Research Article Open Access

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Solution of nonlinear singular initial value problems of generalized Lane-Emden type using block pulse functions in a large interval

DOI 10.1515/wwfaa-2016-0002
Submitted Oct 23, 2015; accepted Jan 12, 2016

Abstract: In this paper, the block pulse functions (BPFs) and their operational matrices of integration and differentiation are used to solve nonlinear singular initial value problems (NSIVPs) of generalized Lane-Emden type in a large interval. In the proposed method, we present a new technique for computing nonlinear terms in such equations. This technique is then utilized to reduce the solution of this nonlinear singular initial value problems to a system of nonlinear algebraic equation whose solution is the coefficients of block pulse expansions of the solution of NSIVPs. Numerical examples are illustrated to show the reliability and efficiency of the proposed method.

Keywords: Lane-Emden equation; Block pulse functions; Nonlinear singular initial value problems; Operational matrices

1 Introduction

Singular initial value problems (SIVPs) occur very frequently in several models of physics, chemistry and mechanics [1–4]. For example, the density profile equation in hydrodynamics may be reduced to a singular nonlinear differential equation in the following form [5]:

$$(p(x)u'(x))' = p(x)f\left(u(x)\right), \quad u(0) = \alpha \;\; u'(0) = \beta, \tag{1}$$

where p and f are known functions which satisfy desired conditions of the under study problems. As another example, theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and theory of thermionic currents are modeled by means of the Lane-Emden equation [6]:

$$u''(t) + \frac{A}{t}u'(t) + R(t)g(u) = h(t), \quad A \geq 0, \qquad u(0) = \alpha, \;\; u'(0) = \beta, \tag{2}$$

where α and β are fixed constants, $g(u)$ is an analytical real valued function, $R(t)$ and $h(t)$ are continuous real valued function in their domains. Because of the singularity behavior at the origin, the linear and nonlinear SIVPs play important role in the physics and engineering fields, and investigating the numerical solution of these kind of problems is very interesting.

The aim of this paper is to introduce a new method for numerical solution of the following generalized Lane-Emden SIVP

$$u''(t) + Q(t)u'(t) + R(t)g(u) = h(t), \quad u(0) = \alpha, \qquad u'(0) = \beta, \tag{3}$$

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where $g(u)$ is an analytic real valued function, $R(t)$ and $h(t)$ are continuous real valued functions in their domains. In addition, we suppose that the function $Q(t)$ can be written as $Q(t) = \frac{p(t)}{q(t)}$, where $p(t) \in C[0, \infty)$ and $q(t)$ is a known function which has some roots in a known large interval.
In recent years, the study of SIVPs has attracted the attention of many mathematicians and physicists (see [7] and references therein) particularly Lane-Emden equation [8–13]. In this study, we propose a new simple method to solve the generalized Lane-Emden SIVPs (3)(examples 1, 2 and 4), in a large interval by using Block Puls Functions (BPFs).
BPFs are easy to use and this simplicity allows one to use them for solving integral equations and differential equations [14, 15]. In this study, we will apply the BPFs to find approximate solution to the nonlinear SIVPs. In the proposed method, both of the operational matrices of integration and differentiation of BPFs have been used mutually, and for the first time, to solve the nonlinear SIVPs. The method is based on reducing the equation to a system of nonlinear algebraic equations by expanding the solution using BPFs with unknown coefficients. The proposed method is simple to understand and easy to implement using suitable computer softwares. Moreover, there are some important points to be noted here. Although, in recent years, BPFs have been used to solve different nonlinear functional equations, but in almost all of these cases, nonlinear terms that has polynomial form and for other nonlinear terms, these terms are approximated by Taylor's expansions and then few terms of these expansions have been used. For this reason the obtained solutions are accurate only in a small interval. In this paper, we have proven that any analytic function $g(u)$, can be expressed in a closed form by BPFs without considering Taylor's expansions (see theorem 2.3). This fact allows us to obtain the numerical solutions of problems in a large interval. In [13], authors have employed BPFs to obtain operational matrix for Bernstein polynomials and then this matrix was used for the numerical solution of the Lane-Emden problem. Finally, in this paper some useful theorem about convergence of BPFs in a large interval are proven.
The structure of this paper is as follows: In section 2 we introduce block pulse functions and their properties. In section 3 the rate of convergence and error analysis of the BPFs expansion are presented. In section 4 we implement the proposed method for the nonlinear SIVPs. In section 5 the error of the proposed method is investigated. In section 6 we apply the proposed method and solve some examples to show the accuracy and efficiency of this method. Finally, a conclusion is drawn in section 7.

2 Block pulse functions (BPFs)

BPFs have been studied by many authors and also have been applied for solving different problms [15–20]. Here, we present a brief review of BPFs and its properties:

Definition 2.1. An m-set of BPFs is defined over the interval $[0, T)$ as:

$$b_i(t) = \begin{cases} 1, & \frac{iT}{m} \le t < \frac{(i+1)T}{m}, \\ 0, & otherwise, \end{cases} \tag{4}$$

where $i = 0, 1, 2, \dots, (m-1)$.

The most important properties of BPFs are disjointness, orthogonality and completeness.
Disjointness: This property can be clearly obtained from the definition of BPFs:

$$b_i(t)b_j(t) = \begin{cases} b_i(t), & i = j, \\ 0, & i \neq j, \end{cases} \tag{5}$$

Orthogonality: It is clear that

$$\int_0^T b_i(\tau)b_j(\tau)d\tau = \frac{T}{m}\delta_{ij}. \tag{6}$$

where δ_{ij} is Kroneker delta.
Completeness: For every $f \in L^2[0, T)$, sequence $\{b_i(t)\}_{i=1}^{\infty}$ is complete if $\int_0^T b_i(t)f(t)dt = 0$ result in $f(t) = 0$ almost everywhere. Because of completeness of $\{b_i(t)\}_{i=1}^{\infty}$, Parseval's identity holds, i.e. we have $\int_0^T [f(t)]^2 dt = \sum_0^{\infty} f_i^2 \|b_i(t)\|^2$, for every real bounded function $f(t) \in L^2[0, T)$ and:

$$f_i = \frac{m}{T}\int_0^T b_i(\tau)f(\tau)d\tau = \frac{m}{T}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} f(\tau)d\tau. \tag{7}$$

2.1 Vector forms

Consider the first m-terms of BPFs and write them concisely as m-vector:

$$B_m(t) = [b_0(t), b_1(t), \dots, b_i(t), \dots, b_{m-1}(t)]^T, \quad t \in [0, T). \tag{8}$$

The above representation and disjointness property yield:

$$B_m(t)B_m^T(t) = \begin{pmatrix} b_0(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{m-1}(t) \end{pmatrix}, \tag{9}$$

$$B_m^T(t)B_m(t) = 1, \tag{10}$$

$$B_m(t)B_m^T(t)V = \tilde{V}B_m(t), \tag{11}$$

where V is an m-vector and $\tilde{V} = diag(V)$.

2.2 Function approximation

Any absolutely integrable function $f(t)$ defined over $[0, T)$ can be expanded in BPFs as:

$$f(t) = \sum_{i=0}^{\infty} f_i b_i(t), \tag{12}$$

where f_i is obtained in (7).
If the infinite series in (12) is truncated, then (12) can be written as

$$f_m(t) \simeq \sum_{i=0}^{m-1} f_i b_i(t) = F_m^T B_m(t), \tag{13}$$

where $F_m = [f_0, f_1, \dots, f_{m-1}]^T$ and $B_m(t)$ is defined in (8).
Also the block pulse coefficients f_i are obtained as (7), such that the mean square error between $f(t)$ and its block pulse expansion (13) in the interval of $t \in [0, T)$ is minimal:

$$\epsilon = \frac{1}{T}\int_0^T \left(f(t) - \sum_{i=0}^{m-1} f_i b_i(t)\right)^2 dt.$$

Lemma 2.2. *Suppose $f_m(t) = F_m^T B_m(t)$ be expansion of $f(t)$ by BPFs, then for any integer $n \geq 2$ we have:*

$$[f(t)]^n = [f_0^n, f_1^n, \dots, f_{m-1}^n]B_m(t). \tag{14}$$

Proof. From (5) and (9) we get:

$$[f(t)]^2 = f(t)f(t)^T = [a_0, a_1, \dots, a_{m-1}]B_m(t)B_m(t)^T[a_0, a_1, \dots, a_{m-1}]^T$$

$$= [a_0, a_1, \dots, a_{m-1}] \begin{pmatrix} b_0(t) & 0 & \dots & 0 \\ 0 & b_1(t) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & b_{m-1}(t) \end{pmatrix} [a_0, a_1, \dots, a_{m-1}]^T$$

$$= f_0^2 b_0(t) + f_1^2 b_1(t) + \dots + f_{m-1}^2 b_{m-1}(t) = [f_0^2, f_1^2, \dots, f_{m-1}^2]B_m(t), \tag{15}$$

Now by induction we have:

$$[u(t)]^n = [a_0^n, a_1^n, \dots, a_{m-1}^n]B_m(t). \quad n \geq 2. \tag{16}$$

This completes the proof.

Theorem 2.3. *Suppose $g(t)$ be an analytic function and $f_m(t)$ be expansion of $f(t)$ by BPFs, Then we have:*

$$g\left(f(t)\right) = [g(a_0), g(a_1), \dots, g(a_{m-1})]B_m(t). \tag{17}$$

Proof. Since g is an analytic function, by considering Maclaurin expansion of g, i.e. $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n$, we get:

$$g(f) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} f^n. \tag{18}$$

From Eqs. (16) and (18) yield:

$$g(f(t)) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}[a_0^n, \dots, a_{m-1}^n]B_m(t) = \left[\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} a_0^n, \dots, \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} a_{m-1}^n\right] B_m(t). \tag{19}$$

Since the series in left hand side is absolutely and uniformly convergent to $g(f)$, therefore each series in right hand side is also absolutely and uniformly convergent to $g(a_i)$ i.e.

$$\left[\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} a_0^n, \dots, \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} a_{m-1}^n\right] = \left[g(a_0), \dots, g(a_{m-1})\right]. \tag{20}$$

Now, from (19) and (20), we have:

$$g(f(t)) = \left[g(a_0), g(a_1), \dots, g(a_{m-1})\right] B_m(t). \tag{21}$$

This completes the proof.

2.3 The operational matrices

Chen and Hsiao [21] introduced the concept of operational matrix in 1975. Kilicman and Al Zhour [22] investigated the generalized integral operational matrix and showed that the integral of the matrix $B_m(t)$ defined in (8), can be approximated by:

$$\int_0^t B_m(\tau)d\tau \simeq PB_m(t), \tag{22}$$

where P is the $m \times m$ operational matrix of one-time integral of $B_m(t)$. Moreover, we can compute the generalized operational matrices P^n of n-times integration of $B_m(t)$ as:

$$\underbrace{\int_0^t \cdots \int_0^t}_{n-times} B_m(\tau)(d\tau)^n \simeq P^n B_m(t). \tag{23}$$

In [22] it is shown that P^n has the following form:

$$P^n = \left(\frac{T}{m}\right)^n \frac{1}{(n+1)!} \begin{pmatrix} 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \dots & \xi_{m-2} \\ 0 & 0 & 1 & \dots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{24}$$

where $\xi_i = (i+1)^{n+1} - 2i^{n+1} + (i-1)^{n+1}$.
Also, the generalized BPFs operational matrices D^n for differentiation can be derived by inverting the P^n matrices, which has the following form [22]:

$$D^n = (n+1)! \left(\frac{m}{T}\right)^n \begin{pmatrix} 1 & d_1 & d_2 & \dots & d_{m-1} \\ 0 & 1 & d_1 & \dots & d_{m-2} \\ 0 & 0 & 1 & \dots & d_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

where $d_i = -\sum_{j=1}^{i} \xi_j d_{i-j}$ for all $i = 1, 2, \dots, m-1$, and $d_0 = 1$.

3 Description of the proposed method

In this section, the BPFs expansion together with its operational matrices of integration and differentiation are used to obtain numerical solutions of equation (3). Let us consider the NSIVP (3). For solving this equation we assume:

$$u''(t) = K_m^T B_m(t), \tag{26}$$

where K_m^T is an unknown vector and $B_m(t)$ is the vector which is defined in (7). By two-times integration of (26) with respect to t and considering the initial conditions, we have:

$$u'(t) = K_m^T P B_m(t) + \beta = K_m^T P B_m(t) + [\beta, \beta, \dots, \beta] B_m(t), \tag{27}$$

and

$$u(t) = K_m^T P^2 B_m(t) + [\beta, \beta, \dots, \beta] P B_m(t) + [\alpha, \alpha, \dots, \alpha] B_m(t). \tag{28}$$

Equation (28) can be written as:

$$u(t) = K_m^T P^2 B_m(t) + [\beta, \dots, \beta] P B_m(t) + [\alpha, \dots, \alpha] B_m(t) = [a_0, a_1, \dots, a_{m-1}] B_m(t). \tag{29}$$

Now from theorem 2.3 we have:

$$g(u) = \left[g(a_0), g(a_1), \dots, g(a_{m-1})\right] B_m(t). \tag{30}$$

By substituting (26), (27) and (30) into (3) and considering (29) we have:

$$\left[K_m^T + Q(t)\left([K_m^T P + [\beta, \ldots, \beta]\right) + R(t)\left[g(a_0), g(a_1), \ldots, g(a_{m-1})\right]\right] B_m(t) = h(t), \tag{31}$$

where

$$K_m^T = \left[[a_0, a_1, \ldots, a_{m-1}] - [\beta, \ldots, \beta]P - [\alpha, \ldots, \alpha]\right] D^2.$$

Equation (31) is a nonlinear algebraic equation. Now, by taking collocation points $t_i = \frac{(2i-1)T}{2m}$, $i = 1, 2, \ldots, m$, this equation is transformed into a nonlinear system of algebraic equations with m unknowns $a_i(i = 0, 1, \ldots, m-1)$. By solving this system and determining $[a_0, a_1, \ldots, a_{m-1}]$, we get the numerical solution of (3) as $u(t) = [a_0, a_1, \ldots, a_{m-1}]B_m(t)$.

4 Convergence and error analysis of BPFs expansion

In this section we review the rate of convergence and error analysis of the BPFs expansion of a continuous function.

Definition 4.1. Suppose $f_m(t) = F_m^T B_m(t)$ be expansion of $f(t)$ by BPFs, then the corresponding error is defined as follows:

$$e_m(t) = f_m(t) - f(t). \tag{32}$$

Theorem 4.2. *Suppose that $f(t)$ satisfies a Lipschitz condition on $[0, T]$, that is:*

$$\exists M > 0,\ \forall \zeta,\ \eta \in [o, T] : |f(\zeta) - f(\eta)| \le M|\zeta - \eta|. \tag{33}$$

Then the BPFs expansion will be convergent in the sense that $e_m(t)$ approaches zero as m closes to infinity. Moreover, the convergence order is one, that is :

$$\|e_m(t)\| = \mathcal{O}(\frac{1}{m}). \tag{34}$$

Proof. By defining the error between $f(t)$ and its BPFs expansion over every subinterval I_i as:

$$e_i(t) = f_i b_i(t) - f(t), \quad t \in I_i\ (i = 0, 1, \ldots, m-1),$$

where $I_i = [\frac{iT}{m}, \frac{(i+1)T}{m})$, we obtain:

$$\|e_i(t)\|^2 = \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} e_i(t)^2 dt = \int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} \left(f_i b_i(t) - f(t)\right)^2 dt = \frac{T}{m}\left(f_i - f(\eta_i)\right)^2,\ \eta_i \in I_i, \tag{35}$$

where we used the mean value theorem for integral.
From (7) and the mean value theorem, we have:

$$f_i = \frac{m}{T}\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} f(t)dt = \frac{m}{T}\frac{T}{m} f(\zeta_i) = f(\zeta_i), \quad \zeta_i \in I_i. \tag{36}$$

By substituting (36) in (35) we obtain:

$$\|e_i(t)\|^2 = \frac{T}{m}\left(f(\zeta_i) - f(\eta_i)\right)^2 \le \frac{TM^2}{m}|\zeta_i - \eta_i|^2 \le \frac{T^3M^2}{m^3}. \tag{37}$$

This leads to:

$$\|e_m(t)\|^2 = \int_0^T e_m(t)^2 dt = \int_0^T \left(\sum_{i=0}^{m-1} e_i(t)\right)^2 dt$$

$$= \int_0^T \left(\sum_{i=0}^{m-1} e_i(t)^2\right) dt + 2\sum_{i\le j} \int_0^T e_i(t)e_j(t)dt. \tag{38}$$

Since for $i \neq j$, $I_i \cap I_j =$, then

$$\|e_m(t)\|^2 = \int_0^T \left(\sum_{i=0}^{m-1} e_i(t)^2\right) dt = \sum_{i=0}^{m-1} \|e_i\|^2. \tag{39}$$

Substituting (37) into (39), we have:

$$\|e_m(t)\|^2 \le \frac{T^3 M^2}{m^2}, \tag{40}$$

or, in other words, $\|e_m(t)\| = \mathcal{O}(\frac{1}{m})$. This completes the proof.

Corollary 4.3. Let $f_m(t)$ be the expansion of $f(t)$ by BPFs and $e_m(t)$ be the corresponding error which is defined in (32), then we have:

$$\|e_m(t)\| \le \frac{MT\sqrt{T}}{m}. \tag{41}$$

Proof. It is an immediate consequence of Theorem 4.2.

Lemma 4.4. *Suppose $f(t)$ is approximated by BPFs on interval $[0, T)$ as:*

$$f_m(t) = \sum_{i=0}^{m-1} f_i b_i(t),$$

and moreover suppose by solving some problems we have found $\hat{f}_i$ as an approximation of f_i and put:

$$\hat{f}_m(t) = \sum_{i=0}^{m-1} \hat{f}_i b_i(t),$$

Then for each $t \in [0, T)$ we obtain:

$$\|\hat{f}_m(t) - f_m(t)\| \le \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

Proof. We have:

$$\|\hat{f}_m(t) - f_m(t)\| = \left(\int_0^T \left(\sum_{i=0}^{m-1} \left(\hat{f}_i - f_i\right) b_i(t)\right)^2 dt\right)^{\frac{1}{2}} = \sum_{i=0}^{m-1} |\hat{f}_i - f_i| \left(\int_{\frac{iT}{m}}^{\frac{(i+1)T}{m}} b_i(t)^2 dt\right)^{\frac{1}{2}}$$

$$= \sum_{i=0}^{m-1} |\hat{f}_i - f_i| \sqrt{\frac{T}{m}} \le \sqrt{\frac{T}{m}} \sum_{i=0}^{m-1} \|\hat{f}_m(t) - f(t)\|_\infty = \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

This completes the proof.

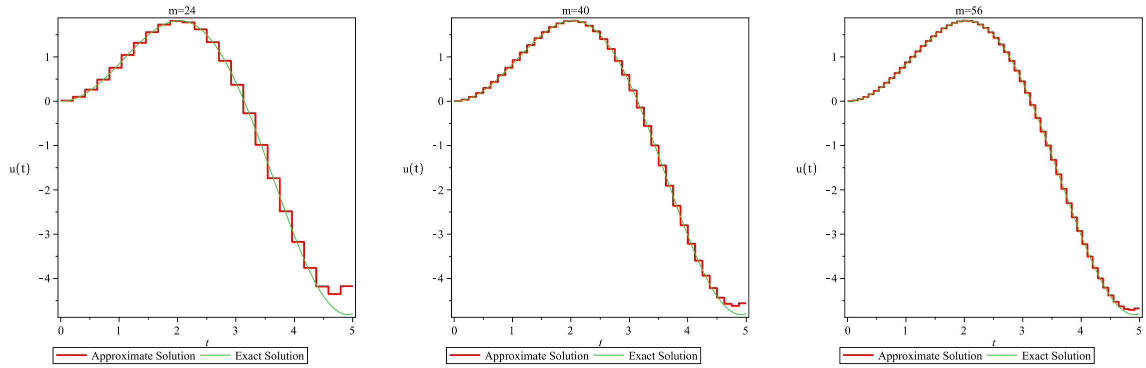

Figure 1: Approximate solutions of example 1, for some values of m.

Corollary 4.5. By solving some problems we obtain $\hat{f}_m(t)$ as the approximation of $f(t)$. Then we obtain:

$$\|\hat{f}_m(t) - f(t)\| \leq \frac{MT\sqrt{T}}{m} + \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

Proof. For every $t \in [0, T)$ we have:

$$\|\hat{f}_m(t) - f(t)\| = \|\hat{f}_m(t) - f(t) - f_m(t) + f_m(t)\| \leq \|f_m(t) - f(t)\| + \|\hat{f}_m(t) - f_m(t)\|.$$

Now from (41) and lemma 4.4 we have:

$$\|\hat{f}_m(t) - f(t)\| \leq \frac{MT\sqrt{T}}{m} + \sqrt{mT}\|\hat{f}_m(t) - f(t)\|_\infty.$$

This completes the proof.

5 Numerical examples

In this section we investigate some examples of NSIVP to show the reliability of the proposed method. Moreover, the obtained results are compared with the results that have been given by other numerical methods.

Example 1. In this example, we consider the NSIVP [23, 24]:

$$u''(t) + \frac{\sin(t)}{t}u'(t) + [u(t)]^2 = h(t), \quad u(0) = 0, \quad u'(0) = 0,$$

where $h(t)$ is compatible to the exact solution, $u(t) = t\sin(t)$.
This equation is now solved by the proposed method. Numerical solutions for some selected values of m are shown in Fig. 1. A comparison between the proposed method for $m = 96$ with 3-th approximate solution ($u_3(t)$) of the MADM [23] and the exact solution are performed in Fig. 2 (left hand side). Also, to see how much increasing number of terms in the MADM can improve the solution of this problem, a comparison between the proposed method for $m = 96$, with $u_7(t)$ of the MADM and the exact solution are performed in Fig. 2 (right hand side).

From Fig. 2, it can be concluded that increasing number of terms for the MADM can not improve the solution of this problem in a large domain. In contrast, by applying more sets of basic block pulse functions (*i.e.* increasing m) a good approximate solution for this problem can be obtained in a large domain.

Example 2. Consider the NSIVP [23, 24]:

$$u''(t) + \frac{(1+5t)}{2t(t+1)}u'(t) + u(t)\ln(u(t)) = h(t), \quad u(0) = 1, \quad u'(0) = 0,$$

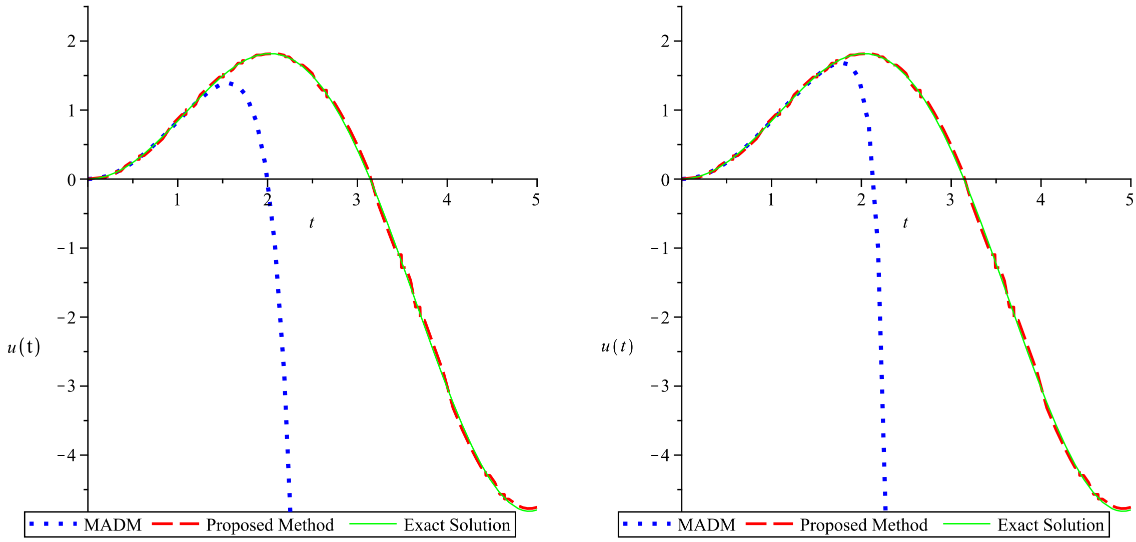

Figure 2: Numerical solutions for example 1.

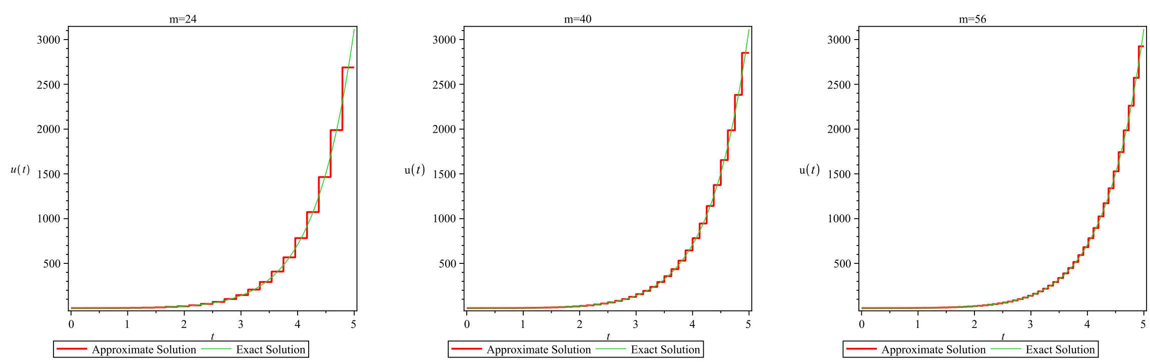

Figure 3: Approximate solution of example 2, for some values of m.

where $h(t)$ is compatible to the exact solution, $u(t) = (1 - t + t^2)e^t$.
To solve this problem by changing the variable $u(t) = 1 + z(t)$ in which $z(t)$ is unknown, we transform this NSIVP to the NSIVP as follows:

$$z''(t) + \frac{(1+5t)}{2t(t+1)}z'(t) + (1 + z(t))\ln(1 + z(t)) = h(t), \quad z(0) = 0, \quad z'(0) = 0. \tag{42}$$

Therefore, $z(t)$ as the solution of (42) can be obtained by the proposed method. Finally $u(t)$ as the solution of original problem is $u(t) = 1 + z(t)$.
In order to show the efficiency of the proposed method, this problem is now solved by the proposed method. Numerical solutions for some selected values of m are shown in Fig. 3. A comparison between the proposed method for $m = 96$ with $u_3(t)$ of the MADM [3] and the exact solution are performed in Fig. 4 (left hand side). Also, a comparison between the proposed method for $m = 96$, with $u_7(t)$ of the MADM and the exact solution are performed in Fig 4 (right hand side). From Fig. 4, it can be seen that increasing the number of terms for the MADM can not improve the solution of this problem in a large domain. But, when applying more sets of basic block pulse functions, a good approximate solution for this problem can be obtained in a large domain.

Example 3. Consider the following nonlinear Lane-Emden type equation [9, 25]:

$$u''(t) + \frac{5}{t}u'(t) + 8(e^{u(t)} + 2e^{\frac{u(t)}{2}}) = 0, \quad u(0) = 0, \quad u'(0) = 0,$$

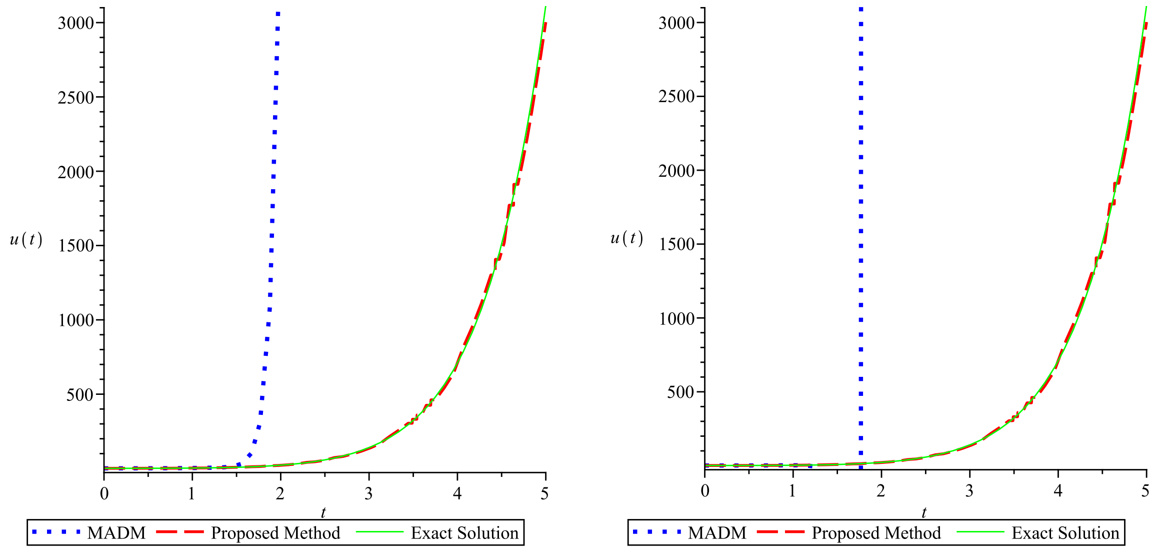

Figure 4: Numerical solutions for example 2.

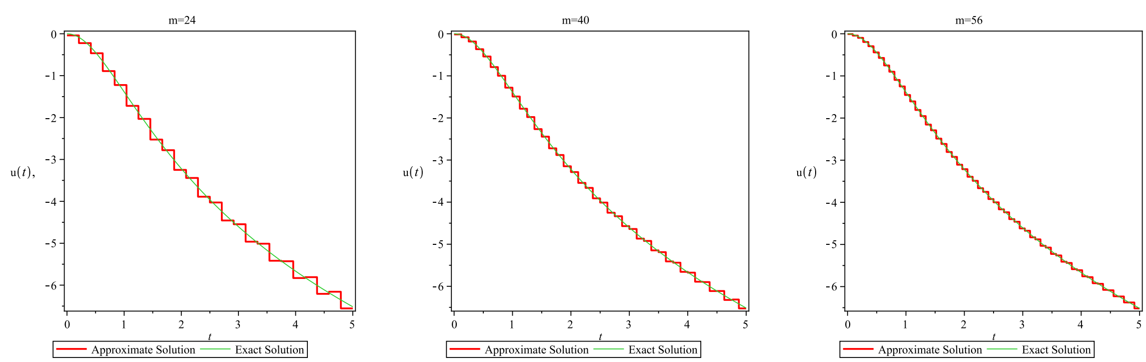

Figure 5: Approximate solutions of example 3, for some values of m.

which has the following analytical solution, $u(t) = -2\ln(1 + t^2)$.
This problem is now solved by the proposed method. Numerical solutions for some selected values of m are shown in Fig. 5. A comparison between the proposed method for $m = 96$ with 3-iteration approximate solution ($u_3(t)$) of the VIM [9] and the exact solution are performed in Fig. 6 (left hand side). Also, to see how much increasing number of iteration in the VIM can improve the solution of this problem, a comparison between the proposed method for $m = 96$, with $u_5(t)$ of the VIM and the exact solution are performed in Fig. 6 (right hand side). Fig. 6 shows that increasing number of iterations for the VIM can not improve the solution of this problem in a large domain. But by applying more sets of basic block pulse functions, a good approximate solution for this problem can be obtained in a large domain.

Example 4. Finally, consider the nonlinear singular initial value problem in hydrodynamic problems form:

$$u''(x) + \frac{\sin(x) - x}{\cos(x)} u'(x) = \arcsin\left(u(x)\right), \qquad x \in [0, 2\pi],$$

with the exact solution $u(x) = \sin(x)$.
The numerical solutions of this problem for different values of $m = 128$ are shown in Figs. 7 and 8.

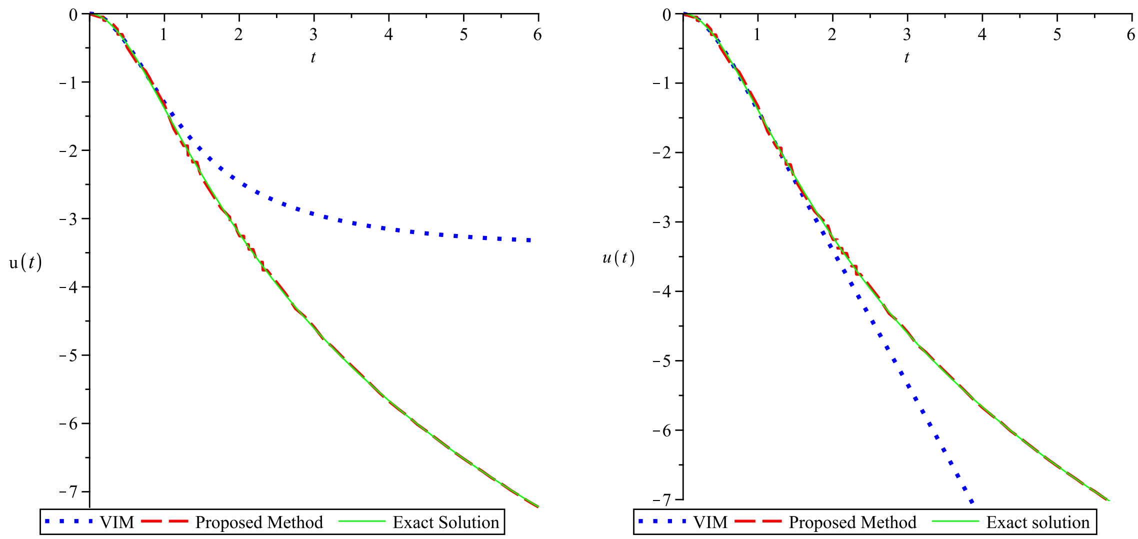

Figure 6: Numerical solutions for example 3.

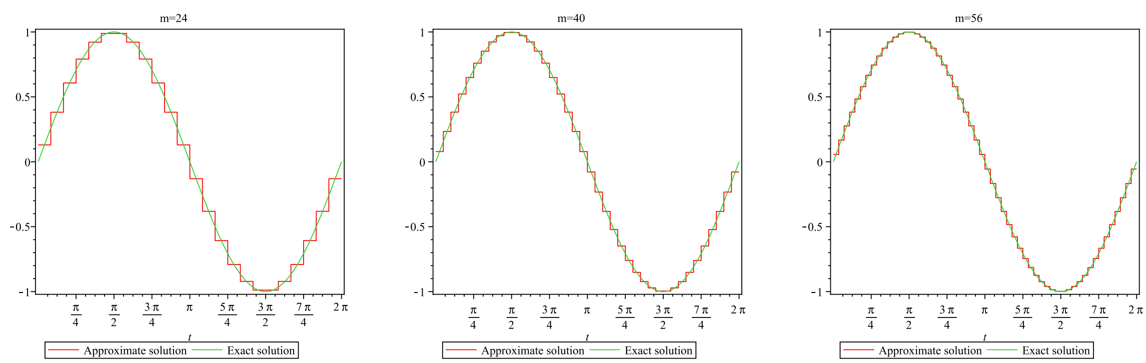

Figure 7: Approximate solutions of example 4, for some values of m.

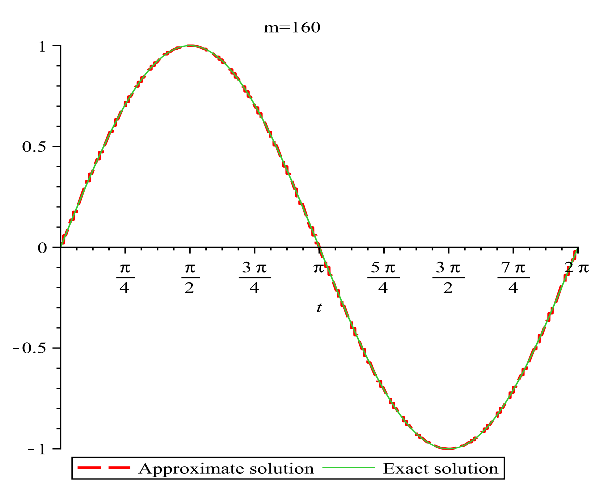

Figure 8: The approximate and exact solutions of example 4.

6 Conclusion

In this paper, we proposed the BPFs method for the numerical solutions of nonlinear singular initial value problems of generalized Lane-Emden type in a large interval. In this method, we proposed a new technique to compute nonlinear terms in such equations. Error estimation of the proposed method for the numerical solution of such equation is investigated. Efficient approximate solutions have been derived for NSIVPs and the results have shown remarkable performance. Moreover, the new method is compared with the MADM and VIM in various NSIVPs. The obtained results show that the proposed method is very efficient and accurate in comparison with the MADM and VIM. Also, the proposed approach can provide a suitable approximate solution in a large interval by using only a few sets of BPFs.

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