ON PRODUCT OF $\mathcal{P}$-FUNCTIONS

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Summary. It is shown that the $\mathcal{P}$-functions for the topological properties $\mathcal{P} = T_0$, T_1, *Hausdorff*, *regular*, introduced by Pasynkov in [PA_2] and $\mathcal{P}=$*Urysohn*, *almost regular*, *semiregular*, introduced by author and Cammaroto in [CN], are productive.

1. INTRODUCTION

During the last two decades the idea to investigate the mappings as objects more general than spaces become rather popular. First approaches to this matter are due to Russian school and particularly to B.A. Pasynkov [PA_1, PA_2].

The concept of $\mathcal{P}$-function, i.e. a continuous function that satisfies a topological property $\mathcal{P}$ was introduced by author and Cammaroto in [CN] to extend the corresponding properties of $\mathcal{P}$-spaces. For a topological property, we define a property $\mathcal{P}$ for a function such that every continuous function on a $\mathcal{P}$-space is always a $\mathcal{P}$-function.

Recently Cammaroto and Fedorchuk [CF] have studied the $\mathcal{P}$-functions for the property $\mathcal{P}$ = *H-closure*. Additional results concerning H-closed functions appear in a Cammaroto and Porter's paper [CP].

In this paper we will show that the $\mathcal{P}$-functions for the topological properties $\mathcal{P} = T_0$, T_1, *Hausdorff*, *regular* introduced by Pasynkov in [PA_2] and $\mathcal{P}=$*Urysohn*, *semiregular*, *almost regular*, introduced by author and Cammaroto in [CN], are productive, i.e. that a product of functions (defined as in Chapter 1 of [PW]) has the property $\mathcal{P}$ iff each function has $\mathcal{P}$.

2. PRELIMINARIES

Throughout this paper, all hypothesized functions are assumed to be continuous unless it is stated otherwise.

For notations, definitions or basic properties not explicitly mentioned here we refer to [E] and [PW].

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Let X be a topological space, in all the sequel, $\tau(X)$ will denote the set of open sets of X and $\sigma(X)$ will denote the set of closed sets of X. If Z is a subset of X, then $\tau(X)_{|Z}$ will denote the relative open topology on Z of $\tau(X)$ while $\sigma(X)_{|Z}$ will denote the relative closed topology on Z of X.

Let X be a topological space, a subset $A \subseteq X$ is said **regular open** if it is the interior of its own closure or, equivalently, if it is the interior of some closed set, while it is said **regular closed** if it is the closure of its own interior or, equivalently, if it is the closure of some open set. We denote by $RO(X)$ and $RC(X)$ respectively the set of regular open subsets of X and the set of regular closed subsets of X.

Let X be a topological space and $A, B, X' \subseteq X$ be subsets. Then A and B are said **separated by neighbourhoods** in X' if the sets $A \cap X'$ and $B \cap X'$ have disjoint neighbourhoods in the topological space X' relative to X, that is there are open sets $U, V \in \tau(X')$ such that $A \cap X' \subseteq U$, $B \cap X' \subseteq V$, $U \cap V = \emptyset$.

A topological space X is said **almost regular** (see [SA]) if any regular closed set and any singleton disjoint from it can be separated by neighbourhoods in the space X.

It is known that the set $RO(X)$ forms an open base for a topology on X. The topological space on X equipped with topology generated by $RO(X)$ is usually denoted by $\mathbf{X(s)}$ and it is called the **semiregularization** of X.

A topological space X is said **semiregular** if the set $RO(X)$ of the regular open subsets of X forms an open base for X, i.e. if $X = X(s)$.

Let X and Y be two topological spaces, and $f \in C(X, Y)$ a function from X to Y, then we will say that:

- f is $\mathbf{T_0}$ if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there is some neighbourhood U of x which doesn't contain y or some neighbourhood U' of y which doesn't contain x;
- f is $\mathbf{T_1}$ if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there is some neighbourhood U of x which doesn't contain y;
- f is **Hausdorff** ($\mathbf{T_2}$) if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there are two disjoint open sets containing x and y;
- f is **Urysohn** ($\mathbf{T_{2\frac{1}{2}}}$) [CN] if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there are some open neighbourhood O of $f(x)$ in Y and open subsets $U, V \in \tau(f^{\leftarrow}(O))$ such that $x \in U$, $y \in V$ and $Cl_{f^{\leftarrow}(O)}(U) \cap Cl_{f^{\leftarrow}(O)}(V) = \emptyset$;
- f is **regular** if for each closed set F and $x \in X \backslash F$ there is some open neighbourhood O of $f(x)$ in Y such that $\{x\}$ and F are separated by neighbourhoods in $f^{\leftarrow}(O)$.
- f is **almost regular** [CN] if for each $C \in RC(X)$ and $x \in X \backslash C$ there is an open neighbourhood O of $f(x)$ in Y such that $\{x\}$ and C are separated by neighbourhoods in $f^{\leftarrow}(O)$;
- f is **semiregular** [CN] if for each $A \in \tau(X)$ and $x \in A$ there are an open neighbourhood O of $f(x)$ in Y and a regular open subset $R \in RO(f^{\leftarrow}(O))$ such that $x \in R \subseteq A \cap f^{\leftarrow}(O)$.

It is immediate to check that each function defined on a T_0, T_1, Hausdorff, Urysohn, regular, almost regular or semiregular space is respectively a T_0, T_1, Hausdorff, Urysohn, regular, almost regular or semiregular function, i.e. that these are well definitions of $\mathcal{P}$-functions. It is easy to prove that every Hausdorff function is a T_1-function and that each T_1-function is a T_0-function. Moreover, in [CN] is shown that every Urysohn function is Hausdorff, that every regular function is almost regular and semiregular and that, in general, all the converses are false. In that same paper, is also proved that every almost regular, Hausdorff function is Urysohn and that a function is regular iff it is almost regular and semiregular.

NOTATION 2.1 [CN]. Let X and Y be two spaces and $f \in C(X,Y)$ a function from X to Y, we consider the set $\mathcal{S} = \{int_X(Cl_X(U)) \cap f^{\leftarrow}(O) \ : U \in \tau(X)\ , O \in \tau(Y)\} = \{int_{f^{\leftarrow}(O)}(Cl_{f^{\leftarrow}(O)}(W)) \ : W \in \tau(f^{\leftarrow}(O))\ , O \in \tau(Y)\}$. It easy to verify that $\mathcal{S}$ forms an open base for a topology on X. We will denote with $X(s,f)$ the topological space on X equipped with topology generated by $\mathcal{S}$.

The following proposition it is easy to verify.

PROPOSITION 2.2 [CN]. *Let X be a space, $U \in \tau(X)$ and $F \in \sigma(X)$. Then:*

(1) if X is Hausdorff, so is $X(s,f)$

(2) $\tau(X(s)) \subseteq \tau(X(s,f)) \subseteq \tau(X)$

(3) $Cl_X(U) = Cl_{X(s,f)}(U)$, $int_X(F) = int_{X(s,f)}(F)$

(4) $int_X(Cl_X(U)) = int_{X(s,f)}(Cl_{X(s,f)}(U))$,
$Cl_X(int_X(F)) = Cl_{X(s,f)}(int_{X(s,f)}(F))$

(5) $RO(X) = RO(X(s,f))$, $RC(X) = RC(X(s,f))$

(6) $(X(s,f))(s,f) = X(s,f)$

Then the space $X(s,f)$ verifies the same properties of the semiregularization $X(s)$ of X (see, for example, [PW]) and for this reason it is natural to call it the **f-semiregularization** of X. We say also that X is **f-semiregular** if $X = X(s,f)$.

NOTATION 2.3 [CN]. Let X and Y be two spaces and $f \in C(X,Y)$ a function between them, we denote by $f(s)$ the function $f : X(s,f) \to Y$. By definition of $\mathcal{S}$, it is clear that $f(s) \in C(X(s,f),Y)$.

In the sequel will be useful the following proposition.

PROPOSITION 2.4 [CN]. *Let $f \in C(X,Y)$ a function from X to Y, then f is semiregular iff X is f-semiregular.*

REMARK 2.5. Clearly $f = f(s)$ iff $X = X(s,f)$ and so, we can to say that f is semiregular iff $f = f(s)$. Hence it is obvious that $f(s)$ is semiregular.

Moreover, we have this important:

PROPOSITION 2.6 [CN]. *Let $f \in C(X,Y)$ a function from X to Y, then f is almost regular iff $f(s)$ is regular.*

Let X and Y be two topological spaces, $f \in C(X,Y)$ a function from X to Y and X' a subset of X, a restriction $f_{|X'} \in C(X',Y)$ of f to X' is said **open** (**dense**) if his domain X' is an open (dense) subset of X.

We will say that a property $\mathcal{P}$ for function is (**open**; **dense**) **hereditary** iff every (open; dense) restriction of a $\mathcal{P}$-function is again a $\mathcal{P}$-function.

PROPOSITION 2.7 [CN].

(1) The T_0, T_1, T_2, $T_{2\frac{1}{2}}$ properties for functions are hereditary.

(2) The regularity for functions is hereditary.

(3) The semiregularity for functions is open hereditary.

(4) The semiregularity for functions is dense hereditary.

(5) The almost-regularity for functions is open hereditary.

(6) The almost-regularity for functions is dense hereditary.

3. QUESTIONS OF PRODUCTIVITY

We start recalling from [PW] the well-known notions of product space and product function.

Let $\{X_i\}_{i\in I}$ be a set of spaces. The **product topology** on the product set $\prod_{i\in I} X_i$ is the open topology which base is the set $\mathcal{B} = \{\bigcap_{i\in F} \Pi_i^{X\leftarrow}(V_i) : F \subseteq I$ finite , $V_i \subseteq \tau(X_i)\}$ (where $\Pi_i^X : X \to X_i$ is the i-th projection function of X onto X_i) or, equivalently, the set $\mathcal{B} = \{\prod_{i\in I} U_i : \exists F \subseteq I$ finite , $U_i \in \tau(X_i)\backslash\{X_i\}\ \forall i \in F$ and $U_i = X_i\ \forall i \in I\backslash F\}$. We call $\mathbf{X} = \prod_{\mathbf{i\in I}} \mathbf{X_i}$, equipped with the product topology, the **product space** of $\{X_i\}_{i\in I}$. The basic closed sets of the product space X are the sets $\prod_{i\in I} F_i$ with $F_i \in \sigma(X_i)$. In all the sequel we will refer to basic open and closed sets without any specification.

PROPOSITION 3.1. *Let $\{A_i\}_{i\in I}$ be a set of subsets $A_i \subseteq X_i$.*
Then $Cl_X\left(\prod_{i\in I} A_i\right) = \prod_{i\in I} Cl_{X_i}(A_i)$.

PROPOSITION 3.2. *Let F be a finite subset of I and $\{A_i\}_{i\in I}$ a set of subsets $A_i \subseteq X_i$ such that $A_i = X_i$ for each $i \in I\backslash F$.*
Then $int_X\left(\prod_{i\in I} A_i\right) = \prod_{i\in I} int_{X_i}(A_i)$

DEFINITION 3.3. Let $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two sets of spaces, $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$ their product spaces and let $\{f_i\}_{i\in I}$ be a set of function $f_i \in F(X_i, Y_i)$. The **product function** $f = \prod_{i\in I} f_i$ is the member of $F(X,Y)$ defined by $\Pi_i^Y \circ f = f_i \circ \Pi_i^X$ for each $i \in I$ (where $\Pi_i^X : X \to X_i$ and $\Pi_i^Y : Y \to Y_i$ are respectively the i-th projection functions of X onto X_i and Y onto Y_i).

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \Pi_i^X \downarrow & & \Pi_i^Y \downarrow \\ X_i & \xrightarrow{f_i} & Y_i \end{array}$$

Then, the product function is explicitly defined by:

$$f(<x_i>_{i\in I}) = \left(\prod_{i\in I} f_i\right)(<x_i>_{i\in I}) = <f_i(x_i)>_{i\in I}$$

for each $x = <x_i>_{i\in I} \in X$.

REMARK 3.4. It is easy to prove (see [PW]) that if every f_i is a continuous function, then so is the product function, i.e. that if for each $i \in I$, $f_i \in C(X_i, Y_i)$, then $f = \prod_{i\in I} f_i \in C(X, Y)$. Thus, we can to consider the product function f as $\mathcal{P}$-function.

REMARK 3.5. We observe that if $\{O_i\}_{i\in I}$ is a set of subsets $O_i \subseteq Y_i$, then

$$f^{\leftarrow}\left(\prod_{i\in I} O_i\right) = \prod_{i\in I} f_i^{\leftarrow}(O_i)$$

Now we prove the productivity of $\mathcal{P}$-functions for the properties $\mathcal{P} = T_0$, T_1, *Hausdorff*, *Urysohn* and *regular*.

THEOREM 3.6. *The product function $f = \prod_{i\in I} f_i$ is T_0 iff each function f_i is T_0.*

Proof. Let $f \in C(X, Y)$ be T_0. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $x, y \in X_k$ such that $x \neq y$ and $f_k(x) = f_k(y)$. Then, picked an arbitrary point $\xi_i \in X_i$ for each $i \in I\backslash\{k\}$, we consider the subspace $Z = \prod_{i\in I} Z_i$ with $Z_k = X_k$ and $Z_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$. Since, from 2.7(1) the T_0 property is hereditary, $f_{|Z} \in C(Z, Y)$ is T_0. Then, the points $p =< p_i >_{i\in I}$, $q =< q_i >_{i\in I} \in Z$, defined by $p_k = x$, $q_k = y$ and $p_i = q_i = \xi_i$ for each $i \in I\backslash\{k\}$, are such that $p \neq q$ and $f_{|Z}(p) = f_{|Z}(q)$. Thus, there is a basic open neighbourhood U of p in Z which doesn't contain q or a basic neighbourhood U' of q in Z which doesn't contain p. Supposed, for example, that exists a basic open neighbourhood $U = \prod_{i\in I} U_i$ of p in Z, then, as $U_k \in \tau(X_k)\backslash\{X_k\}$ and $U_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$, it follows that U_k is an open neighbourhood of x in X_k which doesn't contain y. Since the other case is perfectly analogous, this proves that each function $f_k \in C(X_k, Y_k)$ is T_0.

Conversely, we suppose that each $f_i \in C(X_i, Y_i)$ is T_0. Let $x =< x_i >_{i\in I}$, $y =< y_i >_{i\in I} \in X$ such that $x \neq y$ and $f(x) = f(y)$, i.e. such that $f_i(x_i) = f_i(y_i)$ for each $i \in I$ and there is $j \in I$ such that $x_j \neq y_j$. As, in particular, $f_j \in C(X_j, Y_j)$ is T_0, there is an open neighbourhood U of x_j in X_j which doesn't contain y_j, or an open neighbourhood U of y_j in X_j which doesn't contain x_j. Then, assumed $W = \prod_{i\in I} W_i$ with $W_j = U$ and $W_i = X_i$ for each $i \in I\backslash\{j\}$, W is a basic open neighbourhood of x in X not containing y or a basic open neighbourhood of y in X not containing x. Thus the product function $f \in C(X, Y)$ is T_0. $\square$

THEOREM 3.7. *The product function $f = \prod_{i\in I} f_i$ is T_1 iff each function f_i is T_1.*

Proof. Similar to the proof of 3.6. $\square$

THEOREM 3.8. *The product function $f = \prod_{i\in I} f_i$ is Hausdorff iff each function f_i is Hausdorff.*

Proof. Let $f \in C(X, Y)$ be Hausdorff. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $x, y \in X_k$ such that $x \neq y$ and $f_k(x) = f_k(y)$. Then, picked an arbitrary point $\xi_i \in X_i$ for each $i \in I\backslash\{k\}$, we consider the subspace $Z = \prod_{i\in I} Z_i$ with $Z_k = X_k$ and $Z_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$. Since, from 2.7(1) the T_2 property is hereditary, $f_{|Z} \in C(Z, Y)$ is Hausdorff. Then, the points $p =< p_i >_{i\in I}$, $q =< q_i >_{i\in I} \in Z$, defined by $p_k = x$, $q_k = y$ and $p_i = q_i = \xi_i$ for each $i \in I\backslash\{k\}$, are such that $p \neq q$ and $f_{|Z}(p) = f_{|Z}(q)$. Thus, there are two basic open sets $U = \prod_{i\in I} U_i$, $V = \prod_{i\in I} V_i \in \tau(X)$ such that $p \in U$, $q \in V$, $U \cap V = \emptyset$. Hence, as $U_k, V_k \in \tau(X_k)\backslash\{X_k\}$ and $U_i = V_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$, it follows that U_k and V_k are two open disjoint neighbourhoods of x and y in X_k and so it is proved that $f_k \in C(X_k, Y_k)$ is Hausdorff.

Conversely, we suppose that each $f_i \in C(X_i, Y_i)$ is Hausdorff. Let $x =< x_i >_{i\in I}$, $y =< y_i >_{i\in I} \in X$ such that $x \neq y$ and $f(x) = f(y)$, i.e. such that $f_i(x_i) = f_i(y_i)$ for each $i \in I$ and there is $j \in I$ such that $x_j \neq y_j$. As, in particular, $f_j \in C(X_j, Y_j)$ is Hausdorff, there are $U, V \in \tau(X_j)$ such that $x_j \in U$, $y_j \in V$ and $U \cap V = \emptyset$. Then, assumed $W = \prod_{i\in I} W_i$ with $W_j = U$, $W_i = X_i$ for each $i \in I\backslash\{j\}$ and $M = \prod_{i\in I} M_i$ with $M_j = V$, $M_i = X_i$ for each $i \in I\backslash\{j\}$, W and M are two disjoint open neighbourhoods of x and y in X. Thus the product function $f \in C(X, Y)$ is Hausdorff. □

THEOREM 3.9. *The product function $f = \prod_{i\in I} f_i$ is Urysohn iff each function f_i is Urysohn.*

Proof. Let $f \in C(X, Y)$ be Urysohn. Fixed $k \in I$, we consider the function $f_k \in C(X_k, Y_k)$. Let $x, y \in X_k$ such that $x \neq y$ and $f_k(x) = f_k(y)$. Then, picked an arbitrary point $\xi_i \in X_i$ for each $i \in I\backslash\{k\}$, we consider the subspace $Z = \prod_{i\in I} Z_i$ with $Z_k = X_k$ and $Z_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$. Since, from 2.7(1) the $T_{2\frac{1}{2}}$ property is hereditary, $f_{|Z} \in C(Z, Y)$ is Urysohn. Then, the points $p =< p_i >_{i\in I}$, $q =< q_i >_{i\in I} \in Z$, defined by $p_k = x$, $q_k = y$ and $p_i = q_i = \xi_i$ for each $i \in I\backslash\{k\}$, are such that $p \neq q$ and $f_{|Z}(p) = f_{|Z}(q)$. Thus there are a basic open neighbourhood $O = \prod_{i\in I} O_i$ of $f_{|Z}(p)$ in Y and two basic open sets $U = \prod_{i\in I} U_i$, $V = \prod_{i\in I} V_i \in \tau(f^{\leftarrow}_{|Z}(O)) = \tau(Z)_{|f^{\leftarrow}(O)}$ such that $p \in U$, $q \in V$ and $Cl_{f^{\leftarrow}_{|Z}(O)}(U) \cap Cl_{f^{\leftarrow}_{|Z}(O)}(V) = \emptyset$. By 3.5, $f^{\leftarrow}_{|Z}(O) = f^{\leftarrow}(O) \cap Z = \prod_{i\in I} f^{\leftarrow}_i(O_i) \cap \prod_{i\in I} Z_i = \prod_{i\in I} \left(f^{\leftarrow}_i(O_i) \cap Z_i\right) = \prod_{i\in I} f^{\leftarrow}_{i|Z_i}(O_i)$, and so, by 3.1, we have that $\prod_{i\in I} Cl_{f^{\leftarrow}_{i|Z_i}(O_i)}(U_i) \cap \prod_{i\in I} Cl_{f^{\leftarrow}_{i|Z_i}(O_i)}(V_i) = \emptyset$. Since, obviously $f^{\leftarrow}_{k|Z_k}(O_k) = f^{\leftarrow}_k(O_k)$, $U_k, V_k \in \tau(X_k)\backslash\{X_k\}$ and $U_i = V_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$, we have that O_k is an open neighbourhood of $f_k(x)$ in Y_k and that U_k and V_k are two open neighbourhoods of x and y in $f^{\leftarrow}_k(O_k)$ such that $Cl_{f^{\leftarrow}_k(O_k)}(U_k) \cap Cl_{f^{\leftarrow}_k(O_k)}(V_k) = \emptyset$. Thus $f_k \in C(X_k, Y_k)$ is Urysohn.

Conversely, we suppose that each $f_i \in C(X_i, Y_i)$ is Urysohn. Let $x =< x_i >_{i\in I}$, $y =< y_i >_{i\in I} \in X$ such that $x \neq y$ and $f(x) = f(y)$, i.e. such that $f_i(x_i) = f_i(y_i)$ for each $i \in I$ and there is $j \in I$ such that $x_j \neq y_j$. As, in particular, $f_j \in C(X_j, Y_j)$ is Urysohn, there are an open neighbourhood O of $f_j(x)$ in $\tau(Y_j)$ and two open sets U, $V \in \tau(f^{\leftarrow}_j(O)) = \tau(X_j)_{|f^{\leftarrow}_j(O)}$ such that $x \in U$, $y \in V$ and $Cl_{f^{\leftarrow}_j(O)}(U) \cap Cl_{f^{\leftarrow}_j(O)}(V) = \emptyset$. Then, assumed $A = \prod_{i\in I} A_i$, with $A_j = O$, $A_i = Y_i$ for each $i \in I\backslash\{j\}$; $W = \prod_{i\in I} W_i$ with $W_j = U$, $W_i = X_i$ for each

$i \in I\backslash\{j\}$ and $M = \prod_{i\in I} M_i$ with $M_j = V$, $M_i = X_i$ for each $i \in I\backslash\{j\}$, A is an open neighbourhood of $f(x)$ in Y and that W and M are two open neighbourhoods of x and y in X. Moreover, as from 3.5 $f^{\leftarrow}(A) = \prod_{i\in I} f_i^{\leftarrow}(A_i)$, by 3.1, we have that $Cl_{f^{\leftarrow}(A)}(W) \cap Cl_{f^{\leftarrow}(A)}(M) = \left(\prod_{i\in I} Cl_{f_i^{\leftarrow}(A_i)}(W_i)\right) \cap \left(\prod_{i\in I} Cl_{f_i^{\leftarrow}(A_i)}(M_i)\right) = \emptyset$. So, it is proved that the product function $f \in C(X,Y)$ is Urysohn. □

THEOREM 3.10. *The product function $f = \prod_{i\in I} f_i$ is regular iff each function f_i is regular.*

Proof. Let $f \in C(X,Y)$ be regular. Fixed $k \in I$, we consider the function $f_k \in C(X_k,Y_k)$. Let $F \in \sigma(X_k)$ and $x \in X_k\backslash F$. Then, picked an arbitrary point $\xi_i \in X_i$ for each $i \in I\backslash\{k\}$, we consider the subspace $Z = \prod_{i\in I} Z_i$ with $Z_k = X_k$ and $Z_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$. Since, from 2.7(2), the regularity is hereditary, $f_{|Z} \in C(Z,Y)$ is regular. So, if we consider the point $p =< p_i >_{i\in I} \in Z$, defined by $p_k = x$, and $p_i = \xi_i$ for each $i \in I\backslash\{k\}$, and the basic closed set $C = \prod_{i\in I} C_i$ of the product space Z, defined by $C_k = F$, $C_i = Z_i$ for each $i \in I\backslash\{k\}$, it is clear that $p \in C$. Thus there are a basic open neighbourhood $O = \prod_{i\in I} O_i$ of $f_{|Z}(p)$ in Y and two basic open sets $U = \prod_{i\in I} U_i$, $V = \prod_{i\in I} V_i \in \tau(f_{|Z}^{\leftarrow}(O)) = \tau(Z)_{|f^{\leftarrow}(O)}$ such that $p \in U$, $C \cap f_{|Z}(O) \subseteq V$ and $U \cap V = \emptyset$. As, by 3.5, $f_{|Z}^{\leftarrow}(O) = f^{\leftarrow}(O) \cap Z = \prod_{i\in I} f_i^{\leftarrow}(O_i) \cap \prod_{i\in I} Z_i = \prod_{i\in I}\left(f_i^{\leftarrow}(O_i) \cap Z_i\right) = \prod_{i\in I} f_{i|Z_i}^{\leftarrow}(O_i)$, we have that $\prod_{i\in I} C_i \cap \prod_{i\in I} f_{i|Z_i}^{\leftarrow}(O_i) \subseteq \prod_{i\in I} V_i$ i.e. $\prod_{i\in I}\left(C_i \cap f_{i|Z_i}^{\leftarrow}(O_i)\right) \subseteq \prod_{i\in I} V_i$. Since, obviously $f_{k|Z_k}^{\leftarrow}(O_k) = f_k^{\leftarrow}(O_k)$, $U_k, V_k \in \tau(X_k)\backslash\{X_k\}$ and $U_i = V_i = \{\xi_i\}$ for each $i \in I\backslash\{k\}$, O_k is an open neighbourhood of $f_k(x)$ in Y_k and that U_k and V_k are two sets of X_k such that $x \in U_k$, $F \cap f_k^{\leftarrow}(O_k) \subseteq V_k$ and $U_k \cap V_k = \emptyset$. This proves that $f_k \in C(X_k,Y_k)$ is regular.

Conversely, we suppose that each $f_i \in C(X_i,Y_i)$ is regular. Let $F = \prod_{i\in I} F_i \in \sigma(X)$ a basic closed set of X and $x =< x_i >_{i\in I} \in X\backslash F$. So, there is $j \in I$ such that $x_j \notin F_j$. As, in particular, $f_j \in C(X_j,Y_j)$ is regular, there are an open neighbourhood O of $f_j(x)$ in Y_j and two open sets $U, V \in \tau(f_j^{\leftarrow}(O)) = \tau(X_j)_{|f_j^{\leftarrow}(O)}$ such that $x \in U$, $F_j \cap f^{\leftarrow}(O) \subseteq V$ and $U \cap V = \emptyset$. Then, assumed $A = \prod_{i\in I} A_i$ with $A_j = O$, $A_i = Y_i$ for each $i \in I\backslash\{j\}$ and $W = \prod_{i\in I} W_i$, $M = \prod_{i\in I} M_i$ with $W_j = U$, $M_j = V$ and $W_i = M_i = X_i$ for each $i \in I\backslash\{j\}$, A is an open neighbourhood of $f(x)$ in Y and that W and M are two open sets of X such that $x \in W$ and $W \cap M = \emptyset$. Moreover, by 3.5, $F \cap f^{\leftarrow}(A) = \prod_{i\in I} F_i \cap \prod_{i\in I} f_i^{\leftarrow}(A_i) = \prod_{i\in I}\left(F_i \cap f_i^{\leftarrow}(A_i)\right) \subseteq \prod_{i\in I} M_i = M$. Thus it is proved that the product function $f \in C(X,Y)$ is regular. □

To prove the productivity of $\mathcal{P}$-functions for the properties $\mathcal{P}$=*almost regular* and *semiregular* we need the following:

LEMMA 3.11. $\left(\prod_{i\in I} f_i\right)(s) = \prod_{i\in I} f_i(s)$

Proof. It suffices to prove that $\tau\left(\prod_{i\in I} X_i(s,f_i)\right) = \tau(X(s,f))$. Let $B = \prod_{i\in I} B_i$

a basic open subset of $\prod_{i\in I} X_i(s,f_i)$. By definition of base of product topology, there is $F \subseteq I$ finite such that $B_i \in \tau(X_i(s,f_i))\backslash\{X_i\}$ for each $i \in I$ and $B_i = X_i$ for each $i \in I\backslash F$. Then, there are U_i and O_i such that $B_i = int_{X_i}(Cl_{X_i}(U_i)) \cap f_i^{\leftarrow}(O_i)$, with $U_i \in \tau(X_i)\backslash\{X_i\}$, $O_i \in \tau(Y_i)\backslash\{Y_i\}$ for each $i \in F$ and $U_i = X_i$, $O_i = Y_i$ for each $i \in I\backslash F$. So, by 3.1, 3.2 and 3.5, it follows that $B = \prod_{i\in I} B_i = \prod_{i\in I}(int_{X_i}(Cl_{X_i}(U_i)) \cap f_i^{\leftarrow}(O_i)) = \left(\prod_{i\in I} int_{X_i}(Cl_{X_i}(U_i))\right) \cap \left(\prod_{i\in I} f_i^{\leftarrow}(O_i)\right) = int_X\left(Cl_X\left(\prod_{i\in I} U_i\right)\right) \cap f^{\leftarrow}\left(\prod_{i\in I} O_i\right)$. Since, $\prod_{i\in I} U_i \in \tau(X)$ and $\prod_{i\in I} O_i \in \tau(Y)$, it is clear that $B \in \tau(X(s,f))$.

On the other hand, let $B' = int_X(Cl_X(U)) \cap f^{\leftarrow}(O)$, with $U \in \tau(X)$ and $O \in \tau(Y)$, a basic open set of $X(s,f)$ and let $x \in B'$. Hence, $x \in int_X(Cl_X(U))$ and $f(x) \in O$. By definition of product topology, there are a basic open set $\prod_{i\in I} W_i \in \tau(X)$ such that $x \in \prod_{i\in I} W_i \subseteq U$ and a basic open set $\prod_{i\in I} A_i \in \tau(Y)$ such that $f(x) \in \prod_{i\in I} A_i \subseteq O$. Then, $x \in f^{\leftarrow}\left(\prod_{i\in I} A_i\right) \subseteq f^{\leftarrow}(O)$ and, by 3.5, $x \in \prod_{i\in I} f_i^{\leftarrow}(A_i) \subseteq f^{\leftarrow}(O)$. Moreover, by 3.1 and 3.2, $\prod_{i\in I} int_{X_i}(Cl_{X_i}(W_i)) = int_X\left(Cl_X\left(\prod_{i\in I} W_i\right)\right) \subseteq int_X(Cl_X(U))$. So, $x \in \prod_{i\in I}(int_{X_i}(Cl_{X_i}(W_i)) \cap f_i^{\leftarrow}(A_i)) \subseteq int_X(Cl_X(U)) \cap f^{\leftarrow}(O) = B'$, where $\prod_{i\in I}(int_{X_i}(Cl_{X_i}(W_i)) \cap f_i^{\leftarrow}(A_i))$ belongs to the base of $\tau\left(\prod_{i\in I} X_i(s,f_i)\right)$. Thus, the lemma is proved. □

THEOREM 3.12. *The product function $f = \prod_{i\in I} f_i$ is almost regular iff each function f_i is almost regular.*

Proof. In fact, by 2.6, we have that $\prod_{i\in I} f_i$ is almost regular iff $\left(\prod_{i\in I} f_i\right)(s)$ is regular i.e., by 3.11, iff $\prod_{i\in I} f_i(s)$ is regular and, by 3.10, iff each $f_i(s)$ is regular, that is, by 2.6, iff each f_i is almost regular. □

THEOREM 3.13. *The product function $f = \prod_{i\in I} f_i$ is semiregular iff each function f_i is semiregular.*

Proof. Let $f \in C(X,Y)$ semiregular. Fixed $k \in I$, we consider the function $f_k \in C(X_k,Y_k)$. Let $U \in \tau(X_k)$ and $x \in U$. Then, picked an arbitrary point $\xi_i \in X_i$ for each $i \in I\backslash\{k\}$, we consider the point $p =< p_i >_{i\in I} \in X$, defined by $p_k = x$, and $p_i = \xi_i$ for each $i \in I\backslash\{k\}$, and the basic open set $A = \prod_{i\in I} A_i$ of the product space X, defined by $A_k = U$, $A_i = X_i$ for each $i \in I\backslash\{k\}$. It is clear that $p \in A$. Thus, there are a basic open neighbourhood $O = \prod_{i\in I} O_i$ of $f(p)$ in Y and a basic closed set $F = \prod_{i\in I} F_i \in \sigma(f^{\leftarrow}(O)) = \sigma(X)_{|f^{\leftarrow}(O)}$ such that $p \in int_{f^{\leftarrow}(O)}(F) \subseteq A \cap f^{\leftarrow}(O)$. As, from 3.5, $f^{\leftarrow}(O) = \prod_{i\in I} f_i^{\leftarrow}(O_i)$, by 3.2, we have that $int_{f^{\leftarrow}(O)}(F) = \prod_{i\in I} int_{f_i^{\leftarrow}(O_i)}(F_i)$. Then, $p \in \prod_{i\in I} int_{f_i^{\leftarrow}(O_i)}(F_i) \subseteq \prod_{i\in I} A_i \cap \prod_{i\in I} f_i^{\leftarrow}(O_i) = \prod_{i\in I}(A_i \cap f_i^{\leftarrow}(O_i))$. So, O_k is an open neighbourhoods of $f_k(x)$ in Y_k and F_k is a closed set of $f_k^{\leftarrow}(O_k)$ such that $x \in int_{f_k^{\leftarrow}(O_k)}(F_k) \subseteq U \cap f_k^{\leftarrow}(O_k)$. This proves that $f_k \in C(X_k,Y_k)$ is semiregular.

Conversely, if each f_i is semiregular, by 2.5, we have that $f_i = f_i(s)$ and so, by 3.11, $f = \prod_{i\in I} f_i = \prod_{i\in I} f_i(s) = \left(\prod_{i\in I} f_i\right)(s) = f(s)$. Thus, by 2.5, f is semiregular. □

References

[CF] F. CAMMAROTO, V.V. FEDORCHUK, *On H-closed extension of Hausdorff mappings*, (Submitted), (1993).

[CN] F. CAMMAROTO AND G. NORDO, *On Urysohn, almost regular and semiregular functions*, (Submitted), (1993).

[CP] F. CAMMAROTO, J.R. PORTER, *H-closed functions*, (Preprint), (1993).

[E] R. ENGELKING, *General Topology*, Monografie Matematyczne Warzawa, (1977).

[PA_1] B.A. PASYNKOV, *The dimension and geometry of mappings*, Doul. Akad. Nauk SSSR 221 (1975), 543-546=Soviet Math. Dokl., 16 (1975), 384-388.

[PA_2] B.A. PASYNKOV, *On extension onto mappings of some notions and statements about space* (in Russian) in Mappings and functors, IZD. Moskow Univ., Moskow W (1984), 72-102.

[PW] J.R. PORTER, R.G. WOODS, *Extensions and absolutes of Hausdorff spaces*, Springer-Verlag, 1988.

[SA] M.K. SINGAL, S.P. ARYA, *On almost regular spaces*, Glasnik. Mat. 4 (24) (1969), 89-99.

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