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A Laplace Type Problem for Irregular Lattice with Fundamental Cell Composed by Three Triangles and a Trapetium

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Abstract

In the previous papers, [1], [2], [3], [4],[5], [6], [7], [8], [9], [10], [11], [12], [13] and [14] the authors studies some Laplace problem for different lattices. In this paper paper we determine the probability that a random segment of constant length intersects a side of the lattice with fundamental cell represented in fig. 1.

Let $\Re(a)$ the lattice with fundamental cell C_0 represented in fig. 1

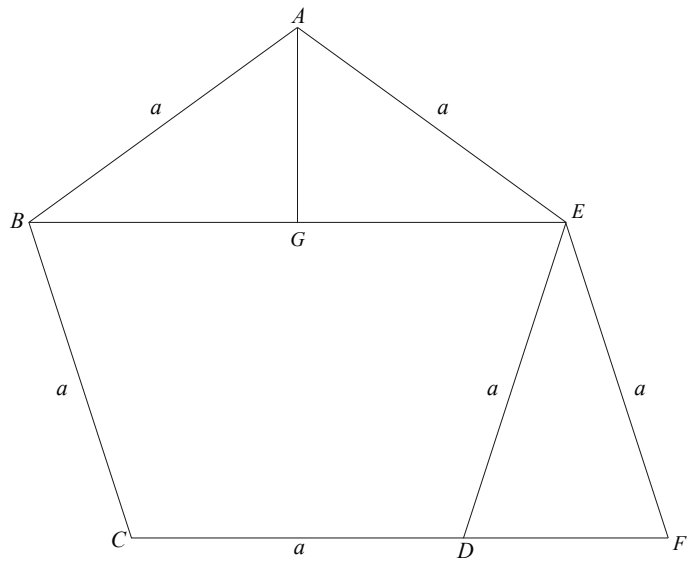

fig.1

With the rotations of this figure we can write that

$$C_0 = C_{01} \cup C_{02} \cup C_{03} \cup C_{04}$$

and

$$\widehat{ABE} = \widehat{AEB} = \widehat{DEF} = \frac{\pi}{5}, \qquad \widehat{CBE} = \widehat{DEB} = \widehat{EDF} = \widehat{EFD} = \frac{2\pi}{5},$$

$$\widehat{BCD} = \widehat{CDE} = \frac{3\pi}{5}, \qquad \widehat{BAG} = \widehat{EAG} = \frac{3\pi}{4}, \tag{1}$$

$$|BG| = |EG| = a\cos\frac{\pi}{5}, \qquad |BE| = 2a\cos\frac{\pi}{5}, \qquad |AG| = a\sin\frac{\pi}{5}, \tag{2}$$

$$|DF| = \frac{a}{2\cos\frac{\pi}{5}}, \qquad areaC_0 = \frac{a^2}{2}\sin\frac{\pi}{5}\left(2\cos\frac{\pi}{5}+1\right)^2. \tag{3}$$

We want to compute the probability that a segment s of random position and of constant length l, with $l < \frac{a}{4\cos\frac{\pi}{5}}$, intersects a side of the lattice $\Re(a)$, i.e. the probability P_{int} that s intersects a side of the fundamental cell C_0. The position of the segment s is determined by center and by the angle φ that it formed with the line CF.

In order to compute the probability P_{int} we consider the limit positions of the segment s, for a fixed value of φ, in the cells C_{0i}, $(i = 1, 2, 3, 4)$. We have the fig. 2

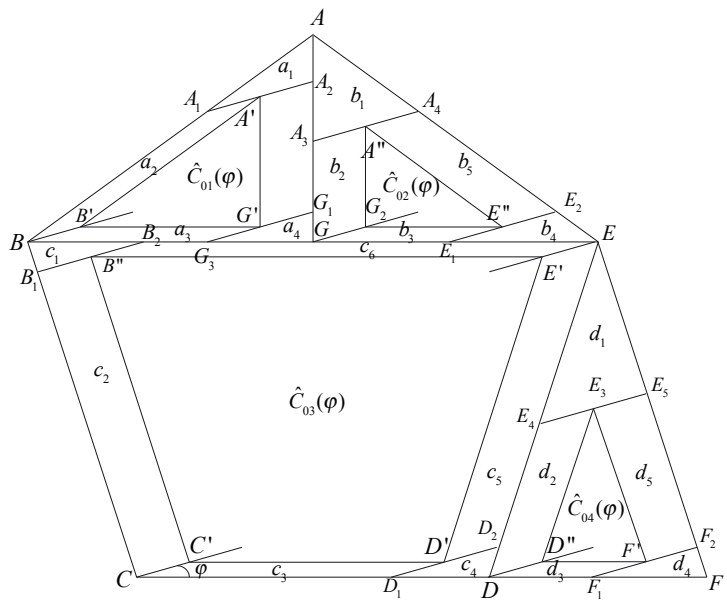

fig.2

and the relations

$$area\widehat{C}_{01}(\varphi) = areaC_{01} - \sum_{j=1}^{5} areaa_j(\varphi), \tag{4}$$

$$area\widehat{C}_{02}(\varphi) = areaC_{02} - \sum_{j=1}^{5} areab_j(\varphi), \tag{5}$$

$$area\widehat{C}_{03}(\varphi) = areaC_{03} - \sum_{j=1}^{5} areac_j(\varphi), \tag{6}$$

$$area\widehat{C}_{04}(\varphi) = areaC_{04} - \sum_{j=1}^{5} aread_j(\varphi). \tag{7}$$

By fig.1 and fig.2 we have that:

$$areaa_4(\varphi) = \frac{l^2}{4}\sin 2\varphi,$$

$$areaa_3(\varphi) = \frac{al}{2}\cos\frac{\pi}{5}\sin\varphi - \frac{l^2}{4}\cos\varphi,$$

$$areaa_1(\varphi) = \frac{l^2\cos\varphi\sin\left(\frac{\pi}{5}-\varphi\right)}{2\cos\frac{\pi}{5}},$$

$$areaa_2(\varphi) = \frac{al}{2}\sin\left(\frac{\pi}{5}-\varphi\right) - \frac{l^2\cos\varphi\sin\left(\frac{\pi}{5}-\varphi\right)}{2\cos\frac{\pi}{5}},$$

$$areaa_5(\varphi) = \frac{al}{2}\sin\frac{\pi}{5}\cos\varphi - \frac{l^2}{4}\sin 2\varphi - \frac{l^2\cos\varphi\sin\left(\frac{\pi}{5}-\varphi\right)}{2\cos\frac{\pi}{5}}.$$

We obtain that:

$$area\widehat{C}_{01} = areaC_{01} - \mathcal{A}_1(\varphi),$$

where

$$\mathcal{A}_1(\varphi) = al\sin\frac{\pi}{5}\cos\varphi - \frac{l^2}{4}tg\frac{\pi}{5}(1+\cos 2\varphi).$$

In order to compute $area\widehat{C}_2(\varphi)$ we have that:

$$areab_4(\varphi) = \frac{l^2\sin\varphi\sin\left(\frac{\pi}{5}+\varphi\right)}{2\sin\frac{\pi}{5}},$$

$$areab_3(\varphi) = \frac{al}{2}\cos\frac{\pi}{5}\sin\varphi - \frac{l^2\sin\varphi\sin\left(\frac{\pi}{5}+\varphi\right)}{2\sin\frac{\pi}{5}},$$

$$areab_1(\varphi) = \frac{l^2\cos\varphi\sin\left(\frac{\pi}{5}+\varphi\right)}{2\cos\frac{\pi}{5}},$$

$$areab_2(\varphi) = \frac{al}{2}\sin\frac{\pi}{5}\cos\varphi - \frac{l^2\cos\varphi\sin\left(\frac{\pi}{5}+\varphi\right)}{\cos\frac{\pi}{5}},$$

$$areab_5(\varphi) = \frac{al}{2}\sin\left(\frac{\pi}{5}+\varphi\right) -$$

$$\frac{l^2}{2\sin\frac{2\pi}{5}}\left(1-\cos\frac{2\pi}{5}\cos 2\varphi + \frac{\sin\frac{2\pi}{5}}{2}\sin 2\varphi\right).$$

We obtain that:

$$area\widehat{C}_{02} = areaC_{02} - \mathcal{A}_2(\varphi),$$

where

$$\mathcal{A}_2(\varphi) = al\sin\left(\frac{\pi}{5}+\varphi\right) - \frac{l^2}{2\sin\frac{2\pi}{5}}\left(1-\cos\frac{2\pi}{5}\cos 2\varphi + \frac{1}{2}\sin\frac{2\pi}{5}\sin 2\varphi\right).$$

To compute $area\widehat{C}_{03}(\varphi)$ we have that

$$areac_4(\varphi) = \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}-\varphi\right)}{2\sin\frac{2\pi}{5}},$$

$$areac_3(\varphi) = \frac{al}{2}\sin\varphi - \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}-\varphi\right)}{2\sin\frac{2\pi}{5}},$$

$$areac_1(\varphi) = \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{2\pi}{5}},$$

$$areac_2(\varphi) = \frac{al}{2}\sin\left(\frac{2\pi}{5}+\varphi\right) - \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{2\pi}{5}},$$

$$areac_5(\varphi) = \frac{al}{2}\sin\left(\frac{2\pi}{5}-\varphi\right) - \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}-\varphi\right)}{2\sin\frac{2\pi}{5}},$$

$$areac_6(\varphi) = al\cos\frac{\pi}{5}\sin\varphi - \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{2\pi}{5}}.$$

We obtain that

$$area\widehat{C}_{03}(\varphi) - \mathcal{A}_3(\varphi),$$

where

$$\mathcal{A}_3(\varphi) = \frac{al}{2}\left[2\sin\frac{2\pi}{5}\cos\varphi + \left(1+2\cos\frac{\pi}{5}\right)\sin\varphi\right] - \frac{l^2}{2}\sin 2\varphi.$$

To compute now the $area\widehat{C}_{04}(\varphi)$.

$$aread_4(\varphi) = \frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{2\pi}{5}},$$

$$aread_3(\varphi) = \frac{al\sin(\varphi)}{4\cos\frac{\pi}{5}} - \frac{l\sin\left(\frac{2\pi}{5}+\varphi\right)}{\sin\frac{2\pi}{5}},$$

$$aread_1(\varphi) = \frac{l^2\sin\left(\frac{2\pi}{5}-\varphi\right)\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{\pi}{5}},$$

$$aread_2(\varphi) = \frac{al}{2}\sin\left(\frac{2\pi}{5}-\varphi\right) - \frac{l^2\sin\left(\frac{2\pi}{5}-\varphi\right)\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{\pi}{5}},$$

$$aread_5(\varphi) = \frac{al}{2}\sin\left(\frac{2\pi}{5}+\varphi\right) - \frac{l^2\sin\left(\frac{2\pi}{5}-\varphi\right)\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{\pi}{5}} -$$

$$\frac{l^2\sin\varphi\sin\left(\frac{2\pi}{5}+\varphi\right)}{2\sin\frac{2\pi}{5}}.$$

We obtain that:

$$area\widehat{C}_{04}(\varphi) = areaC_{04} - \mathcal{A}_4(\varphi),$$

where

$$\mathcal{A}_4(\varphi) = al\left(\sin\frac{2\pi}{5}\cos\varphi + \frac{\sin\varphi}{4\cos\frac{2\pi}{5}}\right) -$$

$$\frac{l^2}{4\sin\frac{2\pi}{5}}\left[\left(2\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right)\cos 2\varphi + \sin\frac{2\pi}{5}\sin 2\varphi + 4\cos^2\frac{\pi}{5} + 1\right].$$

Denoting with M_i, $(i = 1, 2, 3, 4)$, the set of segments s that have the center in the cell C_{0i} and with N_i the set of the segments s completely in C_{0i}, we have that [16]:

$$P_{int} = 1 - \frac{\sum_{i=1}^{4}\mu(N_i)}{\sum_{i=1}^{4}\mu(M_i)}, \tag{8}$$

where μ is the Lebesgue measure in the euclidean plane.

In order to compute $\mu(M_i)$ and $\mu(N_i)$ we use the kinematic measure of Poincaré [15]:

$$dK = dx \wedge dy \wedge d\varphi,$$

where x, y are the coordinates of middle point of s and φ the fixed angle.

We can write:

$$\mu(M_i) = \int_0^{\frac{\pi}{5}} d\varphi \iint_{\{(x,y)\in C_{0i}\}} dxdy = \int_0^{\frac{\pi}{5}} (areaC_{0i})\, d\varphi = \frac{\pi}{5} areaC_{0i}, \qquad (i = 1, 2, 3, 4)$$

and

$$\mu(N_i) = \int_0^{\frac{\pi}{5}} d\varphi \iint_{\left\{(x,y)\in \widehat{C}_{0i}\right\}} dxdy = \int_0^{\frac{\pi}{5}} \left[area\widehat{C}_{0i}(\varphi)\right] d\varphi =$$

$$\int_0^{\frac{\pi}{5}} \left[areaC_{0i} - \mathcal{A}_i(\varphi)\right] d\varphi = \frac{\pi}{5} areaC_{0i} - \int_0^{\frac{\pi}{5}} \left[\mathcal{A}_i(\varphi)\right] d\varphi, \qquad (i = 1, 2, 3, 4).$$

By these two relations give us:

$$\sum_{i=1}^{4}\mu(M_i) = \frac{\pi}{5} areaC_0, \tag{9}$$

and

$$\sum_{i=1}^{4} \mu\left(N_i\right) = \frac{\pi}{5} areaC_0 - \int_0^{\frac{\pi}{5}} \left[\sum_{i=1}^{4} \mathcal{A}_i\left(\varphi\right)\right] d\varphi. \tag{10}$$

follow that:

$$\sum_{i=1}^{4} \mathcal{A}_i\left(\varphi\right) = al\left[\sin\frac{\pi}{5}\left(2+3\cos\frac{\pi}{5}\right)\cos\varphi + \frac{8\cos^2\frac{\pi}{5}+2\cos\frac{\pi}{5}+1}{4\cos\frac{\pi}{5}}\sin\varphi\right] -$$

$$\frac{l^2}{4}\left[\left(tg\frac{\pi}{5} - 3ctg\frac{2\pi}{5} + \frac{1}{\sin\frac{\pi}{5}}\right)\cos 2\varphi + 2\sin 2\varphi + 4\cos^2\frac{\pi}{5} + tg\frac{\pi}{5} + \frac{3}{\sin\frac{2\pi}{5}}\right],$$

then

$$\int_0^{\frac{\pi}{5}} \left[\sum_{i=1}^{4} A_i\left(\varphi\right)\right] d\varphi = al\left(\frac{9}{4} - 3\cos^3\frac{\pi}{5} - 4\cos^2\frac{\pi}{5} - \frac{9}{2}\cos\frac{\pi}{5} + \frac{1}{4\cos\frac{\pi}{5}}\right) -$$

$$\frac{l^2}{4}\left[\frac{9}{2} + \cos\frac{\pi}{5} - 6\cos^2\frac{\pi}{5} + \frac{\pi}{5}\left(4\cos^2\frac{\pi}{5} + tg\frac{\pi}{5} + \frac{3}{\sin\frac{2\pi}{5}}\right)\right]. \tag{11}$$

We obtain that:

$$P_{int} = \frac{10}{\pi a^2 \sin\frac{\pi}{5}\left(2\cos\frac{\pi}{5}+1\right)^2}\left\{al\left(\frac{9}{4} - 3\cos^3\frac{\pi}{5} - 4\cos^2\frac{\pi}{5} - \frac{9}{2}\cos\frac{\pi}{5} + \right.\right.$$

$$\left.\left.\frac{1}{4\cos\frac{\pi}{5}}\right) - \frac{l^2}{4}\left[\frac{9}{2} + \cos\frac{\pi}{5} - 6\cos^2\frac{\pi}{5} + \frac{\pi}{5}\left(4\cos^2\frac{\pi}{5} + tg\frac{\pi}{5} + \frac{3}{\sin\frac{2\pi}{5}}\right)\right]\right\}.$$

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