# ON MAGNETIC RELAXATION EQUATION FOR ANISOTROPIC REACTING FLUID MIXTURES 

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#### Abstract

In some previous papers a linear theory for magnetic relaxation phenomena in magnetizable continuous media was developed, that is based on thermodynamics of irreversible processes with internal variables. Here, we consider magnetizable reacting fluid mixtures, where irreversible microscopic phenomena give rise to magnetic relaxation, and these phenomena are described splitting the total specific magnetization in two irreversible parts and introducing one of these partial specific magnetizations as internal variable in the thermodynamic state vector. The phenomenological equations for these fluid mixtures are derived and, in the linear case, a generalized Snoek equation for magnetic relaxation phenomena is worked out and particular cases are treated. The obtained results have applications in several fields of applied sciences, as, for instance, in nuclear magnetic resonance and in medicine, where complex fluids are taken into consideration.


Keywords: travelling wave solutions.
2010 MSC: 35K57, 45K05,47A53, 47H11.

## 1. INTRODUCTION

In [1]-[7] magnetic relaxation phenomena in magnetizable media were studied using the standard methods of irreversible thermodynamics with internal variables [8]-[13]. In particular, in [1], in the linear approximation, Kluitenberg, assuming that magnetization $\mathbf{M}$ is given by the sum of one reversible part $\mathbf{M}^{(0)}$ and one irreversible part $\mathbf{M}^{(1)}$, derived, for magnetizable isotropic media, the following classical Snoek equation [14]

$$
\begin{equation*}
\chi_{(B M)}^{(0)} \mathbf{B}+\frac{d \mathbf{B}}{d t}=\chi_{(M B)}^{(0)} \mathbf{M}+\chi_{(M B)}^{(1)} \frac{d \mathbf{M}}{d t}, \tag{1}
\end{equation*}
$$

where $\chi_{(B M)}^{(0)}, \chi_{(M B)}^{(0)}$ and $\chi_{(M B)}^{(1)}$ are constant quantities, algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state. Subsequently, in [2], assuming that the total magnetization $\mathbf{M}$ is composed of two irreversible parts, i. e. $\mathbf{M}=\mathbf{M}^{(0)}+\mathbf{M}^{(1)}$, Kluitenberg obtained the following magnetic
relaxation equation,

$$
\begin{equation*}
\chi_{(B M)}^{(0)} \mathbf{B}+\frac{d \mathbf{B}}{d t}=\chi_{(M B)}^{(0)} \mathbf{M}+\chi_{(M B)}^{(1)} \frac{d \mathbf{M}}{d t}+\chi_{(M B)}^{(2)} \frac{d^{2} \mathbf{M}}{d t^{2}} \tag{2}
\end{equation*}
$$

where $\chi_{(B M)}^{(0)} \quad$ and $\quad \chi_{(M B)}^{(k)}(k=0,1,2)$ are constant quantities, algebraic functions of the coefficients occurring in the phenomenological equations and in the equations of state. In [5] one of the authors (LR) and Kluitengerg assumed that an arbitrary number $n$ of microscopic phenomena give rise to the magnetization axial vector $\mathbf{M}$ and that this axial vector can be split in $n+1$ parts, i.e.

$$
\begin{equation*}
\mathbf{M}=\mathbf{M}^{(0)}+\sum_{k=1}^{n} \mathbf{M}^{(k)} \tag{3}
\end{equation*}
$$

where $\mathbf{M}^{(0)}$ and $\mathbf{M}^{(\mathbf{k})}(k=1, \ldots, n)$ have irreversible character. In the isotropic case the following magnetic relaxation equation generalizing Snoek equation was obtained, having the form of a linear relation among the magnetic field $\mathbf{B}$, the first $n$ time derivatives of this field, the total magnetization $\mathbf{M}$ and the first $n+1$ time derivatives of $\mathbf{M}$

$$
\begin{gather*}
\chi_{(B M)}^{(0)} \mathbf{B}+\chi_{(B M)}^{(1)} \frac{d \mathbf{B}}{d t}+\ldots+\chi_{(B M)}^{(n-1)} \frac{d^{n-1} \mathbf{B}}{d t^{n-1}}+\frac{d^{n} \mathbf{B}}{d t^{n}}= \\
\chi_{(M B)}^{(0)} \mathbf{M}+\chi_{(M B)}^{(1)} \frac{d \mathbf{M}}{d t}+\ldots+\chi_{(M B)}^{(n)} \frac{d^{n} \mathbf{M}}{d t^{n}}+\chi_{(M B)}^{(n+1)} \frac{d^{n+1} \mathbf{M}}{d t^{n+1}} \tag{4}
\end{gather*}
$$

where $n$ is the number of phenomena that give rise to the magnetization $\mathbf{M}$ and $\chi_{(B M)}^{(k)}(k=0,1, \ldots, n-1) \quad$ and $\quad \chi_{(M B)}^{(k)}(k=0,1, \ldots, n+1)$ are constant quantities (see also [6], [7]).

Here, we consider anisotropic magnetizable reacting fluid mixtures, where irreversible microscopic phenomena give rise to magnetic relaxation, and these phenomena are described splitting the total magnetization in two irreversible parts and introducing one of these partial magnetizations as internal variable in the thermodynamic state vector (see [4] where reacting fluid mixtures magnetizable and polarizable were studied by the one of the authors (LR) and Kluitenberg). In Sections 2 and 3, the model of the considered media and the equations, that govern all the processes occurring inside them, are presented, and the entropy balance equation is derived. In Sections 4, 5 and 6 the phenomenological equations, the Onsager-Casimir relations and the linear laws of state, with respect to a considered reference state, are obtained. In Section 7, in the linear case, a generalized Snoek equation for magnetic relaxation phenomena is derived. Finally, in Section 8 particular cases are treated. The obtained results have applications in several fields of applied sciences, as, for instance, in medicine and biology, where complex fluids are taken into consideration, in which different types of molecules, having different magnetic susceptibilities and relaxation
times, present magnetic relaxation phenomena and contribute to the total magnetization (as an example such physical situations arise in nuclear magnetic resonance). In [15]-[19] Maugin gave a description of relaxation magnetic phenomena in continuous media with $n$ different ionic species, by means of microscopic considerations and introducing partial magnetizations per unit mass (see also [6], [7]). The continuum theory for magnetizable bodies developed by Maugin gives explanations of internal mechanisms in magnetizable bodies [20]-[24].

## 2. GOVERNING EQUATIONS

The standard Cartesian tensor notation in a rectangular coordinate system is used and the equations governing the behaviour of reacting fluid mixtures, consisting of $n$ chemical components, with magnetic relaxation are considered in a current configuration $\mathcal{K}_{t}$. The model for these media is developed in the framework of classical irreversible thermodynamics with internal variables. In the Galilean approximation all the processes occurring inside them are governed by two groups of laws: the balance equations and Maxwell equations.

The conservation of mass is expressed by the relation

$$
\begin{equation*}
\frac{\partial \varrho}{\partial t}=-\operatorname{div}(\varrho \vec{v}), \tag{5}
\end{equation*}
$$

where $\varrho$ is the total mass density defined by

$$
\begin{equation*}
\varrho=\sum_{k=1}^{n} \rho^{(k)}, \tag{6}
\end{equation*}
$$

where $\rho^{(k)}$ is the mass density of chemical component $k$ and $\vec{v}$ is the barycentric velocity defined by

$$
\begin{equation*}
\vec{v}=\frac{1}{\varrho} \sum_{k=1}^{n} \rho^{(k)} \vec{v}^{(k)} \tag{7}
\end{equation*}
$$

with $\vec{v}^{(k)}$ the velocity of component $k$.
By using (7), Eq. (5) takes the form

$$
\begin{equation*}
\frac{\partial \underline{\varrho}}{\partial t}=-d i v\left(\sum_{k=1}^{n} \rho^{(k)} \vec{v}^{(k)}\right) \tag{8}
\end{equation*}
$$

The mass fractions $c^{(k)}$ are defined by

$$
\begin{equation*}
c^{(k)}=\frac{\rho^{(k)}}{\varrho} \quad(k=1,2, \ldots, n) \tag{9}
\end{equation*}
$$

From this definition and (6) we obtain

$$
\begin{equation*}
\sum_{k=1}^{n} c^{(k)}=1 \tag{10}
\end{equation*}
$$

The diffusion flow of substance $k$ with respect to the barycentric motion is defined by

$$
\begin{equation*}
\vec{J}_{(d i f f)}^{k)}=\rho^{(k)}\left(\vec{v}^{(k)}-\vec{v}\right) \quad(k=1,2 \ldots, n) \tag{11}
\end{equation*}
$$

From this definition, by using Eqs. (6) and (7), we obtain

$$
\begin{equation*}
\sum_{k=1}^{n} \vec{J}_{(d i f f)}^{k}=\overrightarrow{0} \tag{12}
\end{equation*}
$$

which means that only $n-1$ of the $n$ diffusion flows are independent.
The balance equations for $c^{(k)}$ are given by (see [12])

$$
\begin{equation*}
\varrho \frac{d c^{(k)}}{d t}=-\operatorname{div} \vec{J}_{(d i f f)}^{(k)}+\sum_{h=1}^{r} v^{(k h)} J_{c h e m}^{(h)} \quad(k=1,2, \ldots, n), \tag{13}
\end{equation*}
$$

where $v^{(k h)} J_{c h e m}^{(h)}$ is the production of component $k$ per unit volume and per unit time by the $h$-th chemical reaction, the quantity $v^{(k h)}$ divided by the molecular mass $M^{(k)}$ of component $k$ is proportional to the stoichiometric coefficient with which the component $k$ appears in the chemical reaction $h$ and $J_{\text {chem }}^{(h)}$ is the chemical reaction rate of reaction $h$.

Maxwell's equations for the electric and magnetic fields read (in the rationalized Gauss system)

$$
\left\{\begin{array}{l}
\operatorname{rot} \vec{H}-\frac{1}{c} \frac{\partial \vec{E}}{\partial t}=\frac{1}{c} \vec{I},  \tag{14}\\
\operatorname{div} \vec{E}=\varrho^{(e l)}, \\
\operatorname{rot} \vec{E}+\frac{1}{c} \frac{\partial \vec{B}}{\partial t}=\overrightarrow{0}, \\
\operatorname{div} \vec{B}=0,
\end{array}\right.
$$

where $\varrho^{(e l)}$ is the total electric charge per unit volume (electric charge density), $\vec{I}$ is the density of the total electric current, $c$ is the velocity of light, $\vec{E}$ and $\vec{B}$ are the electric and magnetic field strengths, respectively, and $\vec{H}$ is the magnetic displacement field. We indicate by $e^{(k)}$ the charge per unit of mass of component $k$, the total charge $e$ per unit of mass of the system is given by

$$
\begin{equation*}
e=\frac{1}{\varrho} \sum_{k=1}^{n} \rho^{(k)} e^{(k)}=\sum_{k=1}^{n} c^{(k)} e^{(k)} \tag{15}
\end{equation*}
$$

where we have used Eq. (9).

In Eq. $(14)_{1} \vec{I}$ is defined by

$$
\begin{equation*}
\vec{I}=\sum_{k=1}^{n} \rho^{(k)} e^{(k)} \vec{v}^{(k)} \tag{16}
\end{equation*}
$$

and in Eq. $(14)_{2} \varrho^{(e l)}$ is given by

$$
\begin{equation*}
\varrho^{(e l)}=\sum_{k=1}^{n} \rho^{(k)} e^{(k)} \tag{17}
\end{equation*}
$$

and satisfies the charge conservation law

$$
\begin{equation*}
\frac{\partial \varrho^{(e l)} \partial t}{=}-\operatorname{div} \vec{I} \tag{18}
\end{equation*}
$$

By definition (11) and Eq. (17), Eq. (16) can be written in the form

$$
\begin{equation*}
\vec{I}=\varrho^{(e l)} \vec{v}+\sum_{k=1}^{n} e^{(k)} \vec{J}_{(d i f f)}^{k)} \tag{19}
\end{equation*}
$$

where on the right hand side the first contribution $\varrho^{(e l)} \vec{v}$ is the electric current convection, the second contribution is the electric current due to the relative motion of the various components, which is called conduction current $\vec{j}(e l)$, i. e.

$$
\begin{equation*}
\vec{j}^{(e l)}=\sum_{k=1}^{n} e^{(k)} \vec{J}_{(d i f f)}^{k k} \tag{20}
\end{equation*}
$$

Finally, we define the magnetization axial vector $\vec{M}$ by

$$
\begin{equation*}
\vec{M}=\vec{B}-\vec{H} \tag{21}
\end{equation*}
$$

and the specific magnetization axial vector $\vec{m}$ by

$$
\begin{equation*}
\vec{m}=\frac{1}{\varrho} \vec{M} \tag{22}
\end{equation*}
$$

The first law of thermodynamics for magnetizable fluid mixtures in an electromagnetic field, in Galilean approximation, reads [12]

$$
\begin{equation*}
\varrho \frac{d u}{d t}=-\operatorname{div} \vec{J}^{q)}+\tau_{\alpha \beta} \frac{d \varepsilon_{\alpha \beta}}{d t}+\vec{j}^{(e l)} \cdot \vec{E}+\varrho \vec{B} \cdot \frac{d \vec{m}}{d t} \tag{23}
\end{equation*}
$$

In (23) $u$ is the specific internal energy of the system, $\vec{J}^{q}$ ) is the heat flow density, $\tau_{\alpha \beta}$ is the symmetric mechanical stress tensor and $\varepsilon_{\alpha \beta}$ is the small strain tensor defined by

$$
\begin{equation*}
\varepsilon_{\alpha \beta}=\frac{1}{2}\left(u_{\alpha, \beta}+u_{\beta, \alpha}\right) \quad(\alpha, \beta=1,2,3) \tag{24}
\end{equation*}
$$

where $\vec{u}$ is the displacement field. Then, $\frac{d \varepsilon_{\alpha \beta}}{d t}$ is given by

$$
\begin{equation*}
\frac{d \varepsilon_{\alpha \beta}}{d t}=\frac{1}{2}\left(v_{\alpha, \beta}+v_{\beta, \alpha}\right) \quad(\alpha, \beta=1,2,3) . \tag{25}
\end{equation*}
$$

In Eq. (23) all the quantities are per unit of volume and per unit of time. On the right hand side the first term is the heat supply, the second term is the work done by mechanical stress, the third term is the Joule heat, $\varrho \vec{B} \cdot \frac{d \vec{m}}{d t}$ is the work done by the magnetic field to change the magnetization.

## 3. ENTROPY BALANCE EQUATION

We assume that the specific entropy $s$ (i.e. the entropy per unit of mass) for reacting fluid mixtures with magnetic relaxation is a function of the specific internal energy $u$, the strain tensor $\varepsilon_{\alpha \beta}$, the specific magnetization $\vec{m}$, the concentrations $c^{(k)}$ of $n$ components $(k=1, \ldots, n)$ and an axial vector field $\vec{m}^{(1)}$, which represents a thermodynamic internal variable giving rise to magnetic relaxation phenomena

$$
\begin{equation*}
s=s\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right) . \tag{26}
\end{equation*}
$$

Following the general philosophy of CIT (classical irreversible Thermodynamics) (see[8]-[13]) dissipative fluxes, gradients and time derivatives of the physical fields are not included in the state space and the local equilibrium hypothesis for the system is assumed (see[25]-[27]): out of the equilibrium each point of the medium is considered as a thermodynamic cell where the reversible thermodynamics is applicable.

We shall define the equilibrium temperature $T$, the equilibrium stress tensor $\tau_{\alpha \beta}^{(e q)}$, the equilibrium magnetic field $\vec{B}^{(e q)}$, the thermodynamic affinity $\vec{B}^{(1)}$, conjugate to the internal variable $\vec{m}^{(1)}$, and the thermodynamic or chemical potential $\mu^{(k)}$ of component $k$, respectively by

$$
\left\{\begin{array}{l}
T^{-1}=\frac{\partial}{\partial u} s\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right),  \tag{27}\\
\tau_{\alpha \beta}^{(e q)}=-\varrho T \frac{\partial}{\partial \varepsilon_{\alpha \beta}} s\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \\
\vec{B}^{(e q)}=-T \frac{\partial}{\partial \vec{m}} s\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \\
\vec{B}^{(1)}=T \frac{\partial}{\partial m^{(1)}} s\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right), \\
\mu^{(k)}=-T \frac{\partial}{\partial c^{(k)}} s\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right) \quad(k=1, \ldots, n) .
\end{array}\right.
$$

Considering very small deviations with respect to a thermodynamic equilibrium state, we expand the entropy (26) into Taylor's series with respect to this state, and confin-
ing our consideration to the linear terms, we obtain the differential of the entropy $s$ in the following form

$$
\begin{equation*}
T d s=d u-\frac{1}{\varrho} \tau_{\alpha \beta}^{(e q)} d \varepsilon_{\alpha \beta}-\vec{B}^{(e q)} \cdot d \vec{m}+\vec{B}^{(1)} \cdot d \vec{m}^{(1)}-\sum_{k=1}^{n} \mu^{(k)} d c^{(k)} \tag{28}
\end{equation*}
$$

where we have used Eqs. (27) $)_{1}-(27)_{5}$.
An expression of the type (28) is called Gibbs relation.
Let us introduce the definitions of viscous stress tensor $\tau_{\alpha \beta}^{(v i)}$ by

$$
\begin{equation*}
\tau_{\alpha \beta}^{(v i)}=\tau_{\alpha \beta}-\tau_{\alpha \beta}^{(e q)} \quad(\alpha, \beta=1,2,3) \tag{29}
\end{equation*}
$$

of irreversible magnetic field $B^{(i r)}$ by

$$
\begin{equation*}
\vec{B}^{(i r)}=\vec{B}-\vec{B}^{(e q)} \tag{30}
\end{equation*}
$$

and of entropy flow $\vec{J}^{(s)}$ by

$$
\begin{equation*}
\vec{J}^{s)}=\frac{1}{T}\left(\vec{J}^{q)}-\sum_{k=1}^{n} \mu^{(k)} \vec{J}_{(d i f f)}^{k)}\right) \tag{31}
\end{equation*}
$$

We multiply both sides of (13) by $\frac{\mu^{(k)}}{T}$ and sum over $k$. Introducing the so-called chemical affinities $A^{*(h)}$ of each chemical reaction $h$, by the definition

$$
\begin{equation*}
A^{*(h)}=\sum_{k=1}^{n} \mu^{(k)} v^{(k h)} \quad(h=1,2, \ldots, r), \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
A^{(h)}=-A^{*(h)} \quad(h=1,2, \ldots, r) \tag{33}
\end{equation*}
$$

we obtain

$$
\begin{align*}
& \frac{\varrho}{T} \sum_{k=1}^{n} \mu^{(k)} \frac{d c^{(k)}}{d t}=-\operatorname{div}\left(\frac{1}{T} \sum_{k=1}^{n} \mu^{(k)} \vec{J}_{(d i f f)}^{k)}\right)+ \\
& \sum_{k=1}^{n} \vec{J}_{(d i f f)}^{k)} \cdot \operatorname{grad}\left(\frac{\mu^{(k)}}{T}\right)+\frac{1}{T} \sum_{h=1}^{r} A^{*(h)} J_{(c h e m)}^{(h)} \tag{34}
\end{align*}
$$

From (28), deriving the time derivative of $s$ and using Eq. (23) and (31) - (34), we obtain

$$
\varrho \frac{d s}{d t}=-\operatorname{div} \vec{J}^{(s)}+\frac{1}{T}\left(-\frac{1}{T} \vec{J}^{(q)} \cdot \operatorname{gradT}+\tau_{\alpha \beta}^{(v i)} \frac{d \varepsilon_{\alpha \beta}}{d t}+\varrho \vec{B}^{(i r)} \cdot \frac{d \vec{m}}{d t}+\right.
$$

$$
\begin{equation*}
\left.+\varrho \vec{B}^{(1)} \cdot \frac{d \vec{m}^{(1)}}{d t}+\sum_{h=1}^{r} A^{(h)} J_{(c h e m)}^{(h)}\right)-\sum_{k=1}^{n} \vec{J}_{(d i f f)}^{(k)} \cdot \operatorname{grad}\left(\frac{\mu^{(k)}}{T}\right)+\frac{1}{T} \vec{j}^{(e l)} \cdot \vec{E} . \tag{35}
\end{equation*}
$$

Defining the thermodynamic force $\chi[\overrightarrow{2 p t}]^{(k)}$, conjugate to the diffusion flow of matter $\vec{J}_{(d i f f)}^{(k)}$, by

$$
\begin{equation*}
\chi[\overrightarrow{2 p t}]^{(k)}=-\left[T \operatorname{grad}\left(\frac{\mu^{(k)}}{T}\right)-e^{(k)} \vec{E}\right] \quad(k=1, \ldots, n) \tag{36}
\end{equation*}
$$

we obtain

$$
\begin{equation*}
-\sum_{k=1}^{n} \vec{J}_{(d i f f)}^{k k} \cdot \operatorname{grad}\left(\frac{\mu^{(k)}}{T}\right)+\frac{1}{T} \vec{j}^{(e l)} \cdot \vec{E}=\frac{1}{T} \sum_{k=1}^{n} \vec{J}_{(d i f f)}^{k)} \cdot \chi[\overrightarrow{2 p t}]^{(k)} \tag{37}
\end{equation*}
$$

where we have used Eq. (20). With the aid of (12) we obtain the following equality

$$
\begin{equation*}
\sum_{k=1}^{n} \vec{J}_{(d i f f)}^{(k)} \cdot \chi[\overrightarrow{2 p t}]^{(k)}=\sum_{k=1}^{n-1} \vec{J}_{(d i f f)}^{(k)} \cdot\left(\chi[\overrightarrow{2 p t}]^{(k)}-\chi[\overrightarrow{2 p t}]^{(n)}\right) \tag{38}
\end{equation*}
$$

Furthermore, defining the vector $\vec{X}^{(k)}$ by

$$
\begin{equation*}
\vec{X}^{(k)}=\chi[\overrightarrow{2 p t}]^{(k)}-\chi[\overrightarrow{2 p t}]^{(n)} \quad(k=1, \ldots, n-1), \tag{39}
\end{equation*}
$$

Eq. (38) becomes

$$
\begin{equation*}
\sum_{k=1}^{n} \vec{J}_{(d i f f)}^{k)} \cdot \chi[\overrightarrow{2 p t}]^{(k)}=\sum_{k=1}^{n-1} \vec{J}_{(d i f f)}^{k)} \cdot \vec{X}^{(k)} \tag{40}
\end{equation*}
$$

Using definition (20) of $\vec{j}^{(e l)}$ and Eqs. (34), (36), (40), it is easy to write Eq. (35) in the following form

$$
\begin{equation*}
\varrho \frac{d s}{d t}=-\operatorname{div} \vec{J}^{(s)}+\sigma^{(s)} \tag{41}
\end{equation*}
$$

where $\sigma^{(s)}$ is the entropy production per unit volume and per unit time, given by

$$
\begin{gather*}
\sigma^{(s)}=\frac{1}{T}\left(-\frac{1}{T} \vec{J}^{(q)} \cdot \operatorname{grad} T+\tau_{\alpha \beta}^{(v i)} \frac{d \varepsilon_{\alpha \beta}}{d t}+\varrho \vec{B}^{(i r)} \cdot \frac{d \vec{m}}{d t} t+\right. \\
\left.\varrho \vec{B}^{(1)} \cdot \frac{d \vec{m}^{(1)}}{d t}+\sum_{h=1}^{r} A^{(h)} J_{(c h e m)}^{(h)}+\sum_{k=1}^{n-1} \vec{J}_{(d i f f)}^{(k)} \vec{X}^{(k)}\right) \tag{42}
\end{gather*}
$$

## 4. PHENOMENOLOGICAL EQUATIONS

By virtue of the expression (42) for the entropy production and according to the usual procedure of non-equilibrium thermodynamics we obtain for magnetizable media the following phenomenological equations

$$
\begin{align*}
& B_{\alpha}^{(i r)}=\varrho L_{(M) \alpha \beta}^{(0,0)} \frac{d m_{\beta}}{d t}+L_{(M) \alpha \beta}^{(0,1)} B_{\beta}^{(1)}-\frac{1}{T} L_{(M) \alpha \beta}^{(0, q)} T_{, \beta}+ \\
& \sum_{k=1}^{n-1} L_{(M D) \alpha \beta}^{(0, k)} X_{\beta}^{(k)}+\sum_{h=1}^{r} L_{(M C) \alpha}^{(0, h)} A^{(h)}+L_{(M) \alpha \beta \gamma}^{(0, v i)} \frac{d \varepsilon_{\beta \gamma}}{d t},  \tag{43}\\
& \varrho \frac{d m_{\alpha}^{(1)}}{d t}=\varrho L_{(M) \alpha \beta}^{(1,0)} \frac{d m_{\beta}}{d t}+L_{(M) \alpha \beta}^{(1,1)} B_{\beta}^{(1)}-\frac{1}{T} L_{(M) \alpha \beta}^{(1, q)} T_{, \beta}+ \\
& \sum_{k=1}^{n-1} L_{(M D) \alpha \beta}^{(1, k)} X_{\beta}^{(k)}+\sum_{h=1}^{r} L_{(M C) \alpha}^{(1, h)} A^{(h)}+L_{(M) \alpha \beta \gamma}^{(1, v i)} \frac{d \varepsilon_{\beta \gamma}}{d t},  \tag{44}\\
& J_{(\alpha)}^{(q)}=\varrho L_{(M) \alpha \beta}^{(q, 0)} \frac{d m_{\beta}}{d t}+L_{(M) \alpha \beta}^{(q, 1)} B_{\beta}^{(1)}-\frac{1}{T} L_{\alpha \beta}^{(q, q)} T_{, \beta}+\sum_{k=1}^{n-1} L_{(D) \alpha \beta}^{(q, k)} X_{\beta}^{(k)} \\
& +\sum_{h=1}^{r} L_{(C) \alpha}^{(q, h)} A^{(h)}+L_{\alpha \beta \gamma}^{(q, v i)} \frac{d \varepsilon_{\beta \gamma}}{d t},  \tag{45}\\
& J_{(d i f f) \alpha}^{(j)}=\varrho L_{(D M) \alpha \beta}^{(j, 0)} \frac{d m_{\beta}}{d t}+L_{(D M) \alpha \beta}^{(j, 1)} B_{\beta}^{(1)}-\frac{1}{T} L_{(D) \alpha \beta}^{(j, q)} T_{, \beta}+ \\
& \sum_{k=1}^{n-1} L_{(D D) \alpha \beta}^{(j, k)} X_{\beta}^{(k)}+\sum_{h=1}^{r} L_{(D C) \alpha}^{(j, h)} A^{(h)}+L_{(D) \alpha \beta \gamma}^{(j, v i)} \frac{d \varepsilon_{\beta \gamma}}{d t}  \tag{46}\\
& (j=1,2, \ldots, n-1), \\
& J_{(\text {chem })}^{(l)}=\varrho L_{(C M) \beta}^{(l, 0)} \frac{d m_{\beta}}{d t}+L_{(C M) \beta}^{(l, 1)} B_{\beta}^{(1)}-\frac{1}{T} L_{(C) \beta}^{(l, q)} T_{, \beta}+ \\
& \sum_{k=1}^{n-1} L_{(C D) \beta}^{(l, k)} X_{\beta}^{(k)}+\sum_{h=1}^{r} L_{(C C)}^{(l, h)} A^{(h)}+L_{(C) \beta \gamma}^{(l, v i)} \frac{d \varepsilon_{\beta \gamma}}{d t}  \tag{47}\\
& (l=1,2, \ldots, r), \\
& \tau_{\alpha \beta}^{(v i)}=\varrho L_{(M) \alpha \beta \gamma}^{(v i, 0)} \frac{d m_{\gamma}}{d t}+L_{(M) \alpha \beta \gamma}^{(v i, 1)} B_{\gamma}^{(1)}-\frac{1}{T} L_{\alpha \beta \gamma}^{(v i, q)} T_{, \gamma}+ \\
& \sum_{k=1}^{n-1} L_{(D) \alpha \beta \gamma}^{(v i, k)} X_{\gamma}^{(k)}+\sum_{h=1}^{r} L_{(C) \alpha \beta}^{(v i, h)} A^{(h)}+L_{\alpha \beta \gamma \mathrm{d}}^{(v i, v i)} \frac{d \varepsilon_{\gamma \mathrm{d}}}{d t} . \tag{48}
\end{align*}
$$

Eqs. (43) and (44) are connected with irreversible changes in the magnetization. Moreover, the phenomenological equations (45)-(47) are connected with the irreversible processes of heat flow, diffusion flow and chemical reactions. Eq. (48) is a generalization of Newton's law for viscous fluid flow. The quantities $L_{(M) \alpha \beta}^{(0,0)}, L_{(M) \alpha \beta}^{(0,1)}$, $L_{(M) \alpha \beta}^{(0, q)}, \ldots$ which occurr in (43)-(48) are called phenomenological tensors. For instance, $L_{\alpha \beta}^{(q, q)}$ is the heat conductivity polar tensor of order two, $L_{\alpha \beta \gamma \mathrm{d}}^{(v i, v i)}$ is the viscosity polar tensor of order four, $L_{(D D) \alpha \beta}^{(j, k)}(j, k=1,2, \ldots, n-1)$ is a polar tensor of order two connected with the diffusion flow of substance $k, L_{(C C)}^{(l, h)}$ is a scalar connected with the chemical affinity of the reaction $h(l, h=1,2, \ldots, r), L_{(M) \alpha \beta \gamma}^{(v i, 1)}$ is a pseudotensor of order three connected with the influence of the viscous flow on the magnetic relaxation. In principle, all irreversible phenomena described by (43)-(48) can influence each other. For instance, the third, fourth, fifth and sixth term on the right-hand sides of (43) and (44) describe the influences of heat flow, diffusion flow, chemical reactions and viscous flow on magnetic relaxation. Phenomena of this type are called cross effects. The strain tensor $\varepsilon_{\alpha \beta}$ is symmetric, then from (27) 2 and (29) the equilibrium stress tensor $\tau_{\alpha \beta}^{(e q)}$ and the viscous stress tensor $\tau_{\alpha \beta}^{(v i)}$ are also symmetric tensors. Because of the symmetry of $\varepsilon_{\alpha \beta}$ and $\tau_{\alpha \beta}^{(v i)}$ one has

$$
\begin{align*}
& L_{(M) \alpha \beta \gamma}^{(1, v i)}=L_{(M) \alpha \gamma \beta}^{(1, v i)}, \quad L_{(M) \alpha \beta \gamma}^{(v i, 1)}=L_{(M) \beta \alpha \gamma}^{(v i, 1)}, \\
& L_{(M) \alpha \beta \gamma}^{(0, v i)}=L_{(M) \alpha \gamma \beta}^{(0, v i)}, \quad L_{(M) \alpha \beta \gamma}^{(v i, 0)}=L_{(M) \beta \alpha \gamma}^{(v i, 0)}, \\
& L_{\alpha \beta \gamma \gamma}^{(q, v i)}=L_{\alpha \gamma \beta}^{(q, v i)}, \quad L_{\alpha \beta \gamma}^{(v i, q)}=L_{\beta \alpha \gamma}^{(v i, q)}, \\
& L_{(D) \alpha \beta \gamma}^{(j, v i)}=L_{(D) \alpha \gamma \beta}^{(j, v i)}, \quad L_{(D) \alpha \beta \gamma}^{(v i, k)}=L_{(D) \beta \alpha \gamma}^{(v i, k)}(j, k=1,2, \ldots, \ldots, n-1),  \tag{49}\\
& L_{(C) \alpha \beta}^{(l, v i)}=L_{(C) \beta \alpha}^{(l, v i)}, \quad L_{(C) \alpha \beta}^{(v i, h)}=L_{(C) \beta \alpha}^{(v i, h)}(l, h=1,2, \ldots, \ldots, r), \\
& L_{\alpha \beta \gamma \mathrm{d}}^{(v i, v i)}=L_{\alpha \beta \mathrm{d} \gamma}^{(v i v i)}=L_{\beta \alpha \gamma \mathrm{d}}^{(v i, v i)}=L_{\beta \alpha \mathrm{d} \gamma}^{(v i, v i)} .
\end{align*}
$$

Furthermore, $\varrho \frac{d \vec{m}}{d t}, \varrho \frac{d \vec{m}^{(1)}}{d t}, A^{(h)}(h=1,2, \ldots, r), T^{-1} \operatorname{grad} T, \vec{X}^{(k)}(k=1,2, \ldots, n-1)$ and $\tau_{\alpha \beta}^{(v i)}$ are even functions of the microscopic particle velocities, while $\vec{B}^{(i r)}, \vec{B}^{(1)}$, $\vec{J}^{q)}, \quad \vec{J}_{(d i f f)}^{(k)}, \quad(k=1,2, \ldots, n-1), \quad J^{(l)}, \quad(l=1,2, \ldots, r), \frac{d \varepsilon_{\alpha \beta}}{d t}$ are odd functions of these velocities. Hence, according to the usual procedure of non-equilibrium thermodynamics, we have for the phenomenological tensors, which occur in (43)-(48), the following Onsager-Casimir reciprocity relations

$$
L_{(M) \alpha \beta}^{(0,0)}=L_{(M) \beta \alpha}^{(0,0)}, \quad L_{(M) \alpha \beta}^{(1,1)}=L_{(M) \beta \alpha}^{(1,1)}, \quad L_{\alpha \beta}^{(q, q)}=L_{\beta \alpha}^{(q, q)}
$$

$$
\begin{gather*}
L_{(M) \alpha \beta}^{(0,1)}=-L_{(M) \beta \alpha}^{(1,0)}, \quad L_{(M) \alpha \beta}^{(0, q)}=L_{(M) \beta \alpha}^{(q, 0)}, \\
L_{(M D) \alpha \beta}^{(0, k)}=L_{(D M) \beta \alpha}^{(k, 0)}, \quad L_{(D D) \alpha \beta}^{(j, k)}=L_{(D D) \beta \alpha}^{(k, j)} \quad(j, k=1,2, \ldots, n-1), \\
L_{(M C) \alpha}^{(0, h)}=L_{(C M) \beta \alpha}^{(h, 0)}, \quad L_{(D C) \alpha}^{(j, h)}=L_{(C D) \alpha}^{(h, j)}(j=1,2, \ldots, n-1),(h=1,2, \ldots, r), \\
L_{(M D) \alpha \beta}^{(1, k)}=-L_{(M D) \alpha \beta}^{(k, 1)}, \quad L_{(D) \alpha \beta \gamma}^{(k, v i)}=-L_{(D) \beta \gamma \alpha}^{(v i, k)}(k=1,2, \ldots, n-1), \\
L_{(M) \alpha \beta}^{(1, q)}=-L_{(M) \beta \alpha}^{(q, 1)}, \quad L_{(D) \alpha \beta}^{(q, k)}=L_{(D) \beta \alpha}^{(q, k)}(k=1,2, \ldots, n-1) \\
L_{(M C) \alpha}^{(1, h)}=-L_{(C M) \alpha}^{(h, 1)}, \quad L_{(C C)}^{(l, h)}=L_{(C C)}^{(h, l)}(l, h=1,2, \ldots, r) \\
L_{(C) \alpha}^{(q, h)}=L_{(C) \alpha}^{(h, q)}, \quad L_{(C) \alpha \beta}^{(h, v i)}=-L_{(C) \beta \alpha}^{(v i, h)}(h=1,2, \ldots, r), \\
L_{(M) \alpha \beta \gamma}^{(0, v i)}=-L_{(M) \beta \gamma \alpha}^{(v i, 0)}, \quad L_{(M) \alpha \beta \gamma}^{(1, v i)}=L_{(M) \beta \gamma \alpha}^{(v i, 1)}, \\
L_{\alpha \beta \gamma}^{(q, v i)}=-L_{\beta \gamma \alpha}^{(v i, q)}, \quad L_{\alpha \beta \gamma \delta}^{(v i, v i)}=L_{\gamma \delta \alpha \beta}^{(v i v i)} . \tag{50}
\end{gather*}
$$

## 5. REFERENCE STATE AND <br> THERMODYNAMIC EQUILIBRIUM STATE

Now, let us consider a reference state of the medium, with an arbitrary (but fixed) uniform temperature $T_{(0)}$, in which the concentrations $c^{(k)}(k=1,2, \ldots, n)$ of the components of the fluid mixtures assume the fixed values $c_{(0)}^{(k)}$ and the mechanical stress tensor $\tau_{\alpha \beta}$ and the magnetic field $\mathbf{B}$. We also require that this reference state (indicated by the symbol "(0)") is a state of thermodynamic equilibrium. We notice that $\tau_{\alpha \beta}^{(e q)}, \mathbf{B}^{(e q)}$ and $\mathbf{B}^{(1)}$ are functions of the temperature $T_{(0)}$, of the strain tensor $\epsilon_{\alpha \beta}$, the magnetizations $\mathbf{m}$ and $\mathbf{m}^{(1)}$ and the concentrations $c_{(0)}^{(k)}$. We require that in this state the value $\epsilon_{(0) \alpha \beta}$ for the strain tensor and the values $\mathbf{m}_{(0)}$ and $\mathbf{m}_{(0)}^{(1)}$ for the magnetization axial vectors are such that

$$
\begin{gather*}
\tau_{\alpha \beta}^{(e q)}\left(T_{(0)}, \epsilon_{(0) \alpha \beta}, \mathbf{m}_{(0)}, \mathbf{m}_{(0)}^{(1)}, c_{(0)}^{(1)}, \ldots, c_{(0)}^{(n)}\right)=0,  \tag{51}\\
\mathbf{B}^{(e q)}\left(T_{(0)}, \epsilon_{(0) \alpha \beta}, \mathbf{m}_{(0)}, \mathbf{m}_{(0)}^{(1)}, c_{(0)}^{(1)}, \ldots, c_{(0)}^{(n)}\right)=0,  \tag{52}\\
\mathbf{B}^{(1)}\left(T_{(0)}, \epsilon_{(0) \alpha \beta}, \mathbf{m}_{(0)}, \mathbf{m}_{(0)}^{(1)}, c_{(0)}^{(1)}, \ldots, c_{(0)}^{(n)}\right)=0,  \tag{53}\\
\mu^{(k)}\left(T_{(0)}, \epsilon_{(0) \alpha \beta}, \mathbf{m}_{(0)}, \mathbf{m}_{(0)}^{(1)}, c_{(0)}^{(1)}, \ldots, c_{(0)}^{(n)}\right)=0 \\
(k=1, \ldots, n) . \tag{54}
\end{gather*}
$$

Being the tensor $\tau_{\alpha \beta}^{(e q)}$ symmetric, Eqs. (51)-(54) form a set of $12+n$ equations for the values of the 6 independent components of the symmetric strain tensor $\epsilon_{(0) \alpha \beta}$ and the values of the 6 components of the vectors $\mathbf{m}_{(0)}$ and $\mathbf{m}_{(0)}^{(1)}$. We choose the
tensor $\epsilon_{\alpha \beta}$ and the axial vectors $\mathbf{m}, \mathbf{m}^{(1)}$, so that they vanish in the reference state. Thus, $\epsilon_{(0) \alpha \beta}=m_{(0) \alpha}=m_{(0) \alpha}^{(1)}=0$. Furthermore, we choose

$$
\begin{gather*}
\tau_{\alpha \beta}^{(e q)}=0, \quad \mathbf{B}^{(e q)}=0, \quad \mathbf{B}^{(1)}=0, \quad \mu^{(k)}=0 \quad(k=1,2, \ldots, n), \\
\text { for } \quad T=T_{(0)}, \quad \epsilon_{(0) \alpha \beta}=m_{(0) \alpha}=m_{(0) \alpha}^{(1)}=0, \quad c^{(k)}=c_{(0)}^{(k)}(k=1,2, \ldots, n) . \tag{55}
\end{gather*}
$$

A medium is in a state of thermodynamic equilibrium if the entropy production (42) vanishes. It follows that the reference state is a state of thermodynamic equilibrium, provided that $\epsilon_{\alpha \beta}$ and the vectors $\mathbf{m}$ and $\mathbf{m}^{(1)}$ (determined by (51)-(53)) are kept constant. Moreover, the electric field $\mathbf{E}$ must be kept vanishing in this state of thermodynamic equilibrium (see 36), where we assume that there are not chemical reactions. We note that in the reference state the medium has the uniform temperature $T_{(0)}$, and hence $\operatorname{grad} T$ vanishes in this state. Moreover, by virtue of phenomenological equation (48), the viscous stress tensor $\tau_{\alpha \beta}^{(v i)}$ vanishes in the thermodynamic equilibrium and it follows that

$$
\begin{equation*}
\tau_{\alpha \beta}=\tau_{\alpha \beta}^{(e q)} \tag{56}
\end{equation*}
$$

## 6. LINEAR EQUATIONS OF STATE FOR <br> ANISOTROPIC MAGNETIZABLE REACTING FLUID MIXTURES

Let us define the specific free energy $f$ by $f=u-T s$. Using Gibbs relation (28), we obtain the following expression for the differential of $f$,

$$
\begin{equation*}
d f=-s d T+v \tau_{\alpha \beta}^{(e q)} d \epsilon_{\alpha \beta}+\mathbf{B}^{(e q)} \cdot d \mathbf{m}-\mathbf{B}^{(1)} \cdot d \mathbf{m}^{(1)}+\sum_{k=1}^{n} \mu^{(k)} d c^{(k)} \tag{57}
\end{equation*}
$$

where $v=\frac{1}{\rho}$ is the specific volume. Therefore, the following definitions are valid

$$
\begin{gather*}
s=-\frac{\partial}{\partial T} f\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right),  \tag{58}\\
\tau_{\alpha \beta}^{(e q)}=\rho \frac{\partial}{\partial \epsilon_{\alpha \beta}} f\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right),  \tag{59}\\
\mathbf{B}^{(e q)}=\frac{\partial}{\partial \mathbf{m}} f\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right),  \tag{60}\\
\mathbf{B}^{(1)}=-\frac{\partial}{\partial \mathbf{m}^{(1)}} f\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right),  \tag{61}\\
\mu^{(k)}=\frac{\partial}{\partial c^{(k)}} f\left(u, \varepsilon_{\alpha \beta}, \vec{m}, \vec{m}^{(1)}, c^{(1)}, \ldots, c^{(n)}\right) \\
(k=1,2, \ldots, n) . \tag{62}
\end{gather*}
$$

Now, we expand the free energy $f$ into Taylor's series with respect to the considered reference state, considered in Section 5, and we consider very small deviations with respect to this state, confining our consideration to the quadratic terms. We postulate the following form for the specific free energy $f$ of anisotropic reacting fluid mixtures

$$
\begin{equation*}
f=f^{(1)}+f^{(2)} \tag{63}
\end{equation*}
$$

where

$$
\begin{gather*}
f^{(1)}=v_{(0)}\left\{\frac{1}{2} a_{\alpha \beta \gamma \zeta} \epsilon_{\alpha \beta} \epsilon_{\gamma \zeta}+a_{\alpha \beta} \epsilon_{\alpha \beta}\left(T-T_{(0)}\right)+\right. \\
\sum_{k=1}^{n} b^{(k)}\left(c^{(k)}-c_{(0)}^{(k)}\right)\left(T-T_{0}\right)+\sum_{k=1}^{n} b_{\alpha \beta}^{(k)}\left(c^{(k)}-c_{(0)}^{(k)}\right) \varepsilon_{\alpha \beta} \\
\left.+\frac{1}{2} \sum_{i, k=1}^{n} b^{(i, k)}\left(\mathcal{J}-\mathcal{J}_{(0)}\right)\left(c^{(k)}-c_{(0)}^{(k)}\right)\right\}-\varphi(T) \tag{64}
\end{gather*}
$$

and

$$
\begin{gather*}
f^{(2)}=\frac{1}{2} \rho_{(0)}\left\{a_{(M) \alpha \beta}^{(0,0)} m_{\alpha}\left(m_{\beta}-2 m_{\beta}^{(1)}\right)+a_{(M) \alpha \beta}^{(1,1)} m_{\alpha}^{(1)} m_{\beta}^{(1)}\right\}+ \\
\left(T-T_{(0)}\right)\left(a_{(M) \alpha}^{(0)} m_{\alpha}-a_{(M) \alpha}^{(1)} m_{\alpha}^{(1)}\right)+\sum_{k=1}^{n}\left(c^{(k)}-c_{(0)}^{(k)}\right)\left(b_{(M) \alpha}^{(0, k)} m_{\alpha}+b_{(M) \alpha}^{(1, k)} m_{\alpha}^{(1)}\right) \tag{65}
\end{gather*}
$$

In (64) $v_{(0)}$ is the specific volume in the reference state, given by $v_{(0)}=\frac{1}{\rho_{(0)}}$. In the following, we shall replace it by $v=\frac{1}{\rho}$, which is supposed to be a constant. $\varphi(T)$ is some function of the temperature, $b^{(k)}$ is constant and also $a_{\alpha \beta \gamma \zeta}, a_{\alpha \beta}, b_{\alpha \beta}^{(k)}, b^{(i, k)}$ are constant and they satisfy the following symmetry relations

$$
\begin{gather*}
a_{\alpha \beta \gamma \zeta}=a_{\beta \alpha \gamma \zeta}=a_{\alpha \beta \zeta \gamma}=a_{\beta \alpha \zeta \gamma}=a_{\gamma \zeta \alpha \beta}=a_{\gamma \zeta \beta \alpha}=a_{\zeta \gamma \alpha \beta}=a_{\zeta \gamma \beta \alpha}  \tag{66}\\
a_{\alpha \beta}=a_{\beta \alpha}, \quad b_{\alpha \beta}^{(k)}=b_{\beta \alpha}^{(k)}, \quad b^{(i, k)}=b^{(k, i)}(i, k=1,2, \ldots, n)
\end{gather*}
$$

In (65) the vectors $a_{(M) \alpha}^{(0)}, a_{(M) \alpha}^{(1)}, b_{(M) \alpha}^{(0, k)}$ and $b_{(M) \alpha}^{(1, k)}(k=1,2, \ldots, n)$ are constant and also the tensors $a_{(M) \alpha \beta}^{(0,0)}, \quad a_{(M) \alpha \beta}^{(1,1)}$ are constant and they satisfy the following symmetry relations

$$
\begin{equation*}
a_{(M) \alpha \beta}^{(0,0)}=a_{(M) \beta \alpha}^{(0,0)}, \quad a_{(M) \alpha \beta}^{(1,1)}=a_{(M) \beta \alpha}^{(1,1)} . \tag{67}
\end{equation*}
$$

Furthermore, these constants are determined by the physical properties of the medium in the reference state. We have the following form for the specific entropy

$$
\begin{equation*}
s=-\left\{a_{(M) \alpha}^{(0)} m_{\alpha}-a_{(M) \alpha}^{(1)} m_{\alpha}^{(1)}+\sum_{k=1}^{n} b^{(k)}\left(c^{(k)}-c_{(0)}^{(k)}\right)\right\}-v a_{\alpha \beta} \epsilon_{\alpha \beta}+\frac{d \varphi}{d t} \tag{68}
\end{equation*}
$$

We obtain for the equilibrium stress tensor the form

$$
\begin{equation*}
\tau_{\alpha \beta}^{(e q)}=a_{\alpha \beta \gamma \zeta} \epsilon_{\gamma \zeta}+a_{\alpha \beta}\left(T-T_{(0)}\right)+\sum_{k=1}^{n} b_{\alpha \beta}^{(k)}\left(c^{(k)}-c_{(0)}^{(k)}\right) \tag{69}
\end{equation*}
$$

Now, we define the fields $\mathbf{M}^{(0)}$ and $\mathbf{M}^{(1)}$ by

$$
\begin{equation*}
\mathbf{M}^{(0)}=\rho \mathbf{m}^{(0)} \quad \text { and } \quad \mathbf{M}^{(1)}=\rho \mathbf{m}^{(1)} . \tag{70}
\end{equation*}
$$

Finally, from (60), (61) and (63)-(67) we have the following equations of state

$$
\begin{gather*}
B_{\alpha}^{(e q)}=a_{(M) \alpha \beta}^{(0,0)}\left(M_{\beta}-M_{\beta}^{(1)}\right)+a_{(M) \alpha}^{(0)}\left(T-T_{(0)}\right)+\sum_{k=1}^{n} b_{(M) \alpha}^{(0, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right),  \tag{71}\\
B_{\alpha}^{(1)}=a_{(M) \alpha \beta}^{(0,0)} M_{\beta}-a_{(M) \alpha \beta}^{(1,1)} M_{\beta}^{(1)}+a_{(M) \alpha}^{(1)}\left(T-T_{(0)}\right)+\sum_{k=1}^{n} b_{(M) \alpha}^{(1, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right),  \tag{72}\\
\mu^{(k)}=v\left\{b^{(k)}\left(T-T_{0}\right)+b_{\alpha \beta}^{(k)} \varepsilon_{\alpha \beta}+\sum_{i=1}^{n} b^{(i, k)}\left(\mathcal{J}-\mathcal{J}_{(0)}\right)+b_{(M) \alpha}^{(0, k)} M_{\alpha}-b_{(M) \alpha}^{(1, k)} M_{\alpha}^{(1)}\right\} . \tag{73}
\end{gather*}
$$

If all cross effects are neglected, except for possible cross effects among the different types of magnetic relaxation phenomena, we obtain the following equations for the irreversible magnetic relaxation phenomena, the stress tensor, the heat flux, the diffusion flows and chemical reactions, respectively (see Eqs. (43)-(48), (29), (30)and (69)-(70)),

$$
\begin{gather*}
B_{\alpha}=B_{\alpha}^{(e q)}+L_{(M) \alpha \beta}^{(0,0)} \frac{d M_{\beta}}{d t}+L_{(M) \alpha \beta}^{(0,1)} B_{\beta}^{(1)},  \tag{74}\\
\frac{d M_{\alpha}^{(1)}}{d t}=L_{(M) \alpha \beta}^{(1,0)} \frac{d M_{\beta}}{d t}+L_{(M) \alpha \beta}^{(1,1)} B_{\beta}^{(1)},  \tag{75}\\
\tau_{\alpha \beta}=a_{\alpha \beta \gamma \zeta} \epsilon_{\gamma \zeta}+a_{\alpha \beta}\left(T-T_{(0)}\right)+\sum_{k=1}^{n} b_{\alpha \beta}^{(k)}\left(c^{(k)}-c_{(0)}^{(k)}\right)+L_{\alpha \beta \gamma \zeta}^{(v i, v i)} \frac{d \epsilon_{\gamma \zeta}}{d t},  \tag{76}\\
J_{\alpha}^{(q)}=-\frac{1}{T} L_{\alpha \beta}^{(q, q)} T_{, \beta}, \quad J_{(d i f f) \alpha}^{(j)}=\sum_{k=1}^{n-1} L_{(D D) \alpha \beta}^{(j, k)} X_{\beta}^{(k)}(j, k=1,2, \ldots, n-1), \\
J_{(c h e m)}^{(l)}=\sum_{h=1}^{r} L_{(C C)}^{(l, h)} A^{(h)}(l=1,2, \ldots, r), \tag{77}
\end{gather*}
$$

where we have to take into consideration Eqs. (32), (33), (36), (39)-(40) and (73).

## 7. GENERALIZED SNOEK EQUATION FOR ANISOTROPIC MAGNETIZABLE REACTING FLUID MIXTURES

Taking into account (71), (72), Eqs. (74) and (75) may be written, respectively, in the form

$$
\begin{equation*}
c_{\alpha \beta}^{(1)} M_{\beta}^{(1)}=Q_{(0,0) \alpha}^{(1)}, \tag{78}
\end{equation*}
$$

where

$$
\begin{gather*}
c_{\alpha \beta}^{(1)}=a_{(M) \alpha \beta}^{(0,0)}+L_{(M)_{\alpha \gamma}}^{(0,1)} a_{(M) \gamma \beta}^{(1,1)},  \tag{79}\\
Q_{(0,0) \alpha}^{(1)}=\left(a_{(M) \alpha \beta}^{(0,0)}+L_{(M) \alpha \gamma}^{(0,1)} a_{(M) \gamma \beta}^{(0,0)}\right) M_{\beta}+L_{(M) \alpha \beta}^{(0,0)} \frac{d M_{\beta}}{d t}-B_{\alpha}+\left(a_{(M) \alpha}^{(0)}+\right. \\
\left.L_{(M) \alpha \beta}^{(0,1)} a_{(M) \beta}^{(1)}\right)\left(T-T_{(0)}\right)+\sum_{k=1}^{n}\left(b_{(M) \alpha}^{(0, k)}+L_{(M) \alpha \beta}^{(0,1)} b_{(M) \beta}^{(1, k)}\right)\left(c^{(k)}-c_{(0)}^{(k)}\right), \tag{80}
\end{gather*}
$$

and

$$
\begin{equation*}
\frac{d M_{\beta}^{(1)}}{d t}+h_{\beta \gamma} M_{\gamma}^{(1)}=Q_{(1,0) \beta} \tag{81}
\end{equation*}
$$

where

$$
\begin{equation*}
h_{\beta \gamma}=L_{(M) \beta \eta}^{(1,1)} a_{(M) \eta \gamma}^{(1,1)} \tag{82}
\end{equation*}
$$

and

$$
\begin{gather*}
Q_{(1,0) \beta}=L_{(M) \beta \eta}^{(1,1)} a_{(M) \eta \gamma}^{(0,0)} M_{\gamma}+L_{(M) \beta \gamma}^{(1,0)} \frac{d M_{\gamma}}{d t}+ \\
L_{(M) \beta \gamma}^{(1,1)} a_{(M) \gamma}^{(1)}\left(T-T_{(0)}\right)+\sum_{k=1}^{n} L_{(M) \beta \gamma}^{(1,1)} b_{(M) \gamma}^{(1, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right) . \tag{83}
\end{gather*}
$$

If the coefficients in (78) and (79) are constant, it follows from (78) that

$$
\begin{equation*}
c_{\alpha \beta}^{(1)} \frac{d M_{\beta}^{(1)}}{d t}=\frac{d Q_{(0,0) \alpha}^{(1)}}{d t}, \tag{84}
\end{equation*}
$$

provided that all derivatives exist in (84).
Multiplying both sides of Eq. (81) by $\left(c_{\alpha \beta}^{(1)}\right)$ and summing over $\beta$, with the aid of (78), we obtain

$$
\begin{equation*}
c_{\alpha \beta}^{(1)} h_{\beta \gamma} M_{\gamma}^{(1)}=c_{\alpha \beta}^{(1)} Q_{(1,0) \beta}-\frac{d Q_{(0,0) \alpha}^{(1)}}{d t} . \tag{85}
\end{equation*}
$$

Finally, assuming that it is possible to define the inverse matrix $\left(c_{\alpha \beta}^{(1)}\right)^{-1}$, such that

$$
\begin{equation*}
\left(c_{\alpha \beta}^{(1)}\right)^{-1} c_{\beta \gamma}^{(1)}=c_{\alpha \beta}^{(1)}\left(c_{\beta \gamma}^{(1)}\right)^{-1}=\delta_{\alpha \gamma}, \tag{86}
\end{equation*}
$$

inserting in (85) the expression of the internal magnetic field $\mathbf{M}^{(1)}$, obtained by (see (78)), i.e.

$$
\begin{equation*}
M_{\alpha}^{(1)}=\left(c_{\alpha \beta}^{(1)}\right)^{-1} Q_{(0,0) \beta}^{(1)}, \tag{87}
\end{equation*}
$$

one gets the following magnetic relaxation equation (see [7])

$$
\begin{array}{r}
\chi_{(B M) \alpha \beta}^{(0)} B_{\beta}+\frac{d B_{\alpha}}{d t}=\chi_{(M B) \alpha \beta}^{(0)} M_{\beta}+\chi_{(M B) \alpha \beta}^{(1)} \frac{d M_{\beta}}{d t}+\chi_{(M B) \alpha \beta}^{(2)} \frac{d^{2} M_{\beta}}{d t^{2}}+ \\
\chi_{(T) \alpha}^{(0)}\left(T-T_{0}\right)+\chi_{(T) \alpha}^{(1)} \frac{d T}{d t}+\sum_{k=1}^{n} \chi_{(C) \alpha}^{(0, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right)+\frac{d}{d t} \sum_{k=1}^{n} \chi_{(C) \alpha}^{(1, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right) \tag{88}
\end{array}
$$

where

$$
\begin{gather*}
\chi_{(B M) \alpha \beta}^{(0)}=c_{\alpha \gamma}^{(1)} h_{\gamma \zeta}\left(c_{\zeta \beta}^{(1)}\right)^{-1},  \tag{89}\\
\chi_{(M B) \alpha \beta}^{(0)}=c_{\alpha \gamma}^{(1)}\left\{h_{\gamma \zeta}\left(c_{\zeta \eta}^{(1)}\right)^{-1}\left(a_{(M) \eta \beta}^{(0,0)}+L_{(M) \eta \mu}^{(0,1)} a_{(M) \mu \beta}^{(0,0)}\right)-L_{(M)_{\gamma \mu}}^{(1,1)} a_{(M) \mu \beta}^{(0,0)}\right\},  \tag{90}\\
\chi_{(M B) \alpha \beta}^{(1)}=c_{\alpha \gamma}^{(1)}\left\{h_{\gamma \zeta}\left(c_{\zeta \eta}^{(1)}\right)^{-1} L_{(M) \eta \beta}^{(0,0)}-L_{(M) \gamma \beta}^{(1,0)}\right\}+a_{(M) \alpha \beta}^{(0,0)}+L_{(M) \alpha \eta}^{(0,1)} a_{(M) \eta \beta}^{(0,0)},  \tag{91}\\
\chi_{(M B) \alpha \beta}^{(2)}=L_{(M) \alpha \beta}^{(0,0)},  \tag{92}\\
\chi_{(T) \alpha}^{(0)}=c_{\alpha \gamma}^{(1)}\left\{h_{\gamma \zeta}\left(c_{\zeta \eta}^{(1)}\right)^{-1}\left(a_{(M) \eta}^{(0)}+L_{(M) \eta \beta}^{(0,1)} a_{(M) \beta}^{(1)}\right)-L_{(M) \gamma \beta}^{(1,1)} a_{(M) \beta}^{(1)}\right\},  \tag{93}\\
\chi_{(T) \alpha}^{(1)}=a_{(M) \alpha}^{(0)}+L_{(M) \alpha \beta}^{(0,1)} a_{(M) \beta}^{(1)} .  \tag{94}\\
\chi_{(C) \alpha}^{(0, k)}=c_{\alpha \gamma}^{(1)}\left\{h_{\gamma \zeta}\left(c_{\zeta \eta}^{(1)}\right)^{-1}\left(b_{(M) \eta}^{(0, k)}+L_{(M) \eta \beta}^{(0,1)} b_{(M) \beta}^{(1, k)}\right)-L_{(M) \gamma \beta}^{(1,1)} b_{(M) \beta}^{(1, k)}\right\},  \tag{95}\\
\chi_{(C) \alpha}^{(1, k)}=b_{(M) \alpha}^{(0, k)}+L_{(M) \alpha \beta}^{(0,1)} b_{(M) \beta}^{(1, k) .} \tag{96}
\end{gather*}
$$

Hence, it is seen that the linearization of the theory leads to a relaxation equation for anisotropic magnetizable reacting fluid mixtures which has the form of a linear relation among the temperature, the concentrations of the $n$ chemical components, the magnetic field, the total magnetization, the first time derivatives of the temperature, the $n$ concentrations, the magnetic field and the total magnetization, and the second derivative with respect to time of this last axial vector.

## 8. PARTICULAR CASES

In this Section we treat two particular cases of the derived generalized Snoek equation. In the first special case a) where
a) $\quad \mathbf{B}=\mathbf{B}^{\text {eq }}, \quad \mathbf{B}^{\text {ir }}=\mathbf{0}$,
from (74) one gets

$$
\begin{equation*}
L_{(M) \alpha \beta}^{(0,0)}=0, \quad \text { and } \quad L_{(M) \alpha \beta}^{(0,1)}=-L_{(M) \beta \alpha}^{(1,0)}=0 \tag{97}
\end{equation*}
$$

Hence, the magnetic relaxation equation (88) becomes

$$
\begin{gather*}
\chi_{(B M) \alpha \beta}^{(0)} B_{\beta}+\frac{d B_{\alpha}}{d t}=\chi_{(M B) \alpha \beta}^{(0)} M_{\beta}+\chi_{(M B) \alpha \beta}^{(1)} \frac{d M_{\beta}}{d t}+ \\
\chi_{(T) \alpha}^{(0)}\left(T-T_{0}\right)+\chi_{(T) \alpha}^{(1)} \frac{d T}{d t}+\sum_{k=1}^{n} \chi_{(C) \alpha}^{(0, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right)+\frac{d}{d t} \sum_{k=1}^{n} \chi_{(C) \alpha}^{(1, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right) \tag{98}
\end{gather*}
$$

where

$$
\begin{gather*}
\chi_{(B M) \alpha \beta}^{(0)}=a_{(M) \alpha \gamma}^{(0,0)} L_{(M) \gamma \eta}^{(1,1)} a_{(M) \eta \zeta}^{(1,1)}\left(a_{(M) \zeta \beta}^{(0,0)}\right)^{-1},  \tag{99}\\
\chi_{(M B) \alpha \beta}^{(0)}=a_{(M) \alpha \eta}^{(0,0)} L_{(M) \eta \gamma}^{(1,1)}\left(a_{(M) \gamma \beta}^{(1,1)}-a_{(M) \gamma \beta}^{(0,0)}\right),  \tag{100}\\
\chi_{M B) \alpha \beta}^{(1)}=a_{(M) \alpha \beta}^{(0,0)},  \tag{101}\\
\chi_{(T) \alpha}^{(0)}=a_{(M) \alpha \beta}^{(0,0)} L_{(M) \beta \gamma}^{(1,1)}\left\{a_{(M) \gamma \eta}^{(1,1)}\left(a_{(M) \eta \zeta}^{(0,0)}\right)^{-1} a_{(M) \zeta}^{(0)}-a_{(M) \gamma}^{(1)}\right\},  \tag{102}\\
\chi_{(T) \alpha}^{(1)}=a_{(M) \alpha}^{(0)} .  \tag{103}\\
\chi_{(C) \alpha}^{(0, k)}=a_{(M) \alpha \beta}^{(0,0)} L_{(M) \beta \gamma}^{(1,1)}\left\{a_{(M) \gamma \eta}^{(1,1)}\left(a_{(M) \eta \zeta}^{(0,0)}\right)^{-1} b_{(M) \zeta}^{(0, k)}-b_{(M) \gamma}^{(1, k)}\right\},  \tag{104}\\
\chi_{(C) \alpha}^{(1, k)}=b_{(M) \alpha}^{(0, k)} . \tag{105}
\end{gather*}
$$

In this case the magnetization axial vector $\mathbf{M}$ is composed of two parts, $\mathbf{M}=\mathbf{M}^{(0)}+$ $\mathbf{M}^{(1)}$, where the contribution $\mathbf{M}^{(0)}$ has reversible character and the contribution $\mathbf{M}^{(1)}$ has irreversible character. In the case where we have only an isotropic fluid without chemical reactions, eq. (98) reduces to the classical Snoek equation (1) (see [1] and also [6] and [7]).

In the second special case b) where
b) $\quad L_{(M) \alpha \beta}^{(1,1)}=0, \quad$ and $\quad L_{(M) \alpha \beta}^{(1,0)}=-L_{(M) \alpha \beta}^{(0,1)}=0$,

Eqs. (74) and (75) become

$$
\begin{equation*}
B_{\alpha}^{(i r)}=L_{(M) \alpha \beta}^{(0,0)} \frac{d M_{\beta}}{d t} \tag{106}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{d M_{\alpha}^{(1)}}{d t}=0 \tag{107}
\end{equation*}
$$

It is seen that $\mathbf{M}^{(1)}$ is constant and it can be supposed that $\mathbf{M}^{(1)}=0$ (i.e. there is no internal variable) [7]. Equation (88) reduces to a generalized magnetic relaxation equation for De Groot-Mazur media (see [1]-[3])

$$
\begin{equation*}
B_{\alpha}=\chi_{(M B) \alpha \beta}^{(1)} M_{\beta}+\chi_{(M B) \alpha \beta}^{(2)} \frac{d M_{\beta}}{d t}+\chi_{(T) \alpha}^{(1)} \frac{d T}{d t}+\frac{d}{d t} \sum_{k=1}^{n} \chi_{(C) \alpha}^{(1, k)}\left(c^{(k)}-c_{(0)}^{(k)}\right) \tag{108}
\end{equation*}
$$

where

$$
\begin{align*}
\chi_{(M B) \alpha \beta}^{(1)} & =a_{(M) \alpha \beta}^{(0,0)},  \tag{109}\\
\chi_{(M B) \alpha \beta}^{(2)} & =L_{(M) \alpha \beta}^{(0,0)},  \tag{110}\\
\chi_{T) \alpha}^{(1)} & =a_{(M) \alpha}^{(0)} .  \tag{111}\\
\chi_{(C) \alpha}^{(1, k)} & =b_{(M) \alpha}^{(0, k)} . \tag{112}
\end{align*}
$$

In the case where we have only an isotropic fluid without chemical reactions, eq. (113) reduces to the classical De Groot-Mazur equation

$$
\begin{equation*}
B_{\alpha}=\chi_{(M B)}^{(1)} M_{\alpha}+\chi_{(M B)}^{(2)} \frac{d M_{\alpha}}{d t} \tag{113}
\end{equation*}
$$

where $\chi_{(M B)}^{(1)}$ and $\chi_{(M B)}^{(2)}$ are constants.

## 9. CONCLUSIONS

The paper deals with anisotropic magnetizable reacting fluid mixtures, where different types of irreversible microscopic phenomena give rise to magnetic relaxation. The standard procedures of irreversible thermodynamics with internal variables is used. The The obtained results can be applied in several physical situations, in nuclear magnetic resonance, in medicine and biology and other different fields of applied sciences, where complex media, in which different types of molecules, having different magnetic susceptibilities and relaxation times, are taken into consideration. In [15]-[19] Maugin gave a microscopic description of relaxation magnetic phenomena in continuous media with $n$ different ionic species, introducing partial magnetizations per unit mass and illustrating internal mechanisms in magnetizable bodies by internal variables. In this paper the total specific magnetization is supposed sum of two irreversible parts and, linearizing the theory, the magnetic relaxation equation for these magnetizable anisotropic media is derived. The particular cases of classical Snoek and De Groot-Mazur media are treated.

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