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Let R be a prime ring of characteristic $\neq 2$ with a derivation $d \neq 0$, L a noncentral Lie ideal of R such that $[d(u), u]^n$ is central, for all $u \in L$. We prove that R must satisfy s_4 the standard identity in 4 variables. We also examine the case R is a 2-torsion free semiprime ring and $[d([x, y]), [x, y]]^n$ is central, for all $x, y \in R$.

Let R be a prime ring and d a nonzero derivation of R. A well known result of Posner [14] states that if the commutator $[d(x), x] \in Z(R)$, the center of R, for any $x \in R$, then R is commutative.

In [11] C. Lanski generalizes the result of Posner to a Lie ideal. To be more specific, the statement of Lanski's theorem is the following:

Theorem ([11, Theorem 2, page 282])**.** *Let R be a prime ring, L a noncommutative Lie ideal of R and $d \neq 0$ a derivation of R. If $[d(x), x] \in Z(R)$, for all $x \in L$, then either R is commutative, or* $\mathrm{char}(R) = 2$ *and R satisfies s_4, the standard identity in* 4 *variables.*

Here we will examine what happens in case $[d(x), x]^n \in Z(R)$, for any $x \in L$, a noncommutative Lie ideal of R and $n \geq 1$ a fixed integer.

One cannot expect the same conclusion of Lanski's theorem as the following example shows:

Example 1. Let $R = M_2(F)$, the 2×2 matrices over a field F, and take $L = R$ as a noncommutative Lie ideal of R. Since $[x, y]^2 \in Z(R)$, for all $x, y \in R$, then also $[d(x), x]^2 \in Z(R)$, for all $x \in R$.

We will prove that:

Theorem 1.1. *Let R be a prime ring of characteristic different from* 2*, L a noncentral Lie ideal of R, d a nonzero derivation of R such that $[d(u), u]^n \in Z(R)$, for any $u \in L$. Then R satisfies s_4.*

We will proceed by first proving that:

Lemma 1.1. *Let R be a prime ring of characteristic different from* 2*, L a noncentral Lie ideal of R, d a nonzero derivation of R, $n \geq 1$. If d satisfies $[d(u), u]^n = 0$, for any $u \in L$, then R is commutative.*

We then examine the case R is a 2-torsion free semiprime ring. The results we obtain are:

Theorem 2.1. *Let R be a 2-torsion free semiprime ring, d a nonzero derivation of R, n a fixed positive integer, U the left Utumi quotient ring of R and $[d([x,y]),[x,y]]^n = 0$, for any $x,y \in R$. Then there exists a central idempotent element e of U such that on the direct sum decomposition $eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Theorem 2.2. *Let R be a 2-torsion free semiprime ring, d a nonzero derivation of R, n a fixed positive integer, U the left Utumi quotient ring of R and $[d([x,y]),[x,y]]^n \in Z(R)$, for any $x,y \in R$. Then there exists a central idempotent e of U such that, on the direct sum decomposition $U = eU \oplus (1-e)U$, the derivation d vanishes identically on eU and the ring $(1-e)U$ satisfies s_4.*

1. The case: R prime ring.

In all that follows, unless stated otherwise, R will be a prime ring of characteristic $\neq 2$, L a Lie ideal of R, $d \neq 0$ a derivation of R and $n \geq 1$ a fixed integer such that $[d(x),x]^n \in Z(R)$, for all $x \in L$.

For any ring S, $Z(S)$ will denote its center, and $[a,b] = ab - ba$, $[a,b]_2 = [[a,b],b]$, $a,b \in S$. In addition s_4 will denote the standard identity in 4 variables.

We will also make frequent use of the following result due to Kharchenko [8] (see also [12]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f(x_1,\dots,x_n,d(x_1,\dots,x_n))$ a differential identity in I, that is

$$f(r_1,\dots,r_n,d(r_1),\dots,d(r_n)) = 0 \quad \forall r_1,\dots,r_n \in I.$$

One of the following holds:

1) Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d = ad(q)$ and $d(x) = ad(q)\,(x) = [q,x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1,\dots,r_n,[q,r_1],\dots,[q,r_n]) = 0;$$

2) or I satisfies the generalized polynomial identity

$$f(x_1,\dots,x_n,y_1,\dots,y_n) = 0.$$

Lemma 1.1. *Let R be a prime ring of characteristic different from 2, U a noncentral Lie ideal of R, d a nonzero derivation of R and $n \geq 1$. If $([d(u),u])^n = 0$, for any $u \in L$, then R is commutative.*

Proof. Since we assume that $\text{char}(R) \neq 2$, by a result of Herstein [6], $L \supseteq [I, R]$, for some $I \neq 0$, an ideal of R, and also L is not commutative. Therefore we will assume throughout that $L \supseteq [I, R]$. Without loss of generality we can assume $L = [I, I]$.

Hence $[d([x, y]), [x, y]]^n = 0$, for any $x, y \in I$, then I satisfies the differential identity

$$f(x, y, d(x), d(y)) = [[d(x), y] + [x, d(y)], [x, y]]^n = 0.$$

If the derivation d is not inner, by Kharchenko's theorem [8], I satisfies the polynomial identity

$$f(x, y, t, z) = [[z, y] + [x, t], [x, y]]^n = 0$$

and in particular, for $z = 0$,

$$[[x, t], [x, y]]^n = 0.$$

Since the latter is a polynomial identity for I, and so for R too, it is well known that there exists a field F such that R and F_m satisfy the same polynomial identities (see [7, page 57, page 89]). Let e_{ij} the matrix unit with 1 in (i,j)-entry and zero elsewhere. Suppose $m \geq 2$. If we choose $x = e_{11}$, $y = e_{21}$, $t = e_{12}$, then we get the contradiction

$$0 = [[e_{11}, e_{12}], [e_{11}, e_{21}]]^n = [e_{12}, -e_{21}]^n = (-1)^n e_{11} + e_{22} \neq 0.$$

Therefore $m = 1$ and so R is commutative.

Let now d be an inner derivation induced by an element $A \in Q$, the Martindale quotient ring of R. Then, for any $x, y \in I$, $([A, [x, y]]_2)^n = 0$. Since by [2] I and Q satisfy the same generalized polynomial identities, we have $([A, [x, y]]_2)^n = 0$, for any $x, y \in Q$. Moreover, since Q remains prime by the primeness of R, replacing R by Q we may assume that $A \in R$ and C is just the center of R. Note that R is a centrally closed prime C-algebra in the present situation [4], i.e., $RC = R$. By Martindale's theorem in [13], RC (and so R) is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space V over a division ring D. Since R is primitive then there exist a vector space V and the division ring D such that R is dense of D-linear transformation over V.

Assume first that $\dim_D V \geq 3$.

Step 1.

We want to show that, for any $v \in V$, v and Av are linearly D-dependent.

Since if $Av = 0$ then $\{v, Av\}$ is D-dependent, suppose that $Av \neq 0$. If v and Av are D-independent, since $\dim_D V \geq 3$, then there exists $w \in V$ such that v, Av, w are also linearly independent. By the density of I, there exist $x, y \in I$ such that

$$xv = 0, \; xAv = w, \; xw = v$$

$$yv = 0, \; yAv = 0, \; yw = w.$$

These imply that

$$[A,[x,y]]_2 v = -v \text{ and } 0 = ([A,[x,y]]_2)^n v = (-1)^n v,$$

which is a contradiction.

So we can conclude that v are Av are linearly D-dependent, for all $v \in V$.

Step 2.

We show here that there exists $b \in D$ such that $Av = vb$, for any $v \in V$. Now choose $v, w \in V$ linearly independent. Since $\dim_D V \geq 3$, there exists $u \in V$ such that v, w, u are linearly independent. By Step 1, there exist $a_v, a_w, a_u \in D$ such that

$$Av = va_v,\ Aw = wa_w,\ Au = ua_u \text{ that is } A(v+w+u) = va_v + wa_w + ua_u.$$

Moreover $A(v+w+u) = (v+w+u)a_{v+w+u}$, for a suitable $a_{v+w+u} \in D$. Then $0 = v(a_{v+w+u} - a_v) + w(a_{v+w+u} - a_w) + u(a_{v+w+u} - a_u)$ and, because v, w, u are linearly independent, $a_u = a_w = a_v = a_{v+w+u}$. This completes the proof of Step 2.

Let now $r \in R$ and $v \in V$. By Step 2, $Av = vb, r(Av) = r(vb)$, and also $A(rv) = (rv)b$. Thus $0 = [A, r]v$, for any $v \in V$, that is $[A, r]V = 0$. Since V is a left faithful irreducible R-module, $[A, r] = 0$, for all $r \in R$, i.e., $A \in Z(R)$ and $d = 0$, which contradicts our hypothesis.

Therefore $\dim_D V$ must be ≤ 2. In this case R is a simple GPI ring with 1, and so it is a central simple algebra finite dimensional over its center. From Lemma 2 in [**10**] it follows that there exists a suitable field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the same generalized polynomial identity of R.

If we assume $k \geq 3$, by the same argument as in Steps 1 and 2, we get a contradiction.

Obviously if $k = 1$ then R is commutative. Thus we may assume $R \subseteq M_2(F)$, where $M_2(F)$ satisfies $([A,[x,y]]_2)^n = 0$.

Since for any $a, b \in M_2(F)$, $[a,b]^2 \in Z(R)$ then it follows easily that $([A,[x,y]]_2)^2 = 0$, for any $x, y \in M_2(F)$. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If we choose $x = e_{12}, y = e_{21}$ then we get:

$$[A, e_{11} - e_{22}]_2 = \begin{bmatrix} 0 & 4a_{12} \\ 4a_{21} & 0 \end{bmatrix}$$

$$0 = ([A, e_{11} - e_{22}]_2)^2 = \begin{bmatrix} 16(a_{12}a_{21}) & 0 \\ 0 & 16(a_{12}a_{21}) \end{bmatrix}.$$

Therefore either $a_{12} = 0$ or $a_{21} = 0$. Without loss of generality we can pick $a_{12} = 0$.

Now let $[x,y]=[e_{11},e_{12}+e_{21}]=e_{12}-e_{21}$. In this case we have:

$$[A,e_{12}-e_{21}]_2=\begin{bmatrix} 2(a_{22}-a_{11}) & -2a_{21} \\ -2a_{21} & 2(a_{11}-a_{22}) \end{bmatrix}$$

$$\left(\begin{bmatrix} 2(a_{22}-a_{11}) & -2a_{21} \\ -2a_{21} & 2(a_{11}-a_{22}) \end{bmatrix}\right)^2=0$$

that is

$$4(a_{21})^2+4(a_{11}-a_{22})^2=0$$

$$(a_{21})^2=-(a_{22}-a_{11})^2 \quad \mathbf{(1)}.$$

On the other hand if $[x,y]=[e_{11},e_{12}-e_{21}]=e_{12}+e_{21}$ then

$$([A,e_{12}+e_{21}]_2)^2=\begin{bmatrix} 2(a_{11}-a_{22}) & -2a_{21} \\ 2a_{21} & 2(a_{22}-a_{11}) \end{bmatrix}$$

$$\left(\begin{bmatrix} 2(a_{11}-a_{22}) & -2a_{21} \\ 2a_{21} & 2(a_{22}-a_{11}) \end{bmatrix}\right)^2=0$$

that is

$$4(a_{22}-a_{11})^2-4(a_{21})^2=0$$

$$(a_{21})^2=(a_{22}-a_{11})^2 \quad \mathbf{(2)}.$$

(1) and **(2)** imply that $a_{21}=0$ and $a_{11}=a_{22}$ which means that A is a central matrix in $M_2(F)$, $A\in F$ and $d=0$, a contradiction. Therefore $k=1$, i.e., R is commutative. □

Lemma 1.2. *Let $R=M_k(F)$, the ring of $k\times k$ matrices over a field F of characteristic $\neq 2$. If $q\neq 0$ is a noncentral element of R such that $([q,[x,y]]_2)^n\in F$, for any $x,y\in R$, then $k\leq 2$.*

Proof. Suppose $k\geq 3$. Let i,j,r be distinct indices and $q=\sum a_{mn}e_{mn}$, with $a_{mn}\in F$. For simplicity we assume that $i=1,j=2,r=3$. If we choose $[x,y]=[e_{12},e_{23}-e_{31}]=e_{13}+e_{32}$, then

$$[q,[x,y]]_2=a_{21}e_{11}+a_{21}e_{22}-2a_{21}e_{33}+\sum_{n\neq 1}\gamma_n e_{1n}+\sum_{m\neq 2}\delta_m e_{m2}$$

with $\gamma_n,\delta_m\in F$, and

$$([q,[x,y]]_2)^n=(a_{21})^n e_{11}+(a_{21})^n e_{22}+(-2a_{21})^n e_{33}+\sum_{n\neq 1}\alpha_n e_{1n}+\sum_{m\neq 2}\beta_m e_{m2}$$

with $\alpha_n,\beta_m\in F$. Since by assumption $([q,[x,y]]_2)^n\in F$, then $\alpha_n=\beta_m=0$, for all m,n, and $(a_{21})^n=(-2a_{21})^n=0$, i.e., $a_{21}=0$. In a similar way we may conclude that $a_{ij}=0$, for any $i\neq j$. Therefore if $k\geq 3$, q is a diagonal matrix, $q=\sum_t a_{tt}e_{tt}$, with $a_t\in F$.

If we show that q is a central matrix, then we get a contradiction to our assumption and so k must be less or equal than 2.

Let $[x,y]=[e_{ij}-e_{ji},e_{jj}]=e_{ij}+e_{ji}$. Therefore

$$[q,[x,y]]_2=2(a_{ii}-a_{jj})e_{ii}+2(a_{jj}-a_{ii})e_{jj}$$

and

$$([q,[x,y]]_2)^n=2^n(a_{ii}-a_{jj})^n e_{ii}+2^n(a_{jj}-a_{ii})^n e_{jj}.$$

Since $([q,[x,y]]_2)^n\in F$ and $k\geq 3$, it follows that $a_{ii}=a_{jj}$. Thus q is a central matrix.

Notice that if $n=1$ then by using the same argument and choosing $[x,y]=e_{12}$, we get $N=[q,[x,y]]_2=-2e_{12}qe_{12}$, which has rank 1 and so it cannot be central in $M_k(F)$, with $k\geq 2$. This implies that if $n=1$ then $k=1$, and R must be a commutative field. The proof of Lemma 1.2 is now complete. □

Theorem 1.1. *Let R be a prime ring of characteristic different from 2, L a noncentral Lie ideal of R, d a nonzero derivation of R such that $[d(u),u]^n\in Z(R)$, for any $u\in L$. Then R satisfies s_4.*

Proof. Let I be the nonzero two-sided ideal of R such that $0\neq[I,R]\subseteq L$ and J be any nonzero two-sided ideal of R. Then $V=[I,J^2]\subseteq L$ is a Lie ideal of R. If, for every $v\in V$, $[d(v),v]^n=0$, by Lemma 1.1, R is commutative. Otherwise, by our assumptions, $J\cap Z(R)\neq 0$. Let now K be a nonzero two-sided ideal of R_Z, the ring of the central quotients of R. Since $K\cap R$ is an ideal of R then $K\cap R\cap Z(R)\neq 0$, that is K contains an invertible element in R_Z, and so R_Z is simple with 1.

Moreover we may assume $L=[I,I]$. For any $x,y\in I$, $[d([x,y]),[x,y]]^n\in Z(R)$, i.e.,

$$[[d([x,y]),[x,y]]^n,r]=0\quad\text{for any } x\in R.$$

Thus I satisfies the differential identity

$$f(x,y,r,d(x),d(y))=[[[d(x),y]+[x,d(y)],[x,y]]^n,r]=0.$$

If the derivation is not inner, by [8], I satisfies the polynomial identity

$$f(x,y,r,z,t)=[[[t,y]+[x,z],[x,y]]^n,r]=0$$

and in particular, for $z=0$,

$$[[[t,y],[x,y]]^n,r]=0.$$

In this case we know that there exists a field F such that R and F_m satisfy the same polynomial identities. Thus $[[t,y],[x,y]]^n$ is central in F_m. Suppose $m\geq 3$ and choose $x=e_{32},y=e_{33},t=e_{23}$.

$$[t,y]=e_{23},\ \ [x,y]=-e_{32}$$

$$[[t,y],[x,y]]=-e_{22}+e_{33}$$

$$[[t,y],[x,y]]^n=(-1)^n e_{22}+e_{33}\notin Z(R)$$

contrary to our assumptions. This forces $m\leq 2$, i.e., R satisfies s_4.

Notice that in the case $n = 1$, $[[t, y], [x, y]]$ must be central in F_m. But if $m \geq 2$ and $t = e_{11}$, $y = e_{12}$, $x = e_{21}$, we get the contradiction $[[t, y], [x, y]] = 2e_{12} \notin Z(R)$. Therefore m must be equal to 1 and R is commutative.

Now let d be an inner derivation induced by an element $A \in Q$. By localizing R at $Z(R)$ it follows that $([A, [x, y]]_2)^n \in Z(R_Z)$, for all $x, y \in R_Z$.

Since R and R_Z satisfy the same polynomial identities, in order to prove that R satisfies $S_4(x_1, x_2, x_3, x_4)$, we may assume that R is simple with 1 and $[R, R] \subseteq L$.

In this case, $([A, [x, y]]_2)^n \in Z(R)$, for all $x, y \in R$. Therefore R satisfies a generalized polynomial identity and it is simple with 1, which implies that $Q = RC = R$ and R has a minimal right ideal. Thus $A \in R = Q$ and R is simple artinian that is $R = D_k$, where D is a division ring finite dimensional over $Z(R)$ [**13**]. From Lemma 2 in [**10**] it follows that there exists a suitable field F such that $R \subseteq M_k(F)$, the ring of all $k \times k$ matrices over F, and moreover $M_k(F)$ satisfies the generalized polynomial identity $[([A, [x, y]]_2)^n, z] = 0$. By Lemma 1.2, if $n \geq 2$ then $k \leq 2$ and R satisfies s_4, also if $n = 1$ then $k = 1$ and R must be commutative. □

2. The case: R semiprime ring.

In all that follows R will be a 2-torsion free semiprime ring. We cannot expect the same conclusion of previous section to hold, as the following example shows:

Example 2. Let R_1 be any prime ring not satisfying s_4 and $R_2 = M_2(F)$, the ring of 2×2 matrices over the field F. Let $R = R_1 \oplus R_2$, d a nonzero derivation of R such that $d = 0$ in R_1. Consider $L = [R, R]$. It is a non-central Lie ideal of R. Let $r_1, s_1 \in R_1$, $r_2, s_2 \in R_2$, $u = [(r_1, r_2), (s_1, s_2)]$. Therefore $d(u) = (0, d([r_2, s_2]))$ and $[d(u), u] = (0, [d([r_2, s_2]), [r_2, s_2]])$. Since $[d([r_2, s_2]), [r_2, s_2]]^2 \in Z(R_2)$, then

$$[d(u), u]^2 = (0, [d([r_2, s_2]), [r_2, s_2]])^2 = (0, [d([r_2, s_2]), [r_2, s_2]]^2) \in Z(R)$$

but R does not satisfy s_4.

The related object we need to mention is the left Utumi quotient ring U of R. For basic definitions and preliminary results we refer the reader to [**1**], [**5**], [**9**].

In order to prove the main result of this section we will make use of the following facts:

Claim 1 ([**1**, Proposition 2.5.1])**.** Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.

Claim 2 ([**3**, p. 38])**.** If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.

Claim 3 ([**3**, p. 42])**.** Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a B-algebra orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any derivation of R.

We will prove the following:

Theorem 2.1. *Let R be a 2-torsion free semiprime ring, d a nonzero derivation of R, n a fixed positive integer, U the left Utumi quotient ring of R and $[d([x,y]),[x,y]]^n = 0$, for any $x,y \in R$. Then there exists a central idempotent element e of U such that on the direct sum decomposition $eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Proof. Since R is semiprime, by Claim 2, $Z(U) = C$, the extended centroid of R, and, by Claim 1, the derivation d can be uniquely extended on U. Since U and R satisfy the same differential identities (see [**12**]), then $[d([x,y]),[x,y]]^n = 0$, for all $x,y \in U$. Let B be the complete boolean algebra of idempotents in C and M be any maximal ideal of B.

Since U is a B-algebra orthogonal complete (see [**3**, p. 42, (2) of Fact 1]), by Claim 3, MU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/MU$ and $\overline{d}$ the derivation induced by d on $\overline{U}$. For any $\overline{x},\overline{y} \in \overline{U}$, $[\overline{d}([\overline{x},\overline{y}]),[\overline{x},\overline{y}]]^n = 0$. In particular $\overline{U}$ is a prime ring and so, by Lemma 1.1, $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ is commutative. This implies that, for any maximal ideal M of B, $d(U) \subseteq MU$ or $[U,U] \subseteq MU$. In any case $d(U)[U,U] \subseteq MU$, for all M. Therefore $d(U)[U,U] \subseteq \bigcap_M MU = 0$.

By using the theory of orthogonal completion for semiprime rings (see [**1**, Chapter 3]), it follows that there exists a central idempotent element e in U such that on the direct sum decomposition $eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative. □

We come now to our last result:

Theorem 2.2. *Let R be a 2-torsion free semiprime ring, d a nonzero derivation of R, n a fixed positive integer, U the left Utumi quotient ring of R and $[d([x,y]),[x,y]]^n \in Z(R)$, for any $x,y \in R$. Then there exists a central idempotent e of U such that, on the direct sum decomposition $U = eU \oplus (1-e)U$, the derivation d vanishes identically on eU and the ring $(1-e)U$ satisfies s_4.*

Proof. By Claim 2, $Z(U) = C$, and by Claim 1 d can be uniquely defined on the whole U. Since U and R satisfy the same differential identities, then $[d([x,y]),[x,y]]^n \in C$, for all $x, y \in U$. Let B be the complete boolean algebra of idempotents in C and M any maximal ideal of B. As already pointed out in the proof of Theorem 2.1, U is a B-algebra orthogonal complete and by Claim 3, MU is a prime ideal of U, which is d-invariant. Let $\overline{d}$ the derivation induced by d on $\overline{U} = U/MU$. Since $Z(\overline{U}) = (C+MU)/MU = C/MU$, then $[\overline{d}([x,y]),[x,y]]^n \in (C+MU)/MU$, for any $x, y \in \overline{U}$. Moreover $\overline{U}$ is a prime ring, hence we may conclude, by Theorem 1.1, that $\overline{d} = 0$ in $\overline{U}$ or $\overline{U}$ satisfies s_4. This implies that, for any maximal ideal M of B, $d(U) \subseteq MU$ or $s_4(x_1,x_2,x_3,x_4) \subseteq MU$, for all $x_1,x_2,x_3,x_4 \in U$. In any case $d(U)s_4(x_1,x_2,x_3,x_4) \subseteq \bigcap_M MU = 0$. From [**1**, Chapter 3], there exists a central idempotent element e of U, the left Utumi quotient ring of R, such that there exists a central idempotent e of U such that $d(eU) = 0$ and $(1-e)U$ satisfies s_4. □

References

[1] K.I. Beidar, W.S. Martindale and V. Mikhalev, *Rings with generalized identities*, Pure and Applied Math., Dekker, New York, 1996.

[2] C.L. Chuang, *GPI's having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc., **103**(**3**) (1988), 723-728.

[3] ———, *Hypercentral derivations*, J. Algebra, **166** (1994), 34-71.

[4] J.S. Erickson, W.S. Martindale III and J.M. Osborn, *Prime nonassociative algebras*, Pacific J. Math., **60** (1975), 49-63.

[5] C. Faith, *Lecture on Injective Modules and Quotient Rings*, Lecture Notes in Mathematics, **49**, Springer Verlag, New York, 1967.

[6] I.N. Herstein, *Topics in ring theory*, Univ. Chicago Press, 1969.

[7] N. Jacobson, *PI-algebras, an introduction*, Lecture notes in Math., **441**, Springer Verlag, New York, 1975.

[8] V.K. Kharchenko, *Differential identities of prime rings*, Algebra and Logic, **17** (1978), 155-168.

[9] J. Lambek, *Lecture on Rings and Modules*, Blaisdell Waltham, MA, 1966.

[10] C. Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc., **118**(**3**) (1993), 731-734.

[11] ———, *Differential identities, Lie ideals and Posner's theorems*, Pacific J. Math., **134**(**2**) (1988), 275-297.

[12] T.K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sinica, **20**(**1**) (1992), 27-38.

[13] W.S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra, **12** (1969), 576-584.

[14] E.C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc., **8** (1957), 1093-1100.

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