AN ALGORITHM ON NUMBER OF ISOMORPHISM CLASSES OF HYPERGROUPS OF ORDER 3

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Summary. An algorithm which computes the number S_3 of isomorphism classes of hypergroups of order 3 and the cardinality of each class of isomorphism is given.

1. Introduction.

In [2], De Salvo and Freni studied the structure of hypergroups of order 2, finding that – up to isomorphism – there are 8 hypergroups of order 2 included the group and the total hypergroup.

In [3], Migliorato found, by computer, that the total number N_3 of hypergroups of order 3 is 23192 and gave the lower and upper bounds for the number S_3 of non isomorphic hypergroups of order 3, i.e.,

$$\frac{23192}{6} < S_3 < 23192.$$

In the present paper, it is presented an algorithm which computes S_3 and the cardinality of each class of isomorphisms.

For notations and definitions not explicitly mentioned here we refer to [1].

2. The method

Let $\mathcal{H}_3$ be the set of all hypergroupoids of order $n = 3$. Up to isomorphism, we can suppose that they are all defined on the set $H = \{1, 2, 3\}$.

The main idea is to build a bijective correspondence from the set $\mathcal{H}_3$ to a subset X of integer numbers, so that it is possible to order all the hypergroupoids in a finite sequence $\{H_x\}_{x \in X}$ where $X \subseteq \mathbb{N}$. As, it is well known that $|\mathcal{H}_3| = (2^n - 1)^{n^2} = 7^9$, we can set $X = \{x \in \mathbb{N} : 0 \leq x \leq 7^9 - 1\}$.

First, we define a function $\ell : \mathbb{Z}_7 \to \mathbb{P}^*(H)$ setting $\ell(0) = \{1\}$, $\ell(1) = \{2\}$, $\ell(2) = \{3\}$, $\ell(3) = \{1, 2\}$, $\ell(4) = \{2, 3\}$, $\ell(5) = \{1, 3\}$ and $\ell(6) = H$ and a function $g : H^2 \to I_9$ (with $I_9 = \{1, 2, ..., 9\}$ by $g(\alpha, \beta) = n(\alpha - 1) + \beta$ for each $(\alpha, \beta) \in H^2$.

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Now, we consider the *representation function* $r : \mathcal{H}_3 \to \mathbb{Z}_7^9$ which, for any $H \in \mathcal{H}_3$, is defined by $r(H) = \langle r_i \rangle_{i \in I_9}$ where $r_{g(\alpha,\beta)} = \ell^{\leftarrow}(\alpha \cdot \beta)$ for each $(\alpha, \beta) \in H^2$.

Moreover, we introduce the *tag function* $t : \mathbb{Z}_7^9 \to X$ defined by $t(z) = \sum_{i \in I_9} z_i 7^{9-i}$ for each $z = \langle z_i \rangle_{i \in I_9} \in \mathbb{Z}_7^9$.

Since both r and t are bijective functions, so is their composition $t \circ r : \mathcal{H}_3 \to X$ and for any numeric *label* $x \in X$ there is one and only one representation $R = t^{-1}(x) \in \mathbb{Z}_7^9$ of an hypergroupoid $H = (t \circ r)^{-1}(x)$ of $\mathcal{H}_3$. Obviously, from these representations, we can select the hypergroups simply checking if the associative property and the condition of reproducibility are satisfied.

Now, let $p \in \Sigma_H$ be a permutation of H and consider the associated function $\tilde{p} : \mathbb{P}^*(H) \to \mathbb{P}^*(H)$ defined by $\tilde{p}(A) = p[A]$ for each $A \in \mathbb{P}^*(H)$.

The corresponding *label permutation* will be the function $P_p = \ell^{-1} \circ \tilde{p} \circ \ell$ which is a permutation of $\mathbb{Z}_7$.

Moreover, if we consider the product function of p, i.e. $p^2 = p \times p : H^2 \to H^2$ defined by $p^2((\alpha, \beta)) = (p(\alpha), p(\beta))$ for each $(\alpha, \beta) \in H^2$, the *index permutation* will be the function $Q_p = g \circ p^2 \circ g^{-1}$ which is a permutation of I_9.

Thus, for each hypergroup H of representation $r(H)$ and for each permutation $p \in \Sigma_H$ we can find the representation $r(H')$ of the isomorphic hypergroup H' by the corresponding isomorphism $\varphi_p : \mathbb{Z}_7^9 \to \mathbb{Z}_7^9$ which for any $u = \langle u_i \rangle_{i \in I_9} \in \mathbb{Z}_7^9$ is defined by $\varphi_p(u) = \langle v_i \rangle_{i \in I_9}$ where $v_{Q_p(i)} = P_p(u_i)$ for each $i \in I_9$.

Let $\mathbb{H}_3$ be the set of hypergroups of order 3. For each $H \in \mathbb{H}_3$, we denote by $[H] = \{K \in \mathbb{H}_3 : K \simeq H\} = \{r^{-1}(\varphi_p(r(H))) : p \in \Sigma_H\}$.

We choice as representant of the class $[H]$ the hypergroup of minimal label, i.e. the hypergroup

$$H_0 = r^{-1}(x_0) \in \mathbb{H}_3$$

where

$$x_0 = \min\{t(\varphi_p(r(H))) : p \in \Sigma_H\}.$$

So, in order to verify if some hypergroup $(t \circ r)^{-1}(x) \in \mathbb{H}_3$ of label $x \in X$ is the representant of his isomorphism class it suffices to check if $x \leq t(\varphi_p(t^{-1}(x)))$ for each $p \in \Sigma_H$.

Finally, if for each $k \in \{1, \ldots |\Sigma_H|\}$, we say $c_k = |\{[H] : H \in \mathbb{H}_3, |[H]| = k\}|$ the number of classes of isomorphism of cardinality k, it is easy to compute it counting the number of hypergroup labels which cardinality of set of tags of isomorphic image of their representation is exactly k.

3. The algorithm

The following, written in Meta–Pascal, summarizes the algoritm.

```
CONST   n = 3                                   H={1..n};
        base = 2^n − 1;                         max_val_base = base − 1;
        number_hypergroupoids = base^(n^2)      num = number_hypergroupoids − 1;
        num_perm = n! − 1;

TYPE    val_base = 0.. max_val_base;            index = 1..n^2;
        element = 1..n;                         hyperproduct = SET OF element;
        tab = ARRAY [1..n^2] OF val_base;       pack = ARRAY [1..n!] OF integer;

VAR     x, N3, S3 : integer;                    c : pack;
        R : tab;
        ℓ : ARRAY [0.. max_val_base] OF hyperproduct;
```

$$\ell := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \{1\} & \{2\} & \{3\} & \{1,2\} & \{2,3\} & \{1,3\} & H \end{bmatrix}$$

```
        P : ARRAY [1..num_perm, 0..max_val_base] OF val_base;
        Q : ARRAY [1..num_perm, 1..n^2] OF index;
```

$$P[1] := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 1 & 5 & 4 & 3 & 6 \end{bmatrix} \quad Q[1] := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 2 & 7 & 9 & 8 & 4 & 6 & 5 \end{bmatrix}$$

$$P[2] := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 2 & 3 & 5 & 4 & 6 \end{bmatrix} \quad Q[2] := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 6 & 2 & 1 & 3 & 8 & 7 & 9 \end{bmatrix}$$

$$P[3] := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 0 & 4 & 5 & 3 & 6 \end{bmatrix} \quad Q[3] := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 7 & 8 & 3 & 1 & 2 & 6 & 4 & 5 \end{bmatrix}$$

$$P[4] := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & 1 & 5 & 3 & 4 & 6 \end{bmatrix} \quad Q[4] := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 6 & 4 & 8 & 9 & 7 & 2 & 3 & 1 \end{bmatrix}$$

$$P[5] := \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 0 & 4 & 3 & 5 & 6 \end{bmatrix} \quad Q[5] := \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

```
Function g (α, β : element) : index;
Begin
   g := n(α − 1) + β;
End;

Procedure REPRESENTATION (xx : integer; Var TB : tab);
Begin
   a := xx;
   For i := 1 TO n^2 DO
      Begin
         q := a DIV base;          r := a MOD base;
         TB[n^2 + 1 − i] := r;     a := q;
      End;
End;
```

```
Function ASSOCIATIVE(TB : tab) : boolean;   Var xx, yy, aa, bb : hyperproduct;
Begin
   ASSOCIATIVE:=FALSE;
   For i := 1 To n Do
      For j := 1 To n Do
         For k := 1 To n Do
            Begin
               xx := ∅;  yy := ∅;
               aa := ℓ[TB[g(i, j)]];  bb := ℓ[TB[g(j, k)]];
               For w := 1 To n Do
                  Begin
                     If w ∈ aa Then xx := xx ∪ ℓ[TB[g(w, k)]];
                     If w ∈ bb  Then yy := yy ∪ ℓ[TB[g(i, w)]];
                  End;
               If xx ≠ yy Then Exit;
            End;
   ASSOCIATIVE:=TRUE;
End;
```

```
Function REPRODUCIBILITY(TB : tab) : boolean; Var cc_o, cc_v : hyperproduct;
Begin
   REPRODUCIBILITY:=FALSE;
   For i := 1 To n Do
      Begin
         cc_o := ∅;  cc_v := ∅;
         For j := 1 To n Do
            Begin
               cc_o := cc_o ∪ ℓ[TB[g(i, j)]];
               cc_v := cc_v ∪ ℓ[TB[g(j, i)]];
            End;
         If (cc_o ≠ H) OR (cc_v ≠ H) Then Exit;
      End;
   REPRODUCIBILITY:=TRUE;
End;
```

```
Function t (TB:tab):integer;
Begin
   t := 0;
   For i := 1 To n^2 Do
      t := t + TB[i] · base^(n^2−i);
End;
```

```
Procedure φ (k : integer; U : tab; Var V : tab);
Begin
For j := 1 To n^2 Do
   V[Q[k, j]] := P[k, U[j]];
End;
```

```
Function ISOMORPH_PREVIOUS(TB:tab):boolean;                    Var VV : tab;
Begin
   ISOMORPH_PREVIOUS:=FALSE;
   For i := 1 To num_perm Do
      Begin
         φ(i, TB, VV);
         If t(VV) < x Then ISOMORPH_PREVIOUS:=TRUE;
      End;
End;

Procedure COUNT(TB:tab);                               Var W : tab; lab:pack;
Begin
   For i := 1 To num_perm Do
      Begin
         φ(i, TB, W); lab[i] := t(W);
      End;
   For i := 1 To n! Do
         If lab[i] ≠ 0 Then
            Begin
               knt := 1;
               For j := i + 1 To n! Do
                  If lab[i] = lab[j] Then
                     Begin
                        knt := knt + 1; lab[j] := 0;
                     End;
               lab[i] := 0; c[knt] := c[knt] + 1;
            End;
End;
```

```
BEGIN
   N_3 := 0; S_3 := 0; c := 0;
   For x := 0 to num Do
      Begin
         Representation(x, R);
         If Associative(R) AND Reproducibility(R)Then
            Begin
               N_3 := N_3 + 1;
               If NOT isomorph_previous(R) Then
                  Begin
                     [found a new isomorphism class] ;
                     S_3 := S_3 + 1; Count(R);
                  End;
            End;
      End;
End.
```

4. Results

On a total of $|\mathcal{H}_3| = 7^9 = 40353607$ hypergroupoids, a Pascal program written from previous algorithm, has found

$$N_3 = |\mathbb{H}_3| = 23192$$

hypergroups of order 3 and

$$S_3 = \left| \mathbb{H} / \simeq \right| = 3999$$

classes of isomorhism shared on:

- $c_6 = 3739$ classes of cardinality 6,
- $c_3 = 244$ classes of cardinality 3,
- $c_2 = 10$ classes of cardinality 2,
- $c_1 = 6$ classes of cardinality 1, and
- $c_4 = c_5 = 0$ classes of cardinality 4 or 5.

The value of N_3 confirms the results of Migliorato [3] which, at the moment, it is proved by 2 different algorithms.

References

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[3] Migliorato, R., *Ipergruppi di cardinalità* 3 *e isomorfismi di ipergruppoidi commutativi totalmente regolari*, Atti Convegno su Ipergruppi, altre Strutture Multivoche e loro applicazioni, Udine (1985).

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