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0-Gaps on 3D Digital Curves

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Abstract. In digital geometry, gaps are some basic portion of a digital object that a discrete ray can cross without intersecting any voxel of the object itself. Such a notion is quite important in combinatorial image analysis and is strictly connected with some applications in fields as CAD and computer graphics. In this paper we prove that the number of 0-gaps of a 3D digital curve can be expressed as a linear combination of the number of its i-cells ($i = 0, \ldots, 3$).

Keywords: digital geometry, digital curve, 0-gap, i-tandem, i-hub, adjacency relation, grid cell model, free cell.

MSC2010: Primary: 52C35; Secondary: 52C99

1. Introduction

By the word "gap" in digital geometry we mean some basic portion of a digital object that a discrete ray can cross without intersecting any voxel of the object itself. Since such a notion is strictly connected with some applications in the field of Computer graphics (e.g., the rendering of a 3D image by the ray-tracing technique), many papers (see, for example, [1]–[4]) concern the study of 0- and 1-gaps of 3-dimensional objects.

More recently, in [5] and [6] two formulas which express, respectively, the number of 1-gaps of a generic 3D object of dimension $\alpha = 1, 2$ and the number of $(n-2)$-gaps of a generic digital n-object by means of a few simple intrinsic parameters of the object itself were found. Furtermore, in [7] the relationship existing between the dimension of a 2D digital object equipped with an adjacency relation A_α, $\alpha \in \{0, 1\}$, and the number of its gaps were investigated.

In the next section we recall and formalize some basic definitions and properties of the general n-dimensional digital spaces with particular regard to the notions of block, tandem and gap.

In Section 3, we restrict our attention to digital curves in 3D digital spaces, deriving some particular cases of the propositions above recalled in order to prove our main result which states that the number g_0 of 0-gaps of a 3D digital curve γ can be expressed as a linear combination of the number $c - i$ of its i-cells, with $i = 0, \ldots, 3$, and, more precisely, that $g_0 = \sum_{i=0}^{3} (-1)^{i+1} 2^i c_i$.

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2. Preliminaries

Throughout this paper we use the *grid cell model* for representing digital objects, and we adopt the terminology from [8] and [9].

Let $x = (x_1, \dots, x_n)$ be a point of $\mathbb{Z}^n$, $\theta \in \{-1, 0, 1\}^n$ be an n-word over the alphabet $\{-1, 0, 1\}$, and $i \in \{1, \dots, n\}$. We define i-cell related to x and θ and denote it by $e = (x, \theta)$, the Cartesian product in a certain fixed order of $n - i$ singletons $\left\{x_j \pm \frac{1}{2}\right\}$ by i closed sets $\left[x_j - \frac{1}{2}, x_j + \frac{1}{2}\right]$, i.e., we set

$$e = (x, \theta) = \prod_{j=1}^{n} \left[x_j + \frac{1}{2}\theta_j - \frac{1}{2}[\theta_j = 0], x_j + \frac{1}{2}\theta_j + \frac{1}{2}[\theta_j = 0] \right],$$

where $[\bullet]$ denotes the Iverson bracket [10]. The word θ is called the *direction* of the cell (x, θ) related to the point x.

Let us note that an i-cell can be related to different point $x \in \mathbb{Z}^n$ and, once we have fixed it, can be related to different direction. So, when we talk generically about i-cell, we mean one of its possible representation.

The dimension of a cell $e = (x, \theta)$, denoted by $\dim(e) = i$, is the number of non-trivial interval of its product representation, i.e., the number of null components of its direction θ. Thus, $\dim(e) = \sum_{j=1}^{n} [\theta_j = 0]$ or, equivalently, $\dim(e) = n - \theta \cdot \theta$. So, e is an i-cell if and only if it has dimension i.

We denote by $\mathbb{C}_n^{(i)}$ the set of all i-cells of $\mathbb{R}^n$ and by $\mathbb{C}_n$ the set of all cells defined in $\mathbb{R}^n$, i.e., we set $\mathbb{C}_n = \bigcup_{j=0}^{n} \mathbb{C}_n^{(j)}$. An n-cell of $\mathbb{C}_n$ is also called an n-voxel. So, for convenience, an n-voxel is denoted by v, while we use other lower case letter (usually e) to denote cells of lower dimension. A finite collection D of n-voxels is a digital n-object. For any $i = 0, \dots, n$, we denote by $C_i(D)$ the set of all i-cells of the object D, that is, $D \cap \mathbb{C}_n^{(i)}$, and by $c_i(D)$ (or simply by c_i if no confusion arises) its cardinality $|C_i(D)|$.

We say that two n-cells v_1, v_2 are i-adjacent, $i = 0, 1, \dots, n-1$, if $v_1 \neq v_2$ and there exists at least one i-cell $\overline{e}$ such that $\overline{e} \subseteq v_1 \cap v_2$, that is if they are distinct and share at least one i-cell. Two n-cells v_1, v_2 are *strictly* i-adjacent, if they are i-adjacent but not j-adjacent, for any $j > i$, that is, if $v_1 \cap v_2 \in \mathbb{C}_n^{(i)}$. The set of all n-cells that are i-adjacent to a given n-voxel v is denoted by $A_i(v)$ and called the *i-adjacent neighborhoods* of v. Two cells $v_1, v_2 \in \mathbb{C}_n$ are *incident* to each other, and we write $e_1 I e_2$, if $e_1 \subseteq e_2$ or $e_2 \subseteq e_1$.

Definition 1. Let $e_1, e_2 \in \mathbb{C}_n$. We say that e_1 *bounds* e_2 (or that e_2 is *bounded* by e_1), and we write $e_1 < e_2$ if $e_1 I e_2$ and $\dim(e_1) < \dim(e_2)$. The relation $<$ is called *bounding relation*.

Definition 2. An incidence structure (see [11]) is a triple $(V, \mathcal{B}, \mathcal{I})$ where V and $\mathcal{B}$ are any two disjoint sets and $\mathcal{I}$ is a binary relation between V and $\mathcal{B}$, that is, $\mathcal{I} \subseteq V \times \mathcal{B}$. The elements of V are called *points*, those of $\mathcal{B}$ blocks. Instead of $(p, B) \in \mathcal{I}$, we simply write $p\mathcal{I}B$ and say that *the point p lies on the block B* or *p and B are incident*.

If p is any point of V, we denote by (p) the set of all blocks incident to p, i.e., $(p) = \{B \in \mathcal{B} \colon p\mathcal{I}B\}$. Similarly, if B is any block of $\mathcal{B}$, we denote by (B) the set of all points incident to each other B, i.e., $(B) = \{p \in V \colon p\mathcal{I}B\}$. For a point p, the number $r_p = |(p)|$ is called the degree of p, and similarly, for a block B, $k_B = |(B)|$ is the degree of B.

Let us remind the following fundamental proposition of incidence structures.

Proposition 1. *Let $(V, \mathcal{B}, \mathcal{I})$ be an incidence structure. We have*

$$\sum_{p \in V} r_p = \sum_{B \in \mathcal{B}} k_B, \tag{1}$$

where r_p and k_B are the degrees of any point $p \in V$ and any block $B \in \mathcal{B}$, respectively.

Definition 3. Let e be an i-cell, $0 \leqslant i \leqslant n-1$, of $\mathbb{C}_n$. Then an i-block *centered on* e, denoted by $B_i(e)$, is the union of all the n-voxels bounded by e, i.e., $B_i(e) = \bigcup\{v \in \mathbb{C}_n^{(n)} : e < v\}$.

Remark 1. Let us note that, for any i-cell e, $B_i(e)$ is the union of exactly 2^{n-i} n-voxels and $e \in B_i(e)$.

Definition 4. Let v_1, v_2 be two n-voxels of a digital object D and e be an i-cell, $i = 0, \dots, n-1$. We say that $\{v_1, v_2\}$ *forms an i-tandem of D over* e and denote it by $t_i(e)$ if $D \cap B_i(e) = \{v_1, v_2\}$, v_1 and v_2 are strictly i-adjacent and $v_1 \cap v_2 = e$.

Definition 5. Let D be a digital n-object and e be an i-cell, $i = 0, \dots, n-2$. We say that D *has an i-gap over* e if there exists an i-block $B_i(e)$ such that $B_i(e) \setminus D$ is an i-tandem over e. The cell e is called i-hub of the related i-gap. Moreover, we denote by $g_i(D)$ (or simply by g_i if no confusion arises) the number of i-gap of D.

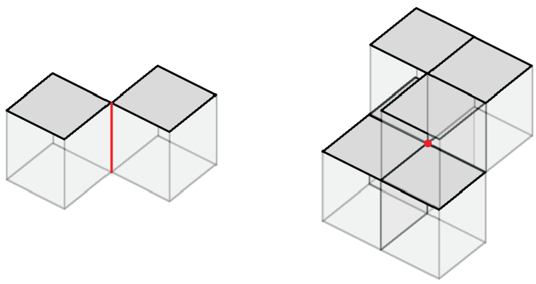

Fig. 1. Configurations of 1- and 0-gaps in $\mathbb{C}_3$

Notation 1. *For any $i = 0, \dots, n-1$, we denote by $\mathcal{H}_i(D)$ (or simply by $\mathcal{H}_i$ if no confusion arises) the sets of all i-hubs of D. Clearly, we have $|\mathcal{H}_i| = g_i$.*

Definition 6. An i-cell e $(i = 0, \dots, n-1)$ of a digital n-object D is *free* if $B_i(e) \nsubseteq D$.

Notation 2. *For any $i = 0, \dots, n-1$, we denote by $C_i^*(D)$ (respectively, by $C_i'(D)$) the set of all free (respectively, non-free) i-cells of the object D. Moreover, we denote by $c_i^*(D)$ (or simply by c_i^*) the number of free i-cells of D and by $c_i'(D)$ (or simply by c_i') the number of non-free cells.*

Remark 2. It is evident that $\{C_i^*(D), C_i'(D)\}$ forms a partition of $C_i(D)$ and that $c_i = c_i^* + c_i'$.

Proposition 2. *Let D be a digital n-object. Then $c_2 = 6c_3 - c'_2$.*

Proof. Let us consider the set $F = \bigcup_{v \in C_n(D)} \{(e, v) \colon e \in C_{n-1}(D), e < v\}$. It is evident that

$$|F| = \left|\{(e, v) \colon e \in C_{n-1}(D), e < v\}\right| |C_n(D)| = c_{n-1 \to n} \cdot c_n = 2nc_n.$$

Let us set

$$F^* = F \cap (C^*_{n-1}(D) \times C_n(D)), \qquad F' = F \cap (C'_{n-1}(D) \times C_n(D)).$$

The map $\phi \colon F^* \to C^*_{n-1}(D)$, defined by $\phi(e, v) = e$, is a bijection. In fact, besides being evidently surjective, it is also injective, since if by contradiction there were two distinct pairs (e, v_1) and $(e, v_2) \in F^*$ associated to e, then $B_{n-1}(e) = \{v_1, v_2\}$ should be an $(n-1)$-block contained in D. This contradicts the fact that the $(n-1)$-cell e is free. Thus $|F^*| = |C^*_{n-1}(D)| = c^*_{n-1}$.

On the other hand, it results that

$$\begin{aligned}
|F'| &= \left| \bigcup_{v \in C_n(D)} \{(e, v) \colon e \in C'_{n-1}(D), e < v\} \right| \\
&= \left| \bigcup_{e \in C'_{n-1}(D)} \{(e, v) \colon v \in C_n(D), e < v\} \right| \\
&= \left|\{(e, v) \colon v \in C_n(D), e < v\}\right| |C'_{n-1}(D)| \\
&= c_{n-1 \leftarrow n} c'_{n-1} = 2c'_{n-1}.
\end{aligned}$$

Since $\{F^*, F'\}$ is a partition of F, we finally have that $|F| = |F^*| + |F'|$, i.e., $2nc_n = c^*_{n-1} + 2c'_{n-1} = c_{n-1} - c'_{n-1} + 2c'_{n-1} = c_{n-1} + c'_{n-1}$, and then the thesis.

Notation 3. *Let i, j be two natural numbers such that $0 \leqslant i < j$. We denote by $c_{i \to j}$ the maximum number of i-cells of $\mathbb{C}_n$ that bound a j-cell. Moreover, we denote by $c_{i \leftarrow j}$ the maximum number of j-cell of $\mathbb{C}_n$ that are bounded by an i-cell.*

The following three propositions were proved in [6].

Proposition 3. *For any $i, j \in \mathbb{N}$ such that $0 \leqslant i < j$, we have*

$$c_{i \to j} = 2^{j-i} \binom{j}{i}.$$

Proposition 4. *For any $i, j \in \mathbb{N}$ such that $0 \leqslant i < j$, we have*

$$c_{i \leftarrow j} = 2^{j-i} \binom{n-i}{j-i}.$$

Proposition 5. *Let D be a digital n-object. Then $c_{n-1} = 2nc_n - c'_{n-1}$.*

Notation 4. *Let e be an i-cell of a digital n-object D and $0 \leqslant i < j$. We denote by $b_j(e, D)$ (or simply by $b_j(e)$ if no confusion arises) the number of j-cells of $bd(D)$ that are bounded by e.*

Let us note that if e is a non-free i-cell, then $b_j(e) = 0$.

Proposition 6. *Let v be an n-voxel and e be one of its i-cells, $i = 0, \ldots, n-1$. Then, for any $i < j \leqslant n$, it results that*

$$b_j(e) = \frac{c_{i\to j} c_{j\to n}}{c_{i\to n}}.$$

3. Gaps and curves in 3D digital space

Throughout the rest of the paper we consider the 3-dimensional digital space $\mathbb{Z}^3$ with the corresponding grid cell model $\mathbb{C}_3$.

Definition 7. A digital object γ of $\mathbb{C}_3$ is called a *digital k-curve* if it satisfies the following two condition:

- $\forall v \in \gamma$ it is $1 \leqslant |A_k(v)| \leqslant 2$,
- for any $v \in \gamma$, if $v_1, v_2 \in A_k(v)$, then $\{v_1, v_2\} \notin A_k(v)$,

that is, for any voxel $v \in \gamma$ there exist at most two voxels k-adjacent to v, and every pair of voxels k-adjacent to a voxel of γ can not be k-adjacent to each other.

The voxels in γ which have only one k-adjacent voxel are called the extreme points of the curve.

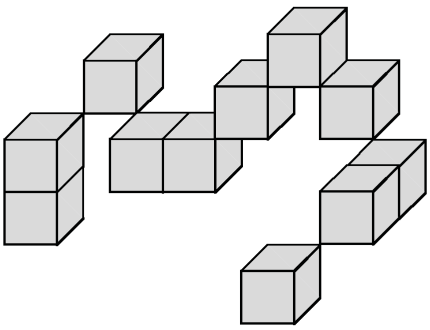

Fig. 2. An example of a digital 0-curve in $\mathbb{C}_3$

We are interested only in a digital 0-curve and, if no confusion arises, we will briefly call it a digital curve.

The following propositions derive from some general ones proved in [6] for the n-dimensional case.

Proposition 7. *Let v be a voxel and e be one of its i-cell, $i = 0, \ldots, 2$. Then, for any $i < j \leqslant 2$, we have*

$$b_j(e) = \binom{3-i}{j-i}.$$

Proposition 8. *Let e be a 2-cell of $\mathbb{C}_3$. Then the number of i-cells, $i = 0, \ldots, 2$, of the 2-block centered on e is*

$$c_i(B_2(e)) = \frac{9+i}{6} c_{i\to 3}.$$

In order to obtain our main result, we need to prove the following result.

Proposition 9. *The number of i-cells, $i = 0, 1$, of an 1-tandem $t_1(e)$ is*

$$c_i(t_1) = \frac{42 + 5i - i^2}{24} c_{i\to 3}.$$

Proof. By definition, $t_1(e)$ is composed of two strictly 1-adjacent voxels. Each of such voxels has exactly $c_{i\to 3}$ i-cells. But some of these cells are repeated onto $t_1(e)$. The number of these repeated i-cells coincides with the number of i-cells of the 1-hub e. Since

$$\binom{1}{i} = \frac{(n-i)(n-i-1)}{n(n-1)} \binom{n}{i},$$

we have

$$\begin{aligned} c_i(t_{n-2}(e)) &= 2c_{i\to n} - c_{i\to n-2} \\ &= 2 \cdot 2^{n-i} \binom{n}{i} - 2^{n-2-i} \binom{n-2}{i} \\ &= \frac{7n^2 - 7n + 2in - i^2 - i}{4n(n-1)} c_{i\to n}. \end{aligned}$$

The following useful proposition was proved in [5].

Proposition 10. *The number of 1-gaps of a digital object D of $\mathbb{C}_3$ is given by*

$$g_1 = 2c_2^* - c_1^*. \tag{2}$$

Proposition 11. *Let e be a free vertex that bounds the center e' of a 2-block $B_2(e')$. Then $b_1(e) = 4$.*

Proof. Let us consider the incidence structure $(C_0(B_2(e')), C_1(B_2(e')), <)$. By Proposition 1, we have

$$\sum_{a\in C_0(B_2(e'))} r_a = \sum_{a\in C_1(B_2(e'))} k_a.$$

Let us note that, by Proposition 8, we have $|C_1(B_2(e'))| = 20$ and $|C_0(B_2(e'))| = 12$. Since, for any $a \in C_1(B_2(e'))$ we have $k_a = c_{0\to 1} = 2$, it follows that

$$\sum_{a\in C_1(B_2(e'))} k_a = 2|C_1(B_2(e'))| = 40. \tag{3}$$

Let us now consider the sets

$$F = \{a \in C_0(B_2(e')) \colon a < e'\}, \qquad G = \{a \in C_0(B_2(e')) \colon a \not< e'\}.$$

Since $\{F, G\}$ forms a partition of $C_0(B_2(e'))$, we can write

$$\sum_{a\in C_0(B_2(e'))} r_a = \sum_{a\in F} r_a + \sum_{a\in G} r_a.$$

For any $a \in F$, let us set $r_a = b_1(e)$. We obtain

$$\sum_{a\in F} r_a = |F| b_1(e) = c_{0\to 2} b_1(e) = 4b_1(e). \tag{4}$$

Instead, by Proposition 7, for any $a \in G$, we have

$$r_a = b_1(a) = \binom{3-0}{1-0} = 3,$$

and so,

$$\sum_{a \in G} r_a = 3|G| = 3(|C_0(B_2(e'))| - c_{0\to 2}) = 3(12-4) = 24. \tag{5}$$

To sum up, by using Equations (3)–(5), we can write $4b_1(e) + 24 = 40$, from which we get the thesis.

Proposition 12. *Let γ be a digital curve of $\mathbb{C}_3$. Then the number of 0-cells that bound some non-free 2-cell is $4c'_2$.*

Proof. Since $c'_2(\gamma)$ coincides with the number of 2-block of γ, and since any non-free 2-cell is bounded by $c_{0\to 2} = 4$ 0-cells, the number of 0-cells that bounds some non-free 2-cell is exactly $4c'_2$.

Proposition 13. *For any $i, j \in \mathbb{N}$ such that $0 \leqslant i < j$, we have*

$$c_{i\leftarrow j} = 2^{j-i}\binom{n-i}{j-i}.$$

Proposition 14. *Let D be a digital object of $\mathbb{C}_3$ and $e \in \mathcal{H}_0$. Then $b_1(e) = 6$.*

Proof. Since the number $b_1(e)$ of 1-cells of D bounded by e coincides with the maximum number of 1-cells bounded by a 0-cell, by Proposition 13, we have

$$b_1(e) = c_{0\leftarrow 1} = 2^{1-0}\binom{3-0}{1-0} = 6.$$

We have the following lemma.

Lemma 1. *The number of 0-cells and 1-cells of a 1-tandem $t_1(e)$ is $c_0(t_1(e)) = 14$ and $c_1(t_1(e)) = 23$, respectively.*

Proof. It directly follows from Proposition 9 for $n = 3$ and $i = 0$ or $i = 1$, respectively.

Proposition 15. *Let e be a 0-cell that bounds a 1-hub. Then $b_1(e) = 5$.*

Proof. Let e' a 1-hub that is bounded by e and $t_1(e')$ the related 1-tandem. Moreover, let us consider the incidence structure $(C_0(t_1(e')), C_1(t_1(e')), <)$. By Proposition 1, we can write

$$\sum_{a \in C_0(t_1(e'))} r_a = \sum_{a \in C_1(t_1(e'))} k_a.$$

By Lemma 1, we have $|C_0(t_1(e'))| = 14$ and $|C_1(t_1(e'))| = 23$. Moreover, since for any $a \in C_1(t_1(e'))$, $k_a = c_{0\to 2} = 2$, it follows that

$$\sum_{a \in C_1(t_1(e'))} k_a = 2|C_1(t_1(e'))| = 46.$$

Now, let us set

$$F = \{a \in C_0(t_1(e')) \colon a < e'\},$$
$$G = \{a \in C_0(t_1(e')) \colon a \not< e'\}.$$

Since $\{F, G\}$ is a partition of $C_0(t_1(e'))$, we have

$$\sum_{a \in C_0(t_1(e'))} r_a = \sum_{a \in F} r_a + \sum_{a \in G} r_a.$$

Let us calculate $\sum_{a \in F} r_a$. If we set $r_a = b_1(e)$, we have

$$\sum_{a \in F} r_a = |F| b_1(e) = c_{o \to 1} b_1(e) = 2b_1(e).$$

Now, let us calculate $\sum_{a \in G} r_a$. By Proposition 7, for any $a \in G$, we have

$$r_a = b_1(a) = 3.$$

Hence, we get

$$\sum_{a \in G} r_a = 3|G| = 3(|C_0(t_1(e'))| - c_{0 \to 1}) = 36.$$

To sum up, we have $2b_1(e) + 36 = 46$, from which we get $b_1(e) = 5$.

Proposition 16. *Let γ be a digital curve of $\mathbb{C}_3$. Then the number of 0-cells that bounds some 1-hub of γ is $2g_1$.*

Proof. Since any 1-hub is bounded by $c_{0 \to 1}$ 0-cell, the number of 0-cells that bounds some 1-hub is exactly $2g_1$.

By applying Proposition 7 with $i = 0$ and $j = 1$, we can easily prove the following proposition.

Proposition 17. *Let e be a 0-cell of a voxel $v \in \mathbb{C}_3$. Then $b_1(e) = 3$.*

Theorem 1. *Let γ be a digital curve of $\mathbb{C}_3$. Then the number of its 0-gaps is given by*

$$g_0 = \sum_{i=0}^{3} (-1)^{i+1} 2^i c_i.$$

Proof. Let us consider the incidence structure $(C_0(\gamma), C_1(\gamma), <)$. By Proposition 1, we have

$$\sum_{a \in C_0(\gamma)} r_a = \sum_{a \in C_1(\gamma)} k_a.$$

Evidently, for any $a \in C_1(\gamma)$, we have that $k_a = 2$. So,

$$\sum_{a \in C_1(\gamma)} k_a = 2|C_1(\gamma)| = 2c_1. \tag{6}$$

Let us denote by $H_i(\gamma)$, $i = 0, 1$, and by $C'_2(\gamma)$ the sets of 0- and 1-hubs and the set of non-free 2-cells of γ, respectively. Let us now calculate $\sum_{a\in\mathbb{C}_0(\gamma)} r_a$. In order to do that, let us consider the following sets of 0-cells:

$$\begin{aligned} A &= \{c \in \mathbb{C}_0(\gamma) \colon c \in H_0(\gamma)\}, \\ B &= \{c \in \mathbb{C}_0(\gamma) \colon c < e,\ e \in H_1(\gamma)\}, \\ C &= \{c \in \mathbb{C}_0(\gamma) \colon c < e,\ e \in \mathbb{C}'_2(\gamma)\}, \\ D &= \mathbb{C}_0(\gamma) \setminus (A \cap B \cap C). \end{aligned}$$

Since $\{A, B, C, D\}$ forms a partition of $\mathbb{C}_0(\gamma)$, we have

$$\sum_{a\in\mathbb{C}_0(\gamma)} r_a = \sum_{a\in A} r_a + \sum_{a\in B} r_a + \sum_{a\in C} r_a + \sum_{a\in D} r_a.$$

Let us calculate $\sum_{a\in A} r_a$. By Proposition 14, for any $a \in A$ it is $r_a = 6$. Evidently $|A| = g_0$. Hence

$$\sum_{a\in A} r_a = r_a|A| = 6g_0. \tag{7}$$

Let us calculate $\sum_{a\in B} r_a$. By Proposition 15, for any $a \in B$, we have $r_a = 5$. Moreover, by Proposition 16, $|B| = 2g_1$. So,

$$\sum_{a\in A} r_a = r_a|B| = 10g_1. \tag{8}$$

Let us calculate $\sum_{a\in C} r_a$. By Proposition 11, for any $a \in C$, we have $r_a = 4$, and by Proposition 12, $|C| = 4c'_2$. It follows that

$$\sum_{a\in A} r_a = r_a|C| = 16c'_2. \tag{9}$$

Finally, let us calculate $\sum_{a\in D} r_a$. By Proposition 17, for any $a \in D$, we have $r_a = 3$. Moreover, $|D| = c_0 - 4c'_2 - 2g_1 - g_0$. So,

$$\sum_{a\in A} r_a = r_a|D| = 3(c_0 - g_0 - 2g_1 - 4c'_2). \tag{10}$$

Combining the Equations (7)–(10), we obtain $6g_0 + 10g_1 + 16c'_2 + 3c_0 - 3g_0 - 6g_1 - 12c'_2 = 2c_1$, that is,

$$3c_0 + 4c'_2 + 4g_1 + 3g_0 = 2c_1. \tag{11}$$

Using Proposition 10, we get $3c_0 + 4c'_2 + 8c^*_2 - 4c_1 + 3g_0 = 2c_1$, since $c_2 = c'_2 + c^*_2$, $3c_0 + 4c_2 + 4c^*_2 + 3g_0 = 6c_1$. Moreover, by Proposition 2, we get $-c'_2 = c_2 - 6c_3$. So, we can write

$$c^*_2 = c_2 - c'_2 = c_2 + c_2 - 6c_3 = 2c_2 - 6c_3.$$

Substituting the last expression in Equation (11), we have

$$3c_0 + 4c_2 + 8c_2 - 24c_2 + 3g_0 = 6c_1,$$

that is,

$$3c_0 + 12c_2 - 24c_3 + 3g_0 = 6c_1,$$

from which we finally get $g_0 = \sum_{i=0}^{3}(-1)^{i+1}2^i c_i$.

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