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To the Editor of Applied Mathematics and Computation

Dear Editor,

we are sending you in attachment the PDF file of the paper *"Follow the Leader" Learning Dynamics on Networks"*, by S. De Lillo, M. Dolfin and G. Fioriti, which is submitted for publication in your Journal.
We believe that our work addresses a topic of interest for the readers of Applied Mathematics and Computation. Indeed, we focus on the modelling of a complex system which may be relevant in the realm of biological and behavioural sciences.
The main aspects of our work are:

- A kinetic model of learning dynamic on network for a complex system is discussed.
- The network is characterized by a special node having the role of "leader".
- The learning process is induced by nonlinear interactions. Binary and multiple interactions are considered.
- Numerical simulations show the effect of the network structure on the emerging system behaviour.

Finally, we hereby state that the paper is not currently submitted to other journals and it will not be submitted during the reviewing process.
Corresponding author for this paper will be G. Fioriti: gioia.fioriti@gmail.com.

Thanking you for your attention.
Yours truly.

Silvana De Lillo
Marina Dolfin
Gioia Fioriti

"Follow the leader" learning dynamics on networks

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Abstract

A learning dynamics on network is introduced, characterized by binary and multiple nonlinear interactions among the individuals distributed in the different nodes. A particular topology of the network is considered by introducing a leader node which influences all the other "follower" nodes without being influenced in turn. Numerical simulations are provided, particularly focusing on the effect of the network structure and of the nonlinear interactions on the emerging behaviour of the system. It turns out that the leader node always exhibits an autonomous evolution, while the follower nodes may have a regression when the interactions with the leader node are switched off. There is instead a remarkable change in the final configurations of the follower nodes, even if only one fo them is connected to the leader: indeed, due to the microscopic interactions among them, all the follower nodes feel a strong "leader effect".

Keywords: Kinetic theory; network; learning dynamics; complex system; mathematical biology.

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1. Introduction

The modeling of intelligent, complex systems is a very active research field which motivated many studies in recent years due to its relevance in life-sciences [1]. In this respect, a very important role is played by collective learning, not only in biological systems but in the field of social sciences as well [2, 3]. In order to describe the complexity of the interactions for large systems of living entities, applied mathematicians developed different approaches ranging from kinetic and statistical mechanics models to stochastic methods and Monte Carlo simulations [4, 5, 6, 7]. In particular, in the framework of the so-called kinetic theory for active particles [KTAP theory] [8], several interesting models were obtained in a variety of different fields such as social systems [9], spread of epidemics [10], micro-scale darwinian evolution [11] and collective learning processes [12, 13, 14]. More recently, the KTAP theory has been extended to model interactions of groups of individuals localized on networks in order to describe social phenomena such as for example, population migration [15], opinion formation [16, 17, 18] or specific problems of political economy [19]. Moreover, a network representation of complex learning systems has been presented in [20, 21] and applied to describe learning processes that take place in a classroom. It is the aim of this paper to analyze the emerging properties in a population of interacting entities characterized by a microscopic learning dynamics and distributed over a network connecting a certain number of nodes. A particular topology of the network is considered by introducing a special node having the role of "leader", i.e. influencing all the other nodes of the network, although not being influenced in turn.

In the mathematical framework of the KTAP theory, living entities are called active particles: their microscopic state is modeled by a scalar variable called activity which is assumed to be heterogeneously distributed among the particles. The activity represents the ability of each individual to express a given strategy, which, in the case under analysis, is the ability to learn.

In the following we divide the complex system in different populations, mathematically characterized as functional subsystems, localized in the different nodes of the network. Conservation of the size of the population on each node is assumed; the case in which migration phenomena among the nodes are allowed [1], is discussed as a research perspectives. In Sec.2 the reference mathematical formalism is introduced and applied to a network of living entities characterized by different abilities to learn. Interactions of the indi-

viduals among them induce a learning process which can improve the level of knowledge of the individuals involved, but also in some cases be misleading and reduce it. Sec.3 is devoted to the solution of the initial value problem for the mathematical model introduced. Next, a specific learning dynamics on the network is discussed in Sec.4. Numerical simulations are provided in Sec.5 expecially focusing on the effect of the network structure on the emerging behaviours of the system. Finally, research perspectives are outlined in Sec.6.

2. Mathematical Framework

In the following we introduce the essential aspects of the reference mathematical formulation and specialize them to the case of living entities interacting on a network. Moreover, we characterize the properties of the network and of the nonlinear interactions involved in the learning dynamics of the complex system under study.

2.1. Mathematical Representation

The hallmarks of the KTAP theory are the following:

- The overall system is subdvided into functional subsystems constituted by entities, called active particles,whose individual state is called activity;
- the state of each functional subsystem is defined by a suitable, time dependent, probability distribution over the activity variable;
- interactions are modeled by games, more precisely stochastic games [17], where the state of the interacting particles and the output of the interactions are known in probability;
- the evolution of the above mentioned probability distributions is obtained by a balance of particles within elementary volumes of the space of the microscopic states,where the dynamics of the inflow and the outflow of particles is determined by the interactions at the microscopic scale.

In particular, in the complex system under analysis, each of the active particles expresses a learning strategy at any given instant of time. The

active particle's strategy is represented by a scalar variable $u \in D_u \subset \mathbb{R}$, called activity, which is heterogeneously distributed among the particles. A discrete normalized domain is assumed for the activity variable on which it takes m discrete states, i.e. $D_u = \left\{u_1 = 0, \ldots, u_r = \dfrac{r-1}{m-1}, \ldots, u_m = 1\right\}$, ranging from the worst learning activity $u_1 = 0$ to the best one $u_m = 1$.

We introduce n subsystems characterized by the same activity variable u, although it will be expressed on each subsystem in different ways by means of different parameters and/or different initial conditions. On each subsystem a time evolving discrete probability on the active variable is introduced

$$f_r^i = f^i(u_r, t) \, : \, D_u \times [0, T_{max}] \to [0, 1], \quad i = 1, \ldots, n, \quad r = 1, \ldots, m, \quad (1)$$

which has been normalized with respect to the constant size of the population; T_{max} is the observation time. The following probability property holds

$$\sum_{r=1}^{m} f_r^i = 1, \quad \forall \, i = 1, \ldots, n \quad \forall t \geq 0$$

on each subsystem in each instant of time. The notation $\mathbf{f}^i = (f_1^i, \ldots, f_m^i)$ will be used in the following for the probability on the $i-th$ node; on each node we introduce the first moment of the distribution

$$\mathbb{M}^{(i)}[f(t)] = \sum_{r=1}^{m} u_r f_r^i(t), \quad \forall i = 1, \ldots, n.$$

In the present model we consider two kinds of interactions:

- binary interactions (between pairs of particles),
- multiple interactions (between a particles and a node).

Binary interactions involve only the microscopic scale: a candidate particle acquires in probability the microstate of the test active particle, representative of the system, due to an interaction with a field active particle. Multiple interactions instead, involve both the microscopic and the macroscopic scale (micro-macro interactions). Indeed in this case the candidate active particles interacts with a node. The state of the latter is represented by the first moment of the probability distribution taken over the particles microstates of the corresponding node, defined in [3]. We may say that at each interaction the active particle plays a stochastic game whose payoff, given only in probability, is given by the acquired microstate.

2.2. Network structure

We introduce now a network structure in the population of active particles by considering interactions among the subsystems [1, 7].

By using the language of graph theory, each subsystem is represented by a node (vertex of the graph) and the interaction between any pair of nodes is represented by an arrow which is bidirectional, whenever the nodes influence each other (undirected graph), or in one direction only whenever one node influence the other but it is not subject to the influence of the other node in turn (directed graph).

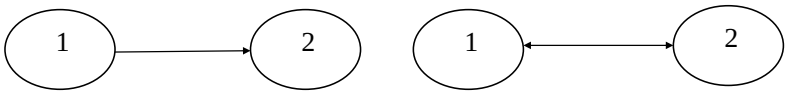

Figure 1: Directed graph (on the left of the figure) and undirected graph (on the right of the figure).

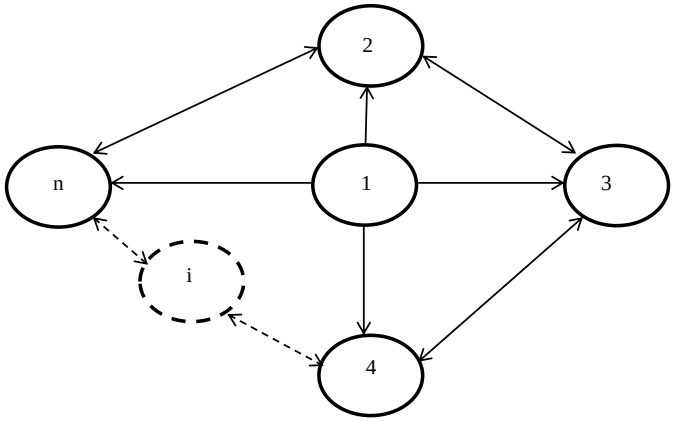

Figure 2: Network topology

Dashed lines are used to represent the general topology of one 'leader' node and $n-1$ 'follower' nodes.

In all the 'follower' nodes, we model the effects on the candidate active particle due to two phenomena:

1. binary interactions with field active particles within the subsystem to which the candidate active particle belongs;
2. influence of the mean activities of the other nodes of the network.

In the 'leader' node, only binary interactions among the field active particles within the subsystem are allowed, modeling the fact that the active particles of this node are not influenced by the activity of the other nodes.

2.3. Interactions

A very important aspect of the learning dynamics [4, 9] is that the interactions are nonlinearly additive: in fact, the result of the interaction between a pair of active particles is nonlinearly dependent not only on the states of the two particles, but also on the states of all the particles in an interaction domain. Due to the topology of our network, we have a different phenomenology of interactions according to whether the node we are considering is the leader node or one of the followers.
In the following we do not take into consideration proliferative and/or destructive interactions. Then we obtain the following general form of kinetic equation governing the evolution in time of the probability (1)

$$\frac{d}{dt}f_r^i(t) = J_r^i[\mathbf{f}^i](t) + \mathcal{J}_r^i[\mathbf{f}^1,\dots,\mathbf{f}^n](t), \quad \forall i = 1,\dots,n,\ \forall r = 1,\dots,m, \quad (2)$$

where each term on the right hand side represents the gain and loss in the microstates due to the two different phenomena listed above, respectively. In particular, the nonlinear operator J_r^i corresponds to interactions involving pairs of active particles in the $i-th$ node and $\mathcal{J}_r^i$ characterizes the influence of the mean activities of all the nodes of the network on the candidate active particle; square parentheses are used to denote the nonlinear dependence of the terms on the probabilities.

In order to derive explicit expressions for the quantities entering Eq.(2) we need to specify:

- η_{hk}^{ij} — encounter rate between pairs of active particles of the $i-th$ node with activity u_h and of $j-th$ node with activity u_k, respectively;

- $\mathcal{B}^{ij}_{hk}(r)[\mathbf{f}^i] = B^{ij}[\mathbf{f}^i](u_h \to u_r \,|\, u_h, u_k)$ — transition probability function, on the $i-th$ node, defining the probability for a candidate active particle characterized by u_h to move to u_r due to an interaction with a field active particle characterized by u_k belonging to the $j-th$ node;
- $\tilde{\eta}^{ij}_h$ — influence rate due to the mean value of the $j-th$ node on an active particle u_h belonging to the $i-th$ node;
- $\tilde{\mathcal{B}}^{ij}_h(r)[\mathbf{f}^i] = \tilde{\mathcal{B}}^{ij}[\mathbf{f}^i](u_h \to u_r; u_h, \mathbb{M}^{(j)})$ — transition probability function on the $i-th$ node, defining the probability for a candidate active particle characterized by u_h to move to u_r due to an influence of the mean activity of the $j-th$ node.

Both the transition probability densities satisfy the probability property

$$\sum_{r=1}^{m} \mathcal{B}^{ij}_{hk}(r) = 1 \quad \forall i, j = 1, \ldots, n, \forall h, k = 1, \ldots, m,$$

$$\sum_{r=1}^{m} \tilde{\mathcal{B}}^{ij}_{h}(r) = 1 \quad \forall i, j = 1, \ldots, n, \forall h = 1, \ldots, m,$$

on each node of the network.

Remark 1. It is worth mentioning here that no migrations among the nodes are modeled.

It there follows that the explicit expression of Eq.(2) takes the form

$$\begin{aligned} \frac{d}{dt} f^i_r(t) = &\sum_{j=1}^{n} \sum_{h,k=1}^{m} \mathcal{B}^{ij}_{hk}(r)[\mathbf{f}^i]\, \eta^{ij}_{hk}\, f^i_h(t)\, f^j_k(t) - f^i_r(t) \sum_{j=1}^{n} \sum_{k=1}^{m} \eta^{ij}_{rk}\, f^j_k(t) \\ &+ \sum_{j=1}^{n} \sum_{h=1}^{m} \tilde{\mathcal{B}}^{ij}_h(r)[\mathbf{f}^i]\, \tilde{\eta}^{ij}_h\, f^i_h(t)\, \mathbb{M}^{(j)} - f^i_r(t) \sum_{j=1}^{n} \tilde{\eta}^{ij}_r\, \mathbb{M}^{(j)}, \\ &i = 1, \ldots, n,\ j = 1, \ldots, m \end{aligned} \tag{3}$$

3. Qualitative analysis

In this section we take into account the initial value (I.V.) problem for Eq. (3) and we show that the solution of such I.V. problem exists and is a

positive, regular function of time, of class $C^1([0,T])$.
We start with

$$\begin{cases} \dfrac{d}{dt} f_{ir}(t) = Q^i[\mathbf{f}](t) = J_r^i[\mathbf{f}^i](t) + \mathcal{J}_r^i[\mathbf{f}^1, \ldots, \mathbf{f}^n](t) \\ f_{ir}(0) = f_i(0, u_r), \qquad i = 1, \ldots, n; \quad r = 1, \ldots, m, \end{cases} \tag{4}$$

where

$$J_r^i[\mathbf{f}^i](t) = \sum_{j=1}^{n} \sum_{h,k=1}^{m} \mathcal{B}_{hk}^{ij}(r)[\mathbf{f}^i]\, \eta_{hk}^{ij}\, f_h^i(t)\, f_k^j(t) - f_r^i(t) \sum_{j=1}^{n} \sum_{k=1}^{m} \eta_{rk}^{ij}\, f_k^j(t),$$

and

$$\mathcal{J}_r^i[\mathbf{f}^1, \ldots, \mathbf{f}^n](t) = \sum_{j=1}^{n} \sum_{h=1}^{m} \tilde{\mathcal{B}}_h^{ij}(r)[\mathbf{f}^i]\, \tilde{\eta}_h^{ij}\, f_h^i(t)\, \mathbb{M}^{(j)} - f_r^i(t) \sum_{j=1}^{n} \tilde{\eta}_r^{ij}\, \mathbb{M}^{(j)},$$

$i = 1, \ldots, n,\ r = 1, \ldots, m$

We introduce the space

$$X = \left\{ f^i : [0,T] \to \mathbb{R},\ f^i \in C^1([0,T]), i = 1, \ldots, n;\ T > 0 \right\}$$

characterized with the norm:

$$\|f^i\|_X = \sum_{r=1}^{m} \left| f_r^i(t) \right|.$$

Moreover, we define the Banach space $\mathbf{X} = X^n$ equipped with the norm:

$$\|\mathbf{f}\|_{\mathbf{X}} = \sum_{i=1}^{n} \|f^i(t)\|_X,$$

and set:

$$\mathbf{X}_+ = \left\{ \mathbf{f} \in \mathbf{X} \mid f^i \geq 0, i = 1, \ldots, n \right\}.$$

The following theorem states a result of local existence and uniqueness for the solution of the I.V. problem (4).

Theorem 3.1. *Consider the I.V. problem (4) with* $\mathbf{f}_0 = \{f^1(0,u), \dots, f^n(0,u)\} \in \mathbf{X}_+$. *Assume that*

$$\eta_{hk}^{ij} \geq 0, \quad \tilde{\eta}_h^{ij} \geq 0, \quad \mathscr{B}_{hk}^{ij}(r) \geq 0, \quad \tilde{\mathscr{B}}_h^{ij}(r) \geq 0, \tag{5}$$

$$\sum_{r=1}^{m} \mathcal{B}_{hk}^{ij}(r) = 1 \quad \forall i,j = 1,\dots,n, \forall h,k = 1,\dots,m,$$

$$\sum_{r=1}^{m} \tilde{\mathcal{B}}_h^{ij}(r) = 1 \quad \forall i,j = 1,\dots,n, \forall h = 1,\dots,m,$$

hold, together with the following hypotheses:

- *The encounter rates* η_{hk}^{ij} *and* $\tilde{\eta}_h^{ij}$ *satisfy the following conditions:*

$$\sum_{r=1}^{m} \eta_{hk}^{ij} \leq C, \qquad \sum_{r=1}^{m} \tilde{\eta}_h^{ij} \leq \tilde{C},$$

  $\forall h,k = 1,\dots,n \quad \forall i,j \in \{1,\dots,m\}$ *and* $\forall \mathbf{f} \in \mathbf{X}$ *with* C *and* $\tilde{C}$ *positive constants;*

- $\forall \mathbf{f} \in \mathbf{X}, \forall \boldsymbol{g} \in \mathbf{X}$ *the probabilities* $\mathscr{B}_{hk}^{ij}(r)$ *and* $\tilde{\mathscr{B}}_h^{ij}(r)$ *and the encounter rates* η_{hk}^{ij} *and* $\tilde{\eta}_h^{ij}$ *are Lipschitz continuous in* X, *that is,* $\forall\ h,k \in \{1,\dots,m\}$ *it results*

$$\sum_{i,j=1}^{n} \sum_{r=1}^{m} \mid \mathscr{B}_{hk}^{ij}(r)\,[\mathbf{f}] - \mathscr{B}_{hk}^{ij}(r)\,[\mathbf{g}] \mid \leq L_1 \|\mathbf{f}-\mathbf{g}\|_{\mathbf{X}},$$

$$\sum_{i,j=1}^{n} \sum_{r=1}^{m} \mid \tilde{\mathscr{B}}_h^{ij}(r)\,[\mathbf{f}] - \tilde{\mathscr{B}}_h^{ij}(r)\,[\mathbf{g}] \mid \leq \tilde{L}_1 \|\mathbf{f}-\mathbf{g}\|_{\mathbf{X}},$$

$$\sum_{i,j=1}^{n} \sum_{r=1}^{m} \mid \eta_{hk}^{ij}\,[\mathbf{f}] - \eta_{hk}^{ij}\,[\mathbf{g}] \mid \leq L_2 \|\mathbf{f}-\mathbf{g}\|_{\mathbf{X}},$$

$$\sum_{i,j=1}^{n} \sum_{r=1}^{m} \mid \tilde{\eta}_h^{ij}\,[\mathbf{f}] - \tilde{\eta}_h^{ij}\,[\mathbf{g}] \mid \leq \tilde{L}_2 \|\mathbf{f}-\mathbf{g}\|_{\mathbf{X}},$$

  with L_1, $\tilde{L}_1$, L_2 *and* $\tilde{L}_2$ *positive constants.*

Then, there exist $T > 0$ *and a unique solution* $\mathbf{f}(t)$ *in* $\mathbf{X}$ *for the I.V. problem (4) on the time interval* $[0,T]$. *Moreover* $\mathbf{f}(t) \in \mathbf{X}_+$, $t \in [0,T]$.

Proof. We start observing that, since the interactions are assumed number conservative, see (5), it results that:

$$\frac{d}{dt}\sum_{r=1}^{m} f_r^i(t) = 0\,,$$

which implies:

$$\|\mathbf{f}(t)\|_{\mathbf{X}} = \|\mathbf{f}(0)\|_{\mathbf{X}},\ \text{for any } t \geq 0. \tag{6}$$

Therefore the solution of (4), if it exists, remains bounded in $\mathbf{X}$ for any time $t \geq 0$. The latter observation assures that the operator $Q^i\left[\mathbf{f}\right](t)$ in the right hand side of (4) is a closed map in $\mathbf{X}$. Let us now prove that $Q^i\left[\mathbf{f}\right](t)$ is Lipschitz continuous in $\mathbf{X}$, i.e. given $\|\mathbf{f}\|_{\mathbf{X}} \leq M$ and $\|\mathbf{g}\|_{\mathbf{X}} \leq M$ it follows that:

$$\|Q^i\left[\mathbf{f}\right](t) - Q^i\left[\mathbf{g}\right](t)\|_{\mathbf{X}} \leq L\|\mathbf{f}-\mathbf{g}\|_{\mathbf{X}} \tag{7}$$

with L a positive constant depending on M.
Following the same lines of the proof of the theorem in [14, 22], considering that $Q^i\left[\mathbf{f}\right](t) = J_r^i[\mathbf{f}^i](t) + \mathcal{J}_r^i[\mathbf{f}^1,\ldots,\mathbf{f}^n](t)$ we can prove (7). Then, the existence of a unique solution $\mathbf{f}(t)$ in $\mathbf{X}$, local in time, to (4) follows. Non negativity of such a solution is easily obtained observing that the components $f_r^i(t)$ of the solution satisfy the condition:

$$f_r^i \geq 0 \quad \forall i = 1,\ldots,n \quad \textit{and} \quad \forall r = 1,\ldots,m \tag{8}$$

when $\mathbf{f}(0) \in \mathbf{X}_+$. We set:

$$R^i(f,f)(t) = \sum_{j=1}^{n}\sum_{h,k=1}^{m}\Big(\eta_{hk}^{ij}\mathscr{B}_{hk}^{ij}(r)\left[\mathbf{f}\right]f_h^i(t)f_k^j(t) + \tilde{\eta}_h^{ij}\tilde{\mathscr{B}}_h^{ij}(r)\left[\mathbf{f}\right]f_h^i(t)f_k^j(t)\Big),$$

$$S(f)(t) = \sum_{k=1}^{n}\sum_{j=1}^{m}\big(\eta_{rk}^{ij}f_k^j(t) + \tilde{\eta}_r^{ij}\mathbb{M}^{(j)}\big).$$

Equation (3) can be rewritten as

$$\frac{d}{dt}f_r^i(t) + f_r^i(t)S(f)(t) = R^i(f,f)(t). \tag{9}$$

Now we call

$$\lambda(t) = \int_0^t S(f)(t^{'})dt^{'}.$$

If f_r^i is solution of (9), it then follows

$$\frac{d}{dt}(\exp(\lambda(t))f_r^i(t)) = \exp(\lambda(t))R^i(f,f)(t)$$

which implies

$$f_r^i(t) = \exp(-\lambda(t))f_r^i(0) + \int_0^t \left[\exp(\lambda(t^{'}))R^i(f,f)(t^{'})\right] dt^{'}. \tag{10}$$

The relation (10) allows us to conclude that, given $\mathbf{f}(0) \in \mathbf{X}_+$ and the positivity of the integral function, the function $f_r^i(t)$ satisfies the condition of non-negativity (8) in its domain of existence. Moreover, when (8) is used together with (6), we obtain that the solution to (4) is uniformly bounded on any compact time interval $[0,T]$, $T > 0$. This latter observation leads immediately to the following result of global existence of the solution in $\mathbf{X}_+$

Theorem 3.2. *Consider the I.V. problem (4) under the assumptions of the theorem 3.1. Then the solution* $\mathbf{f}(t)$ *exists for any finite time* $t \geq 0$.

4. A specific model of learning dynamics on networks

We introduce now a phenomenological learning dynamics in a population structured in a network of three nodes where $i = 1$ denotes the leader node and $i = 2, 3$ denote the follower nodes. For a specific model one needs to introduce given probability transitions characterising the stochastic microscopic interactions among the active particles. Due to the particular network topology we have introduced, we need three probability transitions: on the leader node, on the follower nodes and on the whole network.

4.1. TRANSITION PROBABILITY DENSITIES Binary interactions

We first consider the transition probability densities $\mathscr{B}_{hk}^{ij}[\mathbf{f}^i]$ due to microscopic interactions. Such interactions take place both between particles belonging to the same subsystem and between particles of different subsystems.

The tables below report some of the most relevant interaction rules.

The leader node is characterized by the property that its particles do not change their level of knowledge when interacting with particles belonging to the follower nodes. On the other hand, when a candidate particle u_h of the leader node interacts with a field particle u_k in the same node, in the case $u_h < u_k$, with $u_h \geq \mathbb{M}^{(1)}$, the probability of the transition $u_h \to u_{h+1}$ increases as the distance $|u_h - u_k|$ decreases.
When instead $u_h < u_k$, with $u_h < \mathbb{M}^{(1)}$, the probability of the transition $u_h \to u_{h+1}$ increases as the distances $|\mathbb{M}^{(1)} - u_h|$ and $|u_h - u_k|$ increase. When $u_h \geq u_k$ the microscopic state of the candidate particle does not change. In other words, for the particles belonging to the leader node no regressions of the level of knowledge are allowed. The corresponding tables are shown below.

LEADER - LEADER

$$
\begin{cases}
u_h < u_k \begin{cases}
u_h \geq \mathbb{M}^{(1)} \begin{cases}
\mathscr{B}^{11}_{hk}(r = h) & = |u_h - u_k| \\
\mathscr{B}^{11}_{hk}(r = h+1) & = 1 - |u_h - u_k| \\
\mathscr{B}^{11}_{hk}(r \neq h, r \neq h+1) & = 0
\end{cases} \\
u_h < \mathbb{M}^{(1)} \begin{cases}
\mathscr{B}^{11}_{hk}(r = h) & = 1 - \dfrac{\left|\mathbb{M}^{(1)} - u_h\right| + |u_h - u_k|}{2} \\
\mathscr{B}^{11}_{hk}(r = h+1) & = \dfrac{\left|\mathbb{M}^{(1)} - u_h\right| + |u_h - u_k|}{2} \\
\mathscr{B}^{11}_{hk}(r \neq h, r \neq h+1) & = 0
\end{cases}
\end{cases} \\
u_h \geq u_k \begin{cases}
\mathscr{B}^{11}_{hk}(r = h) & = 1 \\
\mathscr{B}^{11}_{hk}(r \neq h) & = 0
\end{cases}
\end{cases}
$$

LEADER - SUBSYSTEM 2

$$
\begin{cases}
\mathscr{B}^{12}_{hk}(r = h) & = 1 \\
\mathscr{B}^{12}_{hk}(r \neq h) & = 0
\end{cases}
$$

LEADER - SUBSYSTEM 3

$$
\begin{cases}
\mathscr{B}^{13}_{hk}(r = h) & = 1 \\
\mathscr{B}^{13}_{hk}(r \neq h) & = 0
\end{cases}
$$

Next, we consider the microscopic interactions between particles of the node $i = 2$ with particles of the leader node $i = 1$. We always assume that the level of knowledge u_k (leader node) is higher than u_h (follower node). We first look at the case $u_h \geq \mathbb{M}^{(2)}$: if $u_k \geq \mathbb{M}^{(1)}$, then we have the transition $u_h \to u_{h+1}$; if $u_k < \mathbb{M}^{(1)}$, then the probability of the transition $u_h \to u_{h+1}$ decreases as the distance $|\mathbb{M}^{(2)} - u_h|$ increases. Let as now look at the case $u_h < \mathbb{M}^{(2)}$: when $u_k \geq \mathbb{M}^{(1)}$, then the probability of the transition $u_h \to u_{h+1}$ increases as the distance $|\mathbb{M}^{(2)} - u_h|$ increases. When instead $u_k < \mathbb{M}^{(1)}$, the candidate particle is encouraged to improve its level of knowledge by the driving effects of the two distances $|\mathbb{M}^{(2)} - u_h|$ and $|u_h - u_k|$. The related probability transition densities are reported below.

SUBSYSTEM 2 - LEADER

$$
h = m \begin{cases} \mathscr{B}^{21}_{hk}(r = m) & = 1 \\ \mathscr{B}^{21}_{hk}(r \neq m) & = 0 \end{cases}
$$

$$
h \neq m \begin{cases}
u_h \geq \mathbb{M}^{(2)} \begin{cases} u_k \geq \mathbb{M}^{(1)} \begin{cases} \mathscr{B}^{21}_{hk}(r = h+1) & = 1 \\ \mathscr{B}^{21}_{hk}(r \neq h+1) & = 0 \end{cases} \\ u_k < \mathbb{M}^{(1)} \begin{cases} \mathscr{B}^{21}_{hk}(r = h) & = \left|\mathbb{M}^{(2)} - u_h\right| \\ \mathscr{B}^{21}_{hk}(r = h+1) & = 1 - \left|\mathbb{M}^{(2)} - u_h\right| \\ \mathscr{B}^{21}_{hk}(r \neq h, r \neq h+1) & = 0 \end{cases} \end{cases} \\
u_h < \mathbb{M}^{(2)} \begin{cases} u_k \geq \mathbb{M}^{(1)} \begin{cases} \mathscr{B}^{21}_{hk}(r = h) & = 1 - \left|\mathbb{M}^{(2)} - u_h\right| \\ \mathscr{B}^{21}_{hk}(r = h+1) & = \left|\mathbb{M}^{(2)} - u_h\right| \\ \mathscr{B}^{21}_{hk}(r \neq h, r \neq h+1) & = 0 \end{cases} \\ u_k < \mathbb{M}^{(1)} \begin{cases} \mathscr{B}^{21}_{hk}(r = h) & = 1 - \left|\mathbb{M}^{(2)} - u_h\right| \cdot |u_h - u_k| \\ \mathscr{B}^{21}_{hk}(r = h+1) & = \left|\mathbb{M}^{(2)} - u_h\right| \cdot |u_h - u_k| \\ \mathscr{B}^{21}_{hk}(r \neq h, r \neq h+1) & = 0 \end{cases} \end{cases} \\
u_h = u_k \begin{cases} \mathscr{B}^{21}_{hk}(r = h) & = 1 \\ \mathscr{B}^{21}_{hk}(r \neq h) & = 0 \end{cases}
\end{cases}
$$

The next table reports the transition probability densities $\mathscr{B}^{22}_{hk}[\mathbf{f}^i]$ related to interactions between active particles localized in the second node ($i = 2$). In the case $u_h < u_k$, with $u_h \geq \mathbb{M}^{(2)}$ the level of knowledge u_h tends to increase, $u_h \to u_{h+1}$, as the distance $|u_h - u_k|$ decreases and it tends to

remain the same otherwise. For $u_h < u_k$, with $u_h < \mathbb{M}^{(2)}$, the transition $u_h \to u_{h+1}$ is favoured by an increasing value of the two distances $|u_h - u_k|$ and $|\mathbb{M}^{(2)} - u_h|$.
The situation is instead different in the second part of the table, when the level of knowledge of the candidate particle u_h is bigger than that of the field particle u_k. In this case indeed, there is a possibility of a regression, due to misleading informations which can reduce the level of knowledge. For $u_h > u_k$, when $u_h \geq \mathbb{M}^{(2)}$, we see that u_h can either stay the same or decrease: the probability of the transition $u_h \to u_{h-1}$ increases as the distance $|u_h - u_k|$ increases. Instead, when $u_h < \mathbb{M}^{(2)}$, also the distance $|\mathbb{M}^{(2)} - u_h|$ plays a role and the regression $u_h \to u_{h-1}$ is favoured by a decrease of the two distances $|\mathbb{M}^{(2)} - u_h|$ and $|u_h - u_k|$.

SUBSYSTEM 2 - SUBSYSTEM 2

$$
\begin{cases}
u_h < u_k \begin{cases}
u_h \geq \mathbb{M}^{(2)} \begin{cases}
\mathscr{B}^{22}_{hk}(r = h) & = |u_h - u_k| \\
\mathscr{B}^{22}_{hk}(r = h+1) & = 1 - |u_h - u_k| \\
\mathscr{B}^{22}_{hk}(r \neq h, r \neq h+1) & = 0
\end{cases} \\
u_h < \mathbb{M}^{(2)} \begin{cases}
\mathscr{B}^{22}_{hk}(r = h) & = 1 - \dfrac{\left|\mathbb{M}^{(2)} - u_h\right| + |u_h - u_k|}{2} \\
\mathscr{B}^{22}_{hk}(r = h+1) & = \dfrac{\left|\mathbb{M}^{(2)} - u_h\right| + |u_h - u_k|}{2} \\
\mathscr{B}^{22}_{hk}(r \neq h, r \neq h+1) & = 0
\end{cases}
\end{cases} \\
u_h > u_k \begin{cases}
u_h \geq \mathbb{M}^{(2)} \begin{cases}
\mathscr{B}^{22}_{hk}(r = h-1) & = |u_h - u_k| \\
\mathscr{B}^{22}_{hk}(r = h) & = 1 - |u_h - u_k| \\
\mathscr{B}^{22}_{hk}(r \neq h, r \neq h+1) & = 0
\end{cases} \\
u_h < \mathbb{M}^{(2)} \begin{cases}
\mathscr{B}^{22}_{hk}(r = h-1) & = 1 - \dfrac{\left|\mathbb{M}^{(2)} - u_h\right| + |u_h - u_k|}{2} \\
\mathscr{B}^{22}_{hk}(r = h) & = \dfrac{\left|\mathbb{M}^{(2)} - u_h\right| + |u_h - u_k|}{2} \\
\mathscr{B}^{22}_{hk}(r \neq h-1, r \neq h) & = 0
\end{cases}
\end{cases} \\
u_h = u_k \begin{cases}
\mathscr{B}^{22}_{hk}(r = h) & = 1 \\
\mathscr{B}^{22}_{hk}(r \neq h) & = 0
\end{cases}
\end{cases}
$$

When we consider binary interactions between particles belonging to the follower nodes 2 and 3, we assume that the transition probability densities

$\mathscr{B}_{hk}^{23}[f^i]$ only depend on the distance $|u_h - u_k|$.
Indeed, for $u_h < u_k$, a small value of this distance tends to keep u_h fixed; as the distance increases, then the transition $u_h \to u_{h+1}$ is favoured. In the case $u_h \geq u_k$, we see instead the possibility of a regression, $u_h \to u_{h-1}$, which is induced by a growth of the distance $|u_h - u_k|$.
The corresponding table is shown below.

SUBSYSTEM 2 - SUBSYSTEM 3

$$
u_h < u_k \begin{cases} \mathscr{B}_{hk}^{23}(r = h) & = 1 - |u_h - u_k| \\ \mathscr{B}_{hk}^{23}(r = h+1) & = |u_h - u_k| \\ \mathscr{B}_{hk}^{23}(r \neq h, r \neq h+1) & = 0 \end{cases}
$$

$$
u_h \geq u_k \begin{cases} \mathscr{B}_{hk}^{23}(r = h-1) & = |u_h - u_k| \\ \mathscr{B}_{hk}^{23}(r = h) & = 1 - |u_h - u_k| \\ \mathscr{B}_{hk}^{23}(r \neq h, r \neq h-1) & = 0 \end{cases}
$$

The microscopic interactions corresponding to candidate particles belonging to the node 3 can be discussed along the same lines as the previous ones and will not be reported here.
We then turn our attention to the transition probability densities corresponding to the micro-macro interactions between a particle and a node.
Particles belonging to the leader node can change their state u_h only under the influence of their own node. Indeed, the tables below show that the transition probability density $\widetilde{\mathscr{B}}_{hk}^{11}[\mathbf{f}^i]$ depends only on the distance $|\mathbb{M}^{(1)} - u_h|$ when is $u_h < \mathbb{M}^{(1)}$, with the transition $u_h \to u_{h+1}$ which is favoured as the distance increases. Otherwise, the level of knowledge does not change. Similarly there is no change of u_h due to the influence of the follower nodes 2 and 3.

4.2. **TRANSITION PROBABILITY DENSITIES**
Multiple interactions

LEADER - LEADER

$$u_h < \mathbb{M}^{(1)} \begin{cases} \widetilde{\mathscr{B}}^{11}_{hk}(r = h) & = 1 - \left|\mathbb{M}^{(1)} - u_h\right| \\ \widetilde{\mathscr{B}}^{11}_{hk}(r = h+1) & = \left|\mathbb{M}^{(1)} - u_h\right| \\ \widetilde{\mathscr{B}}^{11}_{hk}(r \neq h, r \neq h+1) & = 0 \end{cases}$$

$$u_h \geq \mathbb{M}^{(1)} \begin{cases} \widetilde{\mathscr{B}}^{11}_{hk}(r = h) & = 1 \\ \widetilde{\mathscr{B}}^{11}_{hk}(r \neq h) & = 0 \end{cases}$$

LEADER - SUBSYSTEM 2

$$\begin{cases} \widetilde{\mathscr{B}}^{12}_{hk}(r = h) & = 1 \\ \widetilde{\mathscr{B}}^{12}_{hk}(r \neq h) & = 0 \end{cases}$$

LEADER - SUBSYSTEM 3

$$\begin{cases} \widetilde{\mathscr{B}}^{13}_{hk}(r = h) & = 1 \\ \widetilde{\mathscr{B}}^{13}_{hk}(r \neq h) & = 0 \end{cases}$$

The situation is of course different when we turn our attention to the micro-macro interactions between a candidate particle belonging to the node $i = 2$ and the three nodes in the network.
In the first table below we have reported the transition probability densities due to the influence of the leader node on the candidate particle u_h. For this case no possibility of regression of the level of knowledge is allowed. The transition $u_h \to u_{h+1}$ is favoured by an increase of the distance $|\mathbb{M}^{(1)} - u_h|$ when $u_h \geq \mathbb{M}^{(2)}$; in the case $u_h < \mathbb{M}^{(2)}$, the same transition is induced by an increase of both the distances $|\mathbb{M}^{(1)} - u_h|$ and $|\mathbb{M}^{(2)} - u_h|$.

SUBSYSTEM 2 - LEADER

$$
u_h \geq \mathbb{M}^{(2)} \begin{cases} u_h \geq \mathbb{M}^{(1)} \begin{cases} \widetilde{\mathscr{B}}^{21}_{hk}(r=h) & =1 \\ \widetilde{\mathscr{B}}^{21}_{hk}(r\neq h) & =0 \end{cases} \\ u_h < \mathbb{M}^{(1)} \begin{cases} \widetilde{\mathscr{B}}^{21}_{hk}(r=h) & =1-|\mathbb{M}^{(1)}-u_h| \\ \widetilde{\mathscr{B}}^{21}_{hk}(r=h+1) & =|\mathbb{M}^{(1)}-u_h| \\ \widetilde{\mathscr{B}}^{21}_{hk}(r\neq h, r\neq h+1) & =0 \end{cases} \end{cases}
$$

$$
u_h < \mathbb{M}^{(2)} \begin{cases} u_h \geq \mathbb{M}^{(1)} \begin{cases} \widetilde{\mathscr{B}}^{21}_{hk}(r=h) & =1 \\ \widetilde{\mathscr{B}}^{21}_{hk}(r\neq h) & =0 \end{cases} \\ u_h < \mathbb{M}^{(1)} \begin{cases} \widetilde{\mathscr{B}}^{21}_{hk}(r=h) & =1-\dfrac{|\mathbb{M}^{(1)}-u_h|+|\mathbb{M}^{(2)}-u_h|}{2} \\ \widetilde{\mathscr{B}}^{21}_{hk}(r=h+1) & =\dfrac{|\mathbb{M}^{(1)}-u_h|+|\mathbb{M}^{(2)}-u_h|}{2} \\ \widetilde{\mathscr{B}}^{21}_{hk}(r\neq h, r\neq h+1) & =0 \end{cases} \end{cases}
$$

Next, the candidate particle u_h belonging to the node 2, is under the influence of its own node. Then we have a possible transition $u_h \to u_{h+1}$, in the case $u_h < M^{(2)}$, with a probability which increases as the distance $|M^{(2)} - u_h|$ increases.
We also have a possible transition $u_h \to u_{h-1}$, in the case $u_h > M^{(2)}$, again with a probability which increases as the distance $|M^{(2)} - u_h|$ increases.
Of course in the case $u_h = M^{(2)}$ the level of knowledge does not change. The corresponding table is shown below.

SUBSYSTEM 2 - SUBSYSTEM 2

$$
u_h < \mathbb{M}^{(2)} \begin{cases} \widetilde{\mathscr{B}}^{22}_{hk}(r=h) & =1-\left|\mathbb{M}^{(2)}-u_h\right| \\ \widetilde{\mathscr{B}}^{22}_{hk}(r=h+1) & =\left|\mathbb{M}^{(2)}-u_h\right| \\ \widetilde{\mathscr{B}}^{22}_{hk}(r\neq h, r\neq h+1) & =0 \end{cases}
$$

$$
u_h > \mathbb{M}^{(2)} \begin{cases} \widetilde{\mathscr{B}}^{22}_{hk}(r=h-1) & =\left|\mathbb{M}^{(2)}-u_h\right| \\ \widetilde{\mathscr{B}}^{22}_{hk}(r=h) & =1-\left|\mathbb{M}^{(2)}-u_h\right| \\ \widetilde{\mathscr{B}}^{22}_{hk}(r\neq h, r\neq h+1) & =0 \end{cases}
$$

$$
u_h = \mathbb{M}^{(2)} \begin{cases} \widetilde{\mathscr{B}}^{22}_{hk}(r=h) & =1 \\ \widetilde{\mathscr{B}}^{22}_{hk}(r\neq h) & =0 \end{cases}
$$

The next table refers to the transition probability densities due to the influence of the node i=3 on the candidate particle u_h belonging to the node $i = 2$. It can be discussed along the same lines as the previous table.

SUBSYSTEM 2 - SUBSYSTEM 3

$$
\begin{aligned}
u_h < \mathbb{M}^{(3)} & \begin{cases} \widetilde{\mathscr{B}}^{23}_{hk}(r = h) & = 1 - \left|\mathbb{M}^{(3)} - u_h\right| \\ \widetilde{\mathscr{B}}^{23}_{hk}(r = h+1) & = \left|\mathbb{M}^{(3)} - u_h\right| \\ \widetilde{\mathscr{B}}^{23}_{hk}(r \neq h, r \neq h+1) & = 0 \end{cases} \\
u_h \geq \mathbb{M}^{(3)} & \begin{cases} \widetilde{\mathscr{B}}^{23}_{hk}(r = h-1) & = \left|\mathbb{M}^{(3)} - u_h\right| \\ \widetilde{\mathscr{B}}^{23}_{hk}(r = h) & = 1 - \left|\mathbb{M}^{(3)} - u_h\right| \\ \widetilde{\mathscr{B}}^{23}_{hk}(r \neq h, r \neq h-1) & = 0 \end{cases}
\end{aligned}
$$

Similar tables to the ones reported above, can be obtained for the transition probability densities of a candidate particle belonging to the node $i = 3$ and interacting with the three nodes of the network.
Before closing this section, we need to define the encounter rates for the model under study. We choose them to be characteristic of the nodes but independent of the activity.
Accordingly, for the micro-micro interactions we write:

$$\eta^{ij}_{hk} = \eta^{ij}_0 \qquad \forall\, i,j = 1,\ldots,3 \quad \forall\, h,k = 1,\ldots,m \tag{11a}$$

and for the micro-macro interactions:

$$\tilde{\eta}^{ij}_{h} = \tilde{\eta}^{ij}_0 \qquad \forall\, i,j = 1,\ldots,3 \quad \forall\, h = 1,\ldots,m \tag{11b}$$

5. Simulations and numerical results

The kinetic model defined by Eq.3, characterized by the stochastic interaction rules given through the transition probability densities specified in section 4, is now solved numerically for different initial configurations and for different values of the encounter rate. In the following we consider two sets of simulations. We start by including in the model only the microscopic interactions: in other words, only the first two terms in the right hand side of Eq.3 are taken into account. In Fig.3-5 the initial (left) and final (right) configurations of the distribution functions $f^i(u)$, $i = 1\ldots,3$,

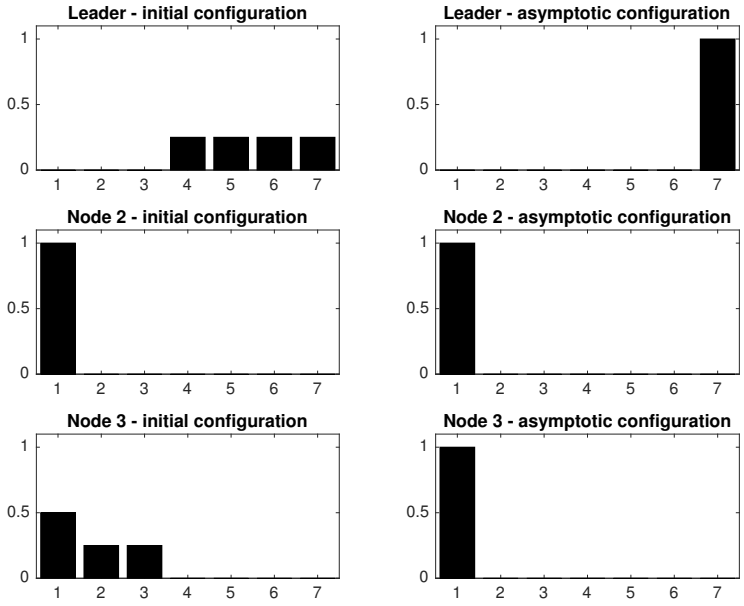

Figure 3: Initial (left) and asymptotic (right) configurations of the distribution functions $f^i(u)$ versus u for $i = 1, \ldots, 3$ from top to bottom. $\eta_0^{11} = \eta_0^{22} = \eta_0^{23} = \eta_0^{32} = \eta_0^{33} = 0.8$, $\eta_0^{12} = \eta_0^{21} = \eta_0^{13} = \eta_0^{31} = 0$ $\widetilde{\eta}_0^{ij} = 0$, $i, j = 1, \ldots, 3$.

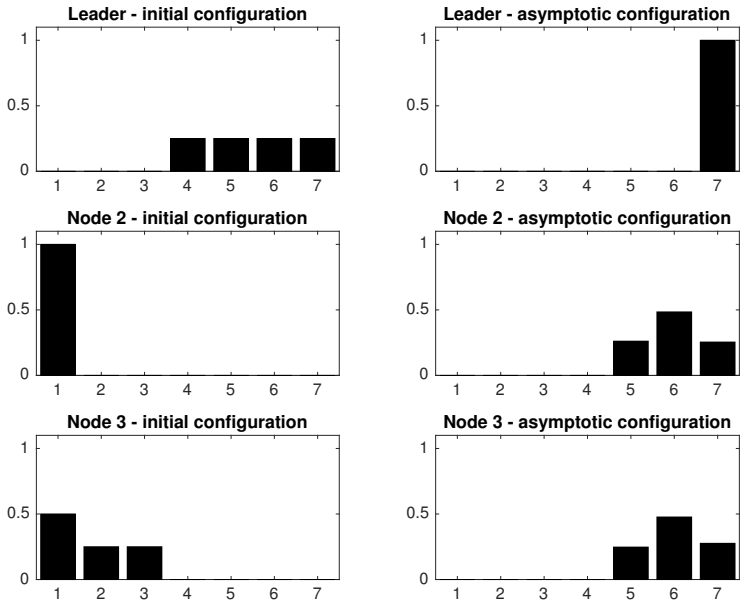

Figure 4: Initial (left) and asymptotic (right) configurations of the distribution functions $f^i(u)$ versus u for $i = 1, \ldots, 3$ from top to bottom. $\eta_0^{11} = \eta_0^{13} = \eta_0^{22} = \eta_0^{23} = \eta_0^{31} = \eta_0^{32} = \eta_0^{33} = 0.8$, $\eta_0^{12} = \eta_0^{21} = 0$,$\widetilde{\eta}_0^{ij} = 0$, $i, j = 1, \ldots, 3$.

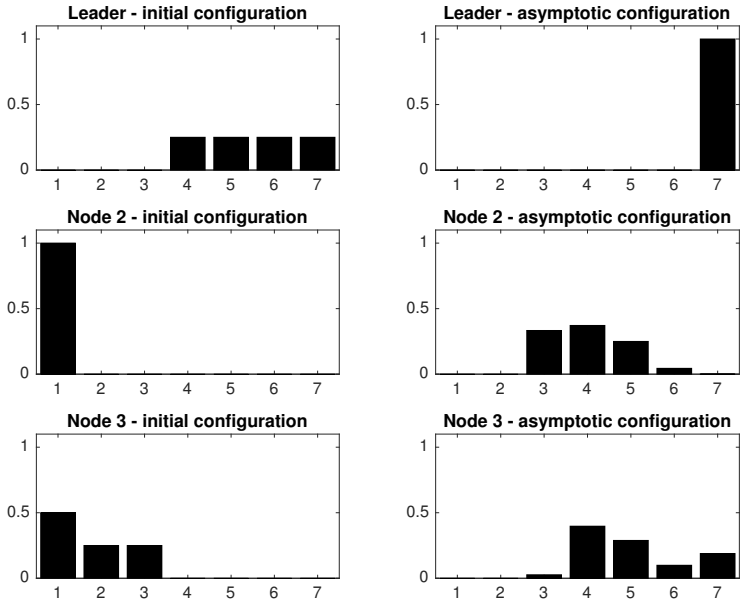

Figure 5: Initial (left) and asymptotic (right) configurations of the distribution functions $f^i(u)$ versus u for $i = 1, \ldots, 3$ from top to bottom. $\eta_0^{11} = \eta_0^{13} = \eta_0^{22} = \eta_0^{31} = \eta_0^{33} = 0.8$, $\eta_0^{12} = \eta_0^{21} = 0$, $\eta_0^{23} = \eta_0^{32} = 0.4$, $\widetilde{\eta}_0^{ij} = 0$, $i, j = 1, \ldots, 3$.

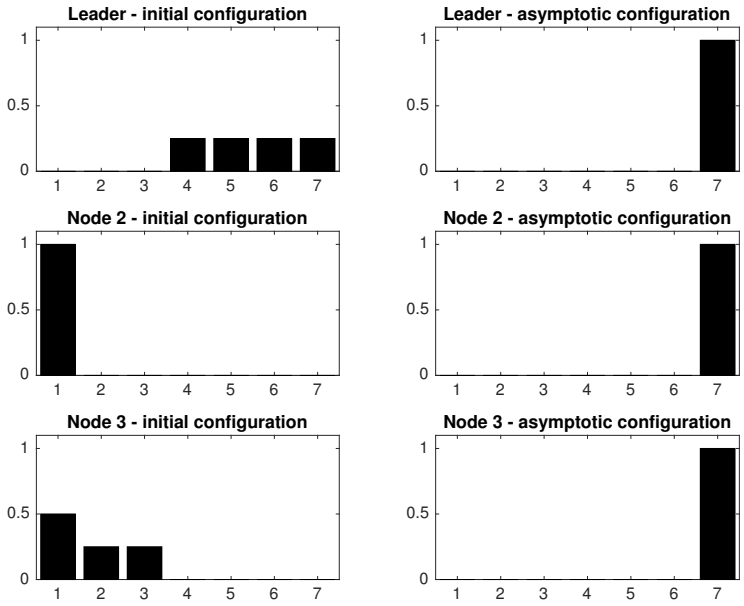

Figure 6: Initial (left) and asymptotic (right) configurations of the distribution functions $f^i(u)$ versus u for $i = 1, \ldots, 3$ from top to bottom. $\eta_0^{ij} = \widetilde{\eta}_0^{ij} = 0.8$, $i, j = 1, \ldots, 3$.

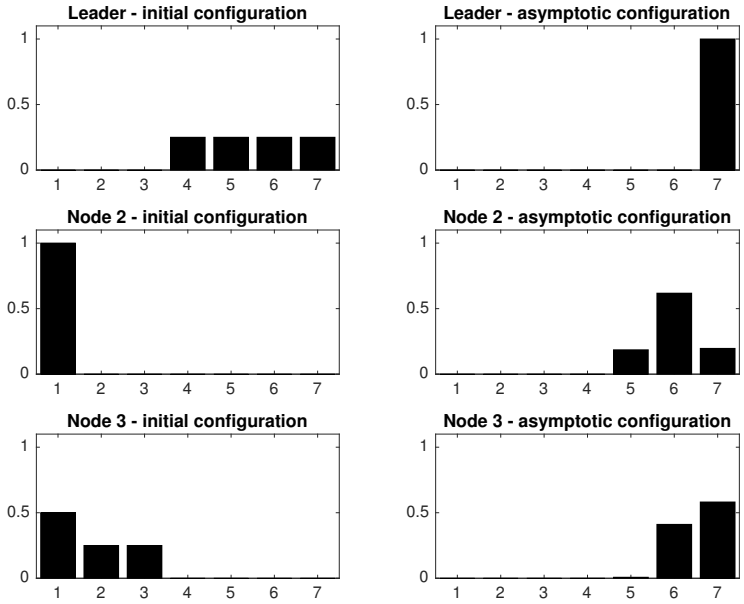

Figure 7: Initial (left) and asymptotic (right) configurations of the distribution functions $f^i(u)$ versus u for $i = 1, \ldots, 3$ from top to bottom. $\eta_0^{11} = \eta_0^{13} = \eta_0^{22} = \eta_0^{31} = \eta_0^{33} = \widetilde{\eta}_0^{11} = \widetilde{\eta}_0^{13} = \widetilde{\eta}_0^{22} = \widetilde{\eta}_0^{31} = \widetilde{\eta}_0^{33} = 0.8$, $\eta_0^{12} = \eta_0^{21} = \widetilde{\eta}_0^{12} = \widetilde{\eta}_0^{21} = 0$, $\eta_0^{23} = \eta_0^{32} = \widetilde{\eta}_0^{23} = \widetilde{\eta}_0^{32} = 0.4$.

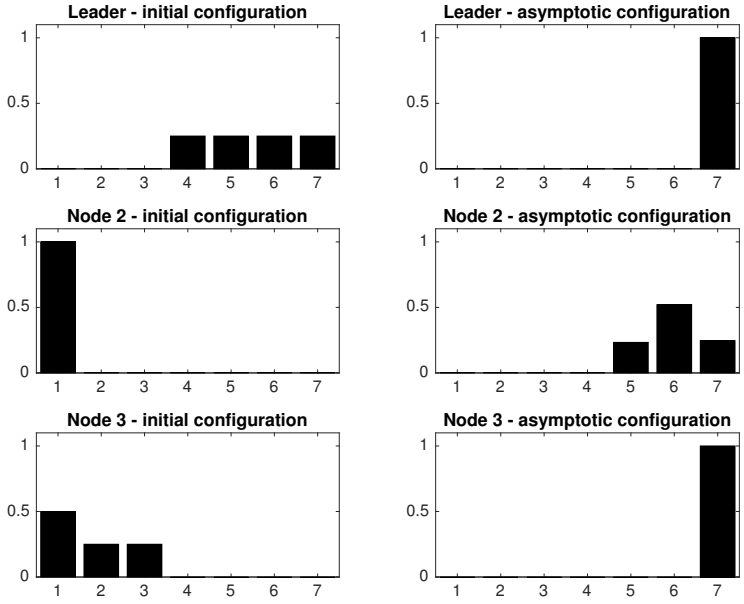

Figure 8: Initial (left) and asymptotic (right) configurations of the distribution functions $f^i(u)$ versus u for $i = 1, \ldots, 3$ from top to bottom. $\eta_0^{11} = \eta_0^{13} = \eta_0^{22} = \eta_0^{31} = \eta_0^{33} = \widetilde{\eta}_0^{11} = \widetilde{\eta}_0^{13} = \widetilde{\eta}_0^{22} = \widetilde{\eta}_0^{31} = \widetilde{\eta}_0^{33} = 0.8$, $\eta_0^{12} = \eta_0^{21} = \widetilde{\eta}_0^{12} = \widetilde{\eta}_0^{21} = 0.2$, $\eta_0^{23} = \eta_0^{32} = \widetilde{\eta}_0^{23} = \widetilde{\eta}_0^{32} = 0$.

are shown versus the level of knowledge u. In Fig.3 we have not considered microscopic interactions between the particles of the follower nodes and those of the leader node. We observe that the leader node has of course its autonomous evolution, while the third node has a regression, due to the microscopic interactions with the particles of the second node. (In this case we put $\eta_0^{23} = \eta_0^{32} = \eta_0^{11} = \eta_0^{22} = \eta_0^{33} = 0.8$, $\eta_0^{12} = \eta_0^{21} = \eta_0^{13} = \eta_0^{31} = 0$). In Fig.4 we show the results obtained when to the interactions of the previous case we add the microscopic interactions between particles of the third node and the leader. In other words we now put $\eta_0^{13} = \eta_0^{31} = 0.8$ leaving all the others η_0^{ij} the same values as in the previous case. We now see a remarkable change in the final configurations of $f^2(u)$ and $f^3(u)$: the third node has a direct connection with the leader and tends to follow it in its evolution, on the other hand the microscopic interaction with the second node (not directly connected) with the leader, has a damping effect on the growth of the level of knowledge of the third node. The final configuration of the two follower nodes is the same, under the strong influence of the leader. A similar situation is presented in Fig.5, where we have still $\eta_0^{13} = \eta_0^{31} = 0.8$ but the encounter rate between the two follower nodes is smaller: $\eta_0^{23} = \eta_0^{32} = 0.4$.
In the second set of simulations we consider the effects of all the interactions involved on the network. In other words the micro-macro interactions are now included which in turn implies that all the terms in the right hand side of Eq.3 are taken into account.
We start with Fig.6 where the evolution of $f^2(u)$ and $f^3(u)$ from the initial to the final configuration shows a very strong "follow the leader" effect. Indeed, in this case al the η_0^{ij}, $i, j = 0, \ldots, 3$ and the $\widetilde{\eta}_0^{ij}$, $i, j = 1 \ldots, 3$ are fixed at the value 0.8. The "follow the leader" effect is still evident in Fig.7, although now $f^2(u)$ and $f^3(u)$ have different final configurations with lower levels of knowledge than in the previous case. Indeed in this case the particles of the node $i = 2$ have no interactions with the leader node ($\eta_0^{12} = \eta_0^{21} = \widetilde{\eta}_0^{12} = \widetilde{\eta}_0^{21} = 0$) but they have interaction rate $\eta_0^{23} = \eta_0^{32} = \widetilde{\eta}_0^{23} = \widetilde{\eta}_0^{32} = 0.4$. All the other interaction rates are the same as in Fig.6.
Finally, in Fig.8 it is shown the case where the two follower nodes are totally disconnected between them ($\eta_0^{23} = \eta_0^{32} = \widetilde{\eta}_0^{23} = \widetilde{\eta}_0^{32} = 0$). Their particles though have microscopic and macro-micro interactions with particles belonging to the leader node. The distribution functions $f^2(u)$ and $f^3(u)$ have different final ("follow the leader") configurations, essentially due to the different values of the encounter rates ($\eta_0^{12} = \eta_0^{21} = \widetilde{\eta}_0^{12} = \widetilde{\eta}_0^{21} = 0.2$) and ($\eta_0^{13} = \eta_0^{31} = \widetilde{\eta}_0^{13} = \widetilde{\eta}_0^{31} = 0.8$).

6. Conclusions and perspectives

The learning dynamics on network described in this paper is determined by the interactions (binary and multiple) among the individuals. The leader has a strong influence on the evolution of the system, since it can induce a change in the level of knowledge of the other individuals, without changing its own.
From the social point of view [23] it would be of interest to develop an application of the model to a problem of support/opposition formation in political choices, where the role of the leader is to influence other's people opinions. In such context, we could of course include the phenomenon of people migration among the nodes of the network, as an effect of opinion changes. Moreover, we look forward the extension of our study in order to describe open systems in the sense of including in the model external effects such as the role of the Media. Also, inspired by Ref.[24], it would be of great interest to consider learning propagation on contact networks characterized by different leaders. In such case of course, we should include in our model also suitable interactions between the two networks. We plan to address this problem in a near future.

7. References

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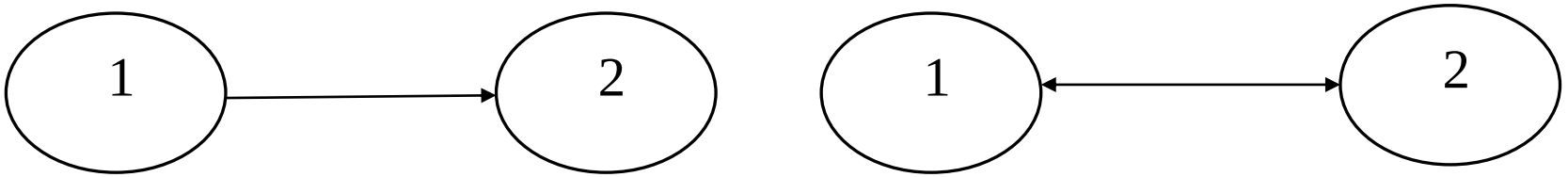

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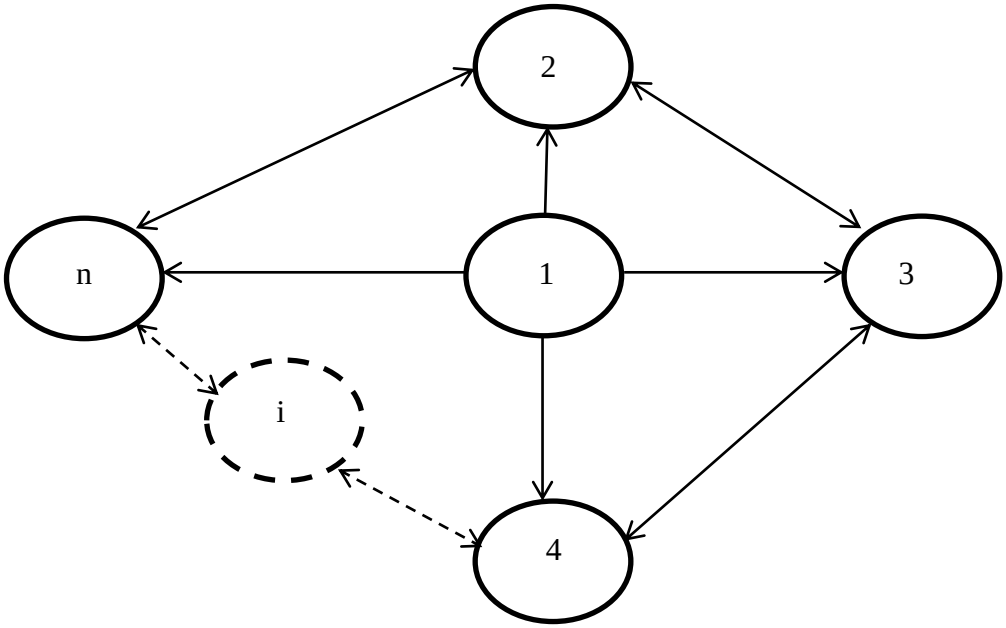

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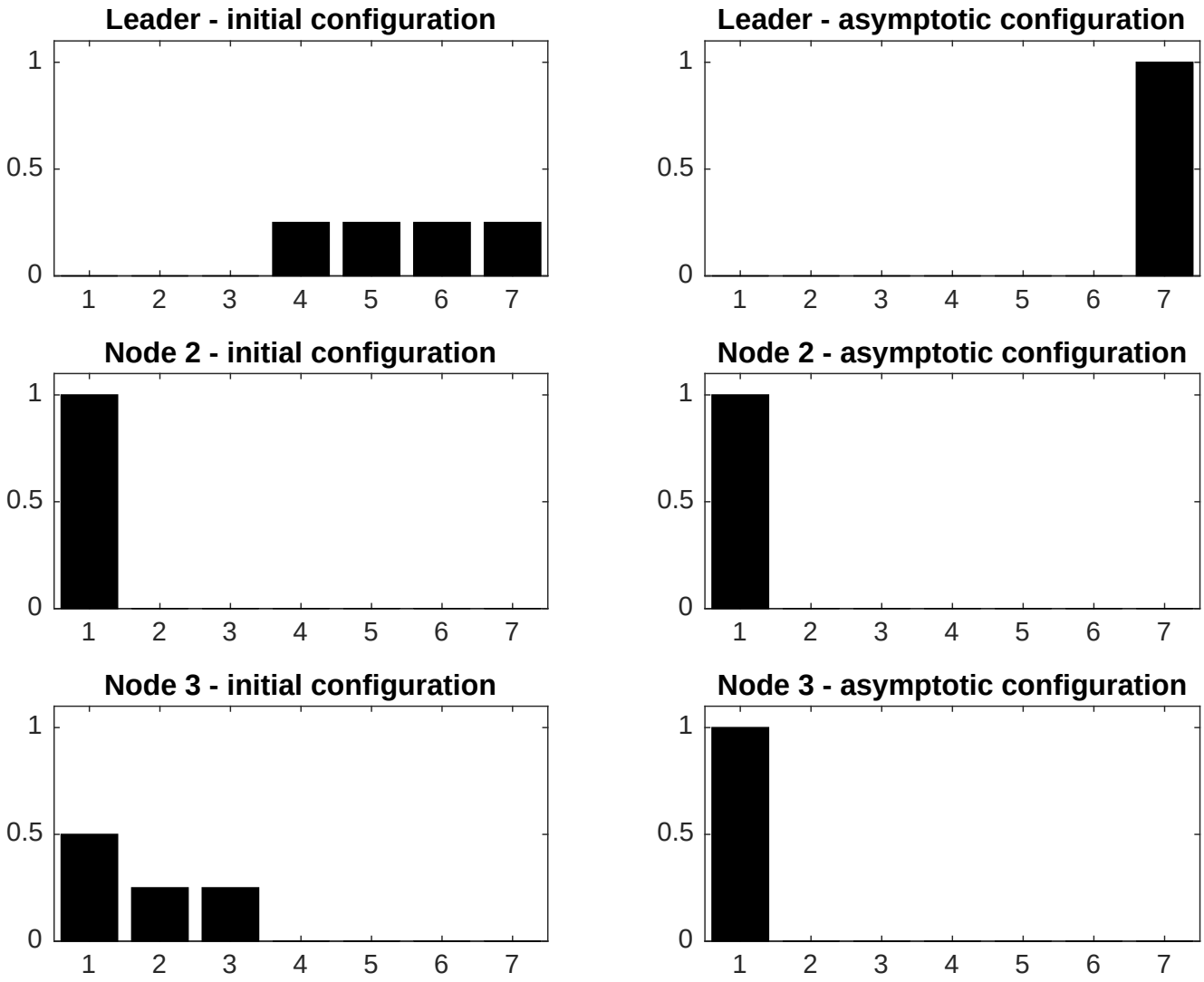

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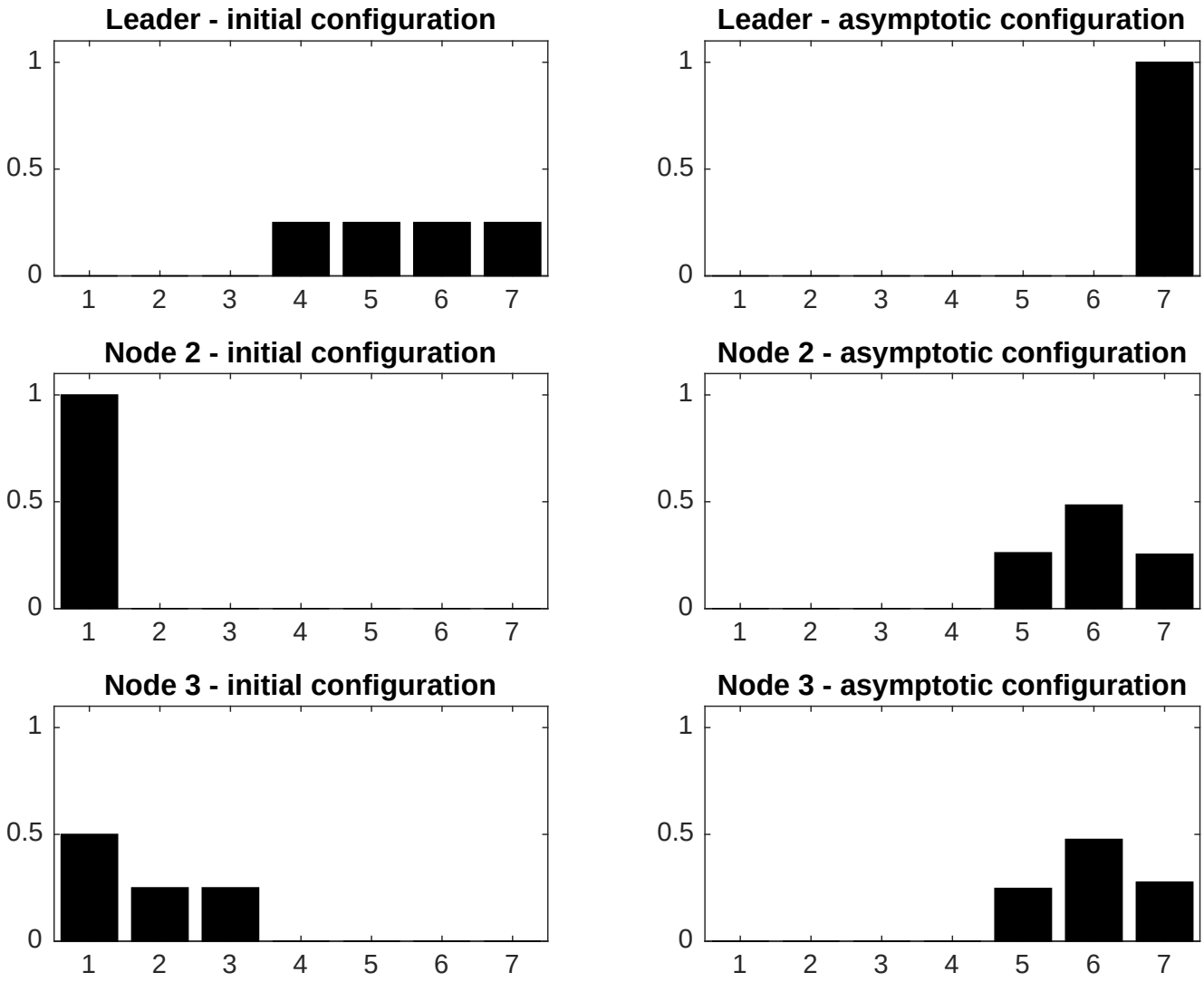

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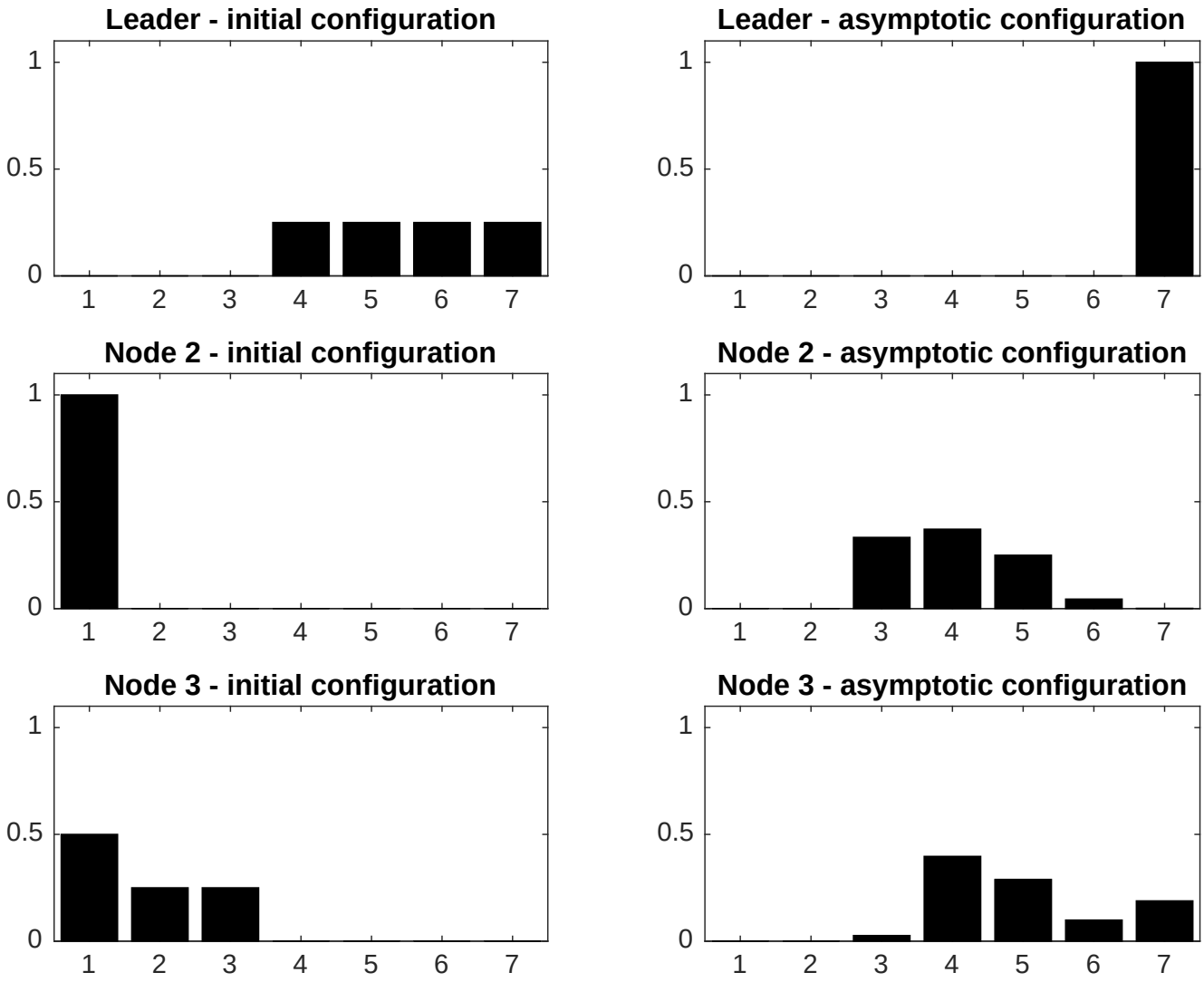

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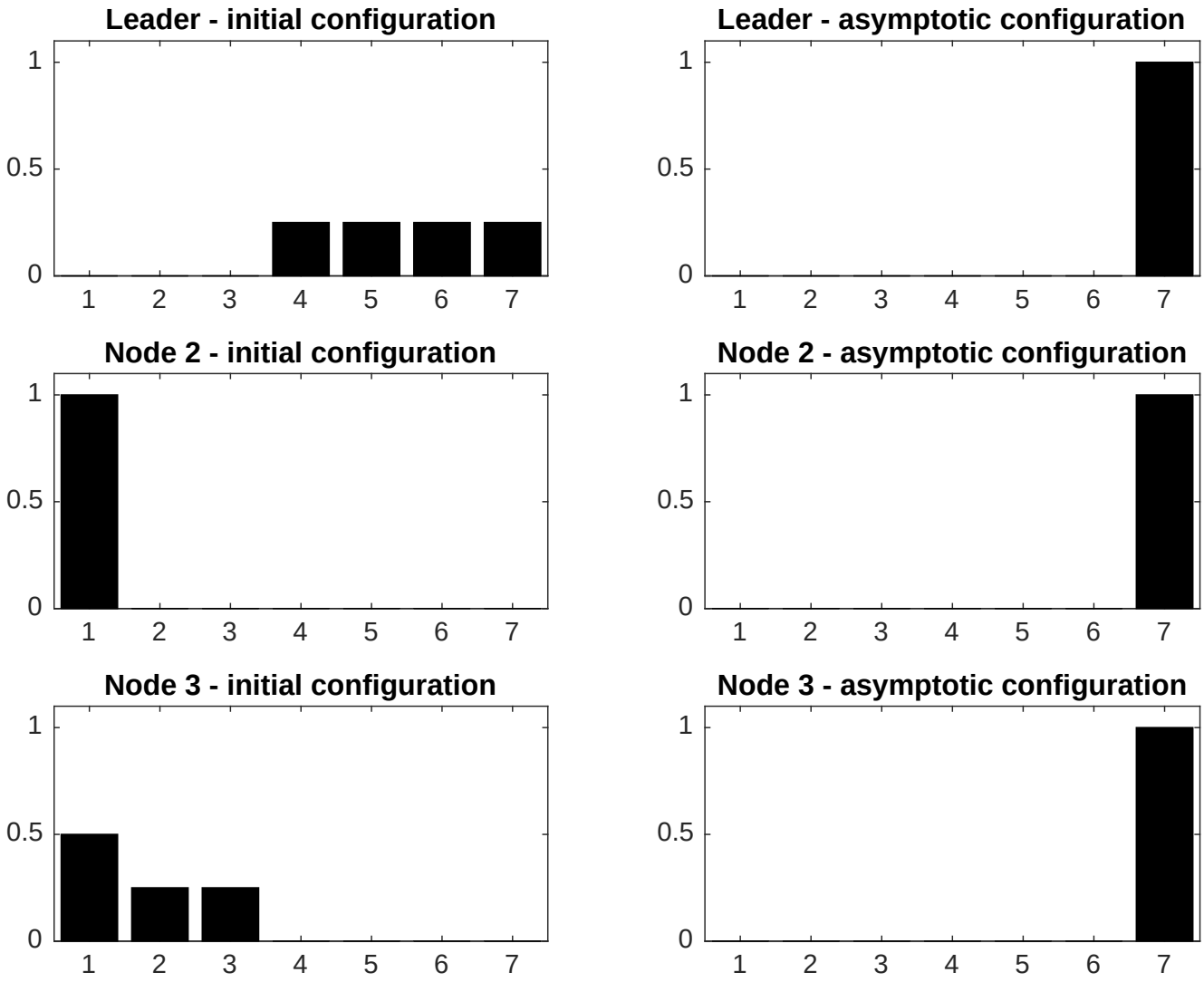

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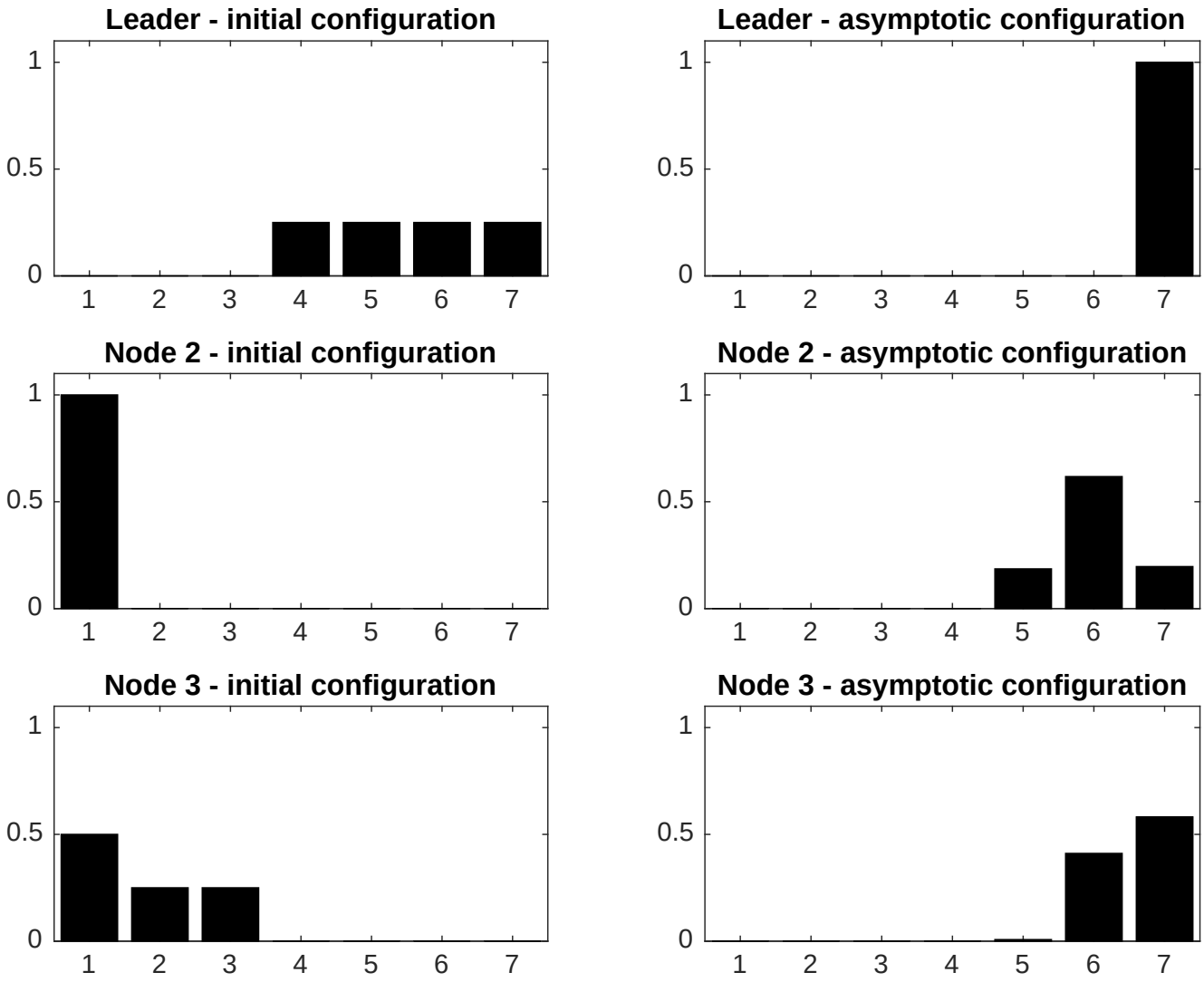

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