] + [x_1,\delta(x_2)]\big) \\ &+ \lambda^2\big([\delta^2(x_1),x_2] + [x_1,\delta^2(x_2)] + 2[\delta(x_1),\delta(x_2)]\big) \\ &+ 2\lambda\big([\delta(x_1),x_2] + [x_1,\delta(x_2)]\big)a + [x_1,x_2]G(c)\Big)[x_1,x_2] \\ &- [x_1,x_2]\Big(G(c)[x_1,x_2] + 2c\Big([\delta(x_1),x_2] + [x_1,\delta(x_2)]\Big) \\ &+ [\delta^2(x_1),x_2] + [x_1,\delta^2(x_2)] + 2[\delta(x_1),\delta(x_2)]\Big)\Big\}^n\end{aligned}$$

that is

$$\begin{aligned}&\Big\{\Big(\lambda\delta(\lambda)\big([y_1,x_2] + [x_1,y_2]\big) \\ &+ \lambda^2\big([t_1,x_2] + [x_1,t_2] + 2[y_1,y_2]\big) \\ &+ 2\lambda\big([y_1,x_2] + [x_1,y_2]\big)a\Big)[x_1,x_2] \\ &- [x_1,x_2]\Big(2c\Big([y_1,x_2] + [x_1,y_2]\Big) \\ &+ [t_1,x_2] + [x_1,t_2] + 2[y_1,y_2]\Big)\Big\}^n\end{aligned}$$

is a generalized identity for Q_r. For $t_2 = y_1 = y_2 = 0$ it follows that Q_r satisfies the polynomial identity

$$\Big\{\lambda^2[t_1,x_2][x_1,x_2] - [x_1,x_2][t_1,x_2]\Big\}^n. \tag{3.22}$$

It is well known that in this case there exists a field K such that Q_r and the matrix ring $M_n(K)$ satisfies the same polynomial identities. Of course we may assume $n \geq 3$. For $x_1 = e_{12}$, $x_2 = e_{22}$, $t_1 = e_{32} - e_{21}$ in (3.22) we get the contradiction $(\lambda^2 e_{22} - e_{11})^n = 0$. Let now $a_4 \in C$, which means that $d = \lambda\delta$ and $F(x) = ax + \lambda\delta(x)$, for any $x \in R$.

In this case (3.21) reduces to

$$\Big\{(a_1 + a_3a_4)[x_1,x_2]^2 + [x_1,x_2]a_2[x_1,x_2]\Big\}^n$$

and, by Lemma 7, it follows $a_1 + a_3a_4 = -a_2 \in C$. Therefore, once again we have

$$0 = a_1 + a_2 + a_3a_4 = F(a) - G(c).$$

Moreover, since $q, \delta(q) \in C$ and $a_2 = q^2 - \lambda\delta(q) - G(c) \in C$, it follows that $G(c) \in C$. Therefore

$$\begin{aligned}
&F^2(x) = a\Big(ax + \lambda\delta(x)\Big) + \lambda\delta\Big(ax + \lambda\delta(x)\Big) = \\
&\lambda\delta(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda a\delta(x) + (a^2 + \lambda\delta(a))x = \\
&\lambda\delta(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda a\delta(x) + G(c)x = \\
&\lambda\delta(\lambda)\delta(x) + \lambda^2\delta^2(x) + 2\lambda a\delta(x) + xG(c).
\end{aligned}$$

By (3.3), one has that Q_r satisfies

$$\begin{aligned}
&\Big\{\Big(\lambda\delta(\lambda)\big([\delta(x_1), x_2] + [x_1, \delta(x_2)]\big) \\
&+ \lambda^2\big([\delta^2(x_1), x_2] + [x_1, \delta^2(x_2)] + 2[\delta(x_1), \delta(x_2)]\big) \\
&+ 2\lambda a\big([\delta(x_1), x_2] + [x_1, \delta(x_2)]\big) + [x_1, x_2]G(c)\Big)[x_1, x_2] \\
&- [x_1, x_2]\Big(G(c)[x_1, x_2] + 2c\Big([\delta(x_1), x_2] + [x_1, \delta(x_2)]\Big) \\
&+ [\delta^2(x_1), x_2] + [x_1, \delta^2(x_2)] + 2[\delta(x_1), \delta(x_2)]\Big)\Big\}^n
\end{aligned}$$

that is

$$\begin{aligned}
&\Big\{\Big(\lambda\delta(\lambda)\big([y_1, x_2] + [x_1, y_2]\big) \\
&+ \lambda^2\big([t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\big) \\
&+ 2\lambda a\big([y_1, x_2] + [x_1, y_2]\big)\Big)[x_1, x_2] \\
&- [x_1, x_2]\Big(2c\Big([y_1, x_2] + [x_1, y_2]\Big) \\
&+ [t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big)\Big\}^n
\end{aligned}$$

is a generalized identity for Q_r. For $t_2 = y_1 = y_2 = 0$ it follows that Q_r satisfies the polynomial identity

$$\Big\{\lambda^2[t_1, x_2][x_1, x_2] - [x_1, x_2][t_1, x_2]\Big\}^n$$

which implies the same above contradiction (see relation (3.22)).

We finally assume that d and δ are linearly C-independent modulo inner derivations of Q_r. Applying Kharchenko's theorem (see Theorem 14) to identity (3.3), it follows that Q_r satisfies

$$\begin{aligned}
&\Big\{\Big(F(a)[x_1, x_2] + 2a\Big([y_1, x_2] + [x_1, y_2]\Big) + [t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big)[x_1, x_2] \\
&- [x_1, x_2]\Big(G(c)[x_1, x_2] + 2c\Big([y_1, x_2] + [x_1, y_2]\Big) + [t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big)\Big\}^n.
\end{aligned} \tag{3.23}$$

In particular, for $y_1 = y_2 = t_1 = t_2 = 0$ in relation (3.23),

$$\Big\{F(a)[x_1, x_2]^2 - [x_1, x_2]G(c)[x_1, x_2]\Big\}^n$$

is a generalized identity for Q_r. Again by Lemma 7, we get $F(a) = G(c) \in C$ and (3.23) reduces to

$$\begin{aligned}&\Big\{\Big(2a\Big([y_1, x_2] + [x_1, y_2]\Big) + [t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big)[x_1, x_2]\\ &- [x_1, x_2]\Big(2c\Big([y_1, x_2] + [x_1, y_2])\Big) + [t_1, x_2] + [x_1, t_2] + 2[y_1, y_2]\Big)\Big\}^n.\end{aligned} \tag{3.24}$$

For $y_1 = y_2 = t_2 = 0$ in (3.24), it follows that Q_r satisfies the polynomial identity

$$\Big\{\Big([t_1, x_2][x_1, x_2] - [x_1, x_2][t_1, x_2]\Big\}^n$$

which leads us to the same previous contradiction. □

Chapter 4

Nil sets defined through generalized derivations having homomorphism-like behavior.

In [4, Theorem 3] Bell and Kappe prove that if d is a derivation of a prime ring R which acts as a homomorphism or anti-homomorphism on a non-zero right ideal of R, then $d = 0$ on R. Later Ali et al. in [1] extend this result to Lie ideals of 2-torsion free prime rings. More precisely they prove that if L is a non-central Lie ideal of R such that $u^2 \in L$, for all $u \in L$ and d acts as a homomorphism or anti-homomorphism on L, then $d = 0$.
In [57] Wang and You eliminate the hypothesis $u^2 \in L, \forall u \in L$ and prove the same result in [1].
Recently in [54] Rehman studies the case when the derivation d is replaced by a generalized derivation g. He proves that if R is a 2-torsion free prime ring and g acts as a homomorphism or an anti-homomorphism on a non-zero ideal of R, then R must be commutative.
In [27] Golbasi and Kaya prove a similar result for Lie ideals. More precisely they proved the following: Let R be a prime ring of characteristic different from 2, g a generalized derivation of R, which is associated with the derivation d, L a Lie ideal of R such that $u^2 \in L$ for all $u \in L$. If g acts as a homomorphism or anti-homomorphism on L, then either $d = 0$ or L is central in R. More recently in [16] De Filippis proves the following result: let R be a prime ring, L a non-central Lie ideal of R and F a non-zero generalized derivation of R. If F acts as a Jordan homomorphism on L, then either $F(x) = x$ for all $x \in R$, or $char(R) = 2$, R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$, L is commutative and $u^2 \in Z(R)$, for any $u \in L$.
Starting from the results just mentioned, in this chapter we intend to continue along this line of research and to prove the following theorem:

Theorem 23. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, L a non-central Lie ideal of R, $n \geq 1$ a fixed integer, F and G two generalized derivations of R. If $\big(F(xy) - G(x)G(y)\big)^n = 0$, for any $x, y \in L$, then there exists $\lambda \in C$ such that $F(x) = \lambda^2 x$ and $G(x) = \lambda x$, for any $x \in R$.*

We firstly fix the following easy result:

Lemma 11. *Assume that*

$$\big\{F(xy) - G(x)G(y)\big\}^n = 0, \ \forall x, y \in [R, R]. \tag{4.1}$$

Then $F = 0$ if and only if $G = 0$.

Proof. If it were $F = 0$ in (4.1) we would have

$$\Big\{G([x_1, x_2])G([y_1, y_2])\Big\}^n = 0$$

in particular, for $[x_1, x_2] = [y_1, y_2]$

$$\Big\{G([x_1, x_2])\Big\}^{2n} = 0.$$

In this case, by [48, Theorems 1 and 3], it follows $G = 0$.
On the other hand, if it were $G = 0$ in (4.1), it would follow that

$$\{F(xy)\}^n = 0, \ \forall x, y \in [R, R].$$

In this case, once again by [48, Theorems 1 and 3], we would conclude $F = 0$. □

4.1 Inner generalized derivations.

In this first section we will prove Theorem 23 in the case F and G are both inner generalized derivations and $L = [R, R]$. Thus we assume

$$F(x) = ax + xb, \qquad G(x) = cx + xu$$

for all $x \in R$, for suitable fixed elements $a, b, c, u \in Q_r$. Under this assumption, R satisfies

$$\Phi(x_1, x_2, y_1, y_2) = \Big\{\Big(a[x_1, x_2][y_1, y_2] + [x_1, x_2][y_1, y_2]b\Big) - \Big(c[x_1, x_2] + [x_1, x_2]u\Big)\Big(c[y_1, y_2] + [y_1, y_2]u\Big)\Big\}^n. \tag{4.2}$$

Here, our aim is to prove the following result

Proposition 9. *If R satisfies (4.2), then $b, c, u \in C$ and $(a + b) = (c + u)^2$.*

Lemma 12. *If R does not satisfy any non-trivial generalized polynomial identity, then $a, b, c, u \in C$ and $(a + b) = (c + u)^2$.*

Proof. Let $T = Q_r *_C C\{X\}$ be the free product over C of the C-algebra Q_r and the free C-algebra $C\{X\}$, with X the set consisting of non-commuting indeterminates x_1, x_2, y_1, y_2. By our hypothesis, $\Phi(x_1, x_2, y_1, y_2)$ is a trivial generalized polynomial identity for R, that is $\Phi(x_1, x_2, y_1, y_2) = 0 \in Q_r *_C C\{X\}$.
We denote

$$\Psi(x_1, x_2, y_1, y_2) = \Big(a[x_1, x_2][y_1, y_2] + [x_1, x_2][y_1, y_2]b\Big) - \Big(c[x_1, x_2] + [x_1, x_2]u\Big)\Big(c[y_1, y_2] + [y_1, y_2]u\Big) \tag{4.3}$$

hence

$$\Phi(x_1, x_2, y_1, y_2) = \Psi(x_1, x_2, y_1, y_2)^n$$

whence R satisfies

$$\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big\{\Big(a[x_1,x_2][y_1,y_2]+[x_1,x_2][y_1,y_2]b\Big)-$$
$$(c[x_1,x_2]+[x_1,x_2]u)(c[y_1,y_2]+[y_1,y_2]u)\Big\}=$$

$$\begin{aligned}&\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big(a[x_1,x_2]-(c[x_1,x_2]+[x_1,x_2]u)c\Big)[y_1,y_2]+\\&\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big([x_1,x_2][y_1,y_2]\Big)b+\\&-\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big(c[x_1,x_2]+[x_1,x_2]u\Big)[y_1,y_2]u.\end{aligned}\tag{4.4}$$

If $\{1,b,u\}$ is linearly C-independent then

$$\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big(c[x_1,x_2]+[x_1,x_2]u\Big)[y_1,y_2]u=0\in T$$

and again

$$\Psi(x_1,x_2,y_1,y_2)^{n-1}=0\in T.$$

Continuing this process, we will have

$$\Psi(x_1,x_2,y_1,y_2)=0\in T$$

namely

$$\begin{aligned}&\Big(a[x_1,x_2]-(c[x_1,x_2]+[x_1,x_2]u)c\Big)[y_1,y_2]+\\&[x_1,x_2][y_1,y_2]b-\Big(c[x_1,x_2]+[x_1,x_2]u\Big)[y_1,y_2]u=0\in T.\end{aligned}\tag{4.5}$$

This leads to the contradiction $[y_1,y_2]u=0\in T$.
Then $\{1,b,u\}$ is linearly C-dependent, that is $u=\lambda b+\mu$, for some $\lambda,\mu\in C$. It follows from (4.4) that

$$\begin{aligned}&\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big(a[x_1,x_2]-(c[x_1,x_2]+[x_1,x_2]u)c\Big)[y_1,y_2]+\\&\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big([x_1,x_2][y_1,y_2]-(c[x_1,x_2]+[x_1,x_2]u)\lambda[y_1,y_2]\Big)b+\\&-\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big(c[x_1,x_2]+[x_1,x_2]u\Big)\mu[y_1,y_2]=0\in T.\end{aligned}\tag{4.6}$$

If $b\notin C$, from (4.6) it follows

$$\Psi(x_1,x_2,y_1,y_2)^{n-1}\cdot\Big([x_1,x_2][y_1,y_2]-(c[x_1,x_2]+[x_1,x_2]u)\lambda[y_1,y_2]\Big)=0\in T$$

hence

$$\Psi(x_1,x_2,y_1,y_2)^{n-1}=0\in T.$$

As above, continuing this process we will have $\Psi(x_1, x_2, y_1, y_2) = 0 \in T$, then we return to (4.5), in which $u = \lambda b + \mu$, that is

$$\Big(a[x_1, x_2] - (c[x_1, x_2] + [x_1, x_2]u)c\Big)[y_1, y_2] +$$
$$\Big([x_1, x_2][y_1, y_2] - (c[x_1, x_2] + [x_1, x_2]u)\lambda[y_1, y_2]\Big)b +$$
$$-\Big(c[x_1, x_2] + [x_1, x_2]u\Big)\mu[y_1, y_2] = 0 \in T.$$

Since $b \notin C$, it follow that

$$[x_1, x_2][y_1, y_2] - (c[x_1, x_2] + [x_1, x_2]u)\lambda[y_1, y_2] = 0 \in T$$

implying the contradiction

$$[x_1, x_2] - \lambda(c[x_1, x_2] + [x_1, x_2]u) = 0 \in T.$$

We can therefore admit $b \in C$, so $u \in C$ as well. The (4.3) reduces to the

$$\Psi(x_1, x_2, y_1, y_2) = a'[x_1, x_2][y_1, y_2] - c'[x_1, x_2]c'[y_1, y_2] \tag{4.7}$$

where $a' = a + b$ and $c' = c + u$. So

$$\begin{aligned} &a'[x_1, x_2][y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1} + \\ &- c'[x_1, x_2]c'[y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1} = 0 \in T. \end{aligned} \tag{4.8}$$

If $\{a', c'\}$ were linearly C-independent we would have

$$c'[x_1, x_2]c'[y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1} = 0 \in T$$

hence

$$\Psi(x_1, x_2, y_1, y_2)^{n-1} = 0 \in T.$$

Continuing, as above, we get

$$\Psi(x_1, x_2, y_1, y_2) = 0 \in T$$

namely

$$a'[x_1, x_2][y_1, y_2] - c'[x_1, x_2]c'[y_1, y_2] = 0 \in T$$

hence

$$a'[x_1, x_2][y_1, y_2] = 0 \in T$$

which would be a contradiction.
So $c' = \eta a'$, for some $\eta \in C$, and (4.8) becomes

$$\begin{aligned} &a'\Big([x_1, x_2][y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1} - \\ &\eta^2[x_1, x_2]a'[y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1}\Big) = 0 \in T. \end{aligned} \tag{4.9}$$

If $a' \notin C$, from (4.9), it follows

$$[y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1} - \eta^2 a'[y_1, y_2] \cdot \Psi(x_1, x_2, y_1, y_2)^{n-1} = 0 \in T$$

which implies

$$\Psi(x_1, x_2, y_1, y_2)^{n-1} = 0 \in T.$$

As previously done, we get

$$\Psi(x_1, x_2, y_1, y_2) = 0 \in T$$

namely

$$a'[x_1, x_2][y_1, y_2] - \eta^2 a'[x_1, x_2]a'[y_1, y_2] =$$
$$a'[x_1, x_2](1 - \eta^2 a')[y_1, y_2] = 0 \in T$$

hence it follows

$$(1 - \eta^2 a')[y_1, y_2] = 0 \in T$$

which contradicts the fact that $a' \notin C$.
Therefore both $a' \in C$ and $c' \in C$. Now, by (4.7),

$$\Psi(x_1, x_2, y_1, y_2) = (a' - c'^2)[x_1, x_2][y_1, y_2].$$

It must be a trivial generalized polynomial identity for R, so that $a' = c'^2$, as required. □

Proof of Proposition 9

In light of the previous argument, we may consider the case when $\Phi(x_1, x_2, y_1, y_2)$ is a non-trivial generalized polynomial identity for R as well as for Q_r. In view of [22, Theorem 2.5 and Theorem 3.5], we know that both Q_r and $Q_r \otimes_C \overline{C}$ are centrally closed, where $\overline{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \otimes_C \overline{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [49], Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [33, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. Moreover we may assume both Q_r and H non-commutative, otherwise also R must be commutative. Notice that H satisfies the identity (4.2), see for example [40, proof of Theorem 1].
Since H is a simple ring then one of the following holds: either H does not contain any non-trivial idempotent element or H is generated by its idempotents.
In this last case, suppose that $e^2 = e \in H$ and let $[x_1, x_2] = [e, r(1-e)] = er(1-e)$, $[y_1, y_2] = [e, s(1-e)] = es(1-e)$, for $r, s \in H$. Hence, by relation (4.2), we have

$$\left\{ \left(cer(1-e) + er(1-e)u \right) \left(ces(1-e) + es(1-e)u \right) \right\}^n = 0. \tag{4.10}$$

Right multiplying by e, it follows $\left(er(1-e)ues(1-e)u \right)^n e = 0$. In particular, this implies $\left((1-e)uer \right)^{2n+1} = 0$, for any $r \in H$. Hence $(1-e)ueH$ is a nil right ideal of bounded index. Thus, by a well known result which is attribuited to Levitzki (for a proof see [24] and pp. 1-2 in [31]), $(1-e)uer = 0$ for any $r \in H$. By the

primeness of H it follows that $(1-e)ue = 0$, for any idempotent element $e \in H$, which implies $u \in C$.
On the other hand, if H does not contain any non-trivial idempotent element, then H is a finite dimensional division algebra over C and $u \in H = RC = Q$. Notice that if C is finite then H is a finite division ring, that is H is a commutative field and so R is commutative too, a contradiction.
Hence C is infinite and $H \otimes_C F \cong M_r(F)$, where F is a splitting field of H. In this case, a Vandermonde determinant argument shows that (4.2) is still an identiy for $M_r(F)$. As above one can see that u commutes with any idempotent element in $M_r(F)$, then $c \in F$.
Therefore, in any case, (4.2) reduces to

$$\left\{\left(a[x_1,x_2][y_1,y_2]+[x_1,x_2][y_1,y_2]b\right)-(c+u)[x_1,x_2](c+u)[y_1,y_2]\right\}^n. \tag{4.11}$$

Since Q_r is not commutative, we may assume $dim_C V \geq 2$. If we suppose there exists $v \in V$ such that $\{v, bv\}$ are linearly C-independent, then, by the density of Q_r, there exist $r_1, r_2, s_1, s_2 \in Q_r$ such that

$$r_1v = 0 \qquad r_2v = bv, \quad r_1bv = v$$

$$s_1v = v, \quad s_2v = v, \quad s_1bv = 0, \quad s_2bv = v$$

which implies the contradiction

$$0 = \left\{a[r_1,r_2][s_1,s_2]+[r_1,r_2][s_1,s_2]b-(c+u)[r_1,r_2](c+u)[s_1,s_2]\right\}^n v = v.$$

Therefore $\{v, bv\}$ are linearly C-dependent, for any $v \in V$, implying that $b \in C$ and Q_r satisfies

$$\left\{(a+b)[x_1,x_2][y_1,y_2]-(c+u)[x_1,x_2](c+u)[y_1,y_2]\right\}^n. \tag{4.12}$$

Once again, let $e^2 = e \in H$ and $[x_1,x_2] = [e, r(1-e)] = er(1-e)$, $[y_1,y_2] = [e, s(1-e)] = es(1-e)$, for $r, s \in H$. Therefore, by (4.12),

$$\left\{(c+u)er(1-e)(c+u)es(1-e)\right\}^n.$$

As above, $\left((1-e)(c+u)er\right)^{2n+1} = 0$, for any $r \in H$, implying that $(1-e)(c+u)e = 0$, for any idempotent element $e \in H$, that is $c+u \in C$. Moreover, by using the same above argument, if H does not contain any non-trivial idempotent element, then H is a finite dimensional division algebra over C and $c+u \in C$ follows again.
We have finally obtained the following identity for Q_r:

$$\left\{\left((a+b)-(c+u)^2\right)[x_1,x_2][y_1,y_2]\right\}^n. \tag{4.13}$$

Notice that, since Q_r is not commutative then the polynomial $[x_1,x_2][y_1,y_2]$ cannot be central valued on Q_r. Therefore $(a+b)-(c+u)^2 = 0$ follows from [48, Theorems 1 and 3].

Proof of Theorem 23

We consider now the more general situation and write $F(x) = ax + d(x)$, $G(x) = cx + \delta(x)$, for all $x \in R$, where $a, c \in Q_r$ and d, δ are derivations of R.
We also recall that, since L is not central and $char(R) \neq 2$, then there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$. Therefore, in light of the hypothesis Theorem 23 we have that there exists a non-central ideal I of R such that

$$\{F(uv) - G(u)G(v)\}^n = 0 \quad \forall u, v \in [I, I].$$

Since R and I satisfy the same generalized differential identities , we may assume that

$$\{F([x_1, x_2][y_1, y_2]) - G([x_1, x_2])G([y_1, y_2])\}^n \tag{4.14}$$

is an identity for R.

We remark that Proposition 9 shows that the result holds in the case the maps F and G are ordinary generalized inner derivations. The goal of this final step is to prove that the above mentioned cases are the only one that can actually arise. For this reason, in the rest of the proof we assume that d, δ are not simultaneously zero and also that d, δ are not simultaneously inner derivations, proving that a number of contradictions occurs. Moreover we may assume that R is not commutative.
By (4.14), R satisfies

$$\left\{\Big(a[x_1, x_2][y_1, y_2] + d([x_1, x_2][y_1, y_2])\Big) - \Big(c[x_1, x_2] + \delta([x_1, x_2])\Big)\Big(c[y_1, y_2] + \delta([y_1, y_2])\Big)\right\}^n \tag{4.15}$$

then R satisfies

$$\begin{aligned}&\Big\{a[x_1, x_2][y_1, y_2] + \Big(d(x_1)x_2 + x_1d(x_2) - d(x_2)x_1 - x_2d(x_1)\Big)[y_1, y_2]\\ &+ [x_1, x_2]\Big(d(y_1)y_2 + y_1d(y_2) - d(y_2)y_1 - y_2d(y_1)\Big)\\ &- \Big(c[x_1, x_2] + \delta(x_1)x_2 + x_1\delta(x_2) - \delta(x_2)x_1 - x_2\delta(x_1)\Big)\cdot\\ &\cdot\Big(c[y_1, y_2] + \delta(y_1)y_2 + y_1\delta(y_2) - \delta(y_2)y_1 - y_2\delta(y_1)\Big)\Big\}^n.\end{aligned} \tag{4.16}$$

In light of Lemma 11, in all that follows we can therefore always assume both $F \neq 0$ and $G \neq 0$.

Let $d \neq 0$ and $\delta \neq 0$ be linearly C-independent modulo D_{int}.

In this case, by (5.63), R satisfies

$$\begin{aligned}&\Big\{a[x_1,x_2][y_1,y_2]+\Big(t_1x_2+x_1t_2-t_2x_1-x_2t_1\Big)[y_1,y_2]\\&+[x_1,x_2]\Big(t_3y_2+y_1t_4-t_4y_1-y_2t_3\Big)\\&-\Big(c[x_1,x_2]+z_1x_2+x_1z_2-z_2x_1-x_2z_1\Big)\cdot\\&\cdot\Big(c[y_1,y_2]+z_3y_2+y_1z_4-z_4y_1-y_2z_3\Big)\Big\}^n.\end{aligned} \tag{4.17}$$

In particular, R satisfies $\{[t_1,x_2][y_1,y_2]\}^n$. In particular R satisfies $[t_1,x_2]^{2n}$, which is a contradiction, since we assume R is not commutative.

Let $d \neq 0$ and $\delta \neq 0$ be $C-$linearly dependent modulo D_{int}.

Here we assume that there exist $\lambda,\mu \in C$ and $q \in Q_r$ such that $\lambda d(x)+\mu\delta(x) = qx-xq$ for all $x \in R$.

- We firstly study the case $0 \neq \lambda \in C$ and $0 \neq \mu \in C$.

Denote $\eta = -\mu^{-1}\lambda$ and $p = \mu^{-1}q$. So $\delta(x) = \eta d(x)+px-xp$ for all $x \in R$. Therefore by (5.63), Q_r satisfies

$$\begin{aligned}&\Big\{a[x_1,x_2][y_1,y_2]+\Big(d(x_1)x_2+x_1d(x_2)-d(x_2)x_1-x_2d(x_1)\Big)[y_1,y_2]\\&+[x_1,x_2]\Big(d(y_1)y_2+y_1d(y_2)-d(y_2)y_1-y_2d(y_1)\Big)\\&-\Big(c[x_1,x_2]+(\eta d(x_1)+px_1-x_1p)x_2+x_1(\eta d(x_2)+px_2-x_2p)\\&-(\eta d(x_2)+px_2-x_2p)x_1-x_2(\eta d(x_1)+px_1-x_1p)\Big)\cdot\\&\cdot\Big(c[y_1,y_2]+(\eta d(y_1)+py_1-y_1p)y_2+y_1(\eta d(y_2)+py_2-y_2p)\\&-(\eta d(y_2)+py_2-y_2p)y_1-y_2(\eta d(y_1)+py_1-y_1p)\Big)\Big\}^n.\end{aligned} \tag{4.18}$$

In case d is an inner derivation of R, then δ is an inner derivation and we get a contradiction.

Thus we assume that d is not inner. By (4.18) Q_r satisfies

$$\begin{aligned}&\Big\{a[x_1,x_2][y_1,y_2]+\Big(t_1x_2+x_1t_2-t_2x_1-x_2t_1\Big)[y_1,y_2]\\&+[x_1,x_2]\Big(z_1y_2+y_1z_2-z_2y_1-y_2z_1\Big)\\&-\Big(c[x_1,x_2]+(\eta t_1+px_1-x_1p)x_2+x_1(\eta t_2+px_2-x_2p)\\&-(\eta t_2+px_2-x_2p)x_1-x_2(\eta t_1+px_1-x_1p)\Big)\cdot\\&\cdot\Big(c[y_1,y_2]+(\eta z_1+py_1-y_1p)y_2+y_1(\eta z_2+py_2-y_2p)\\&-(\eta z_2+py_2-y_2p)y_1-y_2(\eta z_1+py_1-y_1p)\Big)\Big\}^n.\end{aligned} \tag{4.19}$$

In particular, R satisfies $\eta^{2n}[t_1,x_2]^{2n}$. Since $\eta\neq 0$, it means R is commutative, a contradiction.

- Assume now $\lambda=0$.

Hence $\delta(x)=px-xp$ for all $x\in R$, where $p=\mu^{-1}q$ and d is not inner.
Then, by relation (5.63), Q_r satisfies

$$\begin{aligned}&\Big\{a[x_1,x_2][y_1,y_2]+\Big(d(x_1)x_2+x_1d(x_2)-d(x_2)x_1-x_2d(x_1)\Big)[y_1,y_2]\\&+[x_1,x_2]\Big(d(y_1)y_2+y_1d(y_2)-d(y_2)y_1-y_2d(y_1)\Big)\\&-\Big(c[x_1,x_2]+(px_1-x_1p)x_2+x_1(px_2-x_2p)-(px_2-x_2p)x_1-x_2(px_1-x_1p)\Big)\cdot\\&\cdot\Big(c[y_1,y_2]+(py_1-y_1p)y_2+y_1(py_2-y_2p)-(py_2-y_2p)y_1-y_2(py_1-y_1p)\Big)\Big\}^n.\end{aligned} \tag{4.20}$$

Since d is outer, it follows that Q_r also satisfies

$$\begin{aligned}&\Big\{a[x_1,x_2][y_1,y_2]+\Big(t_1x_2+x_1t_2-t_2x_1-x_2t_1\Big)[y_1,y_2]\\&+[x_1,x_2]\Big(z_1y_2+y_1z_2-z_2y_1-y_2z_1\Big)\\&-\Big(c[x_1,x_2]+(px_1-x_1p)x_2+x_1(px_2-x_2p)-(px_2-x_2p)x_1-x_2(px_1-x_1p)\Big)\cdot\\&\cdot\Big(c[y_1,y_2]+(py_1-y_1p)y_2+y_1(py_2-y_2p)-(py_2-y_2p)y_1-y_2(py_1-y_1p)\Big)\Big\}^n.\end{aligned} \tag{4.21}$$

In particular, R satisfies $[y_1,y_2]^{2n}$, which is a contradiction, as above.

- The case $\mu=0$

Here $d(x)=px-xp$ for all $x\in R$, where $p=\lambda^{-1}q$ and δ is not inner. Starting from (5.63), Q_r satisfies

$$\Big\{a[x_1,x_2][y_1,y_2]+\Big((px_1-x_1p)x_2+x_1(px_2-x_2p)-(px_2-x_2p)x_1-x_2(px_1-x_1p)\Big)[y_1,y_2]$$
$$+[x_1,x_2]\Big((py_1-y_1p)y_2+y_1(py_2-y_2p)-(py_2-y_2p)y_1-y_2(py_1-y_1p)\Big)$$
$$-\Big(c[x_1,x_2]+\delta(x_1)x_2+x_1\delta(x_2)-\delta(x_2)x_1-x_2\delta(x_1)\Big)\cdot$$
$$\cdot\Big(c[y_1,y_2]+\delta(y_1)y_2+y_1\delta(y_2)-\delta(y_2)y_1-y_2\delta(y_1)\Big)\Big\}^n. \tag{4.22}$$

Since δ is outer, one has that Q_r satisfies

$$\Big\{a[x_1,x_2][y_1,y_2]+\Big((px_1-x_1p)x_2+x_1(px_2-x_2p)-(px_2-x_2p)x_1-x_2(px_1-x_1p)\Big)[y_1,y_2]$$
$$+[x_1,x_2]\Big((py_1-y_1p)y_2+y_1(py_2-y_2p)-(py_2-y_2p)y_1-y_2(py_1-y_1p)\Big)$$
$$-\Big(c[x_1,x_2]+z_1x_2+x_1z_2-z_2x_1-x_2z_1\Big)\cdot$$
$$\cdot\Big(c[y_1,y_2]+z_3y_2+y_1z_4-z_4y_1-y_2z_3\Big)\Big\}^n. \tag{4.23}$$

In particular, R satisfies $[z_3,y_2]^{2n}$, a contradiction again.
In order to conclude the proof, we have to study the last two case: either $d=0$ or $\delta=0$.
Suppose firstly $d=0$ and δ not inner. In this case, by (5.63), R satisfies

$$\Big\{a[x_1,x_2][y_1,y_2]-\Big(c[x_1,x_2]+z_1x_2+x_1z_2-z_2x_1-x_2z_1\Big)\cdot$$
$$\cdot\Big(c[y_1,y_2]+z_3y_2+y_1z_4-z_4y_1-y_2z_3\Big)\Big\}^n. \tag{4.24}$$

In particular it follows the contradiction that R satisfies $[z_3,y_2]^{2n}$.
Finally suppose $\delta=0$ and d not inner. In this case, by (5.63), R satisfies

$$\Big\{a[x_1,x_2][y_1,y_2]+\Big(z_1x_2+x_1z_2-z_2x_1-x_2z_1\Big)[y_1,y_2]$$
$$+[x_1,x_2]\Big(z_3y_2+y_1z_4-z_4y_1-y_2z_3\Big)-\Big(c^2[x_1,x_2][y_1,y_2]\Big)\Big\}^n. \tag{4.25}$$

Thus, the contradiction that R satisfies $[y_1,y_2]^{2n}$ follows again. □

As consequences of Theorem 23, we have the following two corollaries:

Theorem 24. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, L a non-central Lie ideal of R, $n\geq 1$ a fixed integer, F a non-zero generalized derivation of R. If $\big(F(xy)-F(x)F(y)\big)^n=0$, for any $x,y\in L$, then F is the identity map on R.*

Theorem 25. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, $n \geq 1$ a fixed integer, F and G two generalized derivations of R. If $\big(F(xy) - G(x)G(y)\big)^n = 0$, for any $x, y \in R$, then there exists $\lambda \in C$ such that $F(x) = \lambda^2 x$ and $G(x) = \lambda x$, for any $x \in R$, unless when R is commutative.*

4.2 The semiprime case

The following example illustrates that Theorem 23 does not hold for semiprime rings.

Example 1. Let A be a non-commutative prime ring with unit. We define the sets

$$R = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \ : \ x, y \in A \right\},$$

$$L = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \ : \ x \in A \right\}.$$

It is very easy to prove that R is a prime ring and L is a non-central Lie ideal of R. Moreover, let $F : R \to R$ and $d : R \to R$ be two additive maps defined as:

$$d \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right],$$

$$F \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} + d \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & [q, y] \end{pmatrix}$$

for some $q \in A \setminus Z(A)$.
Notice that d is a derivation of R and F is a generalized derivation of R associated with d, that is $F(x) = ax + d(x)$, for all $x \in \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, with $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.
Now, we define $R' = R \oplus M_2(K)$, where K is a field, and consider the maps

$$d' : R \oplus M_2(K) \to R \oplus M_2(K)$$

$$d'(X, Y) = \big(d(X), 0\big) \quad \forall X \in R,\ Y \in M_2(K)$$

and

$$F' : R \oplus M_2(K) \to R \oplus M_2(K)$$

$$F'(X, Y) = \big(F(X), 0\big) \quad \forall X \in R,\ Y \in M_2(K).$$

By suitable calculations, we get that for all $X_1, X_2 \in R$ and $Y_1, Y_2 \in M_2(K)$:

$$d'\Big((X_1, Y_1) \cdot (X_2, Y_2)\Big) = d'(X_1, Y_1) \cdot (X_2, Y_2) + (X_1, Y_1) \cdot d'(X_2, Y_2)$$

and

$$F'\Big((X_1, Y_1) \cdot (X_2, Y_2)\Big) = F'(X_1, Y_1) \cdot (X_2, Y_2) + (X_1, Y_1) \cdot d'(X_2, Y_2)$$

that is F' is a generalized derivation of R associated with d'. Moreover, if we consider $a' = (a, 0) \in R \oplus M_2(K)$, we can notice that $F'(X) = a'X + d'(X)$, for all $X \in R \oplus M_2(K)$.
Let $N = [M_2(K), M_2(K)]$ be the additive subgroup of $M_2(K)$ generated by $\{[X, Y] : X, Y \in M_2(K)\}$. Since N is a non-central Lie ideal of $M_2(K)$, we obtain that (L, N) is

a non-central Lie ideal of R'.
In particular, for all $X = (X_1, Y_1), Y = (X_2, Y_2) \in (L, N)$, we have

$$F'(X) = F'(X_1, Y_1) = (F(X_1), 0) = (X_1, 0)$$

$$F'(Y) = F'(X_2, Y_2) = (F(X_2), 0) = (X_2, 0)$$

$$F'(XY) = F'\Big((X_1, Y_1)(X_2, Y_2)\Big) = F'\Big((X_1X_2, Y_1Y_2)\Big) = (F(X_1X_2), 0) = (X_1X_2, 0)$$

that is

$$F'(XY) - F'(X)F'(Y) = 0.$$

Therefore the condition of the Theorem 24 is satisfied by the elements of the non-central Lie ideal (L, N) of the semiprime ring R'. Since F' is not the identity map, we get that the previous theorem does not hold for semiprime rings.

In light of the previous example, we now consider the case when R is semiprime of characteristic different from 2.

Theorem 26. *Let R be a semiprime ring of characteristic different from 2, L a Lie ideal of R, F a generalized derivation of R such that $\Big(F(xy) - F(x)F(y)\Big)^n = 0$, for all $x, y \in L$, with $n > 1$ a fixed integer. If $F(x) = ax + d(x)$, for all $x \in R$, for some $a \in Q_r$, where d is a derivation of R, then one of the following holds:*

1. *$L \subseteq Z(R)$;*
2. *$d(L) = (0)$, $(a - a^2)^n = 0$ and $[a, L] = (0)$;*
3. *R contains a central ideal generated by $d(L)$ and $[L, d(R)] = (0)$.*

Remark 7. If one refers to the Example 1, it is possible to notice that the ring introduced in the example in question is semiprime, the generalized derivation F' introduced in the same example satisfies the assumptions of Theorem 26 and furthermore let the element a' that the derivation d', through which the generalized derivation F' is constructed, satisfy the conditions specified in conclusion 2 of Theorem 26, since

- $a' - a'^2 = 0$
- $d'((L, N)) = (d(L), 0) = (0, 0)$
- $[a', (L, N)] = [(a, 0), (L, N)] = ([a, L], [0, N]) = (0, 0)$

The following example demonstrates that R to be semiprime is essential in the hypothesis of Theorem 26.

Example 2. Let A be a non-commutative ring with unit. We define the following ring:

$$R = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in A \right\}.$$

Notice that, for all $X = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot X = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It implies that R is not semiprime. Consider the following set

$$L = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \ : \ x \in A \right\}.$$

It is a non-central Lie ideal. Let $F : R \longrightarrow R$ and $d : R \longrightarrow R$ be two maps so defined:

$$d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$$

$$F\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 2y \\ 0 & 0 \end{pmatrix}.$$

F is a generalized derivation of R with d associated derivation. Then, the following hold:

- $d(L) \neq 0$ and $F(L) \neq 0$
- $XY = 0$, for all $X, Y \in L$
- $F(X)F(Y) = 0$, for all $X, Y \in L$.

They imply that R satisfies the hypothesis of the Theorem 26, even if none of the conclusions occurs.

In this final section we then consider the case when R is a semiprime ring of characteristic different from 2, F is a generalized derivation of R and L a Lie ideal of R such that $\Big(F(xy) - F(x)F(y)\Big)^n = 0$, for all $x, y \in L$, with $n > 1$ a fixed integer. We can assume that there exist $a \in Q_r$ and d a derivation of R such that $F(x) = ax + d(x)$, for all $x \in R$, so that L satisfies

$$\Big(axy + d(xy) - axay - axd(y) - d(x)ay - d(x)d(y)\Big)^n. \tag{4.26}$$

Proof of Theorem 26: Let P be a prime ideal of R such that R/P has characteristic different from 2. Suppose $[L, R] \not\subseteq P$ and denote $\overline{R} = R/P$ and $\overline{L} = L/P$; in this case, $[\overline{L}, \overline{R}] \neq \overline{0}$.
If $d(P) \subseteq P$, then F induces a canonical generalized derivation $\overline{F}$ on $\overline{R}$; therefore, by main assumption of this section, $(\overline{F}(\overline{xy}) - \overline{F}(\overline{x})\overline{F}(\overline{y}))^n = \overline{0}$, for all $\overline{x}, \overline{y} \in \overline{L}$. Since $\overline{L}$ is a non-central Lie ideal of the prime ring $\overline{R}$, by Theorem 23, $\overline{d}(\overline{R}) = 0$ and either $\overline{a}\overline{R} = \overline{0}$ or $\overline{ax} - \overline{x} = \overline{0}$.
On the other hand, since L satisfies (4.26), if $d(P) \not\subseteq P$, for $x = y = [x_1 p, x_2]$, with $x_1 \in R$, $p \in P$ and $x_2 \in L$, we get

$$\left[\overline{x_1}\overline{d(p)}, \overline{x_2}\right]^{2n} = \overline{0} \tag{4.27}$$

for all $x_1 \in R$ and $x_2 \in L$, that is

$$\left[\overline{R}\overline{d(P)}, \overline{L}\right]^{2n} = \overline{0}. \tag{4.28}$$

Since P is a prime ideal, we can assume that $Rd(P) \not\subseteq P$ otherwise either $R \subseteq P$ or $d(P) \subseteq P$ (in both cases a contradiction); in this case, by (4.28), since $\overline{R}$ is a prime

ring and $\overline{R}\overline{d(P)}$ is its ideal, we get $[\overline{L},\overline{L}] = (\overline{0})$, that is $[L,R] \subseteq P$, a contradiction. Therefore, for any prime ideal P, we have the following cases:
Case 1: $d(P) \subseteq P$. In this case, $d(R) \subseteq P$ and, since $d(L) \subseteq d(R)$, we obtain $[d(L),R] \subseteq P$.
Case 2: $[L,R] \subseteq P$. By replacing $x = y[u,r]$ in (4.26), for all $u \in L, r \in R$, we get

$$\left(\overline{d}([\overline{u},\overline{r}])\right)^{2n} = \overline{0}.$$

By our assumption, it implies $[\overline{d}(\overline{u}),\overline{r}] = \overline{0}$, that is $[\overline{d}(\overline{L}),\overline{R}] = \overline{0}$, therefore, also in this case, $[d(L),R] \subseteq P$.
Therefore, for all prime ideal P of R, $[d(L),R] \subseteq P$, that is $[d(L),R] = (0)$. Firstly, assume $d(L) \neq (0)$, then R contains a central ideal generated by $d(L)$. Moreover, both Case 1 and Case 2 lead to $[L,R]d(R) \subseteq P$, for all prime ideal P of R, that is $[L,R]d(R) = (0)$; it implies $[L,d(R)] = (0)$.
On the other hand, if $d(L) = (0)$, the identity (4.26) becomes

$$\left(axy - axay\right)^{n} = 0 \tag{4.29}$$

for all $x,y \in L$. As above, if we denote $\overline{R} = R/P$ and $\overline{L} = L/P$, for a prime ideal P of R, it follows

$$\left(\overline{axy} - \overline{axay}\right)^{n} = \overline{0} \tag{4.30}$$

for all $x,y \in \overline{L}$. If $\overline{L} \subseteq Z(\overline{R})$, then

$$(\overline{a} - \overline{a}^2)^n(\overline{xy})^n = \overline{0}$$

for all $x,y \in \overline{L}$. In this case, $(\overline{a} - \overline{a}^2)^n = \overline{0}$, that is $(a - a^2)^n \in P$.
If $\overline{L} \nsubseteq Z(\overline{R})$, then, by Proposition 9, $\overline{a} \in Z(\overline{R})$ and $\overline{a} = \overline{a}^2$, that is $[a,R] \subseteq P$ and $(a - a^2) \in P$.
Therefore, in any case, for all prime ideal P in R, $(a - a^2)^n \subseteq P$ and either $[L,R] \subseteq P$ or $[a,R] \subseteq P$. It means that, for all prime ideal P of R, we have $(a - a^2)^n \subseteq P$ and $[a,L] \subseteq P$, so that we conclude $[a,L] = (0)$ and $(a - a^2)^n = 0$, as required.

Chapter 5

Nil sets defined through generalized skew derivations having Jordan-like behavior.

Recently, in [39], Koşan and Lee extended the concept of generalized derivations and introduced the definition of b-generalized derivations, with associated derivation d. More precisely, F is a *b-generalized derivation* of R if $F(xy) = F(x)y + bxd(y)$, for all $x, y \in R$, where d is a derivation of R and $b \in Q_r$.
Later, in [19] and [18], the previous definitions are generalized by the concept of *X-generalized skew derivation* $F : R \to R$, with associated term (b, α, d), that is, that additive map F on R such that

$$F(xy) = F(x)y + b\alpha(x)d(y)$$

for all $x, y \in R$, where d is a skew derivation of R with associated automorphism α of R and $b \in Q_r$.
Another class of maps, related to the concept of derivations, is that of Jordan derivations. We say that the additive map $d : R \mapsto R$ is a *Jordan derivation* if $d(x^2) = d(x)x + xd(x)$ holds for all $x \in R$. A well known result by Herstein [29, Theorem 3.1] shows that if R is a prime ring with characteristic different from 2, derivations and Jordan derivations coincide. In [3], Awtar generalizes Herstein's result to Lie ideals. More precisely, if R is a prime ring with characteristic not two, L is a Lie ideal of R and d is an additive mapping such that $d(x^2) = d(x)x + xd(x)$ holds for all $x \in L$, then d is a derivation of R. Recently, in [47, Theorem 2.2] T.K. Lee and J.H. Lin gave a complete characterization of Jordan derivations of prime rings, proving that, for a prime ring R, an additive map $\delta : R \mapsto Q_r$ is a Jordan derivation if and only if there exist a derivation $d : R \mapsto Q_r$ and an additive map $\mu : R \mapsto C$ such that $\delta = d + \mu$ and $\mu(x^2) = 0$, for all $x \in R$. In literature, generalized derivations and generalized skew derivations acting as Jordan-like maps in prime rings have been extensively studied by several authors.
For instance, let $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C and S the set of all the evaluations of $f(x_1, \ldots, x_n)$ in R. In [53], Prajapati, Tiwari and Gupta describe the forms of b-generalized derivations F, G, H of R, under the assumption that $F(x^2) - G(x)x - xH(x) = 0$, for any $x \in S$. More precisely:

Theorem. *Let R be prime ring of characteristic different from 2, Q_r be right Martindale quotient ring of R with extended centroid C and $f(x_1, \ldots, x_n)$ be a non-central multilinear polynomial over C. Suppose that F, G and H are b-generalized derivations on R such that*

$$F(u)u - uG(u) = H(u^2)$$

for all $u \in f(R^n)$ then one of the following holds:

(1) there exist $\alpha \in C$, $a, c, c' \in Q_r$ such that $F(x) = ax + xc + \alpha x$, $G(x) = cx + xc'$ and $H(x) = ax - xc' + \alpha x$ for all $x \in R$,

(2) there exist $a \in Q_r$, $c \in C$ such that $F(x) = ax$, $G(x) = cx$ and $H(x) = (a - c)x$ for all $x \in R$,

(3) $f(x_1, \dots, x_n)^2$ is central valued on R and one of the following holds:

(a) there exist $\alpha \in C$, $a, c, c' \in Q_r$ such that $F(x) = ax + xc + \alpha x$, $G(x) = cx + xc'$ and $H(x) = [t, x] + x(\alpha + a - c')$ for all $x \in R$,

(b) there exist $\alpha' \in C$, $a', b, w \in Q_r$ such that $F(x) = a'x$, $G(x) = \alpha' x$ and $H(x) = b[x, w] + (a' - \alpha')x$ for all $x \in R$.

Recently, in [52] Pandey and Scudo study the complete structure of X-generalized skew derivations F and H of R, in the case $H(x^2) - F(x)x - xF(x) = 0$, for any $x \in S$. More precisely:

Theorem. *Let R be prime ring of characteristic different from 2, Q_r be right Martindale quotient ring of R, C the extended centroid of R, $\alpha \in Aut(R)$, d, δ two skew derivations of R with associated automorphism α. Let F and H be b'-generalized skew derivations on R with associated term (b', α, d) and (b', α, δ) respectively, where α commutes with d and δ. Let $f(x_1, \dots, x_n)$ be a non-central multilinear polynomial of R such that $F(u)u + uF(u) = H(u^2)$ for all $u = f(r_1, \dots, r_n), r_1, \dots, r_n \in R$ then one of the following holds:*

(1) $F = H = 0$.

(2) there exist $a, c \in C$ such that $F(x) = ax$ and $H(x) = cx$ for all $x \in R$ with $2a = c$.

(3) there exist $a, b, c, d \in Q_r$ such that $F(x) = ax + xb$ and $H(x) = cx + xd$ for all $x \in R$ with $a - c$, $b - d$, $a + b \in C$ and $2(a + b) = c + d$.

(4) there exist $a, c \in C$ with $2a = c$ and $b' \in Q_r$ such that $F(x) = ax + b'd'(x)$ and $H(x) = cx + b'd(x)$ for all $x \in R$, where $d(x)$ is a skew derivation associated with automorphism α such that $b'\alpha(x) = xb'$ for all $x \in R$.

(5) $f(x_1, \dots, x_n)^2$ is central valued on R and there exist $a, b, c, d \in Q_r$ such that $F(x) = ax + xb$ and $H(x) = cx + xd$ for all $x \in R$ with $a + b \in C$ and $2(a + b) = c + d$.

Then in [56] Gupta and Tiwari analyze an identity involving three different X-generalized skew derivations F, G, H of R, that generalizes the concept of Jordan derivation. More precisely they give all possible forms of the maps F, G, H in the case $F(x^2) - G(x)x - xH(x) = 0$, for any $x \in S$. More precisely:

Theorem. *Suppose R is a prime ring with a characteristic not equal to 2, and Q_r and C represent the right Martindale quotient ring of R and the extended centroid of R, respectively. If $D_1(P)P + PD_2(P) = D_3(P^2)$ for every $P = \rho(\eta_1, \dots, \eta_n)$ where $\eta_1, \dots, \eta_n \in R$, and D_1, D_2, D_3 are three b-generalized skew derivations on R, and $\rho(\eta_1, \dots, \eta_n)$ is a non-central multilinear polynomial over C, then one of the following scenarios applies:*

(i) there exist $w_1, w_2, w_3, w_4 \in Q_r$ such that $D_1(x) = w_1x + xw_2$, $D_2(x) = w_3x$ and $D_3(x) = w_4x$ with $w_2 + w_3 = -(w_1 - w_4) \in C$;

(ii) there exist $w_1, w_2, w_3, w_4 \in Q_r$ and $\lambda_1, \lambda_2 \in C$ such that $D_1(x) = w_1x + xw_2$, $D_2(x) = (\lambda_1 - w_2)x + xw_4$ and $D_3(x) = (w_1 - \lambda_2)x + xw_3$ with $\lambda_1 + \lambda_2 = w_3 - w_4$;

(iii) *there exist* $w_1, w_3 \in Q_r$, $w_2 \in C$ *such that* $D_1(x) = w_1x$, $D_2(x) = xw_2$ *and* $D_3(x) = w_3x$ *with* $w_1 - w_3 = -w_2 \in C$;

(iv) *there exist* $w_1, w_2, w_3, w_4 \in Q_r$ *such that* $D_1(x) = w_1x$, $D_2(x) = xw_2$ *and* $D_3(x) = w_3x + xw_4$ *with* $w_1 - w_3 = w_2 + w_4 \in C$;

(v) $\rho(\eta_1, \dots, \eta_n)^2$ *is central valued on R, then one of the following holds*

  (a) *there exist* $w_1, w_2, w_3, w_4, q, b, p, w \in Q_r$ *such that* $D_1(x) = w_1x + xw_2$, $D_2(x) = w_3x + xw_4$ *and* $D_3(x) = qx + bpxp^{-1}$ *with* $w_2 + w_3 = q + bw - w_1 - w_4 \in C$;

  (b) *there exist* $w_1, w_2, w_3, w_4, q, b, p, w \in Q_r$ *such that* $D_1(x) = w_1x + xw_2$, $D_2(x) = w_3x + xw_4$ *and* $D_3(x) = qx + bpxp^{-1}$ *with* $w_2 + w_3 = bw - w_1 + w_4 \in C$;

  (c) *there exist* $w_1, w_2, w_3, b, p, w \in Q_r$ *such that* $D_1(x) = w_1x$, $D_2(x) = xw_2$ *and* $D_3(x) = w_3x + bpxp^{-1}w$ *with* $w_1 + +w_2 - w_3 - bw = 0 \in C$;

  (d) *there exist* $w_1, w_2, w_3, p \in Q_r$ *such that* $D_1(x) = w_1x - xp + w_2h(x) + [w_3, x]$, $D_2(x) = px + [w_3, x] + w_2h(x)$ *and* $D_3(x) = w_1x + w_2h(x)$ *where h is a skew derivation associated with the inner automorphism* α *satisfying* $w_2\alpha(x) = xw_2$;

(vi) *there exist* $w_1, w_2, p \in Q_r$ *such that* $D_1(x) = w_1x - xp + w_2h(x)$, $D_2(x) = px + w_2h(x)$ *and* $D_3(x) = w_1x + w_2h(x)$ *where h is a skew derivation associated with the inner automorphism* α *satisfying* $w_2\alpha(x) = xw_2$.

More recently, in [51], the description of two generalized derivations F, G of R is provided, in the case $F(x^2) - G(F(x)x + xF(x)) = 0$, for any $x \in S$. In particular, the result in [51] is the following:

Theorem. *Let R be a prime ring of characteristic different from* 2, Q_r *its right Martindale quotient ring, C its extended centroid and* $f(x_1, \dots, x_n)$ *a multilinear polynomial over C, which is not central valued on R. Suppose that F and G are two generalized derivations of R, such that* $F \neq 0$ *and*

$$F(f(r)^2) = G(F(f(r))f(r) + f(r)F(f(r)))$$

for all $r = (r_1, \dots, r_n) \in R^n$. *Then one of the following holds:*

(1) *there exist* $\lambda, \mu \in C$ *such that* $F(x) = \lambda x$, $G(x) = \mu x$, *for any* $x \in R$, *with* $2\mu = 1$;

(2) *there exist* $p \in C$ *and* $\alpha, \beta \in C$ *such that* $F(x) = [p, x]$ *and* $G(x) = \alpha x + \beta(px + xp)$, *for any* $x \in R$, *with* $\beta p^2 + (\alpha - 1)p \in C$;

(3) $f(x_1, \dots, x_n)^2 \in C$ *and one of the following holds:*

  (i) *there exists* $p \in Q_r$ *such that* $F(x) = [p, x]$, *for any* $x \in R$;

  (ii) *there exist* $p, u \in Q_r$ *and* $\lambda, \mu \in C$ *such that* $F(x) = [p, x] + \lambda x$, $G(x) = [u, x] + \mu x$, *for all* $x \in R$, *with* $2\mu = 1$;

  (iii) *there exist* $u \in Q_r$ *and an outer derivation d of R such that* $F(x) = d(x)$, $G(x) = x + [u, x]$, *for all* $x \in R$;

  (iv) $G = 0$ *and there exists* $q \in Q_r$ *such that* $F(x) = [q, x]$, *for any* $x \in R$;

(4) $R \subseteq M_2(K)$, *the ring of* 2×2 *matrices over a field K, and there exists* $p \in Q_r$ *such that* $F(x) = [p, x]$, *for any* $x \in R$;

(5) G is the identity map on R and F is an ordinary derivation of R.

Here we intend to continue in this direction, thus following the line of research related to the study of additive maps whose behavior generalizes that of Jordan derivations. More specifically, our aim is to study nil sets, defined through generalized skew derivations that emulate the behavior of Jordan derivations. We will prove the following result:

Theorem. *Let R be a prime ring of characteristic different from $2, 3$, Q_r its right Martindale quotient ring, C its extended centroid, F and G nonzero generalized skew derivations of R, associated with the same automorphism α of R and $n \geq 1$ a fixed integer. If*

$$\Big\{F(x^2) - G(x)x - xG(x)\Big\}^n = 0$$

for all $x \in R$, then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one of the following holds:

(1) $F(x) = G(x) = [x, p]$, for all $x \in R$.

(2) $\exists \eta \in C$, $\eta \neq 0$ such that $F(x) = [x, p] + 2\eta x$ and $G(x) = [x, p] + \eta x$, for all $x \in R$.

In order to demonstrate the main result of this chapter, we will first need to prove an intermediate one, referring to the case where the generalized skew derivations are in particular generalized derivations. We will dedicate the first next section to this reduced case.

5.1 The case of generalized derivations acting on a Lie ideal.

In this first Section we will prove the following result:

Theorem 27. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, F and G nonzero generalized derivations of R, L a non-central Lie ideal of R and $n \geq 1$ a fixed integer. If*

$$\Big\{F(x^2) - G(x)x - xG(x)\Big\}^n = 0 \tag{5.1}$$

for all $x \in L$, then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one of the following holds:

(1) $F(x) = G(x) = [x, p]$, for all $x \in R$.

(2) $\exists \eta \in C$, $\eta \neq 0$ and $\exists d : R \to R$ derivation such that $F(x) = d(x) + 2\eta x$ and $G(x) = d(x) + \eta x$, for all $x \in R$.

We start by studying the case when $L = [R, R]$ and both the derivations F and G are inner generalized derivations, respectively defined as follows:

$$F(x) = ax + xb \quad \text{and} \quad G(x) = cx + xp \tag{5.2}$$

where a, b, c, p are fixed element of Q_r.
In what follows, we denote

$$\begin{aligned} \Psi(x_1, x_2) = \Big\{F([x_1, x_2]^2) - G([x_1, x_2])[x_1, x_2] - [x_1, x_2]G([x_1, x_2])\Big\}^n = \\ \Big\{(a - c)[x_1, x_2]^2 - [x_1, x_2](p + c)[x_1, x_2] + [x_1, x_2]^2(b - p)\Big\}^n \end{aligned} \tag{5.3}$$

and assume that R satisfies the generalized identity $\Psi(x_1, x_2)$.

Lemma 13. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, F and G nonzero inner generalized derivations of R, defined as in (5.2). If R does not satisfy any nontrivial generalized polynomial identity, then there exists $p \in C$ such that one of the following holds:*

(1) $F(x) = G(x) = [x, p]$, for all $x \in R$.

(2) $\exists \eta \in C, \eta \neq 0$ such that $F(x) = [x, p] + 2\eta x$ and $G(x) = [x, p] + \eta x$, for all $x \in R$.

Proof. Assume that $\Psi(x_1, x_2)$ is a trivial generalized polynomial identity for R. Let $T = Q_r *_C C\{X\}$ be the free product over C of the C-algebra Q_r and the free C-algebra $C\{X\}$, with X the set consisting of non-commuting indeterminates x_1, x_2. By our hypothesis, $\Psi(x_1, x_2) = 0 \in Q_r *_C C\{X\}$.
We denote

$$\Phi(x_1,x_2) = F([x_1,x_2]^2) - G([x_1,x_2])[x_1,x_2] - [x_1,x_2]G([x_1,x_2] =$$
$$(a-c)[x_1,x_2]^2 - [x_1,x_2](p+c)[x_1,x_2] + [x_1,x_2]^2(b-p)$$

hence

$$\Psi(x_1,x_2) = \Phi(x_1,x_2)^n$$

whence R satisfies

$$\begin{aligned}&\Phi(x_1,x_2)^{n-1}\cdot(a-c)[x_1,x_2]^2+\\&\Phi(x_1,x_2)^{n-1}\cdot[x_1,x_2](p+c)[x_1,x_2]+\\&\Phi(x_1,x_2)^{n-1}\cdot[x_1,x_2]^2(b-p)=0\in T.\end{aligned}$$

This implies that $\{b-p, 1\}$ is linearly C-dependent or

$$\begin{cases}\Phi(x_1,x_2)^{n-1}[x_1,x_2]^2(b-p)=0\in T\\ \Phi(x_1,x_2)^{n-1}(a-c)[x_1,x_2]^2-\Phi(x_1,x_2)^{n-1}[x_1,x_2](p+c)[x_1,x_2]=0\in T.\end{cases}$$

If $b - p \in C$, then $\Phi(x_1,x_2)^{n-1}\cdot(a-c)[x_1,x_2]^2 - [x_1,x_2](2p+c-b)[x_1,x_2]$.
So

$$\Phi(x_1,x_2)^{n-1}\cdot(a-c)[x_1,x_2]^2-\Phi(x_1,x_2)^{n-1}\cdot[x_1,x_2](2p+c-b)[x_1,x_2]=$$
$$\Big\{\Phi(x_1,x_2)^{n-1}\cdot(a-c)[x_1,x_2]-\Phi(x_1,x_2)^{n-1}\cdot[x_1,x_2](2p+c-b)\Big\}[x_1,x_2]=0\in T.$$

Hence

$$\Phi(x_1,x_2)^{n-1}\cdot(a-c)[x_1,x_2]-\Phi(x_1,x_2)^{n-1}\cdot[x_1,x_2](2p+c-b)=0\in T.$$

This implies that $2p + c - b \in C$ or

$$\begin{cases}\Phi(x_1,x_2)^{n-1}[x_1,x_2](a-c)=0\in T\\ \Phi(x_1,x_2)^{n-1}[x_1,x_2]=0\in T\Rightarrow\Phi(x_1,x_2)^{n-1}=0\in T.\end{cases}$$

If $\Phi(x_1,x_2)^{n-1}[x_1,x_2]^2(b-p)$ is a trivial GPI, then $\Phi(x_1,x_2)^{n-1} = 0 \in T$. By reiterating this procedure $n-1$ times, we obtain that

$$\Phi(x_1,x_2)=0\in T\ \ or\ \ \ b-p,2p+c-b\in C.$$

If $\Phi(x_1, x_2) = 0 \in T$, then

$$(a-c)[x_1,x_2]^2 - [x_1,x_2](p+c)[x_1,x_2] + [x_1,x_2]^2(b-p) = 0 \in T.$$

Furthermore,

(a) $\{b-p, 1\}$ is linearly C-dependent $\Rightarrow b-p \in C$

(b1)

$$[x_1,x_2]^2(b-p) = 0 \in T \Rightarrow b-p = 0 \in T$$

and

(b2)

$$\{(a-c)[x_1,x_2]^2 - [x_1,x_2](p+c)[x_1,x_2] + [x_1,x_2]^2(b-p)\}[x_1,x_2] = 0 \in T.$$

In any case, $b-p \in C$, then

$$(a-c)[x_1,x_2]^2 - [x_1,x_2](2p+c-b)[x_1,x_2] = 0 \in T$$
$$(a-c)[x_1,x_2] - [x_1,x_2](2p+c-b) = 0 \in T$$

therefore, $2p+c-b \in C$.
Since $b-p \in C$ and $2p+c-b \in C$, then

$$(a-2c-2p+b)[x_1,x_2]^2 = 0 \in T$$

$$a-2c-2p+b = 0 \in T. \tag{5.4}$$

Furthermore,

$$b-p \in C \Rightarrow b = p+\lambda \ \ (\lambda \in C) \tag{5.5}$$

$$2p+c-b \in C \Rightarrow c = b-2p+\mu = p+\lambda-2p+\mu = \lambda+\mu-p \ \ (\lambda,\mu \in C). \tag{5.6}$$

Considering the definitions of F and G, we obtain

$$F(x) = ax + xp + \lambda x$$
$$G(x) = -px + xp + (\lambda+\mu)x = [x,p] + \eta x$$

with $\eta = \lambda + \mu$.
In particular, if $\eta = 0$ then $c = -p$ for (5.6) and $a = -b$ for (5.4). So, $F(x) = -bx + xb = [x,b] \underset{(5.5)}{=} [x, p+\lambda] \underset{\lambda \in C}{=} [x,p]$ and $G(x) = -px + xp = [x,p]$.
Now, we assume $\eta \neq 0$, then

$$a-2c-2p+b = a+2p-2\eta-2p+p+\lambda = a+p-\lambda-2\mu = 0,$$

so,

$$a+p = \lambda+2\mu.$$

By the definition of Φ, we have

$$a[x_1,x_2]^2 + [x_1,x_2]^2p + \lambda[x_1,x_2]^2 - ([x_1,x_2]p - p[x_1,x_2]) +$$
$$-(\lambda+\mu)[x_1,x_2]^2 - [x_1,x_2]([x_1,x_2]p - p[x_1,x_2]) - (\lambda+\mu)[x_1,x_2]^2 =$$
$$(a+p)[x_1,x_2]^2 + [x_1,x_2]^2(\lambda-\lambda-\mu-\lambda-\mu) =$$
$$(a+p)[x_1,x_2]^2 - [x_1,x_2]^2(\lambda+2\mu).$$

We observe that

$$F(x) = ax + xp + \lambda x =$$
$$(\lambda + 2\mu - p)x + xp + \lambda x = [x, p] + 2(\lambda + \mu)x.$$

So,

$$F(x) = [x, p] + 2\eta x$$
$$G(x) = [x, p] + \eta x.$$

□

Remark 8. In light of Lemma 13, we can suppose that (5.18) is a nontrivial generalized polynomial identity for R.

Proposition 10. *Let R be a prime ring of characteristic different from 2, Q_r its right Martindale quotient ring, C its extended centroid, F and G nonzero inner generalized derivations of R, defined as in (5.2). Let $n \geq 1$ be a fixed integer such that*

$$\left\{F(x^2) - G(x)x - xG(x)\right\}^n = 0$$

for all $x \in [R, R]$, then

- *either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, with $c + p = \eta \in Z(R)$ and $(a + b - 2\eta)^2 = 0$;*
- *or there exists $\mu \in C$ such that $F(x) = [x, p] + 2\mu x$, $G(x) = [x, p] + \mu x$, for any $x \in R$.*

Proof. Since Q_r and R satisfy the same generalized polynomial identities with coefficients in Q_r , the relation (5.18) is also a non-trivial generalized polynomial identity for Q_r. In case C is infinite, we have $\Psi(r_1, r_2) = 0$ for all $r_1, r_2 \in Q_r \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q_r and $Q_r \otimes_C \overline{C}$ are centrally closed (Theorems 2.5 and 3.5 in [22]), we may replace R by Q_r or $Q_r \otimes_C \overline{C}$ according as C is finite or infinite. Thus, without loss of generality, we may consider the case when R is centrally closed over C which is either finite or algebraically closed and $\Psi(r_1, r_2) = 0$ for all $r_1, r_2 \in R$. By Martindale's theorem [49], R is a primitive ring having a nonzero socle with C as the associated division ring. In light of Jacobson's theorem (p. 75 in [34]), R is isomorphic to a dense ring of linear transformations on some vector space V over C.
In case $dim_C V = 2$, then $Q_r \cong M_2(C)$ and (5.18) reduces to

$$\left\{[x_1, x_2]^2(a - c + b - p) + [x_1, x_2](-p - c)[x_1, x_2]\right\}^2. \tag{5.7}$$

Here we denote $A = a - c + b - p = \sum a_{ij}e_{ij}$, $B = -p - c = \sum b_{ij}e_{ij}$, where $a_{ij}, b_{ij} \in C$ and e_{ij} is the usual matrix unit with 1 in the (i, j) entry and zero elsewhere.
Since the matrix $M = [x_1, x_2]^2 A + [x_1, x_2]B[x_1, x_2]$ is nilpotent in $M_2(C)$, it is not invertible and its determinant is zero. We use this condition a number of times to prove our result in this case.
For $[x_1, x_2] = [e_{12} + e_{21}, e_{22}] = e_{12} - e_{21}$ we have

$$M = \begin{bmatrix} -a_{11} - b_{22} & -a_{12} + b_{21} \\ -a_{21} + b_{12} & -a_{22} - b_{11} \end{bmatrix}$$

and

$$0 = det(M) = (a_{11} + b_{22})(a_{22} + b_{11}) - (b_{21} - a_{12})(b_{12} - a_{21}). \tag{5.8}$$

Again for $[x_1, x_2] = [e_{12} - e_{21}, e_{22}] = e_{12} + e_{21}$ we have

$$M = \begin{bmatrix} a_{11} + b_{22} & a_{12} + b_{21} \\ a_{21} + b_{12} & a_{22} + b_{11} \end{bmatrix} \tag{5.9}$$

and

$$0 = det(M) = (a_{11} + b_{22})(a_{22} + b_{11}) - (b_{21} + a_{12})(b_{12} + a_{21}). \tag{5.10}$$

By (1) and (3) we get $2(a_{12}b_{12} + a_{21}b_{21}) = 0$ and since $char(R) \neq 2$ it follows

$$a_{12}b_{12} + a_{21}b_{21} = 0. \tag{5.11}$$

Let now $[x_1, x_2] = [e_{12} + 2e_{21}, -e_{22}] = -e_{12} + 2e_{21}$. Hence

$$M = \begin{bmatrix} -2a_{11} - 2b_{22} & -2a_{12} + b_{21} \\ -2a_{21} + 4b_{12} & -2a_{22} - 2b_{11} \end{bmatrix}$$

and by calculation

$$0 = det(M) = 2(a_{11} + b_{22})(a_{22} + b_{11}) + (2a_{12} - b_{21})(2b_{12} - a_{21}). \tag{5.12}$$

By replacing (3) in (5) and using (4) we have

$$b_{12}(a_{12} - b_{21}) = 0. \tag{5.13}$$

Our aim is to prove that either $b_{12} = 0$ or $b_{21} = 0$. To do this, we suppose that both $b_{12} \neq 0$ and $b_{21} \neq 0$ and we will prove that this assumption leads to a contradiction. By (6) we get $a_{12} = b_{21}$. By using this last in (4) it follows that $b_{21}(b_{12} + a_{21}) = 0$, that is $b_{12} = -a_{21}$. In light of this the matrix in (2) is

$$M = \begin{bmatrix} a_{11} + b_{22} & 2b_{21} \\ 0 & a_{22} + b_{11} \end{bmatrix}$$

and since it is nilpotent it follows that $(a_{11} + b_{22})^n = (a_{22} + b_{11})^n = 0$, that is $a_{11} + b_{22} = a_{22} + b_{11} = 0$. Therefore, if both b_{12} and b_{21} are not zero, then

$$a_{12} - b_{21} = 0, \quad a_{21} + b_{12} = 0, \quad a_{11} + b_{22} = 0, \quad a_{22} + b_{11} = 0. \tag{5.14}$$

Let $\varphi(x) = (1 + e_{12})x(1 - e_{12})$ and $\chi(x) = (1 - e_{12})x(1 + e_{12})$ be inner automorphisms of R. Of course

$$\varphi(([x_1, x_2]^2(a - c + b - p) + [x_1, x_2](-p - c)[x_1, x_2])^n) = 0$$

and

$$\chi(([x_1, x_2]^2(a - c + b - p) + [x_1, x_2](-p - c)[x_1, x_2])^n) = 0$$

that is the matrices $\varphi(A), \varphi(B), \chi(A), \chi(B)$ satisfy the same properties of A and B. Denote by $\varphi(A)_{ij}$ the (i, j)-entry of $\varphi(A)$, $\varphi(B)_{ij}$ the (i, j)-entry of $\varphi(b)$, $\chi(A)_{ij}$ the (i, j)-entry of $\chi(A)$ and $\chi(B)_{ij}$ the (i, j)-entry of $\chi(B)$.
Therefore $\varphi(B)_{21} = \varphi(A)_{12}$ and by (7) it follows

$$a_{12} = b_{21} = \varphi(B)_{21} = \varphi(A)_{12} = a_{12} + a_{22} - a_{11} - a_{21}$$

that is

$$a_{22} - a_{11} - a_{21} = 0. \tag{5.15}$$

On the other hand $\chi(B)_{21} = \chi(A)_{12}$ and by (7) it follows

$$a_{12} = b_{21} = \chi(B)_{21} = \chi(A)_{12} = a_{12} - a_{22} + a_{11} - a_{21}$$

that is

$$-a_{22} + a_{11} - a_{21} = 0. \tag{5.16}$$

By (8) and (9) and since $char(R) \neq 2$, we get the contradiction $b_{12} = -a_{21} = 0$.
This argument proves that one of b_{12} and b_{21} must be zero. Without loss of generality we may assume that $b_{12} = 0$. As above consider the following inner automorphisms of R:

$$\varphi(x) = (1 + e_{12})x(1 - e_{12}) \qquad \text{and} \qquad \chi(x) = (1 - e_{12})x(1 + e_{12}).$$

If either $0 = \varphi(B)_{21} = b_{21}$ or $0 = \chi(B)_{21} = b_{21}$ then B is a diagonal matrix. On the other hand if both $\varphi(B)_{21} = b_{21} \neq 0$ and $\chi(B)_{21} = b_{21} \neq 0$, then by the previous argument we have that $\varphi(B)_{12} = 0$ and also $\chi(B)_{12} = 0$. By calculation one has

$$0 = \varphi(B)_{12} = b_{22} - b_{11} - b_{21}$$

and

$$0 = \chi(B)_{12} = -b_{22} + b_{11} - b_{21}$$

and the last two equalities imply $b_{21} = 0$, since $char(R) \neq 2$.
Thus we may conclude that in any case B must be a diagonal matrix in $M_2(C)$. Using this, we may repeat the same above argument and consider the matrix $\varphi(B)$: it must be a diagonal one. Hence $0 = \varphi(B)_{12} = b_{22} - b_{11}$, which implies $b_{11} = b_{22} = \alpha \in C$. Therefore, if denote by I_2 the identity matrix in $M_2(C)$, we have that $B = \eta \cdot I_2$ is a central matrix in $M_2(C)$. Therefore R satisfies $([x_1, x_2]^2(a + b - 2\eta))^2 = 0$ and since $[x_1, x_2]^2$ is central valued on $M_2(C)$, it follows that $(a + b - 2\eta)^2 = 0$ as required.

Hence in the rest of the proof we assume $dim_C V \geq 3$.

Assume $b - p \notin C$, then there exists $v \in V$ such that $\{(b - p)v, v\}$ is linearly C-independent. Since $\dim_C V \geq 3$ then there exists $w \in V$ such that $\{(b - p)v, v, w\}$ is linearly C-independent. Then by the density of Q_r there exist $r_1, r_2 \in Q_r$ such that

$$r_1 v = 0, \quad r_2 v = 0, \quad r_1(b - p)v = 0, \quad r_2(b - p)v = w, \quad r_1 w = w, \quad r_2 w = -v,$$

then

$$[r_1, r_2]v = 0, \quad [r_1, r_2]w = v, \quad [r_1, r_2](b - p)v = w.$$

Now we get the following contradiction:

$$0 = \{(a - c)[r_1, r_2]^2 - [r_1, r_2](p + c)[r_1, r_2] + [r_1, r_2]^2(b - p)\}v = v \neq 0.$$

Hence, $\boxed{b - p \in C}$ and

$$\{(a - c + b - p)[x_1, x_2]^2 - [x_1, x_2](p + c)[x_1, x_2]\}^n = 0.$$

If $(p + c) \notin C$ then there exists $v' \in V$ such that $\{v', (p + c)v'\}$ is linearly C-independent. So, there exists $w' \in V$ such that $\{v', (p + c)v', w'\}$ is linearly C-independent. Then by the density of R, there exist $s_1, s_2 \in R$ such that

$$s_1 v' = 0, \quad s_2 v' = 0 \quad s_1 w' = v', \quad s_2 w' = w'.$$

Since
$$[s_1,s_2]v' = 0, \quad [s_1,s_2]w = v, \quad [s_1,s_2](p+c)v' = -w'$$
we get the following contradiction:
$$0 = \{(a-c+b-p)[s_1,s_2]^2 - [s_1,s_2](p+c)[s_1,s_2]\}^n w' = w' \neq 0.$$
Hence, $\boxed{p+c \in C}$.
Since $b-p,\ p+c \in C$, then $\{(a+b-2c-2p)[x_1,x_2]^2\}^n = 0$. Let $A = a+b-2c-2p$, we show that $A \in C$. If $A \notin C$ then there exist $v'', w'' \in V$ such that $\{Av'', v'', w''\}$ is linearly C-independent. Then by the density of R, there exist $t_1, t_2 \in R$ such that
$$t_1 v'' = 0, \quad t_2 v'' = w'' \quad t_1 w'' = v'', \quad t_2 w'' = v'',$$
$$t_1 A v'' = -w'', \quad t_2 A v'' = 0.$$
Since
$$[t_1,t_2]Av'' = v'', \quad [t_1,t_2]Av'' = v''.$$
So, it follows the contradiction
$$\{A[t_1,t_2]\}^n v'' = Av'' = 0.$$
The previous argument says that $A \in C$, hence

- $b-p \in C$
- $p+c \in C$
- $A \in C$.

Since $A^n[x_1,x_2]^{2n} = 0$ and $R$is not commutative, it follows $A = 0$, that is

- $b-p = \lambda \in C$
- $p+c = \mu \in C$
- $a+b-2\mu = 0$.

This means that
$$F(x) = (2\mu-b)x + xb = [x,b] + 2\mu x \underset{b=\lambda+p}{=} [x,p+\lambda] + 2\mu x \underset{\lambda \in C}{=} [x,p] + 2\mu x.$$
$$G(x) = (\mu-p)x + xp = [x,p] + \mu x$$
as required. □

We conclude this section by considering the more generale case of generalized derivations in prime rings and write $F(x) = ax + d(x)$, $G(x) = cx + \delta(x)$ for all $x \in R$, where $a, c \in Q_r$ and d, δ are derivations of R. Our goal is to prove Theorem 27.

The proof of Theorem 27

Proof. By (5.1) we have that R satisfies
$$\Big\{F([x_1,x_2]^2) - G([x_1,x_2])[x_1,x_2] - [x_1,x_2]G([x_1,x_2])\Big\}^n. \tag{5.17}$$

Notice that, if d, δ both are inner generalized derivations, then the conclusion follows from Proposition 10.
Therefore, here we consider the only case when $F(x) = ax + d(x)$, $G(x) = cx + \delta x$ are not simultaneously inner, and R satisfies

$$\begin{aligned} &\{a[x_1,x_2]^2 + d([x_1,x_2])[x_1,x_2] + [x_1,x_2]d([x_1,x_2]) + \\ &- c[x_1,x_2]^2 - \delta([x_1,x_2])[x_1,x_2] - [x_1,x_2]c[x_1,x_2] - [x_1,x_2]\delta([x_1,x_2])\}^n. \end{aligned} \tag{5.18}$$

We define the following map $\Delta : R \to R$

$$\Delta(x) = d(x) - \delta(x).$$

Clearly Δ is a derivation.
By (5.18) we can write

$$\begin{aligned} \Psi(x_1,x_2) = \{&(a-c)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2] + \\ &\Delta([x_1,x_2])[x_1,x_2] + [x_1,x_2]\Delta([x_1,x_2])\}^n = 0. \end{aligned} \tag{5.19}$$

If $\Delta(x)$ is inner, that is $\Delta(x) = [x, q]$, $q \in R$ then

$$\begin{aligned} &\{(a-c)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2] + [x_1,x_2]q[x_1,x_2] + \\ &- q[x_1,x_2]^2 + [x_1,x_2]^2 q - [x_1,x_2]q[x_1,x_2]\}^n = 0 \end{aligned} \tag{5.20}$$

$$\{(a-c-q)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2] + [x_1,x_2]^2 q\}^n = 0.$$

Thus, by Proposition 10 $a, q, c \in C$, then $\Delta = 0$ and $\delta = d$. So, by (5.19) we can write

$$\{(a-2c)[x_1,x_2]^2\}^n = 0.$$

Since $a - 2c \in C$, $a - 2c$ is not a zero divisor. So,

$$(a-2c)^n[x_1,x_2]^{2n} = 0 \Rightarrow a - 2c = 0.$$

Therefore $F(x) = 2cx + d(x)$ and $G(x) = cx + d(x)$, for any $x \in R$, with $c \in C$, and we are done.
Finally we assume that $\Delta(x)$ is not inner and show that a contradiction follows. In fact, in this last case, by (5.18), we can write

$$\begin{aligned} &\{(a-c)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2] + \Delta([x_1,x_2])[x_1,x_2] + [x_1,x_2]\Delta([x_1,x_2])\}^n = \\ &\{(a-c)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2] + \\ &[\Delta(x_1),x_2][x_1,x_2] + [x_1,\Delta(x_2)][x_1,x_2] + [x_1,x_2][\Delta(x_1),x_2] + [x_1,x_2][x_1,\Delta(x_2)]\}^n = 0. \end{aligned} \tag{5.21}$$

The second step is due to the fact that if Δ is outer, then it is a derivation of Lie. So, by Kharchenko's theorem (see Theorem 14) and (5.21), for $y_1 = \Delta(x_1)$ and $y_2 = \Delta(x_2)$ we have that R satisfies

$$\begin{aligned} &\{(a-c)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2] + [y_1,x_2][x_1,x_2] + \\ &[x_1,y_2][x_1,x_2] + [x_1,x_2][y_1,x_2] + [x_1,x_2][x_1,y_2]\}^n = 0. \end{aligned} \tag{5.22}$$

If $y_1 = y_2 = 0$ then

$$\{(a-c)[x_1,x_2]^2 - [x_1,x_2]c[x_1,x_2]\}^n = 0.$$

So, $a, c \in C$ once again by Proposition 10, then

$$(a-2c)^n[x_1,x_2]^{2n} = 0.$$

Follows as above, $a = 2c$ and (5.22) reduces to

$$[y_1,x_2][x_1,x_2] + [x_1,y_2][x_1,x_2] + [x_1,x_2][y_1,x_2] + [x_1,x_2][x_1,y_2]\}^n = 0.$$

In particular, for $x_1 = y_1$ and $x_2 = y_2$, we get

$$4^n[x_1,x_2]^{2n} = 0$$

which is a contradiction, since R is not commutative.

□

5.2 The case of generalized skew derivations.

Here we extend in some sense the result obtained in the previous Section to the case of generalized skew derivations of R. More precisely we will prove:

Theorem 28. *Let R be a prime ring of characteristic different from $2, 3$, Q_r its right Martindale quotient ring, C its extended centroid, F and G nonzero generalized skew derivations of R, associated with the same automorphism α of R and $n \geq 1$ a fixed integer. If*

$$\left\{F(x^2) - G(x)x - xG(x)\right\}^n = 0$$

for all $x \in R$, then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or one of the following holds:

(1) there exists $0 \neq \eta \in C$ such that $F(x) = 2\eta x$ and $G(x) = \eta x$, for all $x \in R$;

(1) there exists a non-zero derivation $d : R \to R$ such that $F(x) = G(x) = d(x)$, for all $x \in R$.

(2) there exist $0 \neq \eta \in C$ and a non-zero derivation $d : R \to R$ such that $F(x) = d(x) + 2\eta x$ and $G(x) = d(x) + \eta x$, for all $x \in R$.

We remark that, in case the automorphism α is precisely the identity map, then the conclusion follows from Theorem 27. Therefore, in all that follows we always assume that α is not the identity map on R.

Fact 3. Let R be a prime ring, then the following statements hold:

1. Any automorphism of R can be uniquely extended to Q_r [11, Fact 2].
2. Every generalized skew derivation of R can be uniquely extended to Q_r [8, Lemma 2].
3. A generalized skew derivation having associated automorphism α and skew derivation d assumes the following form:

$$F(x) = ax + d(x) \tag{5.23}$$

for all $x \in R$ (see [8, Lemma 2], [9, Theorem 3.1 and Corollary 3.2]).

Fact 4. [15] If $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for R, where R is a prime ring and D is an outer skew derivation of R, then R also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where x_i and y_i are distinct indeterminates.

Fact 5. [36, Theorem 6.5.9, page 365] Let R be a prime ring satisfying the polynomial identity of type $f(x_j^{\alpha_i \Delta_k}) = 0$, where $f(z_j^{i,k})$ is a generalized polynomial with the coefficients from Q_r, $\Delta_1, \dots, \Delta_n$ are mutually different correct words from a reduced set of skew derivations commuting with all the corresponding automorphisms, and $\alpha_1, \dots, \alpha_m$ are mutually outer automorphisms. In this case the identity $f(z_j^{i,k}) = 0$ is valid on Q_r.

Before proceeding with the proof of our main result of this Section, we also need to recall the following:

Lemma 14. *Let R be a prime ring, $\alpha, \beta \in \mathrm{Aut}(Q_r)$ and $d, \delta : R \to R$ be two skew derivations, associated with the same automorphism α. If there exist $0 \neq \eta \in C$, and $u \in Q_r$ such that*

$$\delta(x) = \Big(ux - \beta(x)u\Big) + \eta d(x), \quad \forall x \in R \tag{5.24}$$

then either $\alpha = \beta$ or $\delta(x) = \eta d(x)$, for all $x \in R$.

Proof. By the definition of δ we have

$$\delta(xy) = uxy - \beta(x)\beta(y)u + \eta d(x)y + \eta\alpha(x)d(y). \tag{5.25}$$

On the other hand, right multiplying relation (5.24) by $y \in R$, it follows that

$$\delta(x)y = uxy - \beta(x)uy + \eta d(x)y \quad \forall x, y \in R. \tag{5.26}$$

Therefore, subtracting relation (5.26) from (5.25), and using again (5.24), we get

$$\{\alpha(x) - \beta(x)\} \cdot \{uy - \beta(y)u\} = 0 \quad \forall x, y \in R. \tag{5.27}$$

Replacing y by yt in (5.27) and then using (5.27) we have

$$\{\alpha(x) - \beta(x)\} \cdot \beta(y) \cdot \{\beta(t)u - ut\} = 0 \quad \forall x, y, t \in R. \tag{5.28}$$

Then, by the primeness of R, above relation yields either $\alpha(x) - \beta(x) = 0$ for any $x \in R$, or $\beta(t)u - ut = 0$ for any $t \in R$. The last case and (5.24) imply $\delta(x) = \eta d(x)$, for all $x \in R$, as required. □

In light of the Fact 3, there are $a, c \in Q_r$ and d, δ skew derivations of R, such that $F(x) = ax + d(x)$ and $G(x) = cx + \delta(x)$, for any $x \in R$.
Hence, by our main hypothesis, R satisfies the functional identity

$$\Big\{(a-c)x^2 + d(x)x + \alpha(x)d(x) - xcx - x\delta(x) - \delta(x)x\Big\}^n. \tag{5.29}$$

Proposition 11. *Let R be a prime ring of characteristic different from 2, 3, Q_r its right Martindale quotient ring, C its extended centroid, F and G nonzero generalized skew inner derivations of R defined as follows:*

$$F(x) = ax + \alpha(x)b \quad \textit{and} \quad G(x) = cx + \alpha(x)u$$

where a, b, c, u are fixed element of Q_r and α is an automorphism of R. Let $n \geq 1$ be a fixed integer such that

$$\Big\{F(x^2) - G(x)x - xG(x)\Big\}^n = 0$$

for all $x \in R$. If α is not the identity map then either $R \subseteq M_2(K)$, the ring 2×2 matrices over a field K, or $\alpha(x)b = bx$ and $\alpha(x)u = ux$, for any $x \in R$, with $a + b = 2(c + u) \in C$.

Proof. We may assume that R does not satisfy s_4, if not we are done. By our assumption, R satisfies the following identity

$$\Psi(x) = \Big\{ax^2 + \alpha(x^2)b - xcx - x\alpha(x)u - cx^2 - \alpha(x)ux\Big\}^n. \tag{5.30}$$

Since α is not the identity map and R satisfies $\Psi(x)$, then by [11] R is a GPI-ring and Q_r is also GPI-ring by [13]. By Martindale's theorem in [49], Q_r is a primitive ring having non-zero socle. Hence Q_r is isomorphic to a dense subring of the ring of linear tranformations of a vector space V over a division ring D, containing non-zero linear transformations of finite rank. Moreover, by [34, p. 79], there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q$.
Hence, Q_r satisfies

$$\Psi(x) = \Big\{ax^2 + Tx^2T^{-1}b - xcx - xTxT^{-1}u - cx^2 - TxT^{-1}ux\Big\}^n. \tag{5.31}$$

We firstly assume that $dim_D V \geq 3$ and suppose there exists $v \in V$ such that $\{v, T^{-1}uv\}$ is linearly D-independent.
If $\{v, T^{-1}uv, T^{-1}bv\}$ is linearly D-independent, then, by the density of Q_r, there exists $x_0 \in Q_r$ such that

$$x_0v = 0 \;\; x_0T^{-1}uv = T^{-1}v \;\; x_0T^{-1}bv = T^{-1}uv.$$

Thus, by (5.31)

$$0 = \Big\{ax_0^2 + Tx_0^2T^{-1}b - x_0cx_0 - x_0Tx_0T^{-1}u - cx_0^2 - Tx_0T^{-1}ux_0\Big\}^n v = v$$

which is a contradiction. Hence $\{v, T^{-1}uv, T^{-1}bv\}$ is linearly D-dependent, that is there exist $\lambda, \mu \in D$ such that $T^{-1}bv = T^{-1}uv\lambda + v\mu$.
In this case and if $\{v, T^{-1}uv, T^{-1}v\}$ is linearly D-independent, as above, there exists $x_1 \in Q_r$ such that

$$x_1v = 0 \;\; x_1T^{-1}uv = T^{-1}v \;\; x_1T^{-1}v = T^{-1}v$$

implying that $x_1T^{-1}bv = T^{-1}v\lambda$. Thus, by (5.31)

$$0 = \Big\{ax_1^2 + Tx_1^2T^{-1}b - x_1cx_1 - x_1Tx_1T^{-1}u - cx_1^2 - Tx_1T^{-1}ux_1\Big\}^n v = Tx_1(T^{-1}v)\lambda = v\lambda$$

that is $\lambda = 0$.
On the other hand, if there exist $\eta, \vartheta \in D$ such that $T^{-1}v = T^{-1}uv\eta + v\vartheta$ and for $x_2 \in Q_r$ such that

$$x_2v = 0 \;\; x_2T^{-1}uv = T^{-1}v$$

we get both $x_2T^{-1}bv = T^{-1}v\lambda$ and $x_2T^{-1}v = T^{-1}v\eta$. Hence, by (5.31)

$$0 = \Big\{ax_2^2 + Tx_2^2T^{-1}b - x_2cx_2 - x_2Tx_2T^{-1}u - cx_2^2 - Tx_2T^{-1}ux_2\Big\}^n v = Tx_2(T^{-1}v)\lambda = v\eta\lambda.$$

Therefore, in any case, either $\lambda = 0$ or $\eta = 0$.
In other words, we have proved that, for any $v \in V$ such that $\{v, T^{-1}uv\}$ is linearly D-independent it follows that

$$\text{either } T^{-1}bv \in < v > \text{ or } T^{-1}v \in < v > . \tag{5.32}$$

On the other hand, if there is $v \in V$ such that both $\{v, T^{-1}uv\}$ is linearly D-independent and $T^{-1}bv \in < v >$, then there exists $\mu \in D$ such that $T^{-1}bv = v\mu$ and, since $dim_D V \geq 3$, there exists $w \in V$ such that $\{v, T^{-1}uv, w\}$ is linearly D-independent. Moreover, by the density of Q_r, there is $x_3 \in Q_r$ such that

$$x_3 v = 0 \quad x_3 T^{-1}uv = T^{-1}w \quad x_3 w = v.$$

Then, by (5.31), it follows the contradiction

$$0 = \left\{ax_3^2 + Tx_3^2T^{-1}b - x_3cx_3 - x_3Tx_3T^{-1}u - cx_3^2 - Tx_3T^{-1}ux_3\right\}^n v = v.$$

The previous contradiction and relation (5.32) imply that, for any $v \in V$ such that $\{v, T^{-1}uv\}$ is linearly D-independent,

$$T^{-1}v \in < v > . \tag{5.33}$$

For the rest of the proof, let $v \in V$ be such that $\{v, T^{-1}uv\}$ is linearly D-independent and $T^{-1}v = v\vartheta_0$, with $\vartheta_0 \in D$ depending on the choice of v.
Our next target is to show that, for any $w \in V$, $T^{-1}w \in < w >$.
Notice that, if $0 \neq w \in V$ is linearly dependent by v, say $w = v\eta_1$, then $\{w, T^{-1}uw\}$ is linearly D-independent. In fact, if we suppose that there exists $0 \neq \eta_2 \in C$ such that $T^{-1}uw = w\eta_2$ then $T^{-1}uv\eta_1 = v\eta_1\eta_2$, which is a contradiction, since $\{v, T^{-1}uv\}$ is linearly D-independent.
Therefore, if $w \in < v >$, then $T^{-1}w \in < w >$.
Let now $0 \neq w \in W$ be such that $\{v, w\}$ is linearly D-independent. We suppose on the contrary that $\{T^{-1}w, w\}$ is linearly D-independent and show that a number of contradictions follows.
If $\{T^{-1}w, w\}$ is linearly D-independent, by the above argument, there exists $\gamma_0 \in D$ such that $T^{-1}uw = w\gamma_0$.
We now analyze the following sets of vectors:

$$S_1 = \{T^{-1}u(v+w), v+w\} \quad S_2 = \{T^{-1}u(v-w), v-w\} \quad S_3 = \{T^{-1}u(v+2w), v+2w\}.$$

If both S_1 and S_2 are linearly dependent, as above, there exist $0 \neq \gamma_1 \in D$ and $0 \neq \gamma_2 \in D$ such that

$$T^{-1}u(v+w) = (v+w)\gamma_1$$

that is

$$T^{-1}uv + w\gamma_0 = v\gamma_1 + w\gamma_1 \tag{5.34}$$

and also

$$T^{-1}u(v-w) = (v-w)\gamma_2$$

that is

$$T^{-1}uv - w\gamma_0 = v\gamma_2 - w\gamma_2. \tag{5.35}$$

Subtracting (5.35) from (5.34), we have

$$2w\gamma_0 = v(\gamma_1 - \gamma_2) + w(\gamma_1 + \gamma_2)$$

which implies that $\gamma_0 = \gamma_1 = \gamma_2$. Then, from (5.34) it follows the contradiction $T^{-1}uv = v\gamma_1$.
This means that at least one of S_1 and S_2 should be linearly D-independent.
Analogously, if both S_1 and S_3 are linearly dependent, there exist $0 \neq \gamma_1 \in D$ and $0 \neq \gamma_3 \in D$ such that

$$T^{-1}u(v+w) = (v+w)\gamma_1$$

that is

$$T^{-1}uv + w\gamma_0 = v\gamma_1 + w\gamma_1 \tag{5.36}$$

and also

$$T^{-1}u(v+2w) = (v+2w)\gamma_3$$

that is

$$T^{-1}uv + 2w\gamma_0 = v\gamma_3 + 2w\gamma_3. \tag{5.37}$$

Subtracting (5.36) from (5.37), we have

$$w\gamma_0 = v(\gamma_3 - \gamma_1) + w(2\gamma_3 - \gamma_1)$$

which implies that $\gamma_0 = \gamma_1 = \gamma_3$. As above, from (5.36), it follows the contradiction $T^{-1}uv = v\gamma_1$.
Therefore, at least one of S_1 and S_3 should be linearly D-independent.
Finally, assume that both S_2 and S_3 are linearly dependent, and there exist $0 \neq \gamma_2$ and $0 \neq \gamma_3$ such that

$$T^{-1}u(v-w) = (v-w)\gamma_2$$

that is

$$T^{-1}uv - w\gamma_0 = v\gamma_2 - w\gamma_2 \tag{5.38}$$

and also

$$T^{-1}u(v+2w) = (v+2w)\gamma_3$$

that is

$$T^{-1}uv + 2w\gamma_0 = v\gamma_3 + 2w\gamma_3. \tag{5.39}$$

Subtracting (5.38) from (5.39), we have

$$3w\gamma_0 = v(\gamma_3 - \gamma_2) + w(2\gamma_3 + \gamma_2)$$

which implies that $\gamma_0 = \gamma_2 = \gamma_3$. In this case (5.38) give the contradiction $T^{-1}uv = v\gamma_1$.
Thus, we conclude that at least one of either S_2 or S_3 should be linearly D-independent.
All the previous arguments say that at least 2 sets among S_1, S_2, S_3 should be linearly D-independent.

Now, we analyse all the possible cases.

- S_1 and S_2 are linearly D-independent.

In this case, in light of relation (5.33), there exist $\vartheta_1, \vartheta_2 \in D$ such that

$$T^{-1}(v+w) = (v+w)\vartheta_1$$

that is

$$v\vartheta_0 + T^{-1}w = v\vartheta_1 + w\vartheta_1 \tag{5.40}$$

and

$$T^{-1}(v-w) = (v-w)\vartheta_2$$

that is

$$v\vartheta_0 - T^{-1}w = v\vartheta_2 - w\vartheta_2. \tag{5.41}$$

Adding (5.40) and (5.41), it follows

$$2v\vartheta_0 = v(\vartheta_1 + \vartheta_2) + w(\vartheta_1 - \vartheta_2)$$

that is $\vartheta_0 = \vartheta_1 = \vartheta_2$ and (5.40) reduces to $T^{-1}w = w\vartheta_1$, a contradiction.

- S_1 and S_3 are linearly D-independent.

By relation (5.33), there exist $\vartheta_1, \vartheta_3 \in D$ such that

$$T^{-1}(v+w) = (v+w)\vartheta_1 \quad \Rightarrow \quad 2T^{-1}(v+w) = 2(v+w)\vartheta_1$$

that is

$$2v\vartheta_0 + 2T^{-1}w = 2v\vartheta_1 + 2w\vartheta_1 \tag{5.42}$$

and

$$T^{-1}(v+2w) = (v+2w)\vartheta_3$$

that is

$$v\vartheta_0 + 2T^{-1}w = v\vartheta_3 + 2w\vartheta_3. \tag{5.43}$$

Subtracting (5.43) from (5.42), it follows

$$v\vartheta_0 = v(2\vartheta_1 - \vartheta_3) + 2w(\vartheta_1 - \vartheta_3)$$

that is $\vartheta_0 = \vartheta_1 = \vartheta_3$ and (5.42) implies the contradiction $T^{-1}w = w\vartheta_1$.

- S_2 and S_3 are linearly D-independent.

By relation (5.33), there exist $\vartheta_2, \vartheta_3 \in D$ such that

$$T^{-1}(v-w) = (v-w)\vartheta_2$$

that is

$$v\vartheta_0 - T^{-1}w = v\vartheta_2 - w\vartheta_2 \tag{5.44}$$

and

$$T^{-1}(v+2w) = (v+2w)\vartheta_3$$

that is

$$v\vartheta_0 + 2T^{-1}w = v\vartheta_3 + 2w\vartheta_3. \tag{5.45}$$

Adding (5.44) and (5.45), we get

$$3v\vartheta_0 = v(\vartheta_3 + 2\vartheta_2) + 2w(\vartheta_3 - \vartheta_2)$$

that is $\vartheta_0 = \vartheta_1 = \vartheta_2$ and the contradiction $T^{-1}w = w\vartheta_2$ follows from (5.44).

Finally, we have proved that, if there exists $v \in V$ such that $\{v, T^{-1}uv\}$ is linearly C-independent, then, for any $w \in V$, there exist $\lambda_w \in D$ such that $T^{-1}w = w\lambda_w$. By a standard argument it follows that there exists a unique $\lambda \in D$ such that $T^{-1}w = w\lambda$, for all $w \in V$ (see for example Lemma 1 in [14]). In this case

$$\alpha(x)w = (TxT^{-1})w = Txw\lambda$$

and

$$\Big(\alpha(x) - x\Big)w = T(xw\lambda) - xw = T(T^{-1}xw) - xw = 0$$

which implies the contradiction that α is the identity map, since V is faithful.
Thanks to the previous argument, we have proved that, for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}uv = v\lambda_v$. By a standard argument it follows that there exists a unique $\lambda \in D$ such that $T^{-1}uv = v\lambda$, for all $v \in V$ (once again we refer the reader to Lemma 1 in [14]). In this case

$$\alpha(x)uv = TxT^{-1}uv = T(xv\lambda) = T((xv)\lambda) = T(T^{-1}uxv) = uxv.$$

Hence, for all $v \in V$,

$$\Big(\alpha(x)u - ux\Big)v = 0$$

which implies $\alpha(x)u = ux$, for all $x \in Q_r$, since V is faithful. As a consequence, for any $x \in R$,

$$cx + \alpha(x)u = (c+u)x.$$

Therefore Q_r satisfies

$$\Psi(x) = \Big\{ax^2 + Tx^2T^{-1}b - x(c+u)x - (c+u)x^2\Big\}^n.$$

Suppose now there exists $v \in V$ such that $\{v, T^{-1}bv\}$ is linearly D-independent. Since $dim_D V \geq 3$, there exists $w \in V$ such that $\{v, T^{-1}bv, w\}$ is linearly D-independent. Moreover, by the density of Q_r, there is $y_0 \in Q_r$ such that

$$y_0 v = 0 \;\; y_0 T^{-1}bv = w \;\; y_0 w = T^{-1}v$$

implying the contradiction

$$0 = \Big\{ay_0^2 + Ty_0^2T^{-1}b - y_0(c+u)y_0 - (c+u)y_0^2\Big\}^n v = v.$$

Hence, for any $v \in V$, $\{v, T^{-1}bv\}$ is linearly D-independent and, as above, $\alpha(x)b = bx$ and $ax + \alpha(x)b = (a+b)x$, for any $x \in R$.
Thus, the required conclusion follows from Proposition 10.
Let now $dim_D V \leq 2$.
Then, by (5.30), Q_r satisfies

$$\Big\{ax^2 + \alpha(x^2)b - xcx - x\alpha(x)u - cx^2 - \alpha(x)ux\Big\}^2. \tag{5.46}$$

In case there is an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for any $x \in R$, then we may write (5.46) as

$$\Big\{ax^2 + qx^2q^{-1}b - xcx - xqxq^{-1}u - cx^2 - qxq^{-1}ux\Big\}^2 \tag{5.47}$$

that is a non-trivial generalized polynomial identity for R as well as for Q_r. In view

of [22, Theorem 2.5 and Theorem 3.5], we know that both Q_r and $Q_r \otimes_C \overline{C}$ are centrally closed, where $\overline{C}$ is the algebraic closure of C. We may replace Q_r by itself or $Q_r \otimes_C \overline{C}$ according as C is finite or infinite. Therefore we may assume that Q_r is centrally closed over C which is either finite or algebraically closed. By Martindale's theorem [49], Q_r is a primitive ring having a non-zero socle H, with C as the associated division ring. In light of Jacobson's theorem [34, page 75], Q_r is isomorphic to a dense ring of linear transformations on some vector space V over C. By using the same above argument, we may conclude that either F and G are both generalized derivations of R (and in this case we conclude by Theorem 27), or $dim_C V \leq 2$, that is $R \subseteq M_2(C)$, which is a contradiction.
Therefore, we may now assume that α is not an inner automorphism of R.
Since α is outer and either $char(R) = 0$ or $char(R) \geq 5$, then the α-word degree in relation (5.46) is lesser than the characteristic p of the ring (in case $p \neq 0$). Thus, by [12, Theorem 3] and (5.46) we have that Q_r satisfies

$$\left\{ax^2 + y^2 b - xcx - xyu - cx^2 - yux\right\}^2. \tag{5.48}$$

In particular both

$$\left(y^2 b\right)^2 \tag{5.49}$$

and

$$\left(xcx + (a-c)x^2\right)^2 \tag{5.50}$$

are satisfied by Q_r. It is well known that (5.49) implies $b = 0$. Moreover, by applying [17, Theorem 2] to relation (5.50), we get $a = 2c \in C$. Thus (5.48) reduces to

$$(xyu - yux)^2. \tag{5.51}$$

For $x = y$ in (5.51), it follows that

$$\left(x^2 u - xux\right)^2 = 0. \tag{5.52}$$

Once again, the application of [17, Theorem 2] to relation (5.52), drives us to the conclusion $u \in C$. Finally, by (5.51) we get

$$u^2 [x, y]^2 \tag{5.53}$$

imlpying that $u = 0$. Therefore $F(x) = 2cx$ and $G(x) = cx$, for any $x \in R$, with $c \in C$, as required. □

The proof of Theorem 28

Proof. Of course, in the rest of the proof, we may assume that α is not the identity map on R and d, δ, the associated skew derivations, are not simultaneoulsy zero, if not we are done by Theorem 27 and Proposition 10, respectively. Moreover, in light of Proposition 11, we assume that F and G are not simultaneously inner generalized skew derivations of R.
In all that follows we will assume that R does not satisfy s_4 and prove that a number of contradictions follows.
We firstly study the subcase $d = 0$ and $\delta \neq 0$, that is $F(x) = ax$ and $G(x) = cx + \delta(x)$, for all $x \in R$. Since $F \neq 0$, we may assume in what follows $a \neq 0$. Moreover δ is not an inner skew derivation of R, otherwise the conclusion follows by the previous

argument. In this situation, by (5.29) and Fact 4 we have that R satisfies

$$\left\{(a-c)x^2 - xcx - xy - yx\right\}^n. \tag{5.54}$$

In particular R satisfies $\{(a-c)x^2 - xcx\}^n$ and, by applying Proposition 10, we get $a = 2c \in C$. Then (5.54) reduces to $(-1)^n(xy+yx)^n = 0$, which is not possible in a non-zero prime ring.
Let now $d \neq 0$ and $\delta = 0$, that is $F(x) = ax + d(x)$, $G(x) = cx$, for all $x \in R$. Then we write (5.29) as follows

$$\left\{(a-c)x^2 + d(x)x + \alpha(x)d(x) - xcx\right\}^n. \tag{5.55}$$

Moreover, as above, we may consider the only case when δ is not inner. Thus, by (5.55) and Fact 4 we have that R satisfies

$$\left\{(a-c)x^2 + yx + \alpha(x)y - xcx\right\}^n. \tag{5.56}$$

Also in this case, R satisfies the blended component $\{(a-c)x^2 - xcx\}^n$ and $a = 2c \in C$ follows as above. Hence (5.56) reduces to

$$\left\{(yx + \alpha(x)y\right\}^n. \tag{5.57}$$

Since α is not the identity map and R satisfies (5.57), then by [11] R is a GPI-ring and Q_r is also GPI-ring by [13]. By Martindale's theorem in [49], Q_r is a primitive ring having non-zero socle. Hence Q_r is isomorphic to a dense subring of the ring of linear tranformations of a vector space V over a division ring D, containing non-zero linear transformations of finite rank.
In case Q_r is a division ring and by (5.57), we have

$$yx + \alpha(x)y = 0 \tag{5.58}$$

for any $x, y \in Q_r$. Replace x by xz in (5.58). Thus

$$yxz + \alpha(x)\alpha(z)y = 0 \tag{5.59}$$

for any $x, y, z \in Q_r$. On the other hand, by using (5.58) in (5.59), it follows

$$yxz - \alpha(x)yz = 0 \tag{5.60}$$

and, by the primeness of Q_r, one has

$$yx - \alpha(x)y = 0. \tag{5.61}$$

Therefore, by comparing (5.58) with (5.61), we get $2yx = 0$ which is a contradiction, since $char(R) \neq 2$.
Thus we must assume $dim_D V \geq 2$. Moreover, by [34, p. 79], there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q$.
Hence, Q_r satisfies

$$\left\{(yx + TxT^{-1}y\right\}^n. \tag{5.62}$$

Firstly we suppose there exists $v \in V$ such that $\{v, T^{-1}v\}$ are linearly D-independent. By the density of Q_r, there exist $x_0, y_0 \in Q_r$ such that $y_0 v = 0$, $x_0 v = T^{-1}v$, $y_0 T^{-1} v = v$, implying the contradiction

$$0 = \left\{(y_0 x_0 + T x_0 T^{-1} y_0\right\}^n v = v \neq 0.$$

Therefore, for any $v \in V$ there exists $\lambda_v \in D$ such that $T^{-1}v = v\lambda_v$. As above remarked, in this case we get the contradiction that α is the identity map.
In light of the previous argument, we may assume both $d \neq 0$ and $\delta \neq 0$. We divide the rest of the proof of Theorem 28 in the two subsections:

Let d, δ be linearly C-independent modulo inner skew derivations

Here, by (5.29) and Fact 5,

$$\left\{(a-c)x^2 + yx + \alpha(x)y - xcx - xz - zx\right\}^n \tag{5.63}$$

is satisfied by R. In particular, $\{(a-c)x^2 - xcx\}^n$ is a generalized identity for R, implying, as above, $a = 2c \in C$. Thus (5.63) reduces to

$$\left\{yx + \alpha(x)y - xz - zx\right\}^n$$

and for $y = 0$ it follows that R should satisfy $(-1)^n(xz + zx)^n$, which is a contradiction.

Let $\{d, \delta\}$ be linearly C-dependent modulo inner skew derivations.

Hence there exist $\lambda, \mu \in C$, $u \in Q_r$ and an automorphism β of R such that $\lambda d(x) + \mu\delta(x) = ux - \beta(x)u$, for any $x \in R$.
If $\lambda = 0$ and $\mu \neq 0$, we write

$$\delta(x) = \left(p_0 x - \beta(x) p_0\right), \quad \forall x \in R$$

where $p_0 = \mu^{-1}u$. Since the automorphism associated with a skew derivation is unique, in this case $\alpha = \beta$.
Since d, δ are not simultaneously inner, we may assume that d is not inner. Thus, by (5.29), R satisfies

$$\left\{(a-c)x^2 + d(x)x + \alpha(x)d(x) - xcx - x(p_0 x - \alpha(x)p_0) - (p_0 x - \alpha(x)p_0)x\right\}^n. \tag{5.64}$$

Since d is outer, by Fact 4 and (5.64), R satisfies

$$\left\{(a-c)x^2 + yx + \alpha(x)y - xcx - x(p_0 x - \alpha(x)p_0) - (p_0 x - \alpha(x)p_0)x\right\}^n. \tag{5.65}$$

In particular

$$\left\{(a-c-p_0)x^2 - x(c+p_0)x + x\alpha(x)p_0 + \alpha(x)p_0x\right\}^n \tag{5.66}$$

is an identity for Q_r. By applying Proposition 11, one has $\alpha(x)p_0 = p_0x$, for any $x \in Q_r$ and by (5.66) it follows that

$$\left\{(a-c)x^2 - xcx\right\}^n$$

is satisfied by Q_r. Then, by Proposition 10, we get $a = 2c \in C$. Thus, by (5.65) reduces to relation (5.57) that is

$$\left\{yx + \alpha(x)y\right\}^n.$$

Hence, by using the same above argument, a contradiction follows.
On the other hand, If $\lambda \neq 0$ and $\mu = 0$, we write

$$d(x) = \Big(p_1x - \beta(x)p_1\Big), \quad \forall x \in R$$

where $p_1 = \lambda^{-1}u$. Also in this case, it is clear that $\alpha = \beta$ and, as above, we may assume that δ is not inner. Thus, by (5.29), R satisfies

$$\left\{(a-c)x^2 + (p_1x - \alpha(x)p_1)x + \alpha(x)(p_1x - \alpha(x)p_1) - xcx - x\delta(x) - \delta(x)x\right\}^n. \tag{5.67}$$

Since δ is outer, it follows that R satisfies

$$\left\{(a-c+p_1)x^2 - \alpha(x^2)p_1 - xcx - xy - yx\right\}^n. \tag{5.68}$$

For $y = 0$, R satisfies

$$\left\{(a-c-p_1)x^2 - \alpha(x)^2p_1 - xcx\right\}^n. \tag{5.69}$$

Also in this case, the application of Proposition 11 implies $\alpha(x)p_1 = p_1x$, for any $x \in Q_r$ and by (5.69) it follows that

$$\left\{(a-c)x^2 - xcx\right\}^n$$

is satisfied by Q_r. As above, we get $a = 2c \in C$ and (5.68) reduces to $(-1)^n(xy + yx)^n = 0$. Once again we abtain a contradiction.
Hence, in the sequel we assume that both $\lambda \neq 0$ and $\mu \neq 0$. We may write

$$\delta(x) = \Big(p_0x - \beta(x)p_0\Big) + \eta d(x), \quad \forall x \in R \tag{5.70}$$

where $\eta = -\lambda\mu^{-1} \neq 0$ and, as above, $p_0 = \mu^{-1}u$. By Lemma 14, either $\alpha = \beta$ or $p_0 = 0$ and $\delta(x) = \eta d(x)$, for all $x \in R$.

Moreover, if d is an inner skew derivation, then also δ is inner and the conclusion follows again from the above argument.
Therefore, in what follows we assume that $0 \neq d$ is outer.
In the case $\delta = \eta d$, (5.29) reduces to

$$\Big\{(a-c)x^2 + d(x)x + \alpha(x)d(x) - xcx - \eta x d(x) - \eta d(x)x\Big\}^n. \tag{5.71}$$

Moreover, since d is outer, R satisfies

$$\Big\{(a-c)x^2 + yx + \alpha(x)y - xcx - \eta xy - \eta yx\Big\}^n. \tag{5.72}$$

In particular $\{(a-c)x^2 - xcx\}^n$ is a generalized identity for R, implying, as above, $a = 2c \in C$. Thus (5.72) reduces to

$$\Big\{yx + \alpha(x)y - \eta xy - \eta yx\Big\}^n. \tag{5.73}$$

As above remarked, since α is not the identity map, Q_r is isomorphic to a dense subring of the ring of linear tranformations of a vector space V over a division ring D, containing non-zero linear transformations of finite rank, and there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in Q_r$.
Hence, Q_r satisfies

$$\Big\{yx + TxT^{-1}y - \eta xy - \eta yx\Big\}^n. \tag{5.74}$$

We firstly assume that $dim_D V \geq 2$. In light of a previous argument, since α is not the identity then we may assume there exists $v \in V$ such that $\{v, T^{-1}v\}$ is linearly D-independent. By the density of Q_r, there exists $x_0, y_0 \in Q_r$ such that

$$x_0 v = 0 \;\; y_0 v = v \;\; x_0 T^{-1} v = T^{-1} v.$$

Hence, by (5.74) we get the contradiction

$$0 = \Big\{y_0x_0 + Tx_0T^{-1}y_0 - \eta x_0 y_0 - \eta y_0 x_0\Big\}^n v = v.$$

This means that $dim_D V = 1$ and Q_r satisfies

$$yx + \alpha(x)y - \eta xy - \eta yx. \tag{5.75}$$

In particular, for $x = 1$ it follows $(2-2\eta)y = 0$, for any $y \in Q_r$, that is $\eta = 1$ and (5.75) reduces to

$$\{\alpha(x) - x\}y = 0 \;\; \forall x, y \in Q_r.$$

The primeness of Q_r leads to the contradiction $\alpha(x) = x$, for any $x \in Q_r$.
Suppose now $\alpha = \beta$. By relations (5.70) and (5.29) R satisfies

$$\begin{aligned}\Big\{&(a-c)x^2 + d(x)x + \alpha(x)d(x) - xcx - x\big(p_0x - \alpha(x)p_0\big) - \eta x d(x) \\ &- \big(p_0x - \alpha(x)p_0\big)x - \eta d(x)x\Big\}^n.\end{aligned} \tag{5.76}$$

Since d is not inner, it follows that

$$\Big\{(a-c)x^2 + yx + \alpha(x)y - xcx - x\big(p_0x - \alpha(x)p_0\big) - \eta xy \\ - \big(p_0x - \alpha(x)p_0\big)x - \eta yx\Big\}^n. \tag{5.77}$$

is a generalized identity for R. Hence R satisfies

$$\Big\{(a-c-p_0)x^2 - x(c+p_0)x + x\alpha(x)p_0 + \alpha(x)p_0x\Big\}^n. \tag{5.78}$$

Again, by the application of Proposition 11, one has $\alpha(x)p_0 = p_0x$, for any $x \in Q_r$ and by (5.78) it follows that

$$\Big\{(a-c)x^2 - xcx\Big\}^n$$

is satisfied by Q_r. Then, by Proposition 10, we get $a = 2c \in C$. Thus, by (5.78) reduces to relation (5.73) that is

$$\Big\{yx + \alpha(x)y - \eta xy - \eta yx\Big\}^n.$$

Hence, by using the same above argument, a contradiction follows. □

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