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The triangle intersection problem for $S(2, 4, v)$ designs[☆]

Yanxun Chang[a], Tao Feng[a], Giovanni Lo Faro[b]

[a] Institute of Mathematics, Beijing Jiaotong University, Beijing 100044, PR China

[b] Department of Mathematics, University of Messina, Contrada Papardo, 31 - 98166, Sant'Agata, Messina, Italy

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Dedicated to the memory of
Lucia Gionfriddo (1973–2008).

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ABSTRACT

In this paper the triangle intersection problem for $S(2, 4, v)$ designs is investigated. Let $t_v = v(v-1)/3$ and $I_T(v) = \{0, 1, \ldots, t_v - 30\} \cup \{t_v - 27, t_v - 24, t_v - 18, t_v\}$. Let $J_T(v) = \{s|$ there exist two $S(2, 4, v)$ designs with s common triangles$\}$. We show that for any positive integer $v \equiv 1, 4 \pmod{12}$, $J_T(v) = I_T(v)$ when $v \geq 121$, and $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v) \subseteq I_T(v)$ when $49 \leq v \leq 112$.

1. Introduction

Let H be a simple graph and G a subgraph of H. A *G-design* of H ((H, G)-design in short) is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is an edge-disjoint decomposition of H into isomorphic copies (called *blocks*) of the graph G. If H is the complete graph K_v, we refer to such a G-design as one of order v. If G is the complete graph K_k, a K_k-design of order v is called a *Steiner system* $S(2, k, v)$.

The *intersection problem* for (H, G)-designs is the determination of all pairs (v, s) such that there exists a pair of (H, G)-designs $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ with $|X| = v$ and $|\mathcal{B}_1 \cap \mathcal{B}_2| = s$. This problem was first considered for $S(2, k, v)$ designs (cf. [9]). A complete solution to the intersection problem for $S(2, 3, v)$ designs was given by Lindner and Rosa [11]. The intersection problem for $S(2, 4, v)$ designs was dealt with by Colbourn et al. [6], apart from three undecided values for $v = 25, 28$ and 37. The intersection problem is also considered for many other different types of combinatorial structures. The interested reader may refer to [2,8], for example.

Theorem 1.1 ([6]). *Let $J(v) = \{s|$ there exist two $S(2, 4, v)$ designs with s common blocks $\}$. Let $b_v = v(v-1)/12$ and $I(v) = \{0, 1, 2, \ldots, b_v\} \setminus \{b_v - 7, b_v - 5, b_v - 4, b_v - 3, b_v - 2, b_v - 1\}$. Then*

(1) $J(v) \subseteq I(v)$ *for all* $v \equiv 1, 4 \pmod{12}$.
(2) $J(v) = I(v)$ *for all* $v \equiv 1, 4 \pmod{12}$ *and* $v \geq 40$.
(3) $J(13) = I(13)$ *and* $J(16) = I(16) \setminus \{7, 9, 10, 11, 14\}$.
(4) $I(25) \setminus \{31, 33, 34, 37, 39, 40, 41, 42, 44\} \subseteq J(25)$ *and* $\{42, 44\} \not\subseteq J(25)$.

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E-mail addresses: yxchang@bjtu.edu.cn (Y. Chang), tfeng@bjtu.edu.cn (T. Feng), lofaro@unime.it (G. Lo Faro).

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(5) $I(28) \setminus \{44, 46, 49, 50, 52, 53, 54, 57\} \subseteq J(28)$.
(6) $I(37) \setminus \{64, 66, 76, 82, 84, 85, 88, 90\text{–}94, 96\text{–}101\} \subseteq J(37)$.

Let B be a simple graph. Denote by $T(B)$ the set of all triangles of the graph B. For example, if B is the graph with vertices a, b, c, d and edges ab, ac, bc, cd (such a graph called a *kite*), then $T(B) = \{\{a, b, c\}\}$. The *triangle intersection problem* for (H, G)-designs is the determination of all pairs (v, s) such that there exists a pair of (H, G)-designs $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ with $|X| = v$ and $|T(\mathcal{B}_1) \cap T(\mathcal{B}_2)| = s$, where $T(\mathcal{B}_i) = \bigcup_{B\in\mathcal{B}_i} T(B)$, $i = 1, 2$.

The triangle intersection problem was introduced by Lindner and Yazici [12], who gave a complete solution to the triangle intersection problem for kite systems (a kite system is a G-design when G is a kite). Recently, Billington et al. [3] discussed the triangle intersection problem for $K_4 - e$ designs.

In this paper we shall investigate the triangle intersection problem for $S(2, 4, v)$ designs. In what follows we always assume that $t_v = v(v-1)/3$, $I_T(v) = \{0, 1, \ldots, t_v - 30\} \cup \{t_v - 27, t_v - 24, t_v - 18, t_v\}$ and $J_T(v) = \{s|$ there exist two $S(2, 4, v)$ designs with s common triangles$\}$. As the main result, we are to prove the following theorem.

Theorem 1.2. (1) *For* $v \equiv 1, 4 \pmod{12}$ *and* $v \ge 121$, $J_T(v) = I_T(v)$; *In particular*, $J_T(40) = I_T(40)$.
(2) *For* $v \equiv 1, 4 \pmod{12}$ *and* $49 \le v \le 112$, $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v) \subseteq I_T(v)$.
(3) $J_T(13) = I_T(13) \setminus \{1, 2, 9\}$ *and* $J_T(16) = I_T(16) \setminus \{37, 39, 41, 43, 45\text{–}50, 53, 62\}$.
(4) $\{0\text{–}122, 124\text{–}131, 134, 135, 137, 140, 143, 146, 155, 158, 164, 200\} \subseteq J_T(25) \subseteq I_T(25) \setminus \{176, 182\}$.
(5) $\{0\text{–}149, 156, 158, 160, 162, 164, 166, 168, 180, 204, 252\} \subseteq J_T(28) \subseteq I_T(28)$.
(6) $\{0\text{–}251, 258\text{–}276, 285\text{–}294, 444\} \subseteq J_T(37) \subseteq I_T(37)$.

2. Necessary conditions

In this section, we establish necessary conditions for $J_T(v)$. A *Steiner* $(4, 2)$ *trade* $\{T_1, T_2\}$ of volume m consists of two disjoint sets T_1 and T_2, each containing m 4-subsets (called *blocks*) of some set V, such that every pair of V occurs in at most one block of T_1, and any pair from V occurs in a block of T_1 if and only if it occurs in a block of T_2.

Lemma 2.1. *Suppose that* $\{T_1, T_2\}$ *is a Steiner* $(4, 2)$ *trade of volume m. If there exists* $b_1 \in T_1$ *such that* $|b_1 \cap e| \le 2$ *for each* $e \in T_2$, *then* $m \ge 10$.

Proof. Suppose that $b_1 = \{1, 2, 3, 4\} \in T_1$ satisfying $|b_1 \cap e| \le 2$ for each $e \in T_2$. Then the pairs $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$ appear in distinct blocks of T_2, and the number of blocks containing i in T_1 for any $i \in \{1, 2, 3, 4\}$ is no less than 3. This means that $|T_1| = |T_2| = m \ge 9$ and T_1 contains 9 blocks of the form

$$\{1, 2, 3, 4\},\quad \{1, *, *, *\},\quad \{1, *, *, *\},\quad \{2, *, *, *\},\quad \{2, *, *, *\},$$
$$\{3, *, *, *\},\quad \{3, *, *, *\},\quad \{4, *, *, *\},\quad \{4, *, *, *\}.$$

If $m = 9$, it is readily checked that the number of blocks containing i in T_j is 3 for each $i \in \{1, 2, 3, 4\}$, $j \in \{1, 2\}$. And we have the fact that there exists $e_1 \in T_2$ such that $|e_1 \cap b| \le 2$ for each $b \in T_1$. Otherwise, for any $c \in T_2$, there exists $c' \in T_1$ such that $|c \cap c'| = 3$. Take $c_1, c_2 \in T_2$, $c_1 \ne c_2$. Then there are $c_1', c_2' \in T_1$ such that $|c_1 \cap c_1'| = 3$ and $|c_2 \cap c_2'| = 3$. Because every pair occurs in at most one block of T_2, we have $c_1' \ne c_2'$. Due to $|T_1| = |T_2|$, there must be a block $d \in T_2$ such that $|d \cap b_1| = 3$. A contradiction occurs.

Considering $|e_1 \cap b_1|$, we have the following three possibilities.

Case 1. $|e_1 \cap b_1| = 0$. Let $e_1 = \{5, 6, 7, 8\}$. Because $|e_1 \cap b| \le 2$ for each $b \in T_1$, the pairs $\{5, 6\}, \{5, 7\}, \{5, 8\}, \{6, 7\}, \{6, 8\}, \{7, 8\}$ must appear in distinct blocks of T_1. Without loss of generality, we may assume that the blocks $\{1, 5, 6, *\}$ and $\{1, 7, 8, *\}$ are contained in T_1. Then the pairs $\{1, 5\}, \{1, 6\}, \{1, 7\}, \{1, 8\}$ must appear in distinct blocks in T_2. Because the number of blocks containing 1 in T_2 is 3, a contradiction occurs.

Case 2. $|e_1 \cap b_1| = 1$. Let $e_1 = \{1, 5, 6, 7\}$. Because $|e_1 \cap b| \le 2$ for each $b \in T_1$, the pairs $\{1, 5\}, \{1, 6\}, \{1, 7\}$ must appear in distinct blocks of T_1. Then there are 4 blocks containing 1 in T_1. That is a contradiction.

Case 3. $|e_1 \cap b_1| = 2$. Let $e_1 = \{1, 2, 5, 6\}$. The pairs $\{1, 5\}, \{1, 6\}, \{2, 5\}, \{2, 6\}, \{5, 6\}$ must appear in distinct blocks of T_1. Without loss of generality, we may assume that $\{3, 5, 6, *\} \in T_1$. Then the pairs $\{3, 5\}, \{3, 6\}$ must appear in distinct blocks of T_2. Because there are only 3 blocks containing 3 in T_2 (i.e., $\{1, 3, *, *\}, \{2, 3, *, *\}, \{3, 4, *, *\}$), a contradiction occurs. This completes the proof. □

Lemma 2.2. *For any positive integer* $v \equiv 1, 4 \pmod{12}$ *and* $v \ge 13$, $J_T(v) \subseteq I_T(v)$. *In particular*, $J_T(16) \subseteq I_T(16) \setminus \{45\text{–}50, 53, 62\}$ *and* $J_T(25) \subseteq I_T(25) \setminus \{176, 182\}$.

Proof. Suppose that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are two $S(2, 4, v)$ designs, which intersect in $t_v - s$ triangles. Consider $\mathcal{D}_1 = \mathcal{B}_1 \setminus \mathcal{B}_2$ and $\mathcal{D}_2 = \mathcal{B}_2 \setminus \mathcal{B}_1$. For any block $D \in \mathcal{D}_1$, $|T(D) \bigcap T(\mathcal{D}_2)| = 0$ or 1. Let $\mathcal{C}_0 = \{D \in \mathcal{D}_1 : |T(D) \bigcap T(\mathcal{D}_2)| = 0\}$ and $\mathcal{C}_1 = \{D \in \mathcal{D}_1 : |T(D) \bigcap T(\mathcal{D}_2)| = 1\}$. Denote by $J(v)$ the set of intersection sizes of $S(2, 4, v)$ designs. It is easy to see that

$$\begin{cases} |\mathcal{D}_1| = |\mathcal{C}_0| + |\mathcal{C}_1|, \\ s = 4|\mathcal{C}_0| + 3|\mathcal{C}_1|, \\ b_v - (|\mathcal{C}_0| + |\mathcal{C}_1|) \in J(v), \end{cases}$$

where b_v is the number of blocks of an $S(2, 4, v)$ design. By Lemma 2.1 when $|\mathcal{D}_1| \leq 9$, we have $|\mathcal{C}_0| = 0$. Combine the results from Theorem 1.1. It is readily checked that the desired results hold. For example, verify $176 \notin J_T(25)$. In this case $t_{25} = 200$ and $s = 24$. Solve the equation $24 = 4|\mathcal{C}_0| + 3|\mathcal{C}_1|$. Due to $|\mathcal{C}_0| + |\mathcal{C}_1| \leq 9$, we have $|\mathcal{C}_0| = 0$ and $|\mathcal{C}_1| = 8$, which implies $42 \in J_T(25)$. That is contradicted to $42 \notin J_T(25)$ from Theorem 1.1(4). □

3. Recursive constructions

In this section we give two recursive constructions for the triangle intersection problem. The concept of GDDs plays an important role in these constructions.

Let K be a set of positive integers. A *group divisible design* (GDD) K-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of a finite set X into subsets (called *groups*); (2) $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality from K, such that a group and a block contain at most one common point; (3) every pair of points from distinct groups occurs in exactly one block.

If $\mathcal{G}$ contains u_i groups of size g_i for $1 \leq i \leq s$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ the *group type* (or *type*) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD. A K-GDD of type 1^v is commonly called a *pairwise balanced design*, denoted by $(v, K, 1)$-PBD. When $K = \{k\}$, a pairwise balanced design is just a Steiner system $S(2, k, v)$, called a *balanced incomplete block design*, denoted by $(v, k, 1)$-BIBD. A K-GDD of type $1^{v-h}h^1$ is commonly called an *incomplete pairwise balanced design*, denoted by $(v, h; K, 1)$-IPBD. When $K = \{k\}$, an incomplete pairwise balanced design is called an *incomplete balanced incomplete block design*, denoted by $(v, h; k, 1)$-IBIBD. Obviously a $(v, h; k, 1)$-IBIBD is also a $((K_v \setminus K_h), K_k)$-design.

A GDD is *resolvable* if its blocks can be partitioned into parallel classes; a parallel class is a set of point-disjoint blocks whose union is the set of all points. The notation K-RGDD is used for a resolvable K-GDD. If $K = \{k\}$, we write $\{k\}$-RGDD as k-RGDD. A 3-RGDD of type 1^v is commonly called a *Kirkman triple system*, denoted by KTS(v). It is well known that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$ [14].

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a partition of a finite set X into subsets (called *holes*), where $|H_i| = n_i$ for $1 \leq i \leq t$. Let $K_{n_1,n_2,\ldots,n_t}$ be the complete multipartite graph on X with the ith part on H_i. A *holely G-design* is a triple $(X, \mathcal{H}, \mathcal{B})$ such that $(X, \mathcal{B})$ is a $(K_{n_1,n_2,\ldots,n_t}, G)$-design. The *hole type* (or *type*) of the holely G-design is $\{n_1, n_2, \ldots, n_t\}$. We usually use an "exponential" notation to describe hole types: the hole type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ denotes u_i occurrences of g_i for $1 \leq i \leq s$.

A pair of holely G-designs $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ are said to *intersect in l triangles* if $|T(\mathcal{B}_1) \cap T(\mathcal{B}_2)| = l$, where $T(\mathcal{B}_i) = \bigcup_{B \in \mathcal{B}_i} T(B)$, $i = 1, 2$. The following construction is a variation of Wilson's Fundamental Construction [16].

Construction 3.1 (*Weighting Construction*)**.** Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a K-GDD, and let $\omega : X \longmapsto Z^+ \cup \{0\}$ be a weight function. For every block $A \in \mathcal{A}$, suppose that there is a pair of holely G-designs of type $\{\omega(x) : x \in A\}$, which intersect in t_A triangles. Then there exists a pair of holely G-designs of type $\{\sum_{x \in G} \omega(x) : G \in \mathcal{G}\}$, which intersect in $\sum_{A \in \mathcal{A}} t_A$ triangles.

Proof. For every $x \in X$, let $S(x)$ be a set of $\omega(x)$ "copies" of x. For any $Y \subseteq X$, let $S(Y) = \bigcup_{x \in Y} S(x)$. For every block $A \in \mathcal{A}$, construct a pair of holely G-designs $\{S(A), \{S(x) : x \in A\}, \mathcal{B}_A\}$ and $\{S(A), \{S(x) : x \in A\}, \mathcal{B}'_A\}$, which intersect in t_A triangles. Then it is readily checked that there exists a pair of holely G-designs $(S(X), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}_A)$ and $(S(X), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}'_A)$, which intersect in $\sum_{A \in \mathcal{A}} t_A$ triangles. □

Construction 3.2 (*Filling Construction*)**.** Let a be a nonnegative integer. Suppose that there exists a pair of holely G-designs of type $\{g_1, g_2, \ldots, g_s\}$, which intersect in t triangles. If there is a pair of $((K_{g_i+a} \setminus K_a), G)$-designs with the same subgraph K_a removed for each $1 \leq i \leq s-1$, which intersect in t_i triangles, and there is a pair of (K_{g_s+a}, G)-designs, which intersect in t_s triangles, then there exists a pair of (K_{v+a}, G)-designs intersecting in $t + \sum_{i=1}^{s} t_i$ triangles, where $v = \sum_{i=1}^{s} g_i$.

Proof. Let $(X, \mathcal{H}, \mathcal{B}_1)$ and $(X, \mathcal{H}, \mathcal{B}_2)$ be a pair of holely G-designs of type $\{g_1, g_2, \ldots, g_s\}$, which intersect in t triangles. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_s\}$ with $|H_i| = g_i$ for $1 \leq i \leq s$, and Y be a set of cardinality a such that $X \cap Y = \emptyset$. For each $1 \leq i \leq s-1$, construct a pair of $((K_{g_i+a} \setminus K_a), G)$-designs $(H_i \bigcup Y, \mathcal{A}_i^1)$ and $(H_i \bigcup Y, \mathcal{A}_i^2)$ with the same subgraph K_a defined on Y removed, which intersect in t_i triangles. By the assumption, we also have a pair of (K_{g_s+a}, G)-designs $(H_s \bigcup Y, \mathcal{A}_s^1)$ and $(H_s \bigcup Y, \mathcal{A}_s^2)$, which intersect in t_s triangles. It is readily checked that there exists a pair of (K_{v+a}, G)-designs $(X \bigcup Y, (\cup_{i=1}^{s} \mathcal{A}_i^1) \bigcup \mathcal{B}_1)$ and $(X \bigcup Y, (\cup_{i=1}^{s} \mathcal{A}_i^2) \bigcup \mathcal{B}_2)$, which intersect in $t + \sum_{i=1}^{s} t_i$ triangles, where $v = \sum_{i=1}^{s} g_i$. □

It is well known that a 5-GDD of type g^5 is equivalent to three mutually orthogonal Latin squares (MOLS) of order g. Thus we quote the following result for later use.

Lemma 3.3 (*[1]*)**.** *There exists a* 5*-GDD of type* g^5 *for any positive integer* $g \geq 4$ *except for* $g \in \{6, 10\}$.

Lemma 3.4 (*[5]*)**.** *The necessary and sufficient conditions for the existence of a* 4*-GDD of type* g^n *are* $(1) n \geq 4$, $(2) (n-1)g \equiv 0 \pmod 3$, $(3) n(n-1)g^2 \equiv 0 \pmod{12}$, *with the exception of* $(g, n) \in \{(2, 4), (6, 4)\}$, *in which case no such GDD exists.*

4. Ingredients

Lemma 4.1. *Let* $J_1(13) = \{s|$ *there exist two* $S(2, 4, 13)$ *designs with s common triangles and at least one common block* $\}$. *Then* $I_T(13) \setminus \{0, 1, \ldots, 7, 19\} \subseteq J_1(13)$.

Proof. Let $X = Z_{13}$ and $\mathcal{B} = \{\{i, 1+i, 3+i, 9+i\} : 0 \le i \le 12\}$. Then $(X, \mathcal{B})$ is an $S(2, 4, 13)$ design. Consider the following permutations on X.

$$\pi_8 = (4\,5\,6)(7\,12\,10\,11\,8), \quad \pi_9 = (4\,5\,6)(7\,12\,10)(8\,11),$$
$$\pi_{10} = (4\,5\,6\,7\,10\,12\,11\,8), \quad \pi_{11} = (4\,5\,6\,7\,12\,10\,11\,8),$$
$$\pi_{12} = (4\,5\,6\,7\,12\,8)(10\,11), \quad \pi_{13} = (4\,5\,6\,7\,12)(10\,11),$$
$$\pi_{14} = (4\,5\,6\,7\,12\,8\,10), \quad \pi_{15} = (4\,5\,6\,8\,7)(11\,12),$$
$$\pi_{16} = (4\,5\,6\,8\,12\,11\,7), \quad \pi_{17} = (4\,5\,6\,8\,12\,11\,10),$$
$$\pi_{18} = (4\,5\,6\,8)(7\,11\,12), \quad \pi_{20} = (7\,8)(11\,12),$$
$$\pi_{21} = (6\,7)(8\,10\,12), \quad \pi_{22} = (6\,7)(8\,12)(10\,11),$$
$$\pi_{25} = (10\,11\,12), \quad \pi_{28} = (8\,10)(11\,12),$$
$$\pi_{34} = (11\,12), \quad \pi_{52} = (1).$$

It is readily checked that $\{0, 1, 3, 9\} \in \pi_j\mathcal{B} \bigcap \mathcal{B}$ and $|\pi_j T(\mathcal{B}) \bigcap T(\mathcal{B})| = j$ for each $j \in I_T(13) \setminus \{0, 1, \ldots, 7, 19\} \subseteq J_1(13)$. □

Lemma 4.2. $I_T(13) \setminus \{1, 2, 19\} \subseteq J_T(13)$.

Proof. Take the $S(2, 4, 13)$ design $(X, \mathcal{B})$ constructed in Lemma 4.1. Consider the following permutations on X.

$$\pi_0 = (3\,4)(6\,7\,8\,10)(9\,11\,12), \quad \pi_3 = (5\,6\,7\,8)(9\,11\,10\,12),$$
$$\pi_4 = (5\,6\,7\,8\,9)(10\,12\,11), \quad \pi_5 = (6\,7\,8\,9)(10\,12\,11),$$
$$\pi_6 = (5\,6)(7\,8\,9)(10\,11\,12), \quad \pi_7 = (6\,7\,8\,9\,10\,12\,11).$$

It is readily checked that $|\pi_i T(\mathcal{B}) \bigcap T(\mathcal{B})| = i$ for each $i \in \{0, 3, 4, 5, 6, 7\}$. Combining the results from Lemma 4.1, we complete the proof. □

Lemma 4.3. *Let* $J_1(16) = \{s|$ *there exist two* $S(2, 4, 16)$ *designs with s common triangles and at least one common block* $\}$. *Then* $I_T(16) \setminus \{0, 1, 2, 3, 37, 39, 41, 43, 45–50, 53, 62\} \subseteq J_1(16)$.

Proof. Construct an $S(2, 4, 16)$ design $(X, \mathcal{B})$ with $X = Z_{16}$. All blocks of $\mathcal{B}$ are listed below, which can be found in Example 1.31 in [13].

{0, 1, 2, 3},	{0, 4, 5, 6},	{0, 7, 8, 9},	{0, 10, 11, 12},	{0, 13, 14, 15},
{1, 4, 7, 10},	{1, 5, 11, 13},	{1, 6, 8, 14},	{1, 9, 12, 15},	{2, 4, 12, 14},
{2, 5, 7, 15},	{2, 6, 9, 11},	{2, 8, 10, 13},	{3, 4, 9, 13},	{3, 5, 8, 12},
{3, 6, 10, 15},	{3, 7, 11, 14},	{4, 8, 11, 15},	{5, 9, 10, 14},	{6, 7, 12, 13}.

Consider the following permutations on X.

$$\pi_4 = (5\,7\,13\,15\,9)(6\,11\,14\,10\,12), \quad \pi_5 = (6\,7\,11\,9\,14\,15\,10\,12\,13\,8),$$
$$\pi_6 = (6\,7\,8\,15\,9\,10)(11\,14\,13\,12), \quad \pi_7 = (6\,7\,8)(9\,14\,11\,10\,12\,13\,15),$$
$$\pi_8 = (6\,7)(8\,9\,10\,13\,14\,11\,12\,15), \quad \pi_9 = (6\,7)(9\,10\,15\,13\,14\,11\,12),$$
$$\pi_{10} = (6\,7)(10\,12\,11\,14\,13\,15), \quad \pi_{11} = (7\,8)(9\,10\,12\,13\,11\,14\,15),$$
$$\pi_{12} = (8\,9\,12\,11)(10\,13\,15\,14), \quad \pi_{13} = (8\,10\,14\,13\,15\,9\,12\,11),$$
$$\pi_{14} = (8\,9\,10)(11\,12\,15)(13\,14), \quad \pi_{15} = (8\,9\,10\,12\,15\,14\,13\,11),$$
$$\pi_{16} = (9\,10\,11\,12\,15\,13\,14), \quad \pi_{17} = (9\,10\,11)(12\,13\,14\,15),$$
$$\pi_{18} = (9\,10\,11\,12)(13\,14\,15), \quad \pi_{19} = (9\,10)(11\,12\,13)(14\,15),$$
$$\pi_{20} = (9\,10)(12\,13\,14\,15), \quad \pi_{21} = (9\,10)(12\,14\,13\,15),$$
$$\pi_{22} = (10\,11\,12\,13\,14\,15), \quad \pi_{23} = (9\,10)(12\,13)(14\,15),$$
$$\pi_{24} = (10\,11)(12\,13\,14\,15), \quad \pi_{25} = (10\,11\,13\,15\,12\,14),$$
$$\pi_{26} = (10\,11)(12\,13)(14\,15), \quad \pi_{27} = (10\,11\,12\,13\,15\,14),$$
$$\pi_{28} = (9\,10)(12\,13\,14), \quad \pi_{29} = (11\,12\,13)(14\,15),$$
$$\pi_{30} = (11\,12\,13\,15\,14), \quad \pi_{31} = (11\,13\,12\,15\,14),$$
$$\pi_{32} = (10\,13\,11\,14\,12\,15), \quad \pi_{33} = (11\,13)(12\,14\,15),$$
$$\pi_{34} = (12\,13\,14\,15), \quad \pi_{35} = (11\,12)(13\,14\,15),$$
$$\pi_{36} = (12\,13)(14\,15), \quad \pi_{38} = (11\,12\,13\,15),$$

$\pi_{40} = (9\ 10)(12\ 13)$, $\pi_{42} = (11\ 12)(14\ 15)$,
$\pi_{44} = (13\ 14\ 15)$, $\pi_{56} = (14\ 15)$,
$\pi_{80} = (1)$.

It is readily checked that the block $\{0, 1, 2, 3\} \in \pi_i\mathcal{B} \bigcap \mathcal{B}$ and $|\pi_i T(\mathcal{B}) \bigcap T(\mathcal{B})| = i$ for each $i \in I_T(16) \setminus \{0, 1, 2, 3, 37, 39, 41, 43, 45–50, 53, 62\}$. □

Lemma 4.4. $I_T(16) \setminus \{37, 39, 41, 43, 45–50, 53, 62\} \subseteq J_T(16)$.

Proof. Take the $S(2, 4, 16)$ design $(X, \mathcal{B})$ constructed in Lemma 4.3. Consider the following permutations on X.

$\pi_0 = (2\ 4)(3\ 8\ 10\ 13\ 15\ 12\ 11\ 6\ 9\ 14)$, $\pi_1 = (3\ 4)(6\ 7\ 12\ 15\ 13\ 10\ 11)(8\ 14\ 9)$,
$\pi_2 = (3\ 4)(6\ 7)(8\ 11\ 10\ 13\ 15)(9\ 14\ 12)$, $\pi_3 = (3\ 4)(7\ 9\ 15\ 13)(8\ 10\ 14\ 11\ 12)$.

It is readily checked that $|\pi_i T(\mathcal{B}) \bigcap T(\mathcal{B})| = i$ for each $i \in \{0, 1, 2, 3\}$. Combining the results from Lemma 4.3, we complete the proof. □

Lemma 4.5. $\{0, 1, 2, 200\} \subseteq J_T(25)$.

Proof. Construct an $S(2, 4, 25)$ design $(X, \mathcal{B})$ with $X = Z_{25}$. All blocks of $\mathcal{B}$ are listed below, which can be found in Table 1.34 in [13] (the 18th design).

{0, 1, 2, 3},	{0, 4, 5, 6},	{0, 7, 8, 9},	{0, 10, 11, 12},	{0, 13, 14, 15},
{0, 16, 17, 18},	{0, 19, 20, 21},	{0, 22, 23, 24},	{1, 4, 7, 10},	{1, 5, 8, 13},
{1, 6, 11, 16},	{1, 9, 17, 19},	{1, 12, 20, 22},	{1, 14, 18, 23},	{1, 15, 21, 24},
{2, 4, 8, 18},	{2, 5, 7, 20},	{2, 6, 19, 24},	{2, 9, 10, 14},	{2, 11, 15, 22},
{2, 12, 16, 23},	{2, 13, 17, 21},	{3, 4, 17, 22},	{3, 5, 12, 21},	{3, 6, 7, 15},
{3, 8, 19, 23},	{3, 9, 11, 13},	{3, 10, 18, 24},	{3, 14, 16, 20},	{4, 9, 12, 24},
{4, 11, 14, 21},	{4, 13, 20, 23},	{4, 15, 16, 19},	{5, 9, 16, 22},	{5, 10, 15, 23},
{5, 11, 18, 19},	{5, 14, 17, 24},	{6, 8, 14, 22},	{6, 9, 21, 23},	{6, 10, 17, 20},
{6, 12, 13, 18},	{7, 11, 17, 23},	{7, 12, 14, 19},	{7, 13, 16, 24},	{7, 18, 21, 22},
{8, 10, 16, 21},	{8, 11, 20, 24},	{8, 12, 15, 17},	{9, 15, 18, 20},	{10, 13, 19, 22}.

Consider the following permutations on X.

$\pi_0 = (0\ 23\ 2\ 14)(1\ 5\ 9\ 11\ 22\ 18\ 24\ 19\ 20\ 6\ 8\ 12)(3\ 21\ 16\ 17\ 13\ 10\ 7\ 4\ 15)$,
$\pi_1 = (0\ 23\ 16\ 24\ 13\ 10\ 7\ 4\ 15\ 3\ 21\ 18\ 20\ 2\ 14)(1\ 5\ 9\ 11\ 22\ 19\ 17\ 6\ 8\ 12)$,
$\pi_2 = (0\ 23\ 2\ 14)(1\ 5\ 9\ 11\ 22\ 18\ 24\ 19\ 20\ 6\ 8\ 12)(3\ 21\ 13\ 10\ 7\ 4\ 15)(16\ 17)$,
$\pi_{200} = (1)$.

It is readily checked that $|\pi_i T(\mathcal{B}) \bigcap T(\mathcal{B})| = i$ for each $i \in \{0, 1, 2, 200\}$. □

Lemma 4.6. *There exists a pair of $S(2, 4, 25)$ designs with exactly one common block and 4 common triangles.*

Proof. Take the $S(2, 4, 25)$ design $(X, \mathcal{B})$ constructed in Lemma 4.5. Consider the permutation $\pi = (4\ 13\ 23\ 18)(5\ 7\ 17\ 10\ 11\ 22\ 12\ 19\ 8\ 15\ 14\ 16\ 9\ 20\ 24)$. It is readily checked that $\{0, 1, 2, 3\} \in \pi\mathcal{B} \bigcap \mathcal{B}$ and $|\pi T(\mathcal{B}) \bigcap T(\mathcal{B})| = 4$. □

Lemma 4.7. $\{0, 252\} \subseteq J_T(28)$.

Proof. Construct an $S(2, 4, 28)$ design $(X, \mathcal{B})$ with $X = Z_{28}$. All blocks of $\mathcal{B}$ are divided into two parts. The first part consists of $\{i, 7 + i, 14 + i, 21 + i\}$, $0 \le i \le 6$. Develop the following base blocks by $+4$ modulo 28 to obtain the second part of $\mathcal{B}$.

{0, 1, 2, 3},	{0, 4, 9, 12},	{0, 6, 11, 22},	{0, 10, 13, 18},
{0, 15, 19, 27},	{0, 17, 23, 26},	{1, 5, 13, 23},	{1, 14, 18, 27}.

Consider the following permutations on X.

$\pi_0 = (0\ 15\ 26\ 25\ 9\ 10)(1\ 12\ 5\ 27\ 18\ 22\ 6\ 7\ 20)(2\ 8\ 14\ 24\ 16\ 21\ 19\ 13\ 23\ 17\ 4\ 3)$,
$\pi_{252} = (1)$.

It is readily checked that $|\pi_j T(\mathcal{B}) \bigcap T(\mathcal{B})| = j$ for each $j \in \{0, 252\}$. □

Lemma 4.8. $\{1, 2\} \subseteq J_T(49)$.

Proof. Construct two $S(2,4,49)$ designs $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$. Only base blocks are listed below. Develop these base blocks by $+1$ modulo 49 to obtain all blocks of $\mathcal{B}_i$, $i = 1, 2$.

$$\begin{array}{lllll} \mathcal{B}_1: & \{0,1,10,22\}, & \{0,2,5,13\}, & \{0,4,20,35\}, & \{0,6,25,32\}. \\ \mathcal{B}_2: & \{0,1,3,8\}, & \{0,4,18,29\}, & \{0,6,21,33\}, & \{0,9,19,32\}. \end{array}$$

Consider the following permutations on X.

$$\pi_1 = (46\ 47\ 48), \qquad \pi_2 = (47\ 48).$$

It is readily checked that $|\pi_j T(\mathcal{B}_2) \bigcap T(\mathcal{B}_1)| = j$ for each $j \in \{1, 2\}$. □

Lemma 4.9. *There exists a pair of 4-GDDs of type 3^4 with i common triangles, $i \in \{9, 12, 18, 36\}$.*

Proof. Take the $S(2,4,13)$ design $(X, \mathcal{B})$ constructed in Lemma 4.1. Delete the point 0 from this design to obtain a 4-GDD of type $3^4(X \setminus \{0\}, \mathcal{G}, \mathcal{B}')$, where $\mathcal{G} = \{\{1,3,9\}, \{4,5,7\}, \{6,10,11\}, \{2,8,12\}\}$, and $\mathcal{B}' = \mathcal{B} \setminus \{B \in \mathcal{B} : 0 \in B\}$. Consider the following permutations on $X \setminus \{0\}$, which keep $\mathcal{G}$ invariant.

$$\pi_9 = (6\ 10\ 11), \qquad \pi_{12} = (8\ 12)(10\ 11),$$
$$\pi_{18} = (10\ 11), \qquad \pi_{36} = (1).$$

It is readily checked that $|\pi_i T(\mathcal{B}') \bigcap T(\mathcal{B}')| = i$ for each $i \in \{9, 12, 18, 36\}$. □

Lemma 4.10. *There exists a pair of 4-GDDs of type 3^5 with i common triangles, $i \in \{0, 60\}$.*

Proof. Take the $S(2,4,16)$ design $(X, \mathcal{B})$ constructed in Lemma 4.3. Delete the point 0 from this design to obtain a 4-GDD of type 3^5 $(X \setminus \{0\}, \mathcal{G}, \mathcal{B}')$, where $\mathcal{G} = \{\{1+3j, 2+3j, 3+3j\} : 0 \le j \le 4\}$, and $\mathcal{B}' = \mathcal{B} \setminus \{B \in \mathcal{B} : 0 \in B\}$. Consider the following permutations on $X \setminus \{0\}$, which keep $\mathcal{G}$ invariant.

$$\pi_0 = (2\ 3)(5\ 6)(7\ 8)(10\ 12)(13\ 15), \qquad \pi_{60} = (1).$$

It is readily checked that $|\pi_i T(\mathcal{B}') \bigcap T(\mathcal{B}')| = i$ for each $i \in \{0, 60\}$. □

Lemma 4.11. *There exists a pair of 4-GDDs of type g^4 without common triangles for $g \in \{4, 5, 9\}$.*

Proof. Let $X = GF(g) \times \{0,1,2,3\}$ and $\mathcal{G} = \{GF(g) \times \{i\} : i \in \{0,1,2,3\}\}$. Let

$$\mathcal{B}_1 = \{\{(j,0), (k,1), (j+\lambda k, 2), (j+\mu k, 3)\} : j, k \in GF(g)\},$$
$$\mathcal{B}_2 = \{\{(j,0), (k,1), (j+\lambda k+\alpha, 2), (j+\mu k+\beta, 3)\} : j, k \in GF(g)\},$$

where $\lambda, \mu, \alpha, \beta \in GF(g)$, $\lambda, \mu \ne 0$ and $\lambda \ne \mu$. Then $(X, \mathcal{G}, \mathcal{B}_1)$ and $(X, \mathcal{G}, \mathcal{B}_2)$ are two 4-GDDs of type g^4.

It is readily checked that if one can choose $\lambda, \mu, \alpha, \beta \in GF(g) \setminus \{0\}$ such that $\lambda \ne \mu, \alpha \ne \beta$ and $\lambda\beta \ne \mu\alpha$, then $|T(\mathcal{B}_1) \bigcap T(\mathcal{B}_2)| = 0$. Thus for $g = 4$, one may take $(\lambda, \mu, \alpha, \beta) = (1, x, x, 1)$, where x is a primitive element of $GF(4)$ satisfying $1 + x + x^2 = 0$. For $g = 5$, take $(\lambda, \mu, \alpha, \beta) = (1, 2, 2, 1)$. For $g = 9$, take $(\lambda, \mu, \alpha, \beta) = (1, 2, 1, x)$, where x is a primitive element of $GF(9)$ satisfying $2 + x + x^2 = 0$. □

Lemma 4.12. *There exists a pair of 4-GDDs of type 4^4 with i common triangles, $i \in \{0, 64\}$.*

Proof. The case of $i = 0$ comes immediately from Lemma 4.11. Take the identity permutation to act on the block sets of two same 4-GDDs of type 4^4 to obtain the case of $i = 64$. □

5. Applying the recursions

Lemma 5.1. *For any positive integer $v \equiv 1, 13 \pmod{48}$ and $v \ge 49$, $I_T(v) \setminus \{1, 2, t_v - 33\} \subseteq J_T(v)$.*

Proof. Let $v = 12u + 1$ with $u \equiv 0, 1 \pmod 4$ and $u \ge 4$. Start from a 4-GDD of type 3^u from Lemma 3.4. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 with α common triangles, $\alpha \in \{0, 64\}$. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 12^u with $\sum_{i=1}^{b} \alpha_i$ common triangles, where $b = 3u(u-1)/4$ and $\alpha_i \in \{0, 64\}$ for $1 \le i \le b$. By Construction 3.2, filling in the holes by a pair of $S(2,4,13)$ designs with β_j $(1 \le j \le u)$ common triangles from Lemma 4.2, we have a pair of $S(2,4,12u+1)$ designs with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u} \beta_j$ common triangles, where $\beta_j \in J_T(13)$ for $1 \le j \le u$. It is readily checked that for any integer $n \in I_T(v) \setminus \{1, 2, t_v - 33\}$, n can be written as the form of $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u} \beta_j$, where $\alpha_i \in \{0, 64\}$ $(1 \le i \le b)$, $\beta_j \in J_T(13)$ $(1 \le j \le u)$. □

Lemma 5.2. $\{1, 2\} \subseteq J_T(97)$.

Proof. There exists a 4-GDD of type $3^4 6^2$ [10]. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type $12^4 24^2$ without common triangles. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs without common triangles from Lemma 4.2, and a pair of $S(2, 4, 25)$ designs with β_j $(1 \leq j \leq 2)$ common triangles from Lemma 4.5, we have a pair of $S(2, 4, 97)$ designs with $\beta_1 + \beta_2$ common triangles, where $\beta_j \in \{0, 1, 2\}$ for $1 \leq j \leq 2$. □

Lemma 5.3. *For any positive integer* $v \equiv 1 \pmod{24}$ *and* $v \geq 121$, $\{1, 2\} \subseteq J_T(v)$.

Proof. For any positive integer $u \geq 5$, there exists a 4-GDD of type 6^u from Lemma 3.4. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 24^u without common triangles. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 25)$ designs with β_j $(1 \leq j \leq u)$ common triangles from Lemma 4.5, we have a pair of $S(2, 4, 24u + 1)$ designs with $\sum_{j=1}^{u} \beta_j$ common triangles, where $\beta_j \in \{0, 1, 2\}$ for $1 \leq j \leq u$. □

Combining the results from Lemmas 4.8 and 5.1–5.3, we have the following

Lemma 5.4. *For any positive integer* $v \equiv 1 \pmod{48}$ *and* $v \geq 49$, $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v)$.

Lemma 5.5. $\{1, 2\} \subseteq J_T(61)$.

Proof. Start from a 4-GDD of type 3^4 from Lemma 3.4. Give each point of the GDD weight 5. By Lemma 4.11, there is a pair of 4-GDDs of type 5^4 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 15^4 without common triangles. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 16)$ designs with β_j $(1 \leq j \leq 4)$ common triangles from Lemma 4.4, we have a pair of $S(2, 4, 61)$ designs with $\sum_{j=1}^{4} \beta_j$ common triangles, where $\beta_j \in \{0, 1, 2\}$ for $1 \leq j \leq 4$. □

Lemma 5.6. $\{1, 2\} \subseteq J_T(109)$.

Proof. Start from a 5-GDD of type 7^5 from Lemma 3.3. Give each point of the GDD weight 3. By Lemma 4.10, there is a pair of 4-GDDs of type 3^5 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 21^5 without common triangles. By Construction 3.2, filling in the first four holes by a pair of $S(2, 4, 25)$ designs with exactly one common block and 4 common triangles from Lemma 4.6, and filling in the last hole by a pair of $S(2, 4, 25)$ designs with β common triangles from Lemma 4.5, we have a pair of $S(2, 4, 109)$ designs with β common triangles, where $\beta \in \{1, 2\}$. □

Lemma 5.7. *There exists a pair of* $S(2, 4, 49)$ *designs containing a common* $S(2, 4, 13)$ *as a subdesign, which have no common triangles except for the triangles in the common* $S(2, 4, 13)$.

Proof. Start from a 4-GDD of type 3^4 from Lemma 3.4. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 12^4 without common triangles. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs without common triangles from Lemma 4.2, we have a pair of $S(2, 4, 49)$ designs containing a common $S(2, 4, 13)$ as a subdesign, which have no common triangles except for the triangles in the common $S(2, 4, 13)$. □

Lemma 5.8. $\{1, 2\} \subseteq J_T(157)$.

Proof. Start from a 4-GDD of type 9^4 from Lemma 3.4. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 36^4 without common triangles. By Construction 3.2, filling in the first three holes by a pair of $S(2, 4, 49)$ designs containing a common $S(2, 4, 13)$ as a subdesign from Lemma 5.7, which have no common triangles except for the triangles in the common $S(2, 4, 13)$, and filling in the last hole by a pair of $S(2, 4, 49)$ designs with β common triangles from Lemma 4.8, we have a pair of $S(2, 4, 157)$ designs with β common triangles, where $\beta \in \{1, 2\}$. □

Lemma 5.9. *There exists a pair of* $S(2, 4, 61)$ *designs containing a common* $S(2, 4, 13)$ *as a subdesign, which have no common triangles except for the triangles in the common* $S(2, 4, 13)$.

Proof. Start from a 5-GDD of type 4^5 from Lemma 3.3. Give each point of the GDD weight 3. By Lemma 4.10, there is a pair of 4-GDDs of type 3^5 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 12^5 without common triangles. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs without common triangles from Lemma 4.2, we have a pair of $S(2, 4, 61)$ designs containing a common $S(2, 4, 13)$ as a subdesign, which have no common triangles except for the triangles in the common $S(2, 4, 13)$. □

Lemma 5.10. *For any positive integer* $v \equiv 13 \pmod{48}$ *and* $v \geq 205$, $\{1, 2\} \subseteq J_T(v)$.

Proof. For any positive integer $u \geq 4$, there exists a 4-GDD of type 12^u from Lemma 3.4. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 without common triangles. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 48^u without common triangles. By Construction 3.2, filling in the fist $u-1$ holes by a pair of $S(2, 4, 61)$ designs containing a common $S(2, 4, 13)$ as a subdesign from Lemma 5.9, which have no common triangles except for the triangles in the common $S(2, 4, 13)$, and filling in the last hole by a pair of $S(2, 4, 61)$ designs with β common triangles from Lemma 5.5, where $\beta \in \{1, 2\}$, we have a pair of $S(2, 4, 48u + 13)$ designs with β common triangles. □

Combining the results from Lemmas 5.1, 5.5, 5.6, 5.8 and 5.10, we have the following

Lemma 5.11. *For any positive integer* $v \equiv 13 \pmod{48}$ *and* $v \geq 61$, $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v)$.

Lemma 5.12 ([4]). *There exists a* $(v, \{4, 7^*\}, 1)$*-PBD with exactly one block of size* 7 *for any positive integer* $v \equiv 7, 10 \pmod{12}$ *and* $v \neq 10, 19$.

Lemma 5.13. *For any positive integer* $v \equiv 25, 37 \pmod{48}$ *and* $v \geq 73$, $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v)$.

Proof. Let $v = 12u + 1$ with $u \equiv 2, 3 \pmod 4$ and $u \geq 7$. There exists a $(3u + 1, \{4, 7^*\}, 1)$-PBD from Lemma 5.12, which contains exactly one block of size 7. Take a point from the block of size 7. Delete this point to obtain a 4-GDD of type $3^{u-2}6^1$. Give each point of the GDD weight 4. By Lemma 4.12, there is a pair of 4-GDDs of type 4^4 with α common triangles, $\alpha \in \{0, 64\}$. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type $12^{u-2}24^1$ with $\sum_{i=1}^{b} \alpha_i$ common triangles, where $b = 3(u^2-u-2)/4$ and $\alpha_i \in \{0, 64\}$ for $1 \leq i \leq b$. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs with β_j $(1 \leq j \leq u - 2)$ common triangles from Lemma 4.2, and a pair of $S(2, 4, 25)$ designs with β_{u-1} common triangles from Lemma 4.5, we have a pair of $S(2, 4, 12u+1)$ designs with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2} \beta_j + \beta_{u-1}$ common triangles, where $\beta_j \in J_T(13)$ for $1 \leq j \leq u - 2$ and $\beta_{u-1} \in J_T(25)$. It is readily checked that for any integer $n \in I_T(v) \setminus \{t_v - 33\}$, n can be written as the form of $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2} \beta_j + \beta_{u-1}$, where $\alpha_i \in \{0, 64\}$ $(1 \leq i \leq b)$, $\beta_j \in J_T(13)$ $(1 \leq j \leq u - 2)$, $\beta_{u-1} \in \{0, 1, 2, 200\}$.

When $v = 73$, start from an $S(2, 5, 25)$. Delete a point from this design to obtain a 5-GDD of type 4^6. Give each point of the GDD weight 3. By Lemma 4.10, there is a pair of 4-GDDs of type 3^5 with α common triangles, $\alpha \in \{0, 60\}$. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 12^6 with $\sum_{i=1}^{24} \alpha_i$ common triangles, where $\alpha_i \in \{0, 60\}$ for $1 \leq i \leq 24$. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs with β_j $(1 \leq j \leq 6)$ common triangles from Lemma 4.2, we have a pair of $S(2, 4, 73)$ designs with $\sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j$ common triangles, where $\beta_j \in J_T(13)$ for $1 \leq j \leq 6$. It is readily checked that for any integer $n \in I_T(73) \setminus \{t_{73} - 33\}$, n can be written as the form of $\sum_{i=1}^{24} \alpha_i + \sum_{j=1}^{6} \beta_j$, where $\alpha_i \in \{0, 60\}$ $(1 \leq i \leq 24)$, $\beta_j \in J_T(13)$ $(1 \leq j \leq 6)$. □

Lemma 5.14. *Let* $E(v) = \{t_v-18, t_v-27, t_v-30, t_v-31, t_v-32, t_v-33, t_v-34, t_v-35, t_v-37, t_v-39, t_v-41, t_v-43\}$. *For any positive integer* $v \equiv 4 \pmod{12}$ *and* $v \geq 52$, $I_T(v) \setminus E(v) \subseteq J_T(v)$.

Proof. We divide the problem into two cases.

Case 1: Let $v = 12u+4$ with $u \equiv 0, 1 \pmod 4$ and $u \geq 4$. By similar arguments as in Lemma 5.1, there is a pair of 4-GDDs of type 12^u with $\sum_{i=1}^{b} \alpha_i$ common triangles, where $b = 3u(u-1)/4$ and $\alpha_i \in \{0, 64\}$ for $1 \leq i \leq b$. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 16)$ designs with β_j $(1 \leq j \leq u - 1)$ common triangles and at least one common block from Lemma 4.3, and a pair of $S(2, 4, 16)$ designs with β_u common triangles from Lemma 4.4, we have a pair of $S(2, 4, 12u + 4)$ designs with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-1}(\beta_j - 4) + \beta_u$ common triangles, where $\beta_j \in J_1(16)$ for $1 \leq j \leq u - 1$ and $\beta_u \in J_T(16)$. It is readily checked that for any integer $n \in I_T(v) \setminus E(v)$, n can be written as the form of $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-1}(\beta_j - 4) + \beta_u$, where $\alpha_i \in \{0, 64\}$ $(1 \leq i \leq b)$, $\beta_j \in J_1(16)$ $(1 \leq j \leq u - 1)$, $\beta_u \in J_T(16)$.

Case 2: Let $v = 12u + 4$ with $u \equiv 2, 3 \pmod 4$ and $u \geq 7$. By similar arguments as in Lemma 5.13, there is a pair of 4-GDDs of type $12^{u-2}24^1$ with $\sum_{i=1}^{b} \alpha_i$ common triangles, where $b = 3(u^2 - u - 2)/4$ and $\alpha_i \in \{0, 64\}$ for $1 \leq i \leq b$. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 16)$ designs with β_j $(1 \leq j \leq u - 2)$ common triangles and at least one common block from Lemma 4.3, and a pair of $S(2, 4, 28)$ designs with β_{u-1} common triangles from Lemma 4.7, we have a pair of $S(2, 4, 12u + 4)$ designs with $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2}(\beta_j - 4) + \beta_{u-1}$ common triangles, where $\beta_j \in J_1(16)$ for $1 \leq j \leq u - 2$ and $\beta_{u-1} \in J_T(28)$. It is readily checked that for any integer $n \in I_T(v) \setminus E(v)$, n can be written as the form of $\sum_{i=1}^{b} \alpha_i + \sum_{j=1}^{u-2}(\beta_j - 4) + \beta_{u-1}$, where $\alpha_i \in \{0, 64\}$ $(1 \leq i \leq b)$, $\beta_j \in J_1(16)$ $(1 \leq j \leq u - 2)$, $\beta_{u-1} \in \{0, 252\}$.

When $v = 76$, start from a 5-GDD of type 5^5 from Lemma 3.3. Give each point of the GDD weight 3. By Lemma 4.10, there is a pair of 4-GDDs of type 3^5 with α common triangles, $\alpha \in \{0, 60\}$. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 15^5 with $\sum_{i=1}^{25} \alpha_i$ common triangles, where $\alpha_i \in \{0, 60\}$ for $1 \leq i \leq 25$. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 16)$ designs with β_j $(1 \leq j \leq 5)$ common triangles from Lemma 4.4, we have a pair of $S(2, 4, 76)$ designs with $\sum_{i=1}^{25} \alpha_i + \sum_{j=1}^{5} \beta_j$ common triangles, where $\beta_j \in J_T(16)$ for $1 \leq j \leq 5$. It is readily checked that for any integer $n \in I_T(76) \setminus E(76)$, n can be written as the form of $\sum_{i=1}^{25} \alpha_i + \sum_{j=1}^{5} \beta_j$, where $\alpha_i \in \{0, 60\}$ $(1 \leq i \leq 25)$, $\beta_j \in J_T(16)$ $(1 \leq j \leq 5)$. □

Lemma 5.15 (*[15]*). *If* $v \equiv 1, 4 \pmod{12}$, $w \equiv 1, 4 \pmod{12}$ *and* $v \geq 3w + 1$, *then there is an* $S(2, 4, v)$ *containing an* $S(2, 4, w)$ *as a subdesign.*

Lemma 5.16. *For any positive integer* $v \equiv 1, 4 \pmod{12}$ *and* $v \geq 40$, $E(v) \subseteq J_T(v)$, *where* $E(v)$ *is defined as in Lemma 5.14.*

Proof. By Lemma 5.15, there is an $S(2, 4, v)$ $(X, \mathcal{B})$ containing an $S(2, 4, 13)$ $(Y, \mathcal{A})$ as a subdesign, where $Y \subseteq X$. By Lemma 4.2, there is a pair of $S(2, 4, 13)$ $(Y, \mathcal{A}_1)$ and $(Y, \mathcal{A}_2)$ such that $|T(\mathcal{A}_1) \bigcap T(\mathcal{A}_2)| = r$, $r \in \{9, 11, 13, 15, 17, 18, 20, 21, 22, 25, 34\}$. It is readily checked that $(X, (\mathcal{B} \setminus \mathcal{A}) \bigcup \mathcal{A}_1)$ and $(X, (\mathcal{B} \setminus \mathcal{A}) \bigcup \mathcal{A}_2)$ are two $S(2, 4, v)$ designs with s common triangles, $s \in E(v)$. □

Combining the results from Lemmas 5.14 and 5.16, we have the following

Lemma 5.17. *For any positive integer* $v \equiv 4 \pmod{12}$ *and* $v \geq 52$, $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v)$.

Lemma 5.18. $I_T(40) \setminus \{0, 1, \ldots, 11, t_{40} - 33\} \subseteq J_T(40)$.

Proof. By Lemma 4.11 there is a pair of 4-GDDs of type 9^4 without common triangles. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs with β_i $(1 \leq i \leq 3)$ common triangles and at least one common block from Lemma 4.1, and a pair of $S(2, 4, 13)$ designs with β_4 common triangles from Lemma 4.2, we have a pair of $S(2, 4, 40)$ designs with $\sum_{i=1}^{4} \beta_i$ common triangles, where $\beta_i \in J_1(13)$ for $1 \leq i \leq 3$ and $\beta_4 \in J_T(13)$. Let $N = \{12, 13, \ldots, 162, 164, 165, 166, 169, 172, 178, 196\}$. It is readily checked that for any integer $n \in N$, n can be written as the form of $\sum_{i=1}^{4} \beta_i$, where $\beta_i \in J_1(13)$ $(1 \leq i \leq 3)$ and $\beta_4 \in J_T(13)$. Thus $N \subseteq J_T(40)$.

Start from a 4-GDD of type 3^4 from Lemma 3.4. Give each point of the GDD weight 4. By Lemma 4.9, there is a pair of 4-GDDs of type 3^4 with α common triangles, $\alpha \in \{9, 12, 18, 36\}$. Then apply Construction 3.1 to obtain a pair of 4-GDDs of type 9^4 with $\sum_{i=1}^{9} \alpha_i$ common triangles, where $\alpha_i \in \{9, 12, 18, 36\}$ for $1 \leq i \leq 9$. By Construction 3.2, filling in the holes by a pair of $S(2, 4, 13)$ designs with β_j $(1 \leq j \leq 3)$ common triangles and at least one common block from Lemma 4.1, and a pair of $S(2, 4, 13)$ designs with β_4 common triangles from Lemma 4.2, we have a pair of $S(2, 4, 40)$ designs with $\sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{4} \beta_i$ common triangles, where $\beta_j \in J_1(13)$ $(1 \leq j \leq 3)$ and $\beta_4 \in J_T(13)$. Let $M = \{93, 94, \ldots, 486, 488, 489, 490, 493, 496, 502, 520\}$. It is readily checked that for any integer $m \in M$, m can be written as the form of $\sum_{i=1}^{9} \alpha_i + \sum_{j=1}^{4} \beta_i$, where $\alpha_i \in \{9, 12, 18, 36\}$ $(1 \leq i \leq 9)$, $\beta_j \in J_1(13)$ $(1 \leq j \leq 3)$ and $\beta_4 \in J_T(13)$. Thus $M \subseteq J_T(40)$. This completes the proof. □

Lemma 5.19. $t_{40} - 33 \in J_T(40)$.

Proof. It is well known that a 3-RGDD of type 9^3 is equivalent to two mutually orthogonal Latin squares (MOLS) of order 9. Thus there exists a 3-RGDD of type 9^3 [1]. Let $X = \{1, 2, \ldots, 27\}$, $G_1 = \{1, 2, \ldots, 9\}$, $G_2 = \{10, 11, \ldots, 18\}$, $G_3 = \{19, 20, \ldots, 27\}$ and $\mathcal{G} = \{G_1, G_2, G_3\}$. Let $(X, \mathcal{G}, \mathcal{B})$ be a 3-RGDD of type 9^3, which has 9 parallel classes $P_1, P_2, \ldots, P_9$. Without loss of generality we assume that P_1 contains 9 blocks of the form

$$\underline{\{7, 10, 19\}}, \quad \underline{\{8, 11, 20\}}, \quad \underline{\{9, 12, 21\}}, \quad \{1, *, *\}, \quad \{2, *, *\}, \quad \{3, *, *\},$$
$$\underline{\{4, *, *\}}, \quad \underline{\{5, *, *\}}, \quad \underline{\{6, *, *\}}.$$

Construct three KTS(9)s on G_1, G_2 and G_3, respectively. Each of them has 4 parallel classes $Q_{i1}, Q_{i2}, Q_{i3}, Q_{i4}$, $i = 1, 2, 3$. Without loss of generality we assume that

$$Q_{11} = \{\underline{\{1, 2, 3\}}, \underline{\{4, 5, 6\}}, \underline{\{7, 8, 9\}}\},$$
$$Q_{12} = \{\underline{\{1, 4, 7\}}, \underline{\{2, 5, 8\}}, \underline{\{3, 6, 9\}}\},$$
$$Q_{13} = \{\{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}\},$$
$$Q_{14} = \{\{2, 4, 9\}, \{3, 5, 7\}, \{1, 6, 8\}\},$$
$$Q_{21} = \{\underline{\{10, 11, 12\}}, \{13, 14, 15\}, \{16, 17, 18\}\},$$
$$Q_{22} = \{\{10, 13, 16\}, \{11, 14, 17\}, \{12, 15, 18\}\},$$
$$Q_{23} = \{\{10, 14, 18\}, \{11, 15, 16\}, \{12, 13, 17\}\},$$
$$Q_{24} = \{\{11, 13, 18\}, \{12, 14, 16\}, \{10, 15, 17\}\},$$
$$Q_{31} = \{\underline{\{19, 20, 21\}}, \{22, 23, 24\}, \{25, 26, 27\}\},$$
$$Q_{32} = \{\{19, 22, 25\}, \{20, 23, 26\}, \{21, 24, 27\}\},$$
$$Q_{33} = \{\{19, 23, 27\}, \{20, 24, 25\}, \{21, 22, 26\}\},$$
$$Q_{34} = \{\{20, 22, 27\}, \{21, 23, 25\}, \{19, 24, 26\}\}.$$

Let $P_{10} = Q_{11} \bigcup Q_{22} \bigcup Q_{32}$, $P_{11} = Q_{12} \bigcup Q_{21} \bigcup Q_{31}$, $P_{12} = Q_{13} \bigcup Q_{23} \bigcup Q_{33}$ and $P_{13} = Q_{14} \bigcup Q_{24} \bigcup Q_{34}$. Obviously for each $j \in \{10, 11, 12, 13\}$, P_j is a partition of X into 9 3-subsets. Construct an $S(2, 4, 13)$ design $(Y, \mathcal{A})$ on $Y =$

$\{\infty_1, \infty_2, \ldots, \infty_{13}\}$. Let $\mathcal{C} = \{B \cup \{\infty_l\} : B \in P_l, l \in \{1, 2, \ldots, 13\}\} \bigcup \mathcal{A}$. It is readily checked that $(X \cup Y, \mathcal{C})$ is an S(2, 4, 40) design.

Pay attention to the blocks with underlines listed above. Take $U \subseteq \mathcal{C}$ consisting of the following 11 blocks

$$\begin{array}{llll} \{\infty_1, 7, 10, 19\}, & \{\infty_1, 8, 11, 20\}, & \{\infty_1, 9, 12, 21\}, & \{\infty_{10}, 1, 2, 3\}, \\ \{\infty_{10}, 4, 5, 6\}, & \{\infty_{10}, 7, 8, 9\}, & \{\infty_{11}, 1, 4, 7\}, & \{\infty_{11}, 2, 5, 8\}, \\ \{\infty_{11}, 3, 6, 9\}, & \{\infty_{11}, 10, 11, 12\}, & \{\infty_{11}, 19, 20, 21\}. & \end{array}$$

Let U' consist of the following 11 4-subsets of $X \cup Y$

$$\begin{array}{llll} \{\infty_1, 7, 8, 9\}, & \{\infty_1, 10, 11, 12\}, & \{\infty_1, 19, 20, 21\}, & \{\infty_{10}, 1, 4, 7\}, \\ \{\infty_{10}, 2, 5, 8\}, & \{\infty_{10}, 3, 6, 9\}, & \{\infty_{11}, 1, 2, 3\}, & \{\infty_{11}, 4, 5, 6\}, \\ \{\infty_{11}, 7, 10, 19\}, & \{\infty_{11}, 8, 11, 20\}, & \{\infty_{11}, 9, 12, 21\}. & \end{array}$$

It is readily checked that (U, U') is a Steiner (4, 2) trade of volume 11. Let $\mathcal{D} = (\mathcal{C} \setminus U) \bigcup U'$. Thus $(X \cup Y, \mathcal{D})$ is also an S(2, 4, 40) design. It is easy to verify that $|T(\mathcal{C}) \bigcap T(\mathcal{D})| = t_{40} - 33$. This completes the proof. □

Lemma 5.20. $0 \in J_T(40)$.

Proof. Construct an $S(2, 4, 40)$ design $(X, \mathcal{B})$ with $X = Z_5 \times \{1, 2, 3, 4, 5, 6, 7, 8\}$. Only base blocks are listed below. All other blocks of $\mathcal{B}$ are obtained by developing these base blocks by $(+1 \bmod 5, -)$. This construction can be found in [7].

$$\begin{array}{lll} \{(2,1)(3,1)(2,3)(0,5)\}, & \{(4,1)(0,3)(2,4)(0,5)\}, & \{(1,1)(0,4)(1,4)(0,5)\}, \\ \{(2,2)(1,3)(3,3)(0,6)\}, & \{(3,2)(4,2)(4,4)(0,6)\}, & \{(1,2)(4,3)(3,4)(0,6)\}, \\ \{(0,1)(3,1)(1,2)(0,7)\}, & \{(1,1)(3,3)(4,3)(0,7)\}, & \{(0,2)(2,2)(2,3)(0,7)\}, \\ \{(4,1)(3,2)(1,4)(0,8)\}, & \{(2,1)(4,2)(3,4)(0,8)\}, & \{(1,3)(2,4)(4,4)(0,8)\}, \\ \{(0,1)(1,6)(0,8)(2,8)\}, & \{(0,1)(2,6)(3,6)(3,7)\}, & \{(0,1)(4,6)(1,7)(4,8)\}, \\ \{(0,2)(1,5)(2,7)(3,8)\}, & \{(0,2)(2,5)(1,7)(0,8)\}, & \{(0,2)(3,5)(4,5)(4,8)\}, \\ \{(0,3)(2,5)(4,7)(1,8)\}, & \{(0,3)(1,5)(4,5)(3,6)\}, & \{(0,3)(0,6)(2,8)(3,8)\}, \\ \{(0,4)(2,5)(0,6)(3,6)\}, & \{(0,4)(1,5)(1,7)(4,7)\}, & \{(0,4)(4,6)(2,7)(3,7)\}, \\ \{(0,1)(0,2)(0,5)(0,6)\}, & \{(0,3)(0,4)(0,7)(0,8)\}. & \end{array}$$

Consider the permutation π on X, such that for any $(a, b) \in B, B \in \mathcal{B}$, π keeps the first component of (a, b) invariant, that is, $\pi : (a, b) \longrightarrow (a, c)$. Thus we only list the action of π on the second component of (a, b) as follows

$$(1)(2\ 3)(4\ 7)(5\ 6\ 8).$$

It is readily checked that $|\pi T(\mathcal{B}) \bigcap T(\mathcal{B})| = 0$. □

Lemma 5.21. $\{1, 2, \ldots, 11\} \subseteq J_T(40)$.

Proof. Construct an $S(2, 4, 40)$ design $(X, \mathcal{B})$ with $X = Z_{40}$. All blocks of $\mathcal{B}$ are divided into two parts. The first part consists of $\{i, 10 + i, 20 + i, 30 + i\}$, $0 \le i \le 9$. Develop the following base blocks by $+1$ modulo 40 to obtain the second part of $\mathcal{B}$.

$$\{0, 1, 4, 13\}, \quad \{0, 2, 7, 24\}, \quad \{0, 6, 14, 25\}.$$

Consider the following permutations on X.

$$\begin{aligned}
\pi_1 &= (1\ 29\ 37\ 7\ 16\ 4\ 33\ 13\ 22\ 26\ 39\ 31\ 12\ 30\ 34\ 8\ 32\ 9\ 27\ 20\ 3\ 35\ 24\ 21\ 28\ 18\ 15) \\
&\quad (0\ 23)(2\ 10)(5\ 6\ 17\ 14)(11\ 36\ 38\ 19\ 25), \\
\pi_2 &= (2\ 34\ 30\ 16\ 39\ 36\ 26\ 35\ 4\ 37\ 27\ 17\ 19\ 8\ 11\ 25\ 5\ 29\ 13\ 20\ 6\ 9\ 12\ 38\ 14\ 33\ 21\ 31) \\
&\quad (0\ 28\ 22)(3\ 10\ 7)(15\ 32\ 18\ 24\ 23), \\
\pi_3 &= (0\ 37\ 5\ 25\ 22\ 38\ 8\ 19\ 4\ 16\ 39\ 12\ 34\ 9\ 32\ 14\ 23\ 17)(1\ 20\ 30\ 7\ 2\ 6\ 28\ 10\ 27\ 11) \\
&\quad (3\ 24\ 29\ 13\ 33\ 15\ 35\ 31\ 18\ 36\ 26\ 21), \\
\pi_4 &= (0\ 6\ 29\ 3\ 33\ 18\ 31\ 21\ 14\ 7\ 38\ 23\ 11\ 12\ 19\ 17\ 25\ 1\ 20)(2\ 35\ 8\ 13\ 37\ 24\ 9\ 16\ 26) \\
&\quad (4\ 27\ 34\ 36\ 32\ 28\ 15\ 10\ 39\ 5), \\
\pi_5 &= (2\ 35\ 36\ 18\ 31\ 21\ 14\ 7\ 38\ 32\ 8\ 13\ 37\ 24\ 9\ 16\ 26)(4\ 27\ 34\ 28\ 15\ 10\ 39\ 5) \\
&\quad (0\ 6\ 29\ 3\ 33\ 23\ 11\ 12\ 19\ 17\ 25\ 1\ 20), \\
\pi_6 &= (0\ 19\ 3\ 18\ 4\ 24\ 38\ 6\ 25\ 33\ 14\ 21\ 2\ 31\ 35\ 29\ 28\ 7\ 10\ 16\ 30\ 8)(5\ 9) \\
&\quad (1\ 15\ 36\ 32\ 20\ 13\ 39\ 22\ 23\ 12\ 34\ 37\ 17\ 27\ 11\ 26), \\
\pi_7 &= (0\ 19\ 3\ 18\ 4\ 24\ 38\ 14\ 21\ 2\ 31\ 35\ 29\ 28\ 7\ 10\ 16\ 30\ 8)(6\ 25\ 33)(12\ 34\ 22\ 23) \\
&\quad (5\ 9)(1\ 15\ 36\ 32\ 20\ 13\ 39\ 37\ 17\ 27\ 11\ 26),
\end{aligned}$$

$$\pi_8 = (0\ 19\ 3\ 18\ 4\ 24\ 38\ 6\ 25\ 33\ 14\ 21\ 2\ 31\ 32\ 35\ 29\ 28\ 7\ 10\ 16\ 30\ 8)(5\ 9)$$
$$(1\ 15\ 36\ 17\ 27\ 11\ 26)(12\ 34\ 37\ 20\ 13\ 39\ 22\ 23),$$
$$\pi_9 = (1\ 34\ 8\ 17\ 18\ 39\ 24\ 19\ 21\ 15\ 2\ 27\ 35\ 37\ 32\ 28\ 7\ 25\ 22\ 12\ 13)(5\ 36\ 20\ 38)$$
$$(0\ 10\ 9\ 3\ 29\ 11\ 30\ 31)(6\ 16\ 14\ 26\ 23\ 33),$$
$$\pi_{10} = (0\ 39\ 19\ 6)(1\ 23\ 5\ 12\ 38)(2\ 22\ 7\ 3\ 21\ 34\ 29\ 32\ 27\ 4\ 15\ 8\ 13\ 26\ 28\ 11)(17\ 37)$$
$$(9\ 33\ 24\ 16\ 35\ 30\ 31\ 18\ 25\ 14\ 10\ 20\ 36),$$
$$\pi_{11} = (0\ 39\ 1\ 23\ 5\ 12\ 38\ 19\ 6)(2\ 22\ 7\ 3\ 21\ 34\ 29\ 32\ 27\ 4\ 15\ 8\ 13\ 26\ 28\ 11)(17\ 37)$$
$$(9\ 33\ 24\ 16\ 35\ 30\ 31\ 18\ 25\ 14\ 10\ 20\ 36).$$

It is readily checked that $|\pi_j T(\mathcal{B}) \bigcap T(\mathcal{B})| = j$ for each $j \in \{1, 2, \ldots, 11\}$. $\square$

Lemma 5.22. *For any positive integer* $v \equiv 1, 4 \pmod{12}$ *and* $v \geq 121$, $t_v - 33 \in J_T(v)$.

Proof. By Lemma 5.15, there is an $S(2, 4, v)(X, \mathcal{B})$ containing an $S(2, 4, 40)(Y, \mathcal{A})$ as a subdesign, where $Y \subseteq X$. By Lemma 5.19, there is a pair of $S(2, 4, 40)(Y, \mathcal{A}_1)$ and $(Y, \mathcal{A}_2)$ such that $|T(\mathcal{A}_1) \bigcap T(\mathcal{A}_2)| = t_{40} - 33$. It is readily checked that $(X, (\mathcal{B} \setminus \mathcal{A}) \bigcup \mathcal{A}_1)$ and $(X, (\mathcal{B} \setminus \mathcal{A}) \bigcup \mathcal{A}_2)$ are two $S(2, 4, v)$ designs with $t_v - 33$ common triangles. $\square$

6. The case of $v = 25, 28, 37$

Lemma 6.1. (1) $\{3\text{–}122, 124\text{–}131, 134, 135, 137, 140, 143, 146, 155, 158, 164\} \subseteq J_T(25)$.
(2) $\{1\text{–}149, 156, 158, 160, 162, 164, 166, 168, 180, 204\} \subseteq J_T(28)$.
(3) $\{0\text{–}251, 258\text{–}276, 285\text{–}294, 444\} \subseteq J_T(37)$.

Proof. (1) Take the $S(2, 4, 25)$ design $(X, \mathcal{B})$ listed in Lemma 4.5. Apply random permutations on X to obtain $\{3, 4, \ldots, 111\} \subseteq J_T(25)$. Take four pairs of $S(2, 4, 25)$ designs listed in Table 6.2 in [6]. For each pair of $S(2, 4, 25)$ designs, apply random permutation to obtain $\{112\text{–}122, 124\text{–}131, 134, 135, 137, 140, 143, 146, 155, 158, 164\} \subseteq J_T(25)$.

(2) Take the $S(2, 4, 28)$ design $(X, \mathcal{B})$ constructed in Lemma 4.7. Apply random permutation on X to obtain $\{1, 2, \ldots, 149, 156, 158, 160, 162, 164, 166, 168, 180, 204\} \subseteq J_T(28)$.

(3) Construct two $S(2, 4, 37)$ designs $(X, \mathcal{B}_i)(i = 1, 2)$ with $X = Z_{37}$. Only base blocks are listed below. Develop these base blocks by $+1$ modulo 37 to obtain all blocks of $\mathcal{B}_i$, $i = 1, 2$.

$\mathcal{B}_1$: $\{0, 1, 3, 24\}$, $\{0, 4, 9, 15\}$, $\{0, 7, 17, 25\}$.
$\mathcal{B}_2$: $\{0, 1, 8, 21\}$, $\{0, 2, 11, 34\}$, $\{0, 4, 19, 31\}$.

One can find suitable random permutations π_j on X to obtain $|\pi_j T(\mathcal{B}_2) \bigcap T(\mathcal{B}_1)| = j$ for each $j \in \{0, 1, \ldots, 35\}$ and $|\pi_j T(\mathcal{B}_1) \bigcap T(\mathcal{B}_1)| = j$ for each $j \in \{36, 37, \ldots, 251, 258\text{–}276, 285\text{–}294, 444\}$.

To save space we do not include these random permutations here. The interested reader may get a copy from the authors. $\square$

7. Conclusion

Proof of Theorem 1.2. (1) Combining the results of Lemmas 2.2, 5.4, 5.11, 5.13, 5.17 and 5.22, we have that for any positive integer $v \equiv 1, 4 \pmod{12}$ and $v \geq 121, J_T(v) = I_T(v)$. By Lemmas 5.18–5.21, we have $J_T(40) = I_T(40)$. (2) Combining the results of Lemmas 2.2, 5.4, 5.11, 5.13 and 5.17, we have that for any positive integer $v \equiv 1, 4 \pmod{12}$ and $49 \leq v \leq 112, J_T(v) \subseteq I_T(v)$ and $I_T(v) \setminus \{t_v - 33\} \subseteq J_T(v)$. (3) By computer exhaustive search, we have that $1, 2, 9 \notin J_T(13)$ and $37, 39, 41, 43 \notin J_T(16)$. Thus by Lemmas 2.2, 4.2 and 4.4, we have that $J_T(13) = I_T(13) \setminus \{1, 2, 9\}$ and $J_T(16) = I_T(16) \setminus \{37, 39, 41, 43, 45\text{–}50, 53, 62\}$.

Combining the results of Lemmas 2.2, 4.5 and 6.1(1), (4) of Theorem 1.2 holds. By Lemmas 2.2, 4.7 and 6.1(2), (5) holds. By Lemma 6.1(3), (6) holds. This completes the proof. $\square$

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