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The spectrum of α-resolvable λ-fold $(K_4 - e)$-designs

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Abstract

A λ-fold G-design is said to be α-resolvable if its blocks can be partitioned into classes such that every class contains each vertex exactly α times. In this paper we study the α-resolvability for λ-fold $(K_4 - e)$-designs and prove that the necessary conditions for their existence are also sufficient, without any exception.

Keywords: α-resolvable G-design, α-parallel class, $(K_4 - e)$-design.

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1 Introduction

For any graph Γ, let $V(\Gamma)$ and $\mathcal{E}(\Gamma)$ be the vertex-set and the edge-set of Γ, respectively, and $\lambda\Gamma$ be the graph Γ with each of its edges replicated λ times. Throughout the paper K_v will denote the complete graph on v vertices, while $K_n \setminus K_h$ will denote the graph with $V(K_n)$ as vertex-set and $\mathcal{E}(K_n) \setminus \mathcal{E}(K_h)$ as edge-set (this graph is sometimes referred to as a complete graph of order n with a *hole* of size h); finally, $K_{n_1,n_2,\dots,n_t}$ will denote the complete multipartite graph with t-parts of sizes $n_1, n_2, \dots, n_t$.

Let G and H be simple finite graphs. A *λ-fold G-design* of H ($(\lambda H, G)$-design in short) is a pair $(X, \mathcal{B})$ where X is the vertex-set of H and $\mathcal{B}$ is a collection of isomorphic copies (called *blocks*) of the graph G, whose edges partition the edges of λH. If $\lambda = 1$, we drop the term "1-fold". If $H = K_v$, we refer to such a λ-fold G-design as one of order v. A $(\lambda H, G)$-design is *balanced* if for every vertex x of H the number of blocks containing x is a costant r.

A $(\lambda H, G)$-design is said to be *α-resolvable* if it is possible to partition the blocks into classes (often referred to as *α-parallel classes*) such that every vertex of H appears in exactly α blocks of each class. When $\alpha = 1$, we simply speak of resolvable design and parallel classes. The existence problem of resolvable G-decompositions has been the subject of an extensive research (see [1, 4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 16, 18, 19, 21, 24]). The α-resolvability, with $\alpha > 1$, has been studied for: $G = K_3$ by D. Jungnickel, R. C. Mullin, S. A. Vanstone [13], Y. Zhang and B. Du [25]; $G = K_4$ by M. J. Vasiga, S. Furino and A.C.H. Ling [22]; $G = C_4$ by M.X. Wen and T.Z. Hong [17].

In this paper we investigate the existence of an α-resolvable λ-fold $(K_4 - e)$-design (where $K_4 - e$ is the complete graph K_4 with one edge removed). In what follows, by $(a, b, c; d)$ we will denote the graph $K_4 - e$ having $\{a, b, c, d\}$ as vertex-set and $\{\{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}\}$ as edge-set. Basing on the definitions given above, we can derive the following necessary conditions:

(1) $\lambda v(v-1) \equiv 0 \pmod{10}$;

(2) $\alpha v \equiv 0 \pmod 4$;

(3) $2\lambda(v-1) \equiv 0 \pmod{5\alpha}$.

Note that, since the number of α-parallel classes of an α-resolvable λ-fold $(K_4 - e)$-design of order v is $\frac{2\lambda(v-1)}{5\alpha}$ and every vertex appears exactly α times in each of them, we have the following theorem.

Theorem 1.1. *Any α-resolvable λ-fold $(K_4 - e)$-design is balanced.*

From Conditions $(1) - (3)$ we can desume minimum values for α and λ, say α_0 and λ_0, respectively. Similarly to Lemmas 2.1, 2.2 in [22], we have the following lemmas.

Lemma 1.2. *If an α-resolvable λ-fold $(K_4 - e)$-design of order v exists, then $\alpha_0 | \, \alpha$ and $\lambda_0 | \, \lambda$.*

Lemma 1.3. *If an α-resolvable λ-fold $(K_4 - e)$-design of order v exists, then a $t\alpha$-resolvable $n\lambda$-fold $(K_4 - e)$-design of order v exists for any positive integers n and t with $t \,|\, \frac{2\lambda(v-1)}{5\alpha}$.*

The above two lemmas imply the following theorem (for the proof see Theorem 2.3 in [22]).

Theorem 1.4. *If an α_0-resolvable λ_0-fold $(K_4 - e)$-design of order v exists and α and λ satisfy Conditions $(1) - (3)$, then an α-resolvable λ-fold $(K_4 - e)$-design of order v exists.*

Therefore, in order to show that the necessary conditions for α-resolvable designs are also sufficient, we simply need to prove the existence of an α_0-resolvable λ_0-fold $(K_4 - e)$-design of order v, for any given v.

2 Auxiliary definitions

A $(\lambda K_{n_1,n_2,\dots,n_t}, G)$-design is known as a λ-fold *group divisible design*, G-GDD in short, of type $\{n_1, n_2, \dots, n_t\}$ (the parts are called the *groups* of the design). We usually use an "exponential" notation to describe group-types: the group-type $1^i 2^j 3^k$... denotes i occurrences of 1, j occurrences of 2, etc. When $G = K_n$ we will call it an n-GDD.

If the blocks of a λ-fold G-GDD can be partitioned into *partial* α-parallel classes, each of which contains all vertices except those of one group, we refer to the decomposition as a λ-fold (α, G)-*frame*; when $\alpha = 1$, we simply speak of λ-fold G-frame (n-frame if additionally $G = K_n$). In a λ-fold (α, G)-frame the number of partial α-parallel classes missing a specified group of size g is $\frac{\lambda g |V(G)|}{2\alpha|\mathcal{E}(G)|}$.

An *incomplete* α-resolvable λ-fold G-design of order $v + h$, $h \geq 1$, with a hole of size h is a $(\lambda(K_{v+h} \setminus K_h), G)$-design in which there are two types of classes, $\frac{\lambda(h-1)|V(G)|}{2\alpha|\mathcal{E}(G)|}$ *partial* classes which cover every vertex α times except those in the hole and $\frac{\lambda v |V(G)|}{2\alpha|\mathcal{E}(G)|}$ *full* classes which cover every vertex of K_{v+h} α times.

3 $v \equiv 0 \pmod 4$

In [4, 5, 23] it was showed that there exists a resolvable $(K_4 - e)$-design of order $v \equiv 16 \pmod{20}$; while, for every $v \equiv 0, 4, 8, 12 \pmod{20}$ Gionfriddo et al. ([7]) proved that there exists a resolvable 5-fold $(K_4 - e)$-design of order v. Hence the necessary conditions are also sufficient.

4 $v \equiv 1 \pmod 2$

4.1 $v \equiv 1 \pmod{10}$

If $v \equiv 1 \pmod{10}$, then $\lambda_0 = 1$ and $\alpha_0 = 4$ and so a solution is given by a cyclic $(K_4 - e)$-design ([2]), where every base block generates a 4-parallel class. If $v = 10k + 1$, $k \geq 4$, the desired design can be obtained by developing in Z_{10k+1} the base blocks listed below:

$(1 + 2i, 4k + 1 + i, 1; 2k + 2), \ \ i = 3, 4, \dots, \lfloor \frac{k}{2} \rfloor$;
$(2k + 3 - 2i, 5k + 2 - i, 1; 2k + 2), \ \ i = 1, 2, \dots, \lceil \frac{k}{2} \rceil$;
$(1, 4k + 1, 3; 6k)$;
$(1, 2k + 2, 5; 6k + 1)$;

where $\lfloor x \rfloor$ (or $\lceil x \rceil$) denote the greatest (or lower) integer that does not exceed (or that exceed) x. If $v = 11, 21, 31$, the base blocks are:

$v = 11$: $(1, 10, 2; 5)$ developed in Z_{11};
$v = 21$: $(1, 11, 3; 15)$, $(1, 7, 2; 10)$ developed in Z_{21};
$v = 31$: $(2, 13, 1; 5)$, $(1, 27, 10; 11)$, $(1, 7, 3; 14)$ developed in Z_{31}.

4.2 $v \equiv 3, 5, 7, 9 \pmod{10}$

If $v \equiv 3, 5, 7, 9 \pmod{10}$, then $\lambda_0 = 5$ and $\alpha_0 = 4$ and so a solution is given by a cyclic 5-fold $(K_4 - e)$-design, where every base block generates a 4-parallel class. The required design is obtained by developing in Z_v the following blocks:

$(1 + i, v - 1 - i, 0; 1), \;\; i = 1, 2, \ldots, \frac{v-3}{2};$
$(0, 1, 2; v - 1).$

5 $v \equiv 2 \pmod{4}$

5.1 $v \equiv 6 \pmod{20}$

If $v \equiv 6 \pmod{20}$, then $\lambda_0 = 1$ and $\alpha_0 = 2$. In order to prove the existence of a 2-resolvable $(K_4 - e)$-design of order v for every $v \equiv 6 \pmod{20}$, preliminarly we need to construct one of order 6.

Lemma 5.1. *There exists a* 2*-resolvable* $(K_4 - e)$*-design of order* 6.

Proof. Let $V = \{0, 1, 2, 3, 4, 5\}$ be the vertex-set and $\{(0, 1, 2; 3), (2, 3, 4; 5), (4, 5, 0; 1)\}$ be the class. □

For constructing a 2-resolvable $(K_4 - e)$-design of any order $v \equiv 6 \pmod{20}$ and for later use, note that starting from a $(K_4 - e)$-frame of type h^n also a λ-fold $(2, K_4 - e)$-frame of type h^n can be obtained for any $\lambda > 0$, since necessarily $h \equiv 0 \pmod{5}$ and so the number of partial parallel classes missing any group is even.

Lemma 5.2. *For every* $v \equiv 6 \pmod{20}$*, there exists a* 2*-resolvable* $(K_4 - e)$*-design of order* v.

Proof. Let $v = 20k + 6$. The case $k = 0$ follows by Lemma 5.1. For $k > 0$, consider a $(2, K_4 - e)$-frame of type 5^{4k+1} ([5]) with groups G_i, $i = 1, 2, \ldots, 4k + 1$ and a new vertex ∞. For each $i = 1, 2, \ldots, 4k + 1$, let P_i the unique partial 2-parallel class which misses the group G_i. Place on $G_i \cup \{\infty\}$ a copy of a 2-resolvable $(K_4 - e)$-design of order 6, which exists by Lemma 5.1, and combine its full class with the partial class P_i so to obtain the desired design. □

5.2 $v \equiv 2, 10, 14, 18 \pmod{20}$

To prove the existence of an α-resolvable λ-fold $(K_4 - e)$-design of order $v \equiv 2, 10, 14, 18 \pmod{20}$, with minimum values $\lambda_0 = 5$ and $\alpha_0 = 2$, we will construct some small examples most of which will be used as ingredients in the constructions given by the following theorems.

Theorem 5.3. *Let* v*,* g*,* u*, and* h *be positive integers such that* $v = gu + h$*. If there exists*

i) *a* 5*-fold* $(2, K_4 - e)$*-frame of type* g^u*;*

ii) *a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* g*;*

iii) *an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $g + h$ *with a hole of size* h*;*

then there exists a 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = gu + h$.

Proof. Take a 5-fold $(2, K_4 - e)$-frame of type g^u with groups G_i, $i = 1, 2, \ldots, u$ and a set H of size h such taht $H \cap (\cup_{i=1}^{u} G_i) = \emptyset$. For $j = 1, 2, \ldots, g$, let $P_{i,j}$ be the j-th 2-partial class which misses the group G_i. Place on $H \cup G_1$ a copy $\mathcal{D}_1$ of a 2-resolvable 5-fold $(K_4 - e)$-design of order $g + h$ having $g + h - 1$ classes $R_{1,1}, R_{1,2}, \ldots, , R_{1,g}, H_{1,1}, H_{1,2}, \ldots, H_{1,h-1}$. For $i = 2, 3, \ldots, u$, place on $H \cup G_i$ a copy $\mathcal{D}_i$ of an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order $g + h$ with H as hole and having $h - 1$ partial classes $H_{i,1}, H_{i,2}, \ldots, H_{i,h-1}$ and g full classes $R_{i,1}, R_{i,2}, \ldots, , R_{i,g}$. Combine the g partial classes $P_{1,j}$ with the full classes $R_{1,1}, R_{1,2}, \ldots, , R_{1,g}$ of $\mathcal{D}_1$ and for $i = 2, 3, \ldots, u$ the g partial classes $P_{i,j}$ of $\mathcal{D}_i$ with the full classes $R_{i,1}, R_{i,2}, \ldots, R_{i,g}$ so to obtain gu 2-parallel classes on $H \cup (\cup_{i=1}^{u} G_i)$. Combine the classes $H_{1,1}, H_{1,2}, \ldots, H_{1,h-1}$ with the partial classes $H_{i,1}, H_{i,2}, \ldots, H_{i,h-1}$ so to obtain $h - 1$ 2-parallel classes. The result is a 2-resolvable 5-fold $(K_4 - e)$-design of order $gu + h$ with $gu + h - 1$ 2-parallel classes. □

The following lemma gives an input design in the construction of Theorem5.5.

Lemma 5.4. *There exists a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3.*

Proof. Let $\{0, 3\}$, $\{1, 4\}$ and $\{2, 5\}$ be the groups and consider the following classes: $P_1 = \{(0, 2, 1; 4), (1, 5, 0; 3), (3, 4, 2; 5)\}$, $P_2 = \{(3, 5, 1; 4), (1, 2, 0; 3), (0, 4, 2; 5)\}$, $P_3 = \{(0, 5, 1; 4), (2, 4, 0; 3), (1, 3, 2; 5)\}$, $P_4 = \{(2, 3, 1; 4), (4, 5, 0; 3), (0, 1, 2; 5)\}$. □

Theorem 5.5. *Let v, g, m, h and u be positive integers such that $v = 2gu + 2m + h$. If there exists*

i) a 3-frame of type $m^1 g^u$;

ii) a 2-resolvable 5-fold $(K_4 - e)$-design of order $2m + h$;

iii) an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order $2g + h$ with a hole of size h;

then there exists a 2-resolvable 5-fold $(K_4 - e)$-design of order $2gu + 2m + h$.

Proof. Let $\mathcal{F}$ be a 3-frame with one group G of cardinality m and u groups G_i, $i = 1, 2, \ldots, u$ of cardinality g; such a frame has $\frac{m}{2}$ partial classes which miss G, each containing $\frac{gu}{3}$ triples, and, for $i = 1, 2, \ldots, u$, $\frac{g}{2}$ partial classes which miss G_i, each containing $\frac{g(u-1)+m}{3}$ triples. Expand each vertex 2 times and add a set H of h new vertices. Place on $H \cup (G \times \{1, 2\})$ a copy $\mathcal{D}$ of a 2-resolvable 5-fold $(K_4 - e)$-design of order $2m + h$ having $2m + h - 1$ classes $R_1, R_2, \ldots, R_{2m}, H_1, H_2, \ldots, H_{h-1}$. For each $i = 1, 2, \ldots, u$ place on $H \cup (G_i \times \{1, 2\})$ a copy $\mathcal{D}_i$ of an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order $2g + h$ with H as hole and having $h - 1$ partial classes $H_{i,j}$ with $j = 1, 2, \ldots, h - 1$ and $2g$ full classes $R_{i,t}$, $t = 1, 2, \ldots, 2g$. For each block $b = \{x, y, z\}$ of a given class of $\mathcal{F}$ place on $b \times \{1, 2\}$ a copy of a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3 from Lemma 5.4, having $\{x_1, x_2\}$, $\{y_1, y_2\}$ and $\{z_1, z_2\}$ as groups. This gives $2m$ partial classes (whose blocks are copies of $K_4 - e$) which miss $G \times \{1, 2\}$ and $2g$ partial classes which miss $G_i \times \{1, 2\}$, $i = 1, 2, \ldots, u$. Combine the $2m$ partial classes which miss the group $G \times \{1, 2\}$ with the classes $R_1, R_2, \ldots, R_{2m}$ so to obtain $2m$ classes. For $i = 1, 2, \ldots, u$ combine the $2g$ partial classes which miss the group $G_i \times \{1, 2\}$ with the full classes of $\mathcal{D}_i$ so to obtain $2gu$ classes. Finally, combine the $h - 1$ classes $H_1, H_2, \ldots, H_{h-1}$ of $\mathcal{D}$ with the partial classes of $\mathcal{D}_i$ so to obtain $h - 1$ classes. This gives a 2-resolvable 5-fold $(K_4 - e)$-design of order v and $v - 1$ 2-parallel classes. □

Theorem 5.6. *Let v, k and h be non-negative integers. If there exists*

$i)$ an incomplete α-resolvable λ-fold $(K_4 - e)$-design of order $v + k + h$ with a hole of size $k + h$;

$ii)$ an incomplete α-resolvable λ-fold $(K_4 - e)$-design of order $k + h$ with a hole of size h;

then there exists an incomplete α-resolvable λ-fold $(K_4 - e)$-design of order $v + k + h$ with a hole of size h.

Lemma 5.7. *There exists a resolvable $(K_4 - e)$-GDD of type $5^2 10^1$.*

Proof. Let $Z_{10} \cup \{\infty_0, \infty_1, \ldots, \infty_9\}$ be the vertex-set and $2Z_{10}$, $2Z_{10} + 1$, $\{\infty_0, \infty_1, \ldots, \infty_9\}$ be the groups. The desired design is obtained by adding $2 \pmod{10}$ to the following base blocks, including the subscripts of ∞: $(0, 1, \infty_0; \infty_1)$, $(2, 5, \infty_0; \infty_1)$, $(4, 9, \infty_0; \infty_1)$, $(6, 3, \infty_0; \infty_1)$, $(8, 7, \infty_0; \infty_1)$. The parallel classes are generate by every base block. □

Lemma 5.8. *There exists a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 10^3.*

Proof. Start with the 2-resolvable 5-fold $(K_4 - e)$-GDD $\mathcal{G}$ of type 2^3 of Lemma 5.4 with groups G_i, $i = 1, 2, 3$. For each block $b = (x, y, z; t)$ of a given 2-parallel class of $\mathcal{G}$ consider a copy of a resolvable $(K_4 - e)$-GDD of type $5^2 10^1$ where $\{x\} \times Z_5$, $\{y\} \times Z_5$, $\{z, t\} \times Z_5$ are the groups. □

Lemma 5.9. *There exists an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 6 with a hole of size 2.*

Proof. On $V = Z_4 \cup H$, where $H = \{\infty_1, \infty_2\}$ is the hole, consider the partial class $\{(1, 3, 0; 2), (0, 2, 1; 3)\}$ and the four full classes obtained by developing $\{(0, 2, \infty_1; \infty_2), (\infty_1, 1, 0; 3), (\infty_2, 2, 3; 1)\}$ in Z_4, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$. □

Lemma 5.10. *There exists an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2.*

Proof. On $V = Z_8 \cup H$, where $H = \{\infty_1, \infty_2\}$ is the hole, consider the partial class $\{(0, 4, 2; 6), (1, 5, 3; 7), (2, 6, 4; 0), (3, 7, 5; 1)\}$ and the eight full classes obtained by developing $\{(0, 1, \infty_1; 3), (2, 3, \infty_2; 7), (\infty_1, 5, 6; 2), (\infty_2, 6, 4; 5), (4, 7, 1; 0)\}$ in Z_8, where $\infty_i + 1 = \infty_i$ for $i = 1, 2$. □

Lemma 5.11. *There exists an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 14 with a hole of size 4.*

Proof. Let $V = Z_{10} \cup H$ be the vertex-set, where $H = \{\infty_1, \infty_2, \infty_3, \infty_4\}$ is the hole. The partial classes are obtained by adding $2 \pmod{10}$ to the base blocks $(2, 6, 9; 5)$, $(5, 9, 2; 8)$, $(8, 7, 6; 9)$, each block generating a partial class; while, the full classes are obtained by adding $2 \pmod{10}$ to the following base blocks partitioned into two full classes, each class generating five full classes: $\{(0, 8, \infty_1; \infty_2)$, $(1, 5, \infty_3; \infty_4)$, $(\infty_1, 4, 0; 9)$, $(\infty_2, 6, 2; 3)$, $(\infty_3, 3, 7; 8)$, $(\infty_4, 9, 1; 4)$, $(2, 7, 6; 5)\}$, $\{(1, 5, \infty_1; \infty_2)$, $(0, 8, \infty_3; \infty_4)$, $(\infty_1, 3, 9; 4)$, $(\infty_2, 9, 7; 0)$, $(\infty_3, 2, 6; 1)$, $(\infty_4, 6, 8; 3)$, $(4, 7, 2; 5)\}$, where $\infty_i + 1 = \infty_i$ for $i = 1, 2, 3, 4$. □

Lemma 5.12. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 14 *with a hole of size* 2.

Proof. On $V = Z_{12} \cup H$, where $H = \{\infty_1, \infty_2\}$ is the hole, consider the partial class $\{(0,6,3;9),(1,7,4;10),(2,8,5;11),(3,9,6;0),(4,10,7;1),(5,11,8;2)\}$ and the twelve full classes obtained by developing $\{(0,1,\infty_1;11),(2,4,\infty_2;10),(\infty_1,10,6;5),(\infty_2,9,2;0),(3,7,8;1),(5,8,7;9),(6,11,3;4)\}$ in Z_{12}, where $\infty_i + 1 = \infty_i$ for $i = 1,2$. □

Lemma 5.13. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 22 *with a hole of size 6.*

Proof. Let $V = Z_{16} \cup H$ be the vertex-set, where $H = \{\infty_1, \infty_2, \ldots, \infty_6\}$ is the hole. In Z_{16} develop the full 2-parallel base class $\{(0,3,\infty_1;12),(1,5,\infty_2;2),(8,13,\infty_3;4),(14,15,\infty_4;11),(6,11,\infty_5;\infty_6),(\infty_1,2,1;3),(\infty_2,4,13;8),(\infty_3,7,0;14),(\infty_4,9,6;10),(\infty_5,10,5;15),(\infty_6,12,7;9)\}$. Additionally, include the partial 2-parallel class $\{(0,8,2;10),(1,9,3;11),(2,10,4;12),(3,11,5;13),(4,12,6;14),(5,13,7;15),(6,14,8;0),(7,15,9;1)\}$ repeated five times. □

As consequence of Lemmas 5.9 and 5.13, by Theorem 5.6 the following lemma follows.

Lemma 5.14. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 22 *with a hole of size* 2.

Lemma 5.15. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 10.

Proof. Let $V = Z_9 \cup \{\infty\}$ be the vertex-set. The required design is obtained by developing the base class $\{(\infty,0,6;5),(1,5,4;3),(7,8,1;\infty),(2,6,7;8),(3,4,2;0)\}$ in Z_9. □

Lemma 5.16. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 30 *with a hole of size* 10.

Proof. Start from a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 10^3 (which exists by Lemma 5.8) having G_i, $i = 1,2,3$, as groups. Fill in the groups G_2 and G_3 with a copy of a 2-resolvable 5-fold $(K_4 - e)$-design of order 10, which exists by Lemma 5.15. This gives an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 30 with G_1 as hole. □

Lemma 5.17. *There exists an incomplete* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 38 *with a hole of size* 12.

Proof. Let $V = Z_{26} \cup H$ be the vertex-set, where $H = \{\infty_1, \infty_2, \ldots, \infty_{12}\}$ is the hole. The partial classes are: $\{(i, 13+i, 2+i; 15+i) : i = 0,1,\ldots,12\}$, repaeated five times; $\{(2i, 10+2i, 3+2i; 7+2i) : i = 0,1,\ldots,12\}$ and $\{(1+2i, 11+2i, 4+2i; 8+2i) : i = 0,1,\ldots,12\}$, repeated twice; $\{(2i, 10+2i, 1+2i; 9+2i) : i = 0,1,\ldots,12\}$; $\{(1+2i, 11+2i, 2+2i; 10+2i) : i = 0,1,\ldots,12\}$. The full classes are obtained by developing in $V = Z_{26}$ the full base class $\{(\infty_1,2,1;7),(\infty_2,12,3;24),(\infty_3,16,4;11),(\infty_4,13,5;25),(\infty_5,15,9;22),(\infty_6,17,11;23),(\infty_7,19,18;20),(\infty_8,14,10;18),(\infty_9,4,0;8),(\infty_{10},9,17;19),(\infty_{11},7,2;12),(\infty_{12},15,3;24),(1,5,\infty_1;\infty_2),(10,20,\infty_3;\infty_4),(6,23,\infty_5;\infty_6),(16,21,\infty_7;\infty_8),(22,25,\infty_9;\infty_{10}),(13,21,\infty_{11};\infty_{12}),(0,14,6;8)\}$. □

As consequence of the existence of a 2-resolvable 5-fold $(K_4 - e)$-design of order $v = 4, 12$ (see Section 3 and Theorem 1.4) and Lemmas 5.1, 5.11, 5.13, 5.16, 5.17, 5.15, by Theorem 5.6 the following lemma follows.

Lemma 5.18. *There exists a* 2*-resolvable* 5*-fold* (K_4-e)*-design of order* $v = 14, 22, 30, 38$.

Lemma 5.19. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = 42, 234$.

Proof. Start with a resolvable 3-GDD of type $3^{\frac{v}{6}}$ ([20]). Expand each vertex 2 times and for each triple b of a given parallel class place on $b \times \{1, 2\}$ a copy of a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4. Finally, fill each group of size 6 with a copy of a 2-resolvable 5-fold $(K_4 - e)$-design of order 6, which exists by Lemma 5.1. □

Lemma 5.20. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = 50, 62$.

Proof. Start from a 3-frame of type $6^{\frac{v-2}{12}}$ ([3]) and apply Contruction 5.5 with $m = g = 6$, $h = 2$ and $u = \frac{v-14}{12}$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order $v = 50, 62$ (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 14, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 14 with a hole of size 2, which exists by Lemma 5.12). □

Lemma 5.21. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* $v = 34, 274$.

Proof. Start from a 3-frame of type $4^{\frac{v-2}{8}}$ ([3]) and apply Theorem 5.5 with $m = g = 4$, $h = 2$ and $u = \frac{v-10}{8}$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order $v = 34, 274$ (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 10, which exists by Lemma 5.15; a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2, which exists by Lemma 5.10). □

Lemma 5.22. *There exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* 70.

Proof. Start from a 3-frame of type 8^4 ([3]) and apply Theorem 5.5 with $m = g = 8$, $h = 6$ and $u = 3$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order 70 (the input designs are; a 2-resolvable 5-fold $(K_4 - e)$-design of order 22, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-RGDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 22 with a hole of size 6, which exists by Lemma 5.13). □

Lemma 5.23. *For every* $v \equiv 2 \pmod{20}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v.

Proof. Let v=$20k + 2$. The case $v = 22, 42, 62$ are covered by Lemmas 5.18, 5.19 and 5.20. For $k \geq 4$, start from a 5-fold $(2, K_4 - e)$-frame of type 20^k ([5]) and apply Theorem 5.3 with $h = 2$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are a 2-resolvable 5-fold $(K_4 - e)$-design of order 22, which exists by Lemma 5.18, and an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 22 with a hole of size 2, which exists by Lemma 5.14). □

Lemma 5.24. *For every* $v \equiv 10 \pmod{20}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v.

Proof. Let v=$20k + 10$. The case $v = 10, 30, 50, 70$ are covered by Lemmas 5.15, 5.18, 5.20 and 5.22. For $k \geq 4$, start from a 5-fold $(2, K_4 - e)$-frame of type 20^k ([5]) and apply Theorem 5.3 with $g = 20$ and $h = 10$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are a 2-resolvable 5-fold $(K_4 - e)$-design of order 10, which exists by Lemma 5.15, and an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 30 with a hole of size 10, which exists by Lemma 5.16). □

Lemma 5.25. *For every* $v \equiv 14 \pmod{20}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let v=$20k + 14$. The case $v = 14, 34, 234, 274$ are covered by Lemmas 5.18, 5.19 and 5.21. For $k \geq 2$, $k \notin \{11, 13\}$, start from a 5-fold $(2, K_4 - e)$-frame of type 10^{2k+1} ([5]), apply Theorem 5.3 with $h = 4$ and proceed as in Lemma 5.24. □

Lemma 5.26. *For every* $v \equiv 18 \pmod{60}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let v=$60k+18$. Take a resolvable 3-GDD of type 3^{10k+3} ([6]). Expand each vertex 2 times and for each block b of a parallel class place on $b \times \{1, 2\}$ a copy of a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3 which exists by Lemma 5.4, so to obtain a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 6^{10k+3}. Finally, fill in each group of size 6 with a copy of a 2-resolvable 5-fold $(K_4 - e)$-design, which exists by Lemma 5.1. □

Lemma 5.27. *For every* $v \equiv 38 \pmod{60}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let $v = 60k + 38$. The case $v = 38$ follows by Lemmas 5.18. For $k \geq 1$, start from a 3-frame of type 6^{5k+3} ([6]) and apply Theorem 5.5 with $m = g = 6$, $h = 2$ and $u = 5k + 2$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 14, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-GDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 14 with a hole of size 2, which exists by Lemma 5.11) □

Lemma 5.28. *For every* $v \equiv 58 \pmod{120}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let $v = 120k + 58$. Start from a 3-frame of type 4^{15k+7} ([6]) and apply Theorem 5.5 with $m = g = 4$, $h = 2$ and $u = 15k + 6$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are: a 2-resolvable $(K_4 - e)$-design of order 10, which exists by Lemma 5.15; a 2-resolvable 5-fold $(K_4 - e)$-RGDD of type 2^3, which exists by Lemma 5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2, which exists by Lemma 5.10). □

Lemma 5.29. *For every* $v \equiv 118 \pmod{120}$*, there exists a* 2*-resolvable* 5*-fold* $(K_4 - e)$*-design of order* v*.*

Proof. Let $v = 120k + 118$. Start from a 3-frame of type $10^1 4^{15k+12}$, $k \geq 0$, ([6]) and apply Theorem 5.5 with $h = 2$ to obtain a 2-resolvable 5-fold $(K_4 - e)$-design of order v (the input designs are: a 2-resolvable 5-fold $(K_4 - e)$-design of order 22, which exists by Lemma 5.18; a 2-resolvable 5-fold $(K_4 - e)$-RGDD of type 2^3, which exists by Lemma

5.4; an incomplete 2-resolvable 5-fold $(K_4 - e)$-design of order 10 with a hole of size 2, which exists by Lemma 5.10). □

6 Main result

The results obtained in the previous sections can be summarized into the following theorem.

Theorem 6.1. *The necessary conditions* $(1) - (3)$ *for the existence of* α*-resolvable* λ*-fold* $(K_4 - e)$*-designs are also sufficient.*

References

[1] J.-C. Bermond, K. Heinrich and M.-L. Yu, Existence of resolvable path designs, *European J. Combin.* **11** (1990), 205–211, doi:10.1016/S0195-6698(13)80120-5, http://dx.doi.org/10.1016/S0195-6698(13)80120-5.

[2] J.-C. Bermond and J. Schönheim, G-decomposition of K_n, where G has four vertices or less, *Discrete Math.* **19** (1977), 113–120.

[3] C. J. Colbourn and J. H. Dinitz (eds.), *Handbook of combinatorial designs*, Discrete Mathematics and its Applications (Boca Raton), Chapman & Hall/CRC, Boca Raton, FL, 2nd edition, 2007.

[4] C. J. Colbourn, D. R. Stinson and L. Zhu, More frames with block size four, *J. Combin. Math. Combin. Comput.* **23** (1997), 3–19.

[5] G. Ge and A. C. H. Ling, On the existence of resolvable $K_4 - e$ designs, *J. Combin. Des.* **15** (2007), 502–510, doi:10.1002/jcd.20155, http://dx.doi.org/10.1002/jcd.20155.

[6] G. Ge, R. Rees and N. Shalaby, Kirkman frames having hole type $h^u m^1$ for small h, *Des. Codes Cryptogr.* **45** (2007), 157–184, doi:10.1007/s10623-007-9105-2, http://dx.doi.org/10.1007/s10623-007-9105-2.

[7] M. Gionfriddo, G. Lo Faro, S. Milici and A. Tripodi, Resolvable $(k_v - e)$ -designs of order v and index λ (to appear).

[8] M. Gionfriddo and S. Milici, On the existence of uniformly resolvable decompositions of K_v and $K_v - I$ into paths and kites, *Discrete Math.* **313** (2013), 2830–2834, doi:10.1016/j.disc.2013.08.023, http://dx.doi.org/10.1016/j.disc.2013.08.023.

[9] M. Gionfriddo and S. Milici, Uniformly resolvable $\mathscr{H}$-designs with $\mathscr{H} = \{P_3, P_4\}$, *Australas. J. Combin.* **60** (2014), 325–332.

[10] H. Hanani, On resolvable balanced incomplete block designs, *J. Combinatorial Theory Ser. A* **17** (1974), 275–289.

[11] H. Hanani, D. K. Ray-Chaudhuri and R. M. Wilson, On resolvable designs, *Discrete Math.* **3** (1972), 343–357.

[12] J. D. Horton, Resolvable path designs, *J. Combin. Theory Ser. A* **39** (1985), 117–131, doi:10.1016/0097-3165(85)90033-0, http://dx.doi.org/10.1016/0097-3165(85)90033-0.

[13] D. Jungnickel, R. C. Mullin and S. A. Vanstone, The spectrum of α-resolvable block designs with block size 3, *Discrete Math.* **97** (1991), 269–277, doi:10.1016/0012-365X(91)90443-6, http://dx.doi.org/10.1016/0012-365X(91)90443-6.

[14] S. Küçükçifçi, G. Lo Faro, S. Milici and A. Tripodi, Resolvable 3-star designs, *Discrete Math.* **338** (2015), 608–614, doi:10.1016/j.disc.2014.11.013, http://dx.doi.org/10.1016/j.disc.2014.11.013.

[15] S. Küçükçifçi, S. Milici and Z. Tuza, Maximum uniformly resolvable decompositions of K_v and $K_v - I$ into 3-stars and 3-cycles, *Discrete Math.* **338** (2015), 1667–1673, doi:10.1016/j.disc.2014.05.016, http://dx.doi.org/10.1016/j.disc.2014.05.016.

[16] G. Lo Faro, S. Milici and A. Tripodi, Uniformly resolvable decompositions of K_v into paths on two, three and four vertices, *Discrete Math.* **338** (2015), 2212–2219, doi:10.1016/j.disc.2015.05.030, http://dx.doi.org/10.1016/j.disc.2015.05.030.

[17] X. W. Ma and Z. H. Tian, α-resolvable cycle systems for cycle length 4, *J. Math. Res. Exposition* **29** (2009), 1102–1106.

[18] S. Milici, A note on uniformly resolvable decompositions of K_v and $K_v - I$ into 2-stars and 4-cycles, *Australas. J. Combin.* **56** (2013), 195–200.

[19] S. Milici and Z. Tuza, Uniformly resolvable decompositions of K_v into P_3 and K_3 graphs, *Discrete Math.* **331** (2014), 137–141, doi:10.1016/j.disc.2014.05.010, http://dx.doi.org/10.1016/j.disc.2014.05.010.

[20] R. Rees and D. R. Stinson, On resolvable group-divisible designs with block size 3, *Ars Combin.* **23** (1987), 107–120.

[21] R. Su and L. Wang, Minimum resolvable coverings of K_v with copies of $K_4 - e$, *Graphs Combin.* **27** (2011), 883–896, doi:10.1007/s00373-010-1003-0, http://dx.doi.org/10.1007/s00373-010-1003-0.

[22] T. M. J. Vasiga, S. Furino and A. C. H. Ling, The spectrum of α-resolvable designs with block size four, *J. Combin. Des.* **9** (2001), 1–16, doi:10.1002/1520-6610(2001)9:1⟨1::AID-JCD1⟩3.0.CO;2-H, http://dx.doi.org/10.1002/1520-6610(2001)9:1<1::AID-JCD1>3.0.CO;2-H.

[23] L. Wang, Completing the spectrum of resolvable $(K_4 - e)$-designs, *Ars Combin.* **105** (2012), 289–291.

[24] L. Wang and R. Su, On the existence of maximum resolvable $(K_4 - e)$-packings, *Discrete Math.* **310** (2010), 887–896, doi:10.1016/j.disc.2009.10.007, http://dx.doi.org/10.1016/j.disc.2009.10.007.

[25] Y. Zhang and B. Du, α-resolvable group divisible designs with block size three, *J. Combin. Des.* **13** (2005), 139–151, doi:10.1002/jcd.20028, http://dx.doi.org/10.1002/jcd.20028.