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Cohen–Macaulayness of Vertex Splittable Monomial Ideals

Marilena Crupi *,†,‡ and Antonino Ficarra †,‡

Department of Mathematics and Computer Sciences, Physics and Earth Sciences, University of Messina, Viale Ferdinando Stagno d'Alcontres 31, 98166 Messina, Italy; antficarra@unime.it
* Correspondence: mcrupi@unime.it
† These authors contributed equally to this work.
‡ These authors are members of GNSAGA of INDAM (Italy).

Abstract: In this paper, we give a new criterion for the Cohen–Macaulayness of vertex splittable ideals, a family of monomial ideals recently introduced by Moradi and Khosh-Ahang. Our result relies on a Betti splitting of the ideal and provides an inductive way of checking the Cohen–Macaulay property. As a result, we obtain characterizations for Gorenstein, level and pseudo-Gorenstein vertex splittable ideals. Furthermore, we provide new and simpler combinatorial proofs of known Cohen–Macaulay criteria for several families of monomial ideals, such as (vector-spread) strongly stable ideals and (componentwise) polymatroidals. Finally, we characterize the family of bi-Cohen–Macaulay graphs by the novel criterion for the Cohen–Macaulayness of vertex splittable ideals.

Keywords: minimal resolutions; graded Betti numbers; Betti splittings; Cohen–Macaulay ideals; vertex splittable ideals

MSC: 13D02; 13F20; 13H10; 13F55; 05C75

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1. Introduction

Let $S = K[x_1, \dots, x_n]$ be the polynomial ring with coefficients in a field K. In [1], Moradi and Khosh-Ahang introduced the notion of a *vertex splittable ideal*, an algebraic analog of the vertex decomposability property of a simplicial complex. In more detail, let Δ be a simplicial complex and let F be a face of Δ. One can associate with Δ two special simplicial complexes: the deletion of F, defined as $\mathrm{del}_\Delta(F) = \{G \in \Delta : F \cap G = \emptyset\}$, and the link of F, defined as $\mathrm{lk}_\Delta(F) = \{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\}$. For $F = \{x\}$, one sets $\mathrm{del}_\Delta(\{x\}) = \mathrm{del}_\Delta(x)$ and $\mathrm{lk}_\Delta(\{x\}) = \mathrm{lk}_\Delta(x)$. The notion of vertex decomposition was introduced by Provan and Billera [2] for a pure simplicial complex, and afterwards, it was extended to nonpure complexes by Bjoërner and Wachs [3]. A *vertex decomposable simplicial complex* Δ is recursively defined as follows: Δ is a simplex or Δ has some vertex x such that (1) both $\mathrm{del}_\Delta\, x$ and $\mathrm{lk}_\Delta\, x$ are vertex decomposable, and (2) there is no face of $\mathrm{lk}_\Delta\, x$ which is also a facet of $\mathrm{del}_\Delta\, x$. An ideal I of S is called *vertex decomposable* if $I = I_\Delta$, with Δ being a vertex decomposable simplicial complex. We recall that I_Δ is the Stanley–Reisner ideal of Δ over K, that is, the ideal of S generated by the squarefree monomial $x_F = \prod_{x_j \in F} x_j$, for all $F \in \Delta$. It is well-known that for a simplicial complex Δ, the following implications hold: *vertex decomposable* $\Rightarrow$ *shellable* $\Rightarrow$ *sequentially Cohen–Macaulay* (see, for instance, [4]). Moreover, there exist characterizations of shellable, sequentially Cohen–Macaulay and Cohen–Macaulay complexes Δ via the Alexander dual ideals $I_{\Delta^\vee}$ (see [5] [Theorem 1.4], [6] [Theorem 2.1], [7] [Theorem 3], respectively), where $\Delta^\vee = \{X \setminus F : F \notin \Delta\}$ is the Alexander dual simplicial complex associated with Δ.

Inspired by the above results, in [1], Moradi and Khosh-Ahang asked and solved the following question: *Is it possible to characterize a vertex decomposable simplicial complex Δ by means of $I_{\Delta^\vee}$?* For this aim, they introduced the notion of the vertex splittable monomial ideal (Definition 1) and proved that a simplicial complex Δ is vertex decomposable if and only if $I_{\Delta^\vee}$ is a vertex splittable ideal [1] [Theorem 2.3]. Moreover, the authors in [1] proved that a vertex splittable ideal has a Betti splitting (see Definition 2 and Theorem 1).

Determining when a monomial ideal is Cohen–Macaulay is a fundamental and challenging problem in commutative algebra. Motivated by this and the results of [1], in this paper, we tackle the Cohen–Macaulayness of vertex splittable ideals. Our main contribution (Theorem 2) is a new characterization of the Cohen–Macaulayness of a vertex splittable ideal in terms of a Betti splitting. This new criterion provides a neat and effective inductive strategy to determine when a vertex splittable ideal is Cohen–Macaulay.

This article is organized as follows. In Section 2.2, we recall relevant definitions and auxiliary results that we will use later on. In Section 3, we state a new criterion for the Cohen–Macaulayness of vertex splittable ideals (Theorem 2). As a consequence, we obtain characterizations for Gorenstein, level and pseudo-Gorenstein Cohen–Macaulay vertex splittable ideals (Corollary 1). The results in this section will be used in the subsequent section (Section 4), where we recover some interesting Cohen–Macaulay classifications of families of monomial ideals: (vector-spread) strongly stable ideals and (componentwise) polymatroidal ideals. Moreover, a new characterization of bi-Cohen–Macaulay graphs is presented (Theorem 6). Finally, Section 5 contains our conclusions and perspectives.

2. Preliminaries

In this section, we recall the basic notions and notations we will use in the body of the paper [1,8].

Let $S = K[x_1,\dots,x_n]$ be a polynomial ring in n variables over a field K with the standard grading, i.e., each $\deg x_i = 1$. For any finitely generated graded S-module M, there exists the unique minimal graded free S-resolution

$$F : 0 \to F_p \xrightarrow{d_p} F_{p-1} \xrightarrow{d_{p-1}} \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} M \to 0,$$

with $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. The numbers $\beta_{i,j} = \beta_{i,j}(M)$ are called the graded Betti numbers of M, while $\beta_i(M) = \sum_j \beta_{i,j}(M)$ are called the total Betti numbers of M. Recall that the *projective dimension* and the *Castelnuovo–Mumford regularity* of M are defined as follows:

$$\begin{aligned} \operatorname{pd} M &= \max\{i : \ \beta_i(M) \neq 0\}, \\ \operatorname{reg} M &= \max\left\{j - i : \beta_{i,j}(M) \neq 0, \text{ for some } i \text{ and } j\right\}. \end{aligned}$$

More precisely, the projective dimension $\operatorname{pd}(M)$ is the length of a minimal graded free resolution of the finitely generated graded S-module M.

2.1. Cohen–Macaulay Property

In this subsection, we introduce the notion of the Cohen–Macaulay ring and some related notions.

Firstly, let $\mathfrak{m} = (x_1,\dots,x_n)$ be the unique maximal homogeneous ideal of S, and let M be a finitely generated graded S-module M.

A sequence $\mathbf{f} = f_1,\dots,f_d$ of homogeneous elements of $\mathfrak{m}$ is called an *M-sequence* if the following criteria are met:

(1) the multiplication map $M/(f_1,\dots,f_{i-1})M \xrightarrow{f_i} M/(f_1,\dots,f_i)M$ is injective for all i.
(2) $M/(\mathbf{f})M \neq 0$.

The length of a maximal homogeneous M-sequence is called the *depth* of M. By the Auslander–Buchsbaum formula (see, for instance, [9]), we have

$$\operatorname{depth} M + \operatorname{pd} M = n. \tag{1}$$

A finitely generated graded S-module M is called a *Cohen–Macaulay module* (CM module for short) if $\operatorname{depth} M = \dim M$, where $\dim M$ is the Krull dimension of M [9]. Let I

be a graded ideal of S; the graded ring S/I is said to be CM if S/I, viewed as an S-module, is CM. The graded ideal I is called a CM ideal.

Let $I \subset S$ be a graded CM ideal, and let $p = \operatorname{pd}(S/I)$ be the projective dimension of S/I. The *Cohen–Macaulay type* (CM type for short) of S/I is defined as the integer $\text{CM-type}(S/I) = \beta_p(S/I)$. It is well-known that S/I is Gorenstein if and only if $\text{CM-type}(S/I) = 1$. We say that I is Gorenstein if S/I is such.

By [10] [Corollary 2.17], the graded Betti number $\beta_{p,p+\operatorname{reg} S/I}(S/I)$ is always nonzero. We say that S/I is *level* if and only if $\beta_p(S/I) = \beta_{p,p+\operatorname{reg} S/I}(S/I)$. Following [11], we say that S/I is *pseudo-Gorenstein* if and only if $\beta_{p,p+\operatorname{reg} S/I}(S/I) = 1$. Hence, S/I is Gorenstein if and only if it is both level and pseudo-Gorenstein.

For more details on this subject, see, for instance, [9,10,12,13].

2.2. Vertex Splittable Monomial Ideals

In this subsection, we discuss the notions of vertex splittable monomial ideals and of Betti splittings.

Let $I \subset S$ be a monomial ideal. We denote by $\mathcal{G}(I)$ the unique minimal monomial generating set of I. We recall the following notion [1] [Definition 2.1].

Definition 1. *The ideal I is called vertex splittable if it can be obtained by the following recursive procedure:*

(i) If u is a monomial and $I = (u)$, $I = 0$ or $I = S$, then I is vertex splittable.

(ii) If there exists a variable x_i and vertex splittable ideals $I_1 \subset S$ and $I_2 \subset K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ such that $I = x_i I_1 + I_2$, $I_2 \subseteq I_1$ and $\mathcal{G}(I)$ is the union of $\mathcal{G}(x_i I_1)$ and $\mathcal{G}(I_2)$, then I is vertex splittable.
In this case, we say that $I = x_i I_1 + I_2$ is a vertex splitting of I and x_i is a splitting vertex of I.

Remark 1. *One can observe that while in general, the Cohen–Macaulayness of S/I depends on the field K ([9] (p. 236)), if I is a vertex splittable ideal, then this is not the case. Indeed, the Krull dimension of S/I, where I is a monomial ideal, does not depend on K. Furthermore, by [1] [Theorem 2.4], vertex splittable ideals have linear quotients. Hence,* depth S/I *is also independent from K.*

In [8], the next concept was introduced.

Definition 2. *Let I, J, L be monomial ideals of S such that $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(L)$. We say that $I = J + L$ is a Betti splitting if*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(L) + \beta_{i-1,j}(J \cap L), \quad \text{for all } i, j. \tag{2}$$

When $I = J + L$ is a Betti splitting, important homological invariants of the ideal I are related to the invariants of the smaller ideals J and L. Indeed, in [8] [Corollary 2.2], it is proved that if $I = J + L$ is a Betti splitting, then

$$\operatorname{pd} I = \max\{\operatorname{pd} J, \operatorname{pd} L, \operatorname{pd}(J \cap L) + 1\}. \tag{3}$$

We quote the next crucial result from [1].

Theorem 1 ([1] [Theorem 2.8])**.** *Let $I = xI_1 + I_2$ be a vertex splitting for the monomial ideal I of S. Then $I = xI_1 + I_2$ is a Betti splitting.*

For recent applications of vertex splittings, see the papers [14,15].

We close this subsection by introducing two families of monomial ideals: the t-spread strongly stable ideals and the (componentwise) polymatroidal ideals. We will show in Section 4 that they are families of vertex splittable ideals (see Propositions 1 and 4.2).

A very meaningful class of monomial ideals of the polynomial ring S is the class of *strongly stable* monomial ideals. They are fundamental in commutative algebra, because if K has the characteristic zero, then they appear as generic initial ideals [16]. In [17], the concept of a strongly stable ideal was generalized to that of the $\mathbf{t}$-*spread strongly stable* ideal.

Let $d \geq 2$, $\mathbf{t} = (t_1, \dots, t_{d-1}) \in \mathbb{Z}_{\geq 0}^{d-1}$ be a $(d-1)$ tuple, and let $u = x_{i_1} \cdots x_{i_\ell} \in S$ be a monomial, with $1 \leq i_1 \leq \cdots \leq i_\ell \leq n$ and $\ell \leq d$. We say that u is $\mathbf{t}$-*spread* if

$$i_{j+1} - i_j \geq t_j \ \text{ for all } \ j = 1, \dots, \ell - 1.$$

A monomial ideal $I \subset S$ is called $\mathbf{t}$-*spread* if $\mathcal{G}(I)$ consists of $\mathbf{t}$-spread monomials. A $\mathbf{t}$-spread ideal $I \subset S$ is called $\mathbf{t}$-*spread strongly stable* if for all $\mathbf{t}$-spread monomials $u \in I$ and all $i < j$ such that x_j divides u and $x_i(u/x_j)$ is $\mathbf{t}$-spread, then $x_i(u/x_j) \in I$. For $\mathbf{t} = (0, \dots, 0)$ and $\mathbf{t} = (1, \dots, 1)$, we obtain the strongly stable and the squarefree strongly stable ideals [13].

Another fundamental family of monomial ideals of S is that of the so-called *polymatroidal ideals*.

Let $I \subset S$ be a monomial ideal generated in a single degree. We say that I is *polymatroidal* if the set of the exponent vectors of the minimal monomial generators of I is the set of bases of a discrete polymatroid [13].

Polymatroidal ideals are characterized by the *exchange property* [13] [Theorem 2.3]. For a monomial $u \in S$, let

$$\deg_{x_i}(u) \ = \ \max\{j \ : \ x_i^j \text{ divides } u\}.$$

Lemma 1. *Let $I \subset S$ be a monomial ideal generated in a single degree. Then I is polymatroidal if and only if the following exchange property holds: for all $u, v \in \mathcal{G}(I)$ and all i such that $\deg_{x_i}(u) > \deg_{x_i}(v)$, there exists j with $\deg_{x_j}(u) < \deg_{x_j}(v)$ such that $x_j(u/x_i) \in \mathcal{G}(I)$.*

An arbitrary monomial ideal I is called *componentwise polymatroidal* if the component $I_{\langle j \rangle}$ is polymatroidal for all j. Here, for a graded ideal $J \subset S$ and an integer j, we denote by $J_{\langle j \rangle}$ the graded ideal generated by all polynomials of degree j belonging to J.

Polymatroidal ideals are vertex splittable [18] [Lemma 2.1]. In Proposition 4.2, we prove the analogous case for componentwise polymatroidal ideals.

3. A Cohen–Macaulay Criterion

In this section, we introduce a new criterion for the Cohen–Macaulayness of vertex splittable ideals. As a result, we obtain characterizations for Gorenstein, level and pseudo-Gorenstein vertex splittable ideals.

The main result in the section is the following.

Theorem 2. *Let $I \subset S$ be a vertex splittable monomial ideal such that $I \subseteq \mathfrak{m}^2$, and let x_i be a splitting vertex of I. Then, the following conditions are equivalent:*

(a) *I is CM.*

(b) *$(I : x_i), (I, x_i)$ are CM, and* $\operatorname{depth} S/(I : x_i) = \operatorname{depth} S/(I, x_i)$.

Proof. We may assume $i = 1$. Let $I = x_1 I_1 + I_2$ be the vertex splitting of I. Since $I = x_i I_1 + I_2$ is a Betti splitting (Theorem 1), then Formula (3) together with the Auslander–Buchsbaum Formula (1), implies

$$\operatorname{depth} S/I \ = \ \min\{\operatorname{depth} S/(x_1 I_1), \operatorname{depth} S/I_2, \operatorname{depth} S/(x_1 I_1 \cap I_2) - 1\}.$$

Notice that $\operatorname{depth} S/x_1I_1 = \operatorname{depth} S/I_1$ and $x_1I_1 \cap I_2 = x_1(I_1 \cap I_2) = x_1I_2$, because $I_2 \subseteq I_1$ and x_1 does not divide any minimal monomial generator of I_2. Consequently, $\operatorname{depth} S/(x_1I_1 \cap I_2) = \operatorname{depth} S/(x_1I_2) = \operatorname{depth} S/I_2$, and so

$$\operatorname{depth} S/I = \min\{\operatorname{depth} S/I_1, \operatorname{depth} S/I_2 - 1\}. \tag{4}$$

We have the short exact sequence

$$0 \to S/(I : x_1) \to S/I \to S/(I, x_1) \to 0.$$

Notice that $(I : x_1) = (x_1I_1 + I_2) : x_1 = (x_1I_1 : x_1) + (I_2 : x_1) = I_1 + I_2 = I_1$, because x_1 does not divide any minimal monomial generator of I_2 and $I_2 \subseteq I_1$. Since $I \subseteq \mathfrak{m}^2$, we have $x_1 \notin I$. Thus, $I_1 \neq S$. Moreover, $(I, x_1) = (x_1I_1 + I_2, x_1) = (I_2, x_1)$, and so we obtain the short exact sequence

$$0 \to S/I_1 \to S/I \to S/(I_2, x_1) \to 0.$$

Hence, $\dim S/I = \max\{\dim S/I_1, \dim S/(I_2, x_1)\}$. Since $S/(I_2, x_1) \cong K[x_2, \dots, x_n]/I_2$, we obtain that $\dim S/(I_2, x_1) = \dim K[x_2, \dots, x_n]/I_2 = \dim S/I_2 - 1$. Hence,

$$\dim S/I = \max\{\dim S/I_1, \dim S/I_2 - 1\}. \tag{5}$$

(a)⇒(b) Suppose that I is CM. By Equations (4) and (5), we have

$$\dim S/I \geq \dim S/I_1 \geq \operatorname{depth} S/I_1 \geq \operatorname{depth} S/I = \dim S/I,$$

and

$$\dim S/I \geq \dim S/I_2 - 1 \geq \operatorname{depth} S/I_2 - 1 \geq \operatorname{depth} S/I = \dim S/I.$$

Hence, S/I_1, S/I_2 are CM and $\operatorname{depth} S/I_1 = \operatorname{depth} S/I = \operatorname{depth} S/I_2 - 1$.

(b)⇒(a) Conversely, assume that S/I_1 and S/I_2 are CM and that $\operatorname{depth} S/I_1 = \operatorname{depth} S/I_2 - 1$. Then,

$$\min\{\operatorname{depth} S/I_1, \operatorname{depth} S/I_2 - 1\} = \operatorname{depth} S/I_1,$$

and

$$\max\{\dim S/I_1, \dim S/I_2 - 1\} = \dim S/I_1.$$

Equations (4) and (5) imply that $\operatorname{depth} S/I = \operatorname{depth} S/I_1 = \dim S/I_1 = \dim S/I$ and so S/I is CM. □

The next important vanishing theorem due to Grothendieck [9] [Theorem 3.5.7] will be crucial to characterize Gorenstein, level and pseudo-Gorenstein vertex splittable ideals. If $(R, \mathfrak{m}, k)$ is a Noetherian local ring and M a finitely generated R-module, we denote by $H^i_{\mathfrak{m}}(M)$ the ith local cohomology module of M with support on $\mathfrak{m}$ [9].

Theorem 3. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and M a finitely generated R-module of depth t and dimension d. Then*

(a) $H^i_{\mathfrak{m}}(M) = 0$ *for $i < t$ and $i > d$.*
(b) $H^t_{\mathfrak{m}}(M) \neq 0$ *and* $H^d_{\mathfrak{m}}(M) \neq 0$.

Corollary 1. *Let $I \subset S$ be a vertex splittable CM ideal such that $I \subseteq \mathfrak{m}^2$, and let $I = x_iI_1 + I_2$ be a vertex splitting of I. Then, the following statements hold:*

(a) $\text{CM-type}(S/I) = \text{CM-type}(S/I_1) + \text{CM-type}(S/I_2)$.
(b) *S/I is Gorenstein if and only if I is a principal ideal.*
(c) *S/I is level if and only if S/I_1 and S/I_2 are level and* $\operatorname{reg} S/I_1 + 1 = \operatorname{reg} S/I_2$.

(d) *S/I is pseudo-Gorenstein if and only if one of the following occurs: Either S/I_1 is pseudo-Gorenstein and $\operatorname{reg} S/I_1 + 1 > \operatorname{reg} S/I_2$ or S/I_2 is pseudo-Gorenstein and $\operatorname{reg} S/I_1 + 1 < \operatorname{reg} S/I_2$.*

(e) $H_{\mathfrak{m}}^{\dim S/I}(S/I)/H_{\mathfrak{m}}^{\dim S/I}(S/(I:x_i)) \cong H_{\mathfrak{m}}^{\dim S/I}(S/(I,x_i))$.

Proof. We may assume that $x_i = x_1$. Since S/I is CM, Theorem 2 guarantees that $S/I_1, S/I_2$ are CM and $\operatorname{depth} S/I_1 = \operatorname{depth} S/I_2 - 1$. Hence, $\operatorname{pd} S/I_1 = \operatorname{pd} S/I_2 + 1$. Let $p = \operatorname{pd} S/I_1$. In particular, we have $p = \operatorname{pd} S/I$. Now, by [1] [Remark 2.10], we have for all j

$$\beta_{p,p+j}(S/I) \;=\; \beta_{p,p+j-1}(S/I_1) + \beta_{p,p+j}(S/I_2) + \beta_{p-1,p-1+j}(S/I_2).$$

Since $\operatorname{pd} S/I_2 = p - 1$, the above formula simplifies to

$$\beta_{p,p+j}(S/I) \;=\; \beta_{p,p+j-1}(S/I_1) + \beta_{p-1,p-1+j}(S/I_2). \tag{6}$$

From this formula, we deduce that

$$\operatorname{reg} S/I \;=\; \max\{\operatorname{reg} S/I_1 + 1 \;\; \operatorname{reg} S/I_2\}.$$

We obtain the following:

(a) The assertion follows immediately from (6).

(b) In the proof of Theorem 2 we noted, that $I_1 \neq 0, S$. Thus, CM-type$(S/I_1) \geq 1$. By (a), it follows that I is Gorenstein if and only if I_1 is Gorenstein, and $I_2 = 0$. Using Formula (4) and Theorem 2 (b),we obtain $\operatorname{depth} S/I = n - 1$. Since $\operatorname{depth} S/I = n - \operatorname{pd} S/I$, we have $\operatorname{pd} S/I = 1$, equivalent to saying that I is a principal ideal.

(c) Assume that S/I is level. Then $\beta_{p,p+j}(S/I) \neq 0$ only for $j = \operatorname{reg} S/I$. Since $\operatorname{reg} S/I = \max\{\operatorname{reg} S/I_1 + 1, \operatorname{reg} S/I_2\}$ and $\beta_{p,p+\operatorname{reg} S/I_1}(S/I_1), \beta_{p-1,p-1+\operatorname{reg} S/I_2}(S/I_2)$ are both nonzero, we deduce from Formula (6) that $\operatorname{reg} S/I_1 + 1 = \operatorname{reg} S/I_2$ and that S/I_1, S/I_2 are level. Conversely, if $\operatorname{reg} S/I_1 + 1 = \operatorname{reg} S/I_2$ and S/I_1, S/I_2 are level, we deduce from Formula (6) that S/I is level.

(d) Assume that S/I is pseudo-Gorenstein. Then $\beta_{p,p+\operatorname{reg} S/I}(S/I) = 1$. Since $\operatorname{reg} S/I = \max\{\operatorname{reg} S/I_1 + 1, \operatorname{reg} S/I_2\}$ and $\beta_{p,p+\operatorname{reg} S/I_1}(S/I_1), \beta_{p-1,p-1+\operatorname{reg} S/I_2}(S/I_2)$ are both nonzero, we deduce from Formula (6) that either S/I_1 is pseudo-Gorenstein and $\operatorname{reg} S/I_1 + 1 > \operatorname{reg} S/I_2$ or S/I_2 is pseudo-Gorenstein and $\operatorname{reg} S/I_1 + 1 < \operatorname{reg} S/I_2$. The converse can be proved in a similar way.

(e) Since S/I is CM, Theorem 2 implies that $S/(I : x_i)$ and $S/(I, x_i)$ are CM and $\dim S/(I : x_i) = \dim S/(I, x_i) = \dim S/I$. As shown in the proof of Theorem 2, we have the short exact sequence

$$0 \to S/(I : x_i) \to S/I \to S/(I, x_i) \to 0.$$

This sequence induces the long exact sequence of local cohomology modules:

$$\cdots \to H_{\mathfrak{m}}^{i-1}(S/(I,x_i)) \to H_{\mathfrak{m}}^{i}(S/(I:x_i)) \to H_{\mathfrak{m}}^{i}(S/I) \to H_{\mathfrak{m}}^{i}(S/(I,x_i)) \to \cdots.$$

Let M be a finitely generated CM S-module. By Theorem 3, $H_{\mathfrak{m}}^{i}(M) \neq 0$ if and only if $i = \operatorname{depth} M = \dim M$. Thus, the above exact sequence simplifies to

$$0 \to H_{\mathfrak{m}}^{\dim S/I}(S/(I:x_i)) \to H_{\mathfrak{m}}^{\dim S/I}(S/I) \to H_{\mathfrak{m}}^{\dim S/I}(S/(I,x_i)) \to 0,$$

and the assertion follows. □

Remark 2. *It is clear that any ideal $I \subset S$ generated by a subset of the variables of S is Gorenstein and vertex splittable. Hence, Corollary 1 implies immediately that the only Gorenstein vertex splittable ideals of S are the principal monomial ideals and the ideals generated by a subset of the variables.*

4. Families of Cohen–Macaulay Vertex Splittable Ideals

In this section, by using Theorem 2, we recover in a simple and very effective manner Cohen–Macaulay criteria for several families of monomial ideals. We use the fact that if $I = x_i I_1 + I_2$ is a vertex splitting, then I_1, I_2 are vertex splittable ideals that, in good cases, belong again to a given family of vertex splittable monomial ideals and to which one may apply inductive arguments.

The first two families were introduced in Section 2.2.

4.1. (Vector-Spread) Strongly Stable Ideals

In [19] [Theorem 4.3], we classified the CM $\mathbf{t}$-spread strongly stable ideals. Here, we recover this result using Theorem 2.

Proposition 1. *Let $I \subset S$ be a $\mathbf{t}$-spread strongly stable ideal such that $I \subseteq \mathfrak{m}^2$. Then*

(a) *I is vertex splittable;*
(b) *I is CM if and only if there exists $\ell \leq d$ such that*

$$x_{n-(t_1+t_2\cdots+t_{\ell-1})}x_{n-(t_2+t_3\cdots+t_{\ell-1})}\cdots x_{n-t_{\ell-1}}x_n \in \mathcal{G}(I).$$

Proof. (a) We proceed by double induction on the number of variables n and the highest degree d of a generator $u \in \mathcal{G}(I)$. If $n = 1$, then I is a principal ideal whether or not the integer d is, and so it is vertex splittable. Suppose $n > 1$. If $d = 1$, then I is an ideal generated by a subset of the variables and it is clearly vertex splittable. Suppose $d > 1$. We can write $I = x_1 I_1 + I_2$, where $\mathcal{G}(I_1) = \{u/x_1 : u \in \mathcal{G}(I),\ x_1 \text{ divides } u\}$ and $\mathcal{G}(I_2) = \mathcal{G}(I) \setminus \mathcal{G}(x_1 I_1)$. It is immediately clear that $I_1 \subset S$ is $(t_2, \ldots, t_{d-1})$-spread strongly stable and that I_2 is a $\mathbf{t}$-spread strongly stable ideal of $K[x_2, \ldots, x_n]$. By induction on n and d, we have that I_1 and I_2 are vertex splittable. Hence, so is I.

(b) We may suppose that x_n divides some minimal generator of I. Otherwise, we can consider I as a monomial ideal of a smaller polynomial ring. If I is principal, then we have $I = (u) = (x_1 x_{1+t_1} \cdots x_{1+t_1+\cdots+t_{\ell-1}})$, with $n = 1 + t_1 + \cdots + t_{\ell-1}$, and $\ell \leq d$. Otherwise, if I is not principal, then $\operatorname{pd} S/I > 1$, and we can write $I = x_1 I_1 + I_2$ as above. By Theorem 2, I_2 is CM and $\operatorname{pd} S/I_2 = \operatorname{pd} S/I + 1 > 1$. Thus, $I_2 \neq 0$. Hence, by induction, there exists $\ell \leq d$ such that

$$x_{n-(t_1+\cdots+t_{\ell-1})}\cdots x_{n-t_{\ell-1}}x_n \in \mathcal{G}(I_2).$$

Since $\mathcal{G}(I_2) \subset \mathcal{G}(I)$, the assertion follows. □

4.2. Componentwise Polymatroidal Ideals

In this subsection, we prove that componentwise polymatroidal ideals are also vertex splittable.

A longstanding conjecture of Bandari and Herzog predicted that componentwise polymatroidal ideals have linear quotients [20]. This conjecture was solved recently in [21] [Theorem 3.1]. Inspecting the proof of this theorem, we obtain the following:

Proposition 2. *Componentwise polymatroidal ideals are vertex splittable.*

Proof. Let $I \subset S$ be a componentwise polymatroidal ideal. We prove the statement by induction on $|\mathcal{G}(I)|$. We may assume that all variables x_i divide some minimal monomial generator of I. Moreover, it holds that for any variable x_i which divides a minimal monomial generator of minimal degree of I, we can write $I = x_i I_1 + I_2$, where $\mathcal{G}(x_i I_1) = \{u \in \mathcal{G}(I) : x_i \text{ divides } u\}$, $\mathcal{G}(I_2) = \mathcal{G}(I) \setminus \mathcal{G}(x_i I_1)$ and the following properties are satisfied (see the proof of [21] [Theorem 3.1]):

(i) $I_2 \subseteq I_1$ as monomial ideals of S.
(ii) $x_i I_1$ is a componentwise polymatroidal ideal of S.
(iii) I_2 is a componentwise polymatroidal ideal of $K[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$.

By induction, it follows that both I_1 and I_2 are vertex splittable. Hence, so is I. $\square$

We have the following corollary.

Corollary 2. *Let $I \subset S$ be a componentwise polymatroidal ideal and let x_i be any variable dividing some minimal monomial generator of least degree of I. Suppose that $I \subseteq \mathfrak{m}^2$. Then, the following conditions are equivalent.*

(a) *I is CM.*
(b) *$(I : x_i), (I, x_i)$ are CM componentwise polymatroidal ideals and* $\operatorname{depth} S/(I : x_i) = \operatorname{depth} S/(I, x_i)$.

Moreover, if $I \subset S$ is a polymatroidal ideal generated in degree $d \geq 2$ and x_i is a variable dividing some monomial of $\mathcal{G}(I)$, then $(I : x_i)$ is polymatroidal. And, in addition, if I is CM, then $(I : x_i)$ is also CM.

Proof. It follows by combining the vertex splitting presented in the proof of Proposition with the facts (ii) and (iii) and with Theorem 2. For the last statement, see [22] [Lemma 5.6]. $\square$

At the moment, to classify all CM componentwise polymatroidal ideals seems a hopeless task. For instance, let J be any componentwise polymatroidal ideal. Let ℓ be the highest degree of a minimal monomial generator of J, and let $d > \ell$ be any integer. It is easy to see that $I = J + \mathfrak{m}^d$ is componentwise polymatroidal. Since $\dim S/I = 0$, then S/I is automatically CM.

Example 1. *Consider the ideal $I = (x_1^2, x_1x_3, x_3^2, x_1x_2x_4, x_2x_3x_4, x_2^2x_4^2)$ of $S = K[x_1, x_2, x_3, x_4]$, see [21] [Example 3.2]. One can easily check that I is a CM componentwise polymatroidal ideal. Indeed, it is not difficult to check that $I_{\langle j \rangle}$ is polymatroidal for $j = 0, 1, 2, 3, 4$. For $j \geq 5$, the ideal $I_{\langle j \rangle} = \mathfrak{m}^{j-4} I_{\langle 4 \rangle}$ is polymatroidal because it is the product of two polymatroidal ideals [16] [Theorem 12.6.3]. Notice that in this case,* $\dim S/I > 0$.

Nonetheless, if I is generated in a single degree, that is, if I is actually polymatroidal, then Herzog and Hibi [23] [Theorem 4.2] showed that I is CM if and only if (i) I is a principal ideal, (ii) I is a *squarefree Veronese ideal* $I_{n,d}$, that is, it is generated by all squarefree monomials of S of a given degree $d \leq n$, or (iii) I is a *Veronese ideal*, that is, $I = \mathfrak{m}^d$ for some integer $d \geq 1$.

The proof presented by Herzog and Hibi is based on the computation of $\sqrt{I}$. We now present a different proof based on the criterion for Cohen–Macaulayness proved in Theorem 2.

Corollary 3. *A polymatroidal $I \subset S$ is CM if and only if I is one of the following:*

(a) *A principal ideal.*
(b) *A Veronese ideal.*
(c) *A squarefree Veronese ideal.*

For the proof, we need the following well-known identities. For the convenience of the reader, we provide a proof that uses the vertex splittings technique.

Lemma 2. *Let $n, d \geq 1$ be positive integers. Then*

$$\operatorname{pd} S/\mathfrak{m}^d = n, \quad \textit{and} \quad \operatorname{pd} S/I_{n,d} = n + 1 - d.$$

Proof. Since $\dim S/\mathfrak{m}^d = 0$, we have $\operatorname{depth} S/\mathfrak{m}^d = 0$ and $\operatorname{pd} S/\mathfrak{m}^d = n$. If $d = 1$, then $I_{n,1} = \mathfrak{m}$ and $\operatorname{pd} S/I_{n,1} = n$. If $1 < d \leq n$, we notice that $I_{n,d} = x_n I_{n-1,d-1} + I_{n-1,d}$ is a vertex splitting. By Formula (4) and induction on n and d,

$$\begin{aligned}\operatorname{pd} S/I_{n,d} &= \min\{\operatorname{pd} S/I_{n-1,d-1}, \operatorname{pd} S/I_{n-1,d} + 1\}\\ &= \min\{n-1+1-(d-1), n-1+1-d+1\}\\ &= n+d-1,\end{aligned}$$

as wanted.

□

We are now ready for the proof of Corollary 3.

Proof of Corollary 3. Let I be a polymatroidal ideal. We proceed by induction on $|\mathcal{G}(I)|$. If $|\mathcal{G}(I)| = 1$, then I is principal and it is CM. Now, let $|\mathcal{G}(I)| > 1$. If $I = \mathfrak{m}$, then I is CM. Thus, we assume that I is generated in degree $d \geq 2$, that all variables x_i divide some minimal monomial generator of I and that the greatest common divisor of the minimal monomial generators of I is one. By Proposition 4.2 and Corollary 2, we have a vertex splitting $I = x_i I_1 + I_2$ for each variable x_i I_1 and I_2 are CM polymatroidal ideals with $\operatorname{depth} S/I_1 = \operatorname{depth} S/I_2 - 1$. Thus, $\operatorname{pd} S/I_1 = \operatorname{pd} S/I_2 + 1$. We may assume that $x_i = x_n$.

Since $|\mathcal{G}(x_n I_1)|, |\mathcal{G}(I_2)|$ are strictly less than $|\mathcal{G}(I)|$, by induction, it follows that I_1 is either a principal ideal, a Veronese ideal or a squarefree Veronese ideal, and the same possibilities occur for I_2. We distinguish the various possibilities.

Case 1. Let I_2 be a principal ideal, then $\operatorname{pd} S/I_2 = 1$. Thus, $\operatorname{pd} S/I_1 = 2$.

Under this assumption, I_1 cannot be principal because $\operatorname{pd} S/I_1 = 2 \geq 1$.

Assume that I_1 is a Veronese ideal in m variables. Since all variables of S divide some monomial of $\mathcal{G}(I)$ and $I_2 \subset I_1$, then $I_1 = \mathfrak{m}^d$ or $I_1 = \mathfrak{n}^d$, where $\mathfrak{m} = (x_1, \dots, x_n)$ and $\mathfrak{n} = (x_1, \dots, x_{n-1})$. Thus, $m = n$ or $m = n-1$. Lemma 2 implies $m = \operatorname{pd} S/I_1 = 2$. So, $n = 2$ or $n = 3$.

If $n = 2$, then $I = x_2(x_1, x_2)^{d-1} + (x_1^d) = (x_1, x_2)^d$, which is CM and Veronese, or $I = x_2(x_1^{d-1}) + (x_1^d) = x_1^{d-1}(x_1, x_2)$, which is not CM.

Otherwise, if $n = 3$, then $I = x_3(x_1, x_2, x_3)^{d-1} + (u)$ or else $I = x_3(x_1, x_2)^{d-1} + (u)$ where $u \in K[x_1, x_2]$ is a monomial of degree d. If $d = 2$, then one easily sees that only in the second case and for $u = x_1 x_2$ we have that I is a CM polymatroidal ideal, which is the squarefree Veronese $I_{3,2}$. Otherwise, suppose $d \geq 3$. We may assume that x_1^2 divides u. In the first case, $(I : x_1) = x_3(x_1, x_2, x_3)^{d-2} + (u/x_1)$ is not principal, nor Veronese, nor squarefree Veronese. Thus, by induction, $(I : x_1)$ is not a CM polymatroidal ideal, and by Corollary 2, we deduce that I is also not CM. Similarly, in the second case, we see that $(I : x_1) = x_3(x_1, x_2)^{d-1} + (u/x_1)$, and thus also I, is not a CM polymatroidal ideal.

Assume now that I_1 is a squarefree Veronese ideal in m variables. Then as argued in the case 1.2 we have $m = n$ or $m = n-1$. Lemma 2 gives $\operatorname{pd} S/I_1 = m + 1 - (d-1) = 2$. Thus, $d = m$. Hence $d = n$ or $d = n-1$. So $I = x_n I_{n,n-1} + (u)$ or $I = x_n I_{n-1,n-2} + (u)$ where $u \in K[x_1, \dots, x_{n-1}]$ is a monomial of degree $d = n$ in the first case or $d = n-1$ in the second case. In the first case there is i such that x_i^2 divides u. Say $i = 1$. Then $(I : x_1) = x_n(I_{n,n-1} : x_1) + (u/x_1)$ is not a principal ideal, neither a Veronese ideal, neither a squarefree Veronese. Therefore, by induction $(I : x_1)$ we see that is not a CM polymatroidal ideal, and by Corollary 2 I is also not a CM polymatroidal ideal. Similarly, in the second case, if $u = x_1 \cdots x_{n-1}$, then $I = I_{n,n-1}$ is a CM squarefree Veronese ideal. Otherwise, x_i^2, say with $i = 1$, divides u, and then, arguing as before, we see that I is not a CM polymatroidal ideal.

Case 2. Let I_2 be a Veronese ideal in m variables, then $\operatorname{pd} S/I_2 = m \leq n-1$.

Under this assumption, I_1 cannot be principal because $\operatorname{pd} S/I_1 = m + 1 > 1$.

Assume that I_1 is a Veronese ideal in ℓ variables. Then $\ell = n$ or $\ell = n-1$. Lemma 2 implies that $\ell = \operatorname{pd} S/I_1 = \operatorname{pd} S/I_2 + 1 = m+1$. Thus, $\ell = n$ and $m = n-1$ or $\ell = n-1$ and $m = n-2$. In the first case, $I = x_n\mathfrak{m}^{d-1} + (x_1,\dots,x_{n-1})^d = \mathfrak{m}^d$ is a CM Veronese ideal. In the second case, up to relabeling, we can write $I = x_n(x_1,\dots,x_{n-1})^{d-1} + (x_1,\dots,x_{n-2})^d$. However, this ideal is not polymatroidal. Otherwise, by the exchange property (Lemma 1) applied to $u = x_n x_{n-1}^{d-1}$ and $v = x_{n-2}^d$, we should have $x_{n-2}x_{n-1}^{d-1} \in I$, which is not the case.

Assume now that I_1 is a squarefree Veronese ideal in ℓ variables.Then $\ell = n$ or $\ell = n-1$. Lemma 2 implies that $\operatorname{pd} S/I_1 = \ell + 1 - (d-1) = \ell + 2 - d = \operatorname{pd} S/I_2 + 1 = m+1$. Thus, either $m = n+1-d$ or $m = n-d$. Up to relabeling, we have either $I = x_n I_{n,d-1} + (x_1,\dots,x_{n+1-d})^d$ or $I = x_n I_{n-1,d-1} + (x_1,\dots,x_{n-d})^d$. If $d=2$, then these ideals become either $I = (x_1,\dots,x_n)^2$, which is a CM Veronese ideal, or $I = x_n(x_1,\dots,x_{n-1}) + (x_1,\dots,x_{n-2})^2$, which is not polymatroidal because the exchange property does not hold for $u = x_n x_{n-1}$ and $v = x_{n-2}^2$ since $x_{n-1}x_{n-2} \notin I$. If $d \geq 3$, then the above ideals are not polymatroidal. In the first case, the exchange property does not hold for $u = x_{n+1-d}^d \in I_2$ and $v = (x_{n+2-d}\cdots x_n)x_n \in x_n I_1$, otherwise $x_j x_{n+1-d}^{d-1} \in I$ for some $n+2-d \leq j \leq n$, which is not the case.

Case 3. Let I_2 be a squarefree Veronese in m variables, $m \leq n-1$. Then Lemma 2 implies $\operatorname{pd} S/I_2 = m+1-d$ with $d \leq m$. Hence, $\operatorname{pd} S/I_1 = m+2-d$.

In such a case, the ideal I_1 cannot be principal because $\operatorname{pd} S/I_1 = m+2-d > 1$.

Assume that I_1 is a Veronese ideal in ℓ variables. Then $\ell = n$ or $\ell = n-1$. Lemma 2 implies that $\ell = \operatorname{pd} S/I_1 = m+2-d$. Hence, either $d = m-n+2$ or $d = m-n+3$. Since $m \leq n-1$, either $d \leq 1$ or $d \leq 2$. Only the case $d=2$ is possible. If $d=2$, then $\ell = m = n-1$ and we have $I = x_n(x_1,\dots,x_{n-1}) + (x_1,\dots,x_{n-1})^2$. This ideal is not CM, otherwise it would be height-unmixed. Indeed, $(I : x_n) = (x_1,\dots,x_{n-1})$ and $(I : x_{n-1}) = (x_1,\dots,x_n)$ are two associated primes of I having different heights.
Finally, assume that I_1 is a squarefree Veronese ideal in ℓ variables. Then $\ell = n$ or $\ell = n-1$. Lemma 2 implies that $\operatorname{pd} S/I_1 = \ell+1-(d-1) = \ell+2-d = m+2-d$. Thus, $\ell = m$ and so either $\ell = m = n$ or $\ell = m = n-1$. The first case is impossible because $m \leq n-1$. In the second case, we have $I = x_n I_{n-1,d-1} + I_{n-1,d} = I_{n,d}$ which is a CM squarefree Veronese ideal. □

4.3. Bi-Cohen–Macaulay Graphs

Let $I \subset S$ be a squarefree monomial ideal. Then I may be seen as the Stanley–Reisner ideal of a unique simplicial complex on the vertex set $\{1,\dots,n\}$. Attached to I is the Alexander dual $I^\vee$, which is again a squarefree monomial ideal. We say that I is *bi-Cohen–Macaulay* (bi-CM for short) if both I and $I^\vee$ are CM. By the Eagon–Reiner criterion [16] [Theorem 8.1.9], I has a linear resolution if and only if $I^\vee$ is CM. Hence, I is bi-CM if and only if it is CM with linear resolution.

Let G be a finite simple graph on the vertex set $V(G) = \{1,\dots,n\}$ with edge set $E(G)$. The *edge ideal* $I(G)$ of G is the squarefree monomial ideal of S generated by the monomials $x_i x_j$ with $\{i,j\} \in E(G)$ [4]. The Alexander dual of $I(G)$ is the squarefree monomial ideal of S generated by the squarefree monomial $x_{i_1}\cdots x_{i_t}$ such that $\{i_1,\dots,i_t\}$ is a minimal vertex cover of G [4]. Such an ideal is denoted by $J(G)$ and, since its definition, it is often called the *cover ideal* of G.

We say that G is a bi-CM graph if $I(G)$ is bi-CM.

Let G be a graph. The *open neighborhood* of $i \in V(G)$ is the set

$$N_G(i) = \{j \in V(G) : \{i,j\} \in E(G)\}.$$

A graph G is called *chordal* if it has no induced cycles of a length bigger than three. A *perfect elimination order* of G is an ordering $v_1,\dots,v_n$ of its vertex set $V(G)$ such that $N_{G_i}(v_i)$ induces a complete subgraph on G_i, where G_i is the induced subgraph of G on the vertex set $\{i, i+1,\dots,n\}$. Hereafter, if $1,2,\dots,n$ is a perfect elimination order of G, we highlight it by $x_1 > x_2 > \cdots > x_n$. For a *complete* graph G, we mean a graph satisfying the property that every set $\{i,j\}$ with $i,j \in V(G)$, $i \neq j$ is an edge of G.

Theorem 4 ([24])**.** *A finite simple graph G is chordal if and only if G admits a perfect elimination order.*

The edge ideals with linear resolution were classified by Fröberg [25]. Recall that the *complementary graph* G^c of G is the graph with vertex set $V(G^c) = V(G)$ and where $\{i,j\}$ is an edge of G^c if and only if $\{i,j\} \notin E(G)$. A graph G is called *cochordal* if and only if G^c is chordal.

Theorem 5 ([25] [Theorem 1])**.** *Let G be a finite simple graph. Then $I(G)$ has a linear resolution if and only if G is cochordal.*

We quote the next fundamental result which was proved by Moradi and Khosh-Ahang [1] [Theorem 3.6, Corollary 3.8].

Proposition 3. *Let G be a finite simple graph. Then $I(G)$ has linear resolution if and only if $I(G)$ is vertex splittable. Furthermore, if $x_1 > \cdots > x_n$ is a perfect elimination order of G^c, then*

$$I(G) \;=\; x_1(x_j : j \in N_G(1)) + I(G \setminus \{1\})$$

is a vertex splitting of $I(G)$.

Combining the above result with Theorem 2, we obtain the next characterization of the bi-CM graphs.

Theorem 6. *For a finite simple graph G, the following conditions are equivalent.*

(a) *G is a bi-CM graph.*

(b) *G^c is a chordal graph with perfect elimination order $x_1 > \cdots > x_n$ and*

$$|N_G(i) \cap \{i,\dots,n\}| \;=\; |N_G(j) \cap \{j,\dots,n\}| + (j-i),$$

for all $1 \le i \le j \le n$ such that $|N_G(i) \cap \{i,\dots,n\}|, |N_G(j) \cap \{j,\dots,n\}| > 0$.

In particular, if any of the equivalent conditions hold, then

$$\operatorname{pd} S/I(G) \;=\; |N_G(i) \cap \{i,\dots,n\}| + (i-1),$$

for any $1 \le i \le n$ such that $|N_G(i) \cap \{i,\dots,n\}| > 0$.

Proof. We proceed by induction on $n = |V(G)|$. By Theorem 5, G must be cochordal. Fix $x_1 > \cdots > x_n$, which is a perfect elimination order of G^c. By Proposition 3, $I(G) = x_1(x_j : j \in N_G(1)) + I(G \setminus \{1\})$ is a vertex splitting. Applying Theorem 2, $I(G)$ is CM if and only if $J = (x_j : j \in N_G(1))$ and $I(G \setminus \{1\})$ are CM and $\operatorname{pd} S/I(G) = \operatorname{pd} S/J = \operatorname{pd} S/I(G \setminus \{1\}) + 1$. J is CM because it is an ideal generated by variables and $\operatorname{pd} S/J = |N_G(1)|$. Notice that $x_2 > \cdots > x_n$ is a perfect elimination order of $(G \setminus \{1\})^c$.

If $I(G \setminus \{1\}) = 0$, then $\operatorname{pd} S/I(G) = \operatorname{pd} S/J = \operatorname{pd} S/I(G \setminus \{1\}) + 1 = 1$ and $I(G)$ is principal, say $I(G) = (x_1x_2)$. In this case, the thesis holds.

Suppose now that $I(G \setminus \{1\}) \neq 0$. Then, by induction on n, $I(G \setminus \{1\})$ is CM if and only if

$$|N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}| \;=\; |N_{G\setminus\{1\}}(j) \cap \{j,\dots,n\}| + (j-i), \tag{7}$$

for all $2 \le i \le j \le n$ such that $|N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}|, |N_{G\setminus\{1\}}(j) \cap \{j,\dots,n\}| > 0$ and moreover

$$\operatorname{pd} S/I(G \setminus \{1\}) \;=\; |N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}| + (i-2), \tag{8}$$

for any $2 \le i \le n$ such that $|N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}| > 0$.

Notice that $N_G(i) \cap \{i,\dots,n\} = N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}$ for all $2 \le i \le n$. Thus, by combining (7) and (8) with the equality $\operatorname{pd} S/I(G) = \operatorname{pd} S/J = \operatorname{pd} S/I(G \setminus \{1\}) + 1$, we see that $I(G)$ is CM if and only if

$$\begin{aligned} |N_G(1)| &= |N_G(1) \cap \{1,\dots,n\}| = |N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}| + (i-2) + 1 \\ &= |N_G(i) \cap \{i,\dots,n\}| + (i-1), \end{aligned}$$

for all $2 \le i \le n$ such that $|N_{G\setminus\{1\}}(i) \cap \{i,\dots,n\}| > 0$.

Thus, we deduce that $|N_G(i) \cap \{i,\dots,n\}| = |N_G(j) \cap \{j,\dots,n\}| + (j-i)$ for all $1 \le i \le j \le n$ such that $|N_G(i) \cap \{i,\dots,n\}|, |N_G(j) \cap \{j,\dots,n\}| > 0$, as desired. The inductive proof is complete. □

Notice that in the above characterization, the field K plays no role. In other words, the bi-CM property of edge ideals does not depend on the field K. This also follows from the work of Herzog and Rahimi [26] [Corollary 1.2 (d)], where other classifications of the bi-CM graphs are given.

We end this paper with a couple of examples of a bi-CM and a non-bi-CM graph.

Examples 1. *(a) Consider the graph G on five vertices and its complementary graph G^c depicted below in Figure 1.*

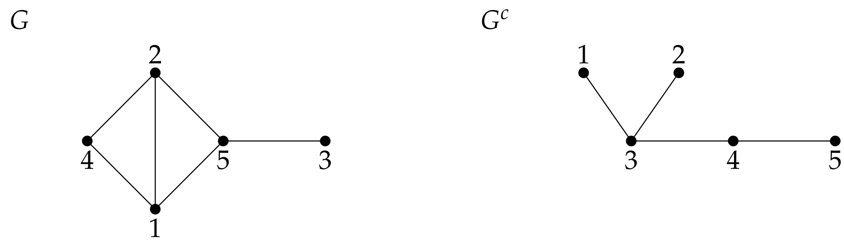

Figure 1. A bi-CM graph.

Notice that $x_1 > x_2 > x_3 > x_4 > x_5$ is a perfect elimination order of G^c, so that G^c is chordal (Theorem 4). We have $|N_G(i) \cap \{i,\dots,5\}| > 0$ only for $i = 1,2,3$. It is easy to see that condition (b) of Theorem 6 is verified. Hence G is bi-CM, as one can also verify by using Macaulay2 [27].

(b) Consider the graph H and its complementary graph H^c depicted below in Figure 2.

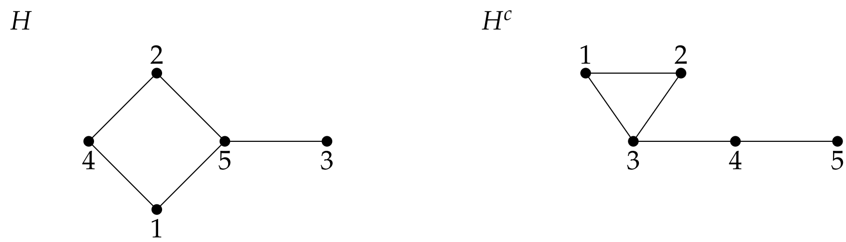

Figure 2. A not bi-CM graph.

As before, $x_1 > x_2 > x_3 > x_4 > x_5$ is a perfect elimination order of H^c, and H^c is chordal. We have $|N_H(i) \cap \{i,\dots,5\}| > 0$ only for $i = 1,2,3$. However, condition (b) of Theorem 6 is not verified. Indeed,

$$|N_H(1) \cap \{1,\dots,5\}| = |N_H(2) \cap \{2,\dots,5\}| = |\{4,5\}| = 2,$$

but $|N_H(1) \cap \{1,\dots,5\}| \neq |N_H(2) \cap \{2,\dots,5\}| + 1$. Hence, H is not bi-CM. We can also verify this by using Macaulay2 [27]. Indeed $J(H)$, the cover ideal of H, is not CM.

5. Conclusions and Perspectives

In view of our main Theorem 2, one can ask for a similar criterion for the sequentially Cohen–Macaulayness of vertex splittable monomial ideals.

Question 1. *Let $I \subset S$ be a vertex splittable ideal, and let $I = x_iI_1 + I_2$ be a vertex splitting of I. Can we characterize the sequentially Cohen–Macaulayness of I in terms of I_1 and I_2?*

This question could have interesting consequences for the theory of polymatroidal ideals. Indeed, a classification of the sequentially Cohen–Macaulay polymatroidal ideals has long been elusive.

On the computational side, to check if a monomial ideal is vertex splittable is far easier than to check if it admits a Betti splitting. Indeed, it is enough to check recursively Definition 1. It could be useful to write a package in *Macaulay2* [27] that checks the vertex splittable property and some related properties.

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