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Closed orders and closed graphs

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Abstract

The class of closed graphs by a linear ordering on their sets of vertices is investigated. A recent characterization of such a class of graphs is analyzed by using tools from the proper interval graph theory.

Introduction

Let G be a simple graph with finite vertex set $V(G)$ and edge set $E(G)$. In the last years, several authors have focused their attention on the class of *closed ideals* (see, for instance, [2, 3, 4, 9, 12] and the references therein). Let $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a polynomial ring with coefficients in a field K. The closed graphs were introduced in [12] in order to characterize those graphs, which, for suitable labeling of their edges, do have a quadratic Gröbner basis with respect to the lexicographic order induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Such a class of graphs is strictly related to the so-called *binomial edge ideal* [12]. The binomial edge ideal of a labeled graph G, denoted by J_G, is the ideal of S generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ such that $i < j$ and $\{i, j\}$ is an edge of G. In [3], the authors have shown that the existence of a quadratic Gröbner basis for J_G is not related to the lexicographic order on S. Indeed, one of the main results in the paper implies that the closed graphs are the only graphs for which J_G has a quadratic Gröbner basis for some monomial order on S. Afterwards, the same authors have proved that the class of closed graphs is isomorphic to the well known class of *proper interval graphs* [4]. Such a class of graphs has been strongly studied from both the theoretical and

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algorithmic point of view and many linear-time algorithms for proper interval graph recognition have been developed (see, for instance [1, 6, 10, 11, 16] and the references therein). Hence, the isomorphism in [4] implies the existence of linear-time algorithms also for the closed graph recognition. In this note we study the behavior of closed graphs by using methods typical of the class of proper interval graphs.

The plan of the paper is the following. Section 1 contains some preliminary notions that will be used in the paper. In Section 2, we discuss some results on proper interval graphs in order to compare such a class of graphs with the class of closed graphs. We introduce the *closed orderings* (Definition 2.7) and observe that closed orderings and *proper interval orderings* (Definition 2.1) coincide; as a consequence, we recover the isomorphism between the class of closed graphs and the class of proper interval graphs (Corollary 2.9). In Section 3, we show how a recent characterization of closed graphs, due to Cox and Erskine [2], can be obtained via some properties of proper interval graphs (Theorem 3.6).

1 Preliminaries

In this Section, we collect some notions that will be useful in the development of the paper.

Let G be a simple, finite graph. Denote by $V(G)$ the set of vertices of G and by $E(G)$ its edge set. Let $v, w \in V(G)$. A *path* π of *length* n from v to w is a sequence of vertices $v = v_0, v_1, \ldots, v_n = w$ such that $\{v_i, v_{i+1}\}$ is an edge of the underlying graph. A path $v_0, v_1, \ldots, v_n$ is *closed* if $v_0 = v_n$. A graph G is *connected* if for every pair of vertices u and v there is a path from u to v. The *distance* $d(u, v)$ between vertices u, v of a graph G is the length of the shortest path connecting them, and the *diameter* $\operatorname{diam}(G)$ of G is the maximum distance between two vertices of G. A *cycle* of *length* n is a closed path $v_0, v_1, \ldots, v_n$ in which $n \geq 3$. G is *chordal* or *triangulated* if its cycles of four or more vertices have a *chord*, which is an edge joining two non adjacent vertices of the cycle. Finally, G is *claw-free* (or *net-free*, or *tent-free*, respectively) if G does not contain as induced subgraph the *claw* (or the *net*, or the *tent*, respectively):

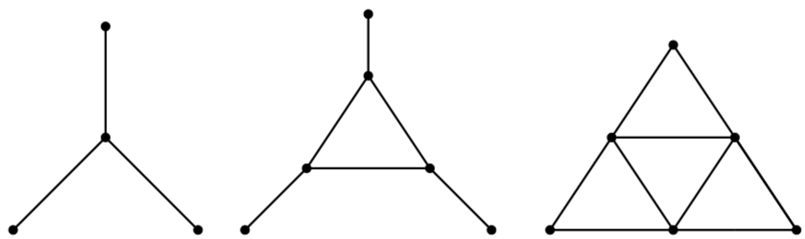


Figure 1: *The claw* (left), *the net* (middle), *the tent* (right).

2 Closed orderings

In this Section, we analyze the isomorphism between the class of closed graphs and the class of proper interval graphs [4] by vertex orderings.

Set $[n] = \{1, \ldots, n\}$. A *vertex ordering* (also called a *labeling*) of a n-vertex graph G is a bjection $\sigma : V(G) \to [n]$. We write $u <_\sigma v$ to mean that $\sigma(u) < \sigma(v)$, for $u, v \in V(G)$. If σ is a vertex ordering of a n-vertex graph G, the vertices of G can be ordered as $v_1, v_2, \ldots, v_n$ such that

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_n.$$

Definition 2.1. Let σ be a vertex ordering of a graph G. Ordering σ is called a *proper interval ordering* if for every triple u, v, w of vertices of G where $u <_\sigma v <_\sigma w$ and $\{u, w\} \in E(G)$, one has $\{u, v\}, \{v, w\} \in E(G)$.

Remark 2.2. Let σ be a vertex ordering of a graph G and let

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_n$$

be the ordering of its vertices. σ is a *proper interval ordering* if for every triple v_i, v_j, v_k of vertices of G, with $i < j < k$ and $\{v_i, v_k\} \in E(G)$, one has $\{v_i, v_j\}, \{v_j, v_k\} \in E(G)$.

The vertex ordering above defined gives an important characterization of the so-called *proper interval graphs*.

Definition 2.3. A graph G is an *interval graph* if to each vertex $v \in V(G)$ it is possible to associate a closed interval $I_v = [\ell_v, r_v]$ of the real line such that two distinct vertices $u, v \in V(G)$ are adjacent if and only if $I_u \cap I_v \neq \emptyset$.

The family $\{I_v\}_{v \in V(G)}$ is *an interval representation* of G.

Definition 2.4. A graph G is a *proper interval graph* if there is an interval representation of G in which no interval properly contains other intervals.

Theorem 2.5. [13] *A graph G is a proper interval graph if and only if G has a proper interval ordering.*

Proper interval graphs are strictly related to *closed graphs*, introduced by Herzog et al. in [12]. Recently, in [4], the authors have shown that closed graphs and proper interval graphs are synonyms via a graph isomorphism involving some technical results contained in [9]. Here, we show that the above connection between these classes of graphs can be obtained by vertex orderings.

Definition 2.6. Let G be a graph on the vertex set $[n]$, G is *closed with respect to the given labeling*, if the following condition is satisfied:
for all $\{i,j\},\{k,\ell\} \in E(G)$ with $i<j$ and $k<\ell$, one has $\{j,\ell\} \in E(G)$ if $i=k$ but $j \neq \ell$, and $\{i,k\} \in E(G)$ if $j=\ell$, but $i \neq k$.

G is a *closed* graph if there exists a labeling for which is closed.

Definition 2.7. Let σ be a vertex ordering of a graph G and let

$$v_1 <_\sigma v_2 <_\sigma \cdots <_\sigma v_n,$$

be the ordering of its vertices. Ordering σ is called *closed* if for all edges $\{v_i,v_j\}$ and $\{v_k,v_\ell\}$ with $i<j$ and $k<\ell$, one has $\{v_j,v_\ell\} \in E(G)$ if $i=k$, but $j \neq \ell$ and $\{v_i,v_k\} \in E(G)$ if $j=\ell$, but $i \neq k$.

Hence, one immediately obtains the following characterization of a closed graph by a closed ordering (see also [2]).

Proposition 2.8. *A graph G is a closed graph if and only if G has a closed ordering.*

The above proposition yields to the following corollary.

Corollary 2.9. *Let G be a graph. The following statements are equivalent:*

1. *G is a closed graph;*
2. *G has a closed ordering;*
3. *G has a proper interval ordering;*
4. *G is a proper interval graph.*

Proof. 1. $\Longleftrightarrow$ 2. follows from Theorem 2.5;
2. $\Longleftrightarrow$ 3. can be found in [14, Proposition 1.8];
3. $\Longleftrightarrow$ 4. follows from Proposition 2.8. □

Remark 2.10. In [4], the isomorphism between the class of closed graphs and the class of proper interval graphs has been proved by using the relevant characterization of Herzog et al. [9, Theorem 2.2] of a closed graph by its clique complex. In this paper, we recover the isomorphism only by vertex orderings. Moreover, one can see that the *characterization* of Herzog et al. is contained in the constructive proof of the Roberts characterization of a proper interval graph in [6, Theorem 1].

3 Closed graphs via proper interval graphs

In this Section, we discuss a recent result on closed graphs [2] via some properties of proper interval graphs.

In order to accomplish this task we need to recall some notions from the graph theory.

Let G be a graph. Denote by $\{u, v\}$ an indirected edge of G and by (u, v) a directed edge (arrow) of G. A graph G is called *mixed* if G has some directed edges (arrows) and some undirected edges such that if G contains the directed edge (x, y), then it contains neither the directed edge (y, x) nor the undirected edge $\{x, y\}$. The *inset*, respectively the *outset*, of a vertex v in a mixed graph G is the set of all vertices $u \in V(G)$ for which (u, v), respectively (v, u), is a directed edge of G. Note that the class of mixed graphs without directed edges is precisely the class of undirected graphs and the class of mixed graphs without undirected edges is precisely the class of oriented graphs. Let D be a mixed graph. If (x, y) is a directed edge of D, then we say that x dominates y and write $x \to y$. A graph D is an *orientation* of an undirected graph G if D is obtained from G by orienting each edge $\{u, v\} \in E(G)$ as an arrow (u, v) or (v, u). A directed graph is called an *oriented* graph if it is the orientation of an undirected graph. A *straight enumeration* of an oriented graph D is a linear ordering $\{v_1, v_2, \dots, v_n\}$ of its vertices such that for each i there exist nonnegative integers h and k such that the vertex v has inset $\{v_{i-1}, v_{i-2}, \dots, v_{i-h}\}$ and outset $\{v_{i+1}, v_{i+2}, \dots, v_{i+k}\}$. An oriented graph which admits a straight enumeration is called *straight*. An undirected graph is said to have a *straight orientation* if it admits an orientation which is a straight oriented graph.

Let G be an undirected graph. For any vertex v, let $N(v)$ be the neighborhood of v, *i.e.*, the set of vertices which are adjacent to v. The closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. We define an equivalence relation on $V(G)$ in which a and b are equivalent just if $N[a] = N[b]$. If the vertices a and b of an edge $\{a, b\}$ are equivalent, we call the edge $\{a, b\}$ *balanced*; otherwise, $\{a, b\}$ is an unbalanced edge. We say that G is *reduced* if there are no balanced edges, *i.e.*, if distinct vertices have distinct closed neighborhoods. The underlying graph of a mixed graph D is the undirected graph $G(D)$ with the vertex set $V(D)$ in which $\{x, y\}$ is an edge of $G(D)$ only if it is a directed or undirected edge of D. We say that D is connected if $G(D)$ is connected. We say that D is reduced if $G(D)$ is reduced. A *straight mixed graph* H is a mixed graph obtained from a reduced straight oriented graph R by the substitution operation which replaces each vertex v of R by a complete graph T_v, with each vertex of T_v dominating each vertex of T_u if and only if $v \to u$ in R. Finally, the *full reversal* of an oriented graph D is the operation of reversing the directions of all oriented edges (arrow) of D. More details on this subject can be found in [5].

We quote the next results from [5].

Proposition 3.1. [5, Corollary 2.2] *A graph is a proper interval graph if and only if it has a straight orientation.*

Proposition 3.2. [5, Corollary 2.5, Proposition 4.2] *Let G be a connected proper interval graph.*

1. *G is uniquely orientable as a straight mixed graph up to full reversal.*
2. *If H is a straight mixed orientation of a connected subgraph of G, and $v \in V(G)$ but $v \notin V(H)$. Then the subgraph of G induced by v and the vertices of H is a proper interval graph.*

In [2], the authors have proved a new characterization of the class of closed graphs by a property that they have called *narrow*. Given vertices v, w of G satisfying $d(v, w) = \mathrm{diam}(G)$, a shortest path connecting v and w is called a *longest shortest path* of G.

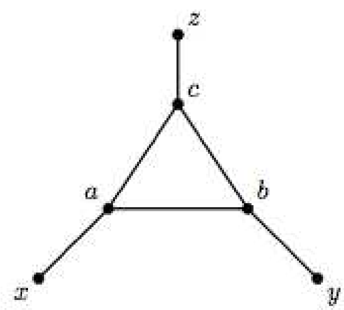
Definition 3.3. A graph G is *narrow* if for every $v \in V(G)$ and every longest shortest path P of G, either $v \in V(P)$ or there is $w \in V(P)$ with $\{v, w\} \in E(G)$.

In other words, a connected graph is narrow if every vertex is distance at most one from every longest shortest path.

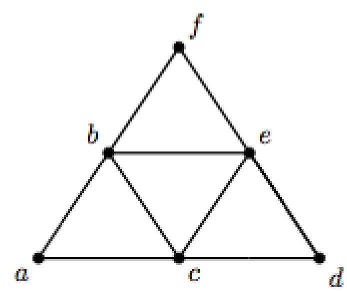
Theorem 3.4. [2, Corollary 1.5] *Let G be a graph. G is closed if and only if it is chordal, claw-free and narrow.*

Proposition 3.5. *A narrow graph G is both net-free and tent-free.*

Proof. In fact, if G contains a copy of the net



as an induced subgraph, then the narrowness fails since the vertex x is distance 2 from the longest shortest path z, c, b, y. If G contains a copy of the tent



as an induced subgraph, then the narrowness fails since the vertex a is distance 2 from the longest shortest path d, e, f.

□

The next result underlines once again the isomorphism between closed graphs and proper interval graphs.

Theorem 3.6. *Let G be a chordal claw-free graph. Then G is narrow if and only if it is net-free or tent-free.*

Proof. The necessary condition follows from Proposition 3.5. Conversely, suppose there exists a graph G which is chordal, claw-free, net-free, tent-free and not narrow. Assume also that G has the minimal numbers of vertices. Therefore, G is connected. Let v be a vertex of G such that $G - \{v\}$ remains connected. Since $G - \{v\}$ is chordal, claw-free, net-free, or tent-free, the minimality of $|V(G)|$ implies that $G - \{v\}$ is narrow and so $G - \{v\}$ is a closed graph (Theorem 3.4) or, equivalently, a proper interval graph. Hence, from Proposition 3.2, $G - \{v\}$ can be oriented as a straight mixed graph and G itself is orientable as a straight mixed graph. Therefore, from Proposition 3.1, G is a proper interval graph (*i.e.*, closed), and consequently a narrow graph. A contradiction. □

Remark 3.7. One can observe that combining Theorem 3.4 with Theorem 3.6, one gets that *a graph G is closed if and only if it is chordal, claw-free, net-free, or tent-free*, and this is one of the *classical* characterizations of a proper interval graph [7].

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