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EXACT SOLUTIONS AND WAVE INTERACTIONS FOR A VISCOELASTIC MEDIUM

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ABSTRACT. A rate-type first order hyperbolic model describing viscoelastic or viscoplastic media is considered. A class of double wave solutions of the governing system at hand is determined and some boundary value problems of interest in viscoelasticity are solved. Finally an exact description of soliton-like wave interactions is given.

1. Introduction

The equations describing a continuum medium in the one–dimensional case, in absence of external forces and neglecting thermal effects are

$$\frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0 \tag{2}$$

where v is the lagrangian velocity, σ the stress, ε the strain while t and x denote, respectively, time and lagrangian space coordinates. In such a framework, relation (1) is the equation of motion while (2) characterizes a compatibility condition. To the system (1), (2) a stress–strain constitutive relation characterizing the material under interest must be added. For instance, for elastic media the constitutive law

$$\sigma = \sigma(\varepsilon) \tag{3}$$

has to be given. In the linear case relation (3) specializes to the well known Hooke's law $\sigma = E\varepsilon$ where E is the Young modulus. If some further viscous effects must be also taken into account, then for a such viscoelastic or viscoplastic medium a rate–type stress–strain law must be considered. One of the simpler as well as of the first stress–strain law which takes both elastic and viscous effects into account is the Maxwell equation

$$\frac{\partial \sigma}{\partial t} - E\frac{\partial \varepsilon}{\partial t} = -\frac{1}{\tau}\sigma \tag{4}$$

where τ is a relaxation time. Starting from the pioneering Maxwell equation, along the years many rate-type stress-strain laws have been proposed (see Cristescu 2007, and references quoted therein). Within such a theoretical framework, here we consider the following rate–type equation

$$\frac{\partial\sigma}{\partial t} - \Phi(\varepsilon,\sigma)\frac{\partial\varepsilon}{\partial t} = \Psi(\varepsilon,\sigma) \tag{5}$$

where the material response functions $\Phi(\varepsilon,\sigma)$ and $\Psi(\varepsilon,\sigma)$ measure, respectively, the instantaneous and the non instantaneous response of the medium.

The model (1), (2) and (5) has been widely adopted in the literature in order to describe viscoelastic and/or viscoplastic processes where memory effects are present (for an exhaustive review on this subject see, for instance: Cristescu and Suliciu 1984; Cristescu 2007). Many results have been obtained for the system (1), (2) and (5) concerning energy estimates and phase transformation fenomena (Suliciu 1989; Făciu 1991a,b; Suliciu 1992; Făciu and Suliciu 1994; Făciu 1996a,b; Tang *et al.* 2006), moving boundary problems (Frydrychowicz and Singh 1985; Fazio 1992), traveling waves and similarity solutions (Suliciu *et al.* 1973), reduction procedures (Fusco and Manganaro 1994b, 1996, 2008), numerical experiments (Cristescu 1972; Schuler and Nunziato 1974).

The rate-type equation (5) generalizes different models proposed in literature. For instance, if

$$\Phi = E, \quad \Psi = -\frac{1}{\tau}\sigma \tag{6}$$

it reduces to the Maxwell's equation, while if

$$\Phi = E, \quad \Psi = -\frac{1}{\tau}(\sigma - \sigma_e(\varepsilon)) \tag{7}$$

it specializes to the Malvern's model (Malvern 1951a,b), where $\sigma = \sigma_e(\varepsilon)$ denotes the equilibrium stress-strain curve characterized by

$$\Psi(\varepsilon, \sigma_e(\varepsilon)) = 0. \tag{8}$$

If the function $\Psi(\varepsilon,\sigma)$ is zero along a unique curve then the process is viscoelastic, while if $\Psi(\varepsilon,\sigma) = 0$ in a suitable domain, viscoplastic processes occur (Cristescu 1972).

Furthermore, Herrmann and Nunziato (1972) proved that the celebrated integro-differential equation proposed by Colemann and Noll (1961) within the framework of finite linear viscoelasticity is equivalent to (5) with

$$\Phi = \sigma_i'(\varepsilon) + \alpha(\varepsilon)(\sigma - \sigma_i(\varepsilon)) \tag{9}$$

$$\Psi = -\frac{1}{\tau}(\sigma - \sigma_e(\varepsilon)) \tag{10}$$

where $\alpha(\varepsilon)$ is a constitutive function while $\sigma = \sigma_i(\varepsilon)$ characterizes the instantaneous stress-strain curve defined by

$$\frac{d\sigma_i}{d\varepsilon} = \Phi(\sigma_i(\varepsilon), \varepsilon). \tag{11}$$

The material response function characterized in (10) is widely used in literature. In particular Gurtin *et al.* (1980) proved that if Ψ is smooth in a neighborhood of a point $(\varepsilon_0, \sigma_0)$ of the equilibrium curve $\sigma = \sigma_e(\varepsilon)$, then there exists a constant $k(\varepsilon_0) \geq 0$ such that

$$\Psi = -k(\sigma - \sigma_e(\varepsilon)) + O(\delta)$$

as

$$\delta = |\varepsilon - \varepsilon_0| + |\sigma - \sigma_0|$$

approaches to zero. Furthermore, it can be proved (Gurtin *et al.* 1980) that the model (1), (2) and (5) admits a free energy $\psi(\varepsilon,\sigma)$ if and only if the following relations hold

$$\frac{\partial \psi}{\partial \varepsilon}+\Phi\frac{\partial \psi}{\partial \sigma}=\sigma, \qquad \Psi\frac{\partial \psi}{\partial \sigma}\leq 0. \tag{12}$$

In passing we notice also that when $\tau \to 0$ system (1), (2) and (5) supplemented by (10) reduces to the celebrated p-system

$$\begin{cases} \varepsilon_t - v_x = 0 \\ v_t + \left(p(\varepsilon)\right)_x = 0 \end{cases} \tag{13}$$

where $p(\varepsilon) = -\sigma_e(\varepsilon)$ is the pressure-like function.

Recently, Currò and Manganaro (2017) proposed a reduction procedure useful for determining double wave solutions to first order hyperbolic homogeneous or nonhomogeneous system of PDEs. The main idea was to reduce the problem of integrating the full set of governing equations to that of solving a suitable 2×2 model. This makes easier the analysis under interest because for 2×2 hyperbolic homogeneous or nonhomogeneous systems a large body of results are known (Jeffrey 1997). In particular Riemann problems and generalized Riemann problems are solved by Currò *et al.* (2011, 2012a) and Currò and Manganaro (2013, 2016), while nonlinear soliton-like wave interactions are analyzed by Currò *et al.* (2012b, 2013). Within such a theoretical framework, as far as the system (1), (2) and (5) is concerned, classes of exact solutions have been determined by Currò and Manganaro (2017) while some Riemann problems were solved by Manganaro (2017).

Following the approach proposed by Currò and Manganaro (2017), the main aim of this paper is to develop a reduction procedure for determining a new class of double wave exact solutions of (1), (2) and (5) as well as to give an exact analitical description of nonlinear waves interaction admitted by the governing system under interest. The paper is organized as follows. In section 2 we characterize a double wave reduction of (1), (2) and (5) and some boundary value problems are solved. In section 3 we develop a procedure aimed at analizing different nonlinear waves interaction problems. Some conclusions and final remarks are given in section 3.

2. Double wave solutions

The main aim of this section is to develop a reduction procedure aimed at determining exact double wave solutions of (1), (2) and (5). To this end and for further convenience we notice that the characteristic speeds of the system (1), (2) and (5) are

$$\lambda^{(1)} = -\sqrt{\Phi}, \quad \lambda^{(2)} = 0, \quad \lambda^{(3)} = \sqrt{\Phi} \tag{14}$$

so that it results to be strictly hyperbolic provided that $\Phi > 0$.

Therefore, following the approach proposed by Currò and Manganaro (2017), we look for the double wave reduction

$$\sigma = f(v,\varepsilon) \tag{15}$$

where the function $f(v,\varepsilon)$ will be determined during the process. Substituting the ansatz (15) into (1) and (2) we obtain

$$\frac{\partial v}{\partial t}-\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}-\frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x}=0 \tag{16}$$

$$\frac{\partial \varepsilon}{\partial t}-\frac{\partial v}{\partial x}=0 \tag{17}$$

while from (5) we get

$$\left(\left(\frac{\partial f}{\partial v}\right)^2+\frac{\partial f}{\partial \varepsilon}-\Phi\right)\frac{\partial v}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x}=\Psi. \tag{18}$$

The next step is to require the equation (18) is a differential constraint of the system (16), (17). For simplicity in the appendix we sketch the main steps of the method of differential constraints and we address the interested reader to the paper by Yanenko (1964) (see also Fusco and Manganaro 1994a; Manganaro and Meleshko 2002; Manganaro and Pavlov 2014, for some applications of the method to 2×2 hyperbolic systems). In the present case a differential constraint of (16) and (17) must adopt necessarily the form

$$\mu\frac{\partial v}{\partial x}-\frac{\partial f}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x}=q(v,\varepsilon) \tag{19}$$

where q is an unknown function while

$$\mu=\mu^{(\mp)}=\frac{1}{2}\left(-\frac{\partial f}{\partial v}\mp\sqrt{\left(\frac{\partial f}{\partial v}\right)^2+4\frac{\partial f}{\partial \varepsilon}}\right) \tag{20}$$

is a characteristic speed of the system (16), (17). Therefore, owing to (19), in order that (18) can be a differential constraint of (16), (17) we require

$$\left(\frac{\partial f}{\partial v}\right)^2+\frac{\partial f}{\partial \varepsilon}-\Phi=\pi\mu \tag{21}$$

$$\frac{\partial f}{\partial v}\frac{\partial f}{\partial \varepsilon}=-\pi\frac{\partial f}{\partial \varepsilon} \tag{22}$$

where π is a proportional factor. From (21) and (22) two possible cases arise.
i) If $\frac{\partial f}{\partial \varepsilon}=0$, then (18) specializes to

$$\frac{\partial v}{\partial x}=\frac{\Psi}{\left(f'(v)\right)^2-\Phi} \tag{23}$$

so that the compatibility between (16), (17) and (23) leads to

$$\sigma=f(v)=k_0 v \tag{24}$$

$$\Psi=\alpha(\sigma)\left(k_0^2-\Phi(\varepsilon,\sigma)\right) \tag{25}$$

where k_0 is a constant and $\alpha(\sigma)$ is a constitutive function. Furthermore equations (16), (17) and (23) reduce, respectively, to

$$\frac{\partial v}{\partial t} = k_0 \alpha \left(k_0 v \right) \tag{26}$$

$$\frac{\partial \varepsilon}{\partial t} = \alpha \left(k_0 v \right) \tag{27}$$

$$\frac{\partial v}{\partial x} = \alpha \left(k_0 v \right) \tag{28}$$

whose integration gives

$$v = V(z); \qquad z = x + k_0 t \tag{29}$$

$$\varepsilon = \frac{1}{k_0} \int \alpha \left(k_0 V(z) \right) dz + g(x) \tag{30}$$

along with

$$\frac{dV}{dz} = \alpha \left(k_0 V \right). \tag{31}$$

In (30) $g(x)$ is an arbitrary function which can be specified once an initial or boundary value problem is given. Therefore, in the present case, if the constitutive relation (25) holds, then an exact solutions of the governing model at hand is given by (24), (29) and (30).

ii) We assume $\frac{\partial f}{\partial \varepsilon} \neq 0$ so that from (21), (22) we obtain

$$\Phi = \left(h \frac{\partial f}{\partial \varepsilon} \right)^2 \tag{32}$$

where, for the sake of convenience, we set

$$h(v,\varepsilon) = \frac{\frac{\partial f}{\partial v} + R}{2 \frac{\partial f}{\partial \varepsilon}}; \qquad R = \mp \sqrt{\left(\frac{\partial f}{\partial v} \right)^2 + 4 \frac{\partial f}{\partial \varepsilon}} \tag{33}$$

while (16), (17) and (18) assume, respectively, the form

$$\frac{\partial v}{\partial t} - h \frac{\partial f}{\partial \varepsilon} \frac{\partial v}{\partial x} = p \tag{34}$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0 \tag{35}$$

$$\frac{1}{2} \frac{\partial f}{\partial v} \left(\frac{\partial f}{\partial v} - R \right) \frac{\partial v}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial f}{\partial \varepsilon} \frac{\partial \varepsilon}{\partial x} = \Psi. \tag{36}$$

In (34), for simplicity, we set

$$p(v,\varepsilon) = \frac{\Psi}{\frac{\partial f}{\partial v}}. \tag{37}$$

Then, the compatibility between (34), (35) and (36) leads to the pair of equations

$$h \frac{\partial h}{\partial \varepsilon} - \frac{\partial h}{\partial v} = 0 \tag{38}$$

$$\frac{\partial p}{\partial \varepsilon} + h \frac{\partial f}{\partial \varepsilon} \frac{\partial p}{\partial v} = h \frac{\partial^2 f}{\partial v \partial \varepsilon} p. \tag{39}$$

Once h and p are determined according to (38) and (39), then integration of (34), (35) and (36) along with (33) leads to an exact double wave solution of the governing model at hand in terms of one arbitrary function.

Altough the general solution of equation (38) can be easily determined, here, as an example, we will consider the particular solution

$$h(v,\varepsilon) = -\frac{\varepsilon}{v+v_1}. \tag{40}$$

with v_1 an arbitrary constant. Therefore, from (33) and (39) we obtain, respectively,

$$\sigma = f(v,\varepsilon) = \frac{v(v+v_1)}{\varepsilon}; \qquad p(v,\varepsilon) = \frac{(v+v_1)\varepsilon}{2}\psi_0(w) \tag{41}$$

where $\psi_0(w)$ is an unspecified function and

$$w = \frac{v}{\varepsilon} \tag{42}$$

while from (32) and (37) we have, respectively,

$$\Phi = \frac{v^2}{\varepsilon^2}, \qquad \Psi = (v+v_1)(v+\frac{v_1}{2})\psi_0(w) \tag{43}$$

where $v = v(\sigma,\varepsilon)$ is defined through $(41)_1$.

Next, for the sake of simplicity, we consider the case $v_1 = 0$ so that (42) and (43) specialize to

$$\Phi = \frac{\sigma}{\varepsilon}, \qquad \Psi = \varepsilon\sigma\psi_0(w), \qquad w = \sqrt{\frac{\sigma}{\varepsilon}} \tag{44}$$

whereas in the present case, taking (40), (41) and (44) into account, the reduced 2×2 system (34) assume the diagonal form

$$\frac{\partial w}{\partial t} - w\frac{\partial w}{\partial x} = 0 \tag{45}$$

$$\frac{\partial \varepsilon}{\partial t} - w\frac{\partial \varepsilon}{\partial x} = \frac{\varepsilon^2}{2}\psi_0(w) \tag{46}$$

while the differential constraint (36) specializes to

$$\frac{\partial w}{\partial x} = \frac{\varepsilon}{2}\psi_0(w). \tag{47}$$

Now, in order to integrate equations (45), (46) and (47) let us we consider the boundary value problem

$$v(0,t) = v_0(t), \qquad \varepsilon(0,t) = \varepsilon_0(t), \qquad \sigma(0,t) = \sigma_0(t) \tag{48}$$

so that

$$w(0,t) = w_0(t) = \frac{v_0(t)}{\varepsilon_0(t)}. \tag{49}$$

Moreover, owing to $(41)_1$, the functions $v_0(t)$, $\varepsilon_0(t)$ and $\sigma_0(t)$ are related by

$$\sigma_0(t) = \frac{v_0^2(t)}{\varepsilon_0(t)}. \tag{50}$$

Integration of (45) and (46), taking $(41)_1$ and (42) into account, leads to

$$v = \frac{2v_0^2(\tau)}{2v_0(\tau) + \psi_0\,(w_0\,(\tau))\,x\varepsilon_0^2} \tag{51}$$

$$\varepsilon = \frac{2v_0(\tau)\varepsilon_0(\tau)}{2v_0(\tau) + \psi_0\,(w_0\,(\tau))\,x\varepsilon_0^2} \tag{52}$$

$$\sigma = \frac{2v_0^3(\tau)}{\varepsilon_0\left(2v_0(\tau) + \psi_0\,(w_0\,(\tau))\,x\varepsilon_0^2\right)} \tag{53}$$

where

$$\tau = t + \frac{x}{w_0\,(\tau)} \tag{54}$$

while (47) specializes to

$$\frac{d}{d\tau}\left(\frac{v_0}{\varepsilon_0}\right) = \frac{1}{2}v_0\psi\left(\frac{v_0}{\varepsilon_0}\right). \tag{55}$$

Relations (51) and (52) characterize a double wave solution of (1), (2) and (5) supplemented by (44) along with the boundary data (48) provided that condition (55) is fulfilled. Therefore the differential constraint (55) selects the class of boundary value problems which can be solved in the present case. In passing we notice that the double wave reduction $(41)_1$ requires the solution at hand satisfies the further condition $\sigma\varepsilon > 0$ which is trivially fulfilled by (52) and (53).

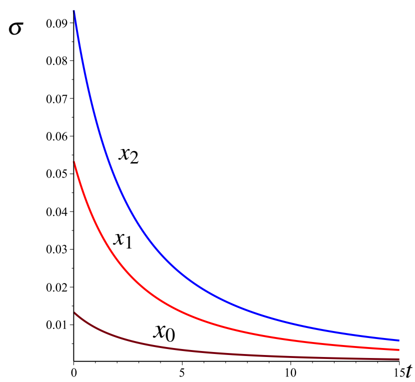

FIGURE 1. Plot of σ characterized by (53) along with (56) versus t for $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. In (56) we choose $\varepsilon_0 = 3$ and $c = -5$.

Finally, we consider some particular boundary value problems which can be of a certain interest in viscoelasticity. To this end we choose $\psi_0 = 2w$. First we point out our attention on a relaxation phenomenon which occurs when a constant strain is applied at the boundary. In such a case, according to (48) and (50), we get

$$\varepsilon(0,t) = \varepsilon_0, \quad v(0,t) = \frac{1}{c-t}, \quad \sigma(0,t) = \frac{1}{\varepsilon_0\,(c-t)^2} \tag{56}$$

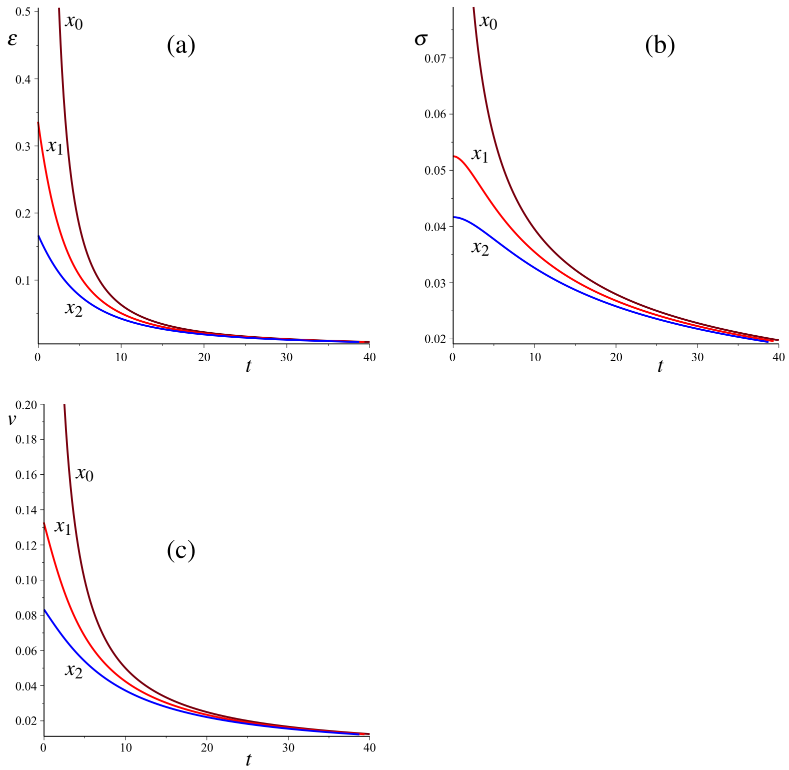

FIGURE 2. Plot of ε (a), σ (b) and v (c), given by (51)–(53) supplemented by (57), versus t, for $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. In (57) we choose $k_0 = 0.5$ and $k_1 = 2$.

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where c is an arbitrary constant, while ε_0 is the constant applied strain. The corresponding plot of the stress versus t at different x given in Fig. 1 is in a good agreement with the sperimental evidence describing the situation under interest.

Next we consider a standard impact problem characterized by a very high value of the velocity at the boundary. In such a case we have

$$v(0,t) = \frac{k_0}{t}, \quad \varepsilon(0,t) = k_1 t^{-(1+k_0)}, \quad \sigma(0,t) = \frac{k_0^2}{k_1} t^{k_0-1} \tag{57}$$

where k_0 and k_1 are arbitrary constants. In Fig. 2 we represent the plot, respectively, of ε, σ and v versus t at different x in the case where the parameter $k_0 < 1$. When $k_0 = 1$, it can be easily verified that the behaviour of the strain and of the velocity does not change qualitatively from that described in the case $k_0 < 1$. Conversely, the plot of the stress depends strongly on k_0 as it is shown in Fig. 3.

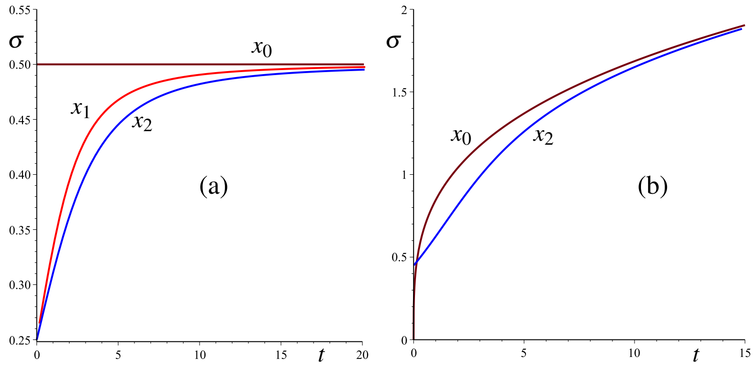

FIGURE 3. Plot of σ given by (53) supplemented by (57) versus t for $x_0 = 0$, $x_1 = 1$, $x_2 = 2$. In (57) we choose $k_1 = 2$ while $k_0 = 1$ in figure (a) and $k_0 = 1.3$ in figure (b).

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3. Initial value problem and wave interaction

In the previous section exact solutions of boundary value problems associated with the original set of the governing equations (1), (2) and (5) are characterized by solving corresponding boundary value problems for the reduced 2×2 model (16), (17) endowed with (18). Of course the same approach can be useful for solving initial value problems. However many results concerning exact solutions as well as nonlinear wave propagation problems are known for 2×2 quasilinear hyperbolic systems.

Within such a context, following the analytical approach proposed first by Seymour and Varley (1982) and Currò and Fusco (1987) and later, in a more systematic way, by Currò *et al.* (2013) for classes of 2×2 strictly hyperbolic and homogeneous systems, here our aim is to give an exact description of nonlinear wave interaction processes ruled by (1), (2) and (5) supplemented by (43). To this aim we will solve initial value problems associated to the reduced system (16), (17) along with the constraint (18). Therefore, owing to (41), the equations (16), (17) assume the form

$$\frac{\partial v}{\partial t} - \left(\frac{2v+v_1}{\varepsilon}\right)\frac{\partial v}{\partial x} + \left(\frac{v(v+v_1)}{\varepsilon^2}\right)\frac{\partial \varepsilon}{\partial x} = 0 \tag{58}$$

$$\frac{\partial \varepsilon}{\partial t} - \frac{\partial v}{\partial x} = 0 \tag{59}$$

whereas, taking (43) into account, the constraint equation (18) becomes

$$\frac{\partial}{\partial x}\left(\frac{v}{\varepsilon}\right) = \frac{\varepsilon}{2}\psi_0\left(\frac{v}{\varepsilon}\right). \tag{60}$$

Under the assumption $v_1 \neq 0$ the system (58)–(59) is strictly hyperbolic with characteristic speeds

$$\mu^{(1)} = -\frac{v+v_1}{\varepsilon}, \qquad \mu^{(2)} = -\frac{v}{\varepsilon} \tag{61}$$

to which there correspond left and right eigenvectors

$$\mathbf{l}_i = \left[\begin{array}{cc} \mu^{(i)}, & \mu^{(1)}\mu^{(2)} \end{array}\right], \qquad \mathbf{d}_i = \left[\begin{array}{c} \mu^{(i)} \\ -1 \end{array}\right] \qquad i = 1,2. \tag{62}$$

In passing, we notice that according to the results obtained by Currò and Manganaro (2017), taking $(43)_1$ into account, $\mu^{(2)} = \lambda^{(1)}$ and the equation (60) is, in fact, the differential constraint associated to $\mu^{(1)}$. Furthermore, it is straightforward to see that the reduced system (58)–(59) is CEX (Boillat 1996) being both characteristic speeds $\mu^{(1)}$ and $\mu^{(2)}$ linearly degenerate (Lax 1957)

$$\nabla\mu^{(1)} \cdot \mathbf{d}_1 = \nabla\mu^{(2)} \cdot \mathbf{d}_2 = 0. \tag{63}$$

It turns out that the Riemann invariants (Courant and Friedrichs 1976; Jeffrey 1997), in the present case, are given by

$$R(v,\varepsilon) = \frac{v+v_1}{\varepsilon}, \qquad S(v,\varepsilon) = \frac{v}{\varepsilon} \tag{64}$$

so that, via the well known hodograph transformation

$$x = x(R,S), \qquad t = t(R,S), \qquad \frac{\partial(x,t)}{\partial(R,S)} \neq 0 \tag{65}$$

the system (58)–(59) can be reduced to the pair of linear equations

$$x_S + S t_S = 0, \qquad x_R + R t_R = 0 \tag{66}$$

whose integration yields

$$x + Rt = (R-S)\frac{\mathrm{d}M}{\mathrm{d}S} + L(R) + M(S) \tag{67}$$

$$x + St = (S-R)\frac{\mathrm{d}L}{\mathrm{d}R} + L(R) + M(S). \tag{68}$$

The arbitrary functions $L(R)$ and $M(S)$ involved in (67), (68) are determined once initial or boundary data, selected by condition (60), are given.

Along the lines of the approach worked out by Currò *et al.* (2013) we consider the characteristic curves $C^{\mu^{(1)}}$ and $C^{\mu^{(2)}}$ defined, respectively, by

$$C^{\mu^{(1)}} : \frac{\mathrm{d}x}{\mathrm{d}t} = \mu^{(1)}, \qquad C^{\mu^{(2)}} : \frac{\mathrm{d}x}{\mathrm{d}t} = \mu^{(2)} \tag{69}$$

and we denote with $\alpha(x,t)$ and $\beta(x,t)$ the characteristic parameters satisfying

$$\frac{\partial\alpha}{\partial t} - \mu^{(1)}\frac{\partial\alpha}{\partial x} = 0, \quad \frac{\partial\beta}{\partial t} - \mu^{(2)}\frac{\partial\beta}{\partial x} = 0, \qquad \alpha(x,0) = \beta(x,0) = x \tag{70}$$

Then, owing to the invariance of the Riemann variables, along the associated characteristic curves, we have $S = S(\alpha)$ and $R = R(\beta)$ and, in turn, from (67) and (68) the following representation of the solution in terms of the characteristic parameters is obtained

$$x + R(\beta)t = (R(\beta) - S(\alpha))\frac{\mathrm{d}M}{\mathrm{d}S}(\alpha) + L(\beta) + M(\alpha) \tag{71}$$

$$x + S(\alpha)t = (S(\alpha) - R(\beta))\frac{\mathrm{d}L}{\mathrm{d}R}(\beta) + L(\beta) + M(\alpha) \tag{72}$$

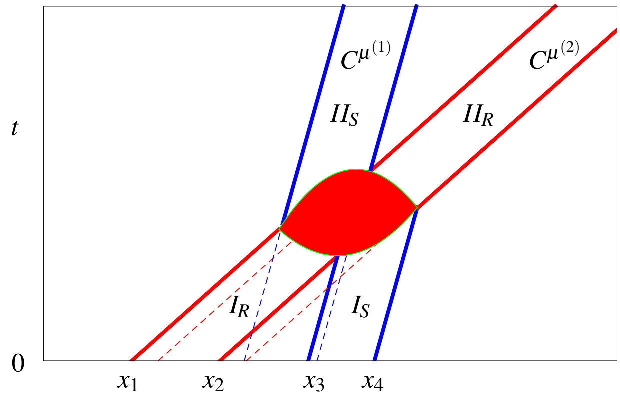

FIGURE 4. Qualitative behavior in the (x,t)-plane of the interaction between two simple waves travelling along different characteristic curves. The initial data for $\mathscr{R}(x)$ and $\mathscr{S}(x)$ are as in (76).

where, as usual, for a generic function $f(S,R)$ we denote $f(\alpha,\beta) = f(S(\alpha),R(\beta))$. As far as the Cauchy problem is concerned, after (70), the initial data for R and S assume the form

$$R(x,0) = \mathscr{R}(x), \qquad S(x,0) = \mathscr{S}(x), \qquad -\infty < x < +\infty$$

and consequently, from (71) and (72) the functions $M(\alpha), L(\beta), \dfrac{\mathrm{d}M}{\mathrm{d}S}(\alpha), \dfrac{\mathrm{d}L}{\mathrm{d}R}(\beta)$ are determined as

$$M(\alpha) = -\int_0^\alpha \frac{\mathscr{S}(x) - \mathscr{S}(\alpha)}{\mathscr{R}(x) - \mathscr{S}(x)}\,\mathrm{d}x, \quad \frac{\mathrm{d}M}{\mathrm{d}S}(\alpha) = \int_0^\alpha \frac{\mathrm{d}x}{\mathscr{R}(x) - \mathscr{S}(x)} \tag{73}$$

$$L(\beta) = \int_0^\beta \frac{\mathscr{R}(x) - \mathscr{R}(\beta)}{\mathscr{R}(x) - \mathscr{S}(x)}\,\mathrm{d}x, \quad \frac{\mathrm{d}L}{\mathrm{d}R}(\beta) = -\int_0^\beta \frac{\mathrm{d}x}{\mathscr{R}(x) - \mathscr{S}(x)}. \tag{74}$$

Moreover, owing to the differential constraint (60), the initial data $\mathscr{R}(x)$ and $\mathscr{S}(x)$ must satisfy the condition

$$(\mathscr{R}(x) - \mathscr{S}(x))\frac{\mathrm{d}\mathscr{S}}{\mathrm{d}x} = \frac{v_1}{2}\psi_0(\mathscr{S}(x)). \tag{75}$$

Next we describe in the (x,t) plane the interaction of two simple waves travelling along characteristic curves belonging to different families. To this aim, withouth loss of generality, we assume $\mu^{(2)} > \mu^{(1)} > 0$ so that at $t = 0$ the pulse travelling along $C^{\mu^{(2)}}$ occupies the region $x_1 \leq x \leq x_2$ whereas the pulse travelling along $C^{\mu^{(1)}}$ the region $x_3 \leq x \leq x_4$ (see Fig. 4). Both waves propagate into a region of constant state where $R = R_0$ and $S = S_0$. We also require that $\mathscr{R}(x)$ and $\mathscr{S}(x)$ are continuous. Therefore at $t=0$ we have

$$\mathscr{R}(x)=\begin{cases}\omega(x) & x_1\le x\le x_2\\ R_0 & \text{otherwise}\end{cases}$$
$$\mathscr{S}(x)=\begin{cases}\zeta(x) & x_3\le x\le x_4\\ S_0 & \text{otherwise}\end{cases}\tag{76}$$

$$\omega(x_1)=\omega(x_2)=R_0,\quad \zeta(x_3)=\zeta(x_4)=S_0.$$

where, taking (75) into account, we require $\psi_0(S_0)=0$. In the (x,t) plane there are several distinct regions where $x-\mu^{(i)}t \quad (i=1,2)$ can be explicitly calculated and, in turn, the behavior of the emerging simple waves can be fully described. In particular the simple wave regions I_R, II_R and I_S, II_S are adjacent, respectively, to the constant state $S=S_0$ and $R=R_0$ (Courant and Friedrichs 1976). Therefore we have

- **:** Simple wave $S=S_0, \quad R=\omega(\beta)$

$$\begin{cases}\text{REGION}\, I_R\\ x_1\le\beta\le x_2,\quad \alpha\le x_3\\ x-\mu^{(2)}t=x+S_0t=\beta\end{cases}\qquad \begin{cases}\text{REGION}\, II_R\\ x_1\le\beta\le x_2,\quad \alpha\ge x_4\\ x-\mu^{(2)}t=x+S_0t=\beta+J_s\end{cases}\tag{77}$$

- **:** Simple wave $R=R_0, \qquad S=\zeta(\alpha)$

$$\begin{cases}\text{REGION}\, I_S\\ x_3\le\alpha\le x_4,\ \beta\ge x_2\\ x-\mu^{(1)}t=x+R_0t=\alpha\end{cases}\qquad \begin{cases}\text{REGION}\, II_S\\ x_3\le\alpha\le x_4,\quad \beta\le x_1\\ x-\mu^{(1)}t=x+R_0t=\alpha+J_r\end{cases}\tag{78}$$

with

$$J_r=\int_{x_1}^{x_2}\frac{R_0-\omega(x)}{S_0-\omega(x)}\,\mathrm{d}x,\qquad J_s=\int_{x_3}^{x_4}\frac{S_0-\zeta(x)}{R_0-\zeta(x)}\,\mathrm{d}x.\tag{79}$$

A direct inspection of (77)–(79) confirms that both pulses are unaffected by the interaction process. In fact the pulse travelling along $C^{\mu^{(1)}}$ traverses region I_S, it interacts with the $C^{\mu^{(2)}}$ travelling pulse and emerges in the region II_S as a simple wave identical with that produced by the following initial conditions at $t=0$

$$\mathscr{R}(x)=\begin{cases}\omega(x) & x_1\le x\le x_2\\ R_0 & \text{otherwise}\end{cases}$$
$$\mathscr{S}(x)=\begin{cases}\zeta(x+J_r) & x_3-J_r\le x\le x_4-J_r\\ S_0 & \text{otherwise}\end{cases}\tag{80}$$

With a similar argument the pulse travelling along $C^{(\mu^{(2)})}$ traverses region I_R and emerges in the region II_R as a simple wave identical with that produced by the initial conditions at $t = 0$

$$\mathscr{R}(x) = \begin{cases} \omega(x+J_s) & x_1 - J_s \le x \le x_2 - J_s \\ R_0 & \text{otherwise} \end{cases}$$
$$\mathscr{S}(x) = \begin{cases} \zeta(x) & x_3 \le x \le x_4 \\ S_0 & \text{otherwise} \end{cases} \tag{81}$$

Therefore these pulses evolve as hyperbolic waves but in the interaction process exhibit a soliton-like behavior being the only effect of the interaction a change in the origins of the original pulses (Seymour and Varley 1982).

In order to illustrate the wave behavior described hitherto as well as to confirm the analytical wave interaction description characterized above, now we consider the numerical solutions of the system (16), (17) with initial data for the lagrangian velocity $v(x,t)$ and the strain $\varepsilon(x,t)$ obtained from

$$v(x,0) = \frac{v_1\,\mathscr{S}(x)}{\mathscr{R}(x) - \mathscr{S}(x)}, \quad \varepsilon(x,0) = \frac{v_1}{\mathscr{R}(x) - \mathscr{S}(x)}, \tag{82}$$

satisfying (75), and simulating two simple waves travelling along different families of characteristic curves. It is worth to notice that, to the initial value problem (82) for the auxiliary 2×2 reduced model (58), (59) there corresponds an initial value problem for the full governing system (16)–(18) endowed with the response functions characterized by (43) where the initial datum for the stress σ is obtained from

$$\sigma(x,0) = \frac{v(x,0)}{\varepsilon(x,0)}\,(v(x,0) + v_1). \tag{83}$$

Although it is well known that because the exceptionality condition (63) both simple waves evolving along the related characteristic curves do not distort in time, our results prove that such waves behave linearly also in the nonlinear interaction process herein described.

Figures 5 and 6 show two regular pulses travelling in the positive $x-$direction ($\mu^{(2)} > \mu^{(1)} > 0$). That simulates a situation where two strain pulses, characterizing localized deformations of the viscoelastic medium and proportional to particle velocities, propagate through the specimen.

4. Conclusions

Double wave exact solutions are degenerate hodograph solution of rank 2 which are determined in terms of wave parameters (Meleshko 2000). In order to determine r-multiple waves we are led to solve an overdetermined set of PDE's for the wave parameters whose analysis is, in general, an hard task without any further assumptions. Within such a theoretical framework, here, following the approach proposed by Currò and Manganaro (2017), a special double wave reduction procedure is developed for the quasilinear nonhomogeneous first order hyperbolic system (1), (2) and (5) describing a viscoelastic medium where short memory effects are taken into account. Moreover classes of boundary value problems, as for instance a standard impact problem, are solved. In such a case a direct inspection of (53) and (57) (see also figures 2(c) and 3) shows that the behaviour of the stress depends

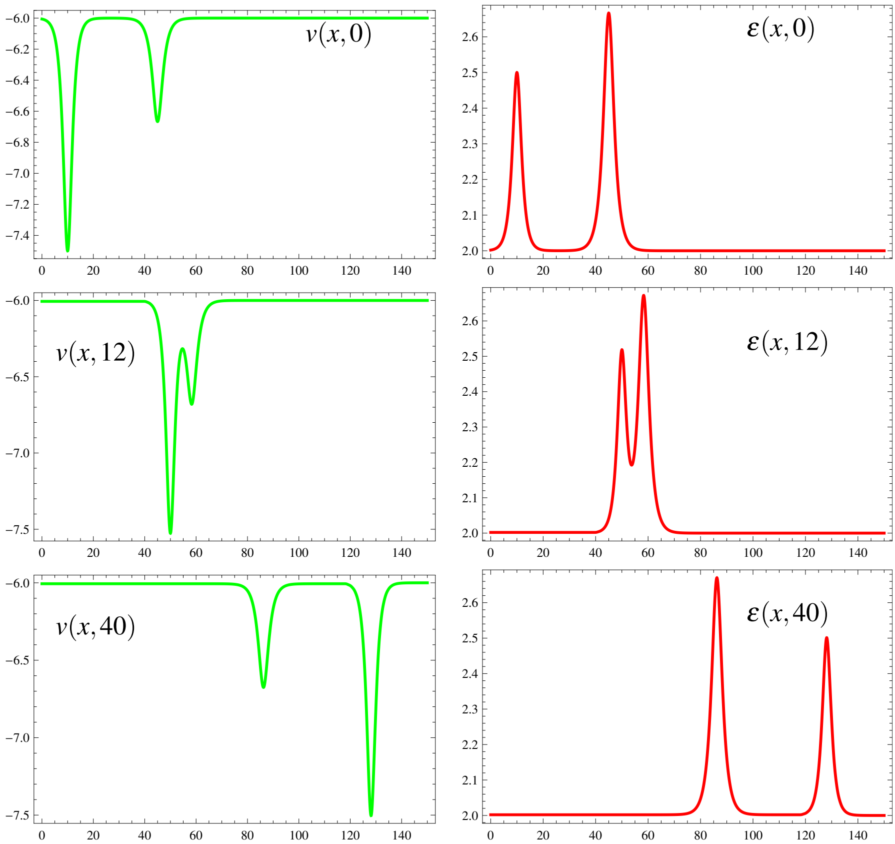

FIGURE 5. Simulation of two interacting simple waves travelling in the positive x-direction. The numerical solution of equations (58)–(59) with $v_1 = 4$ is obtained with initial data from (82) and $\mathscr{S}(x) = -3+0.5\,\text{sech}(0.5(x-45))$, $\mathscr{R}(x) = -1-0.4\,\text{sech}(0.6(x-10))$.

stronlgly on the impact parameter k_0. In particular the stress decreases or increases if $k_0 < 1$ or $k_0 \geq 1$, respectively.

Since the procedure at hand permits to reduce the original governing model to a suitable 2×2 system for which nonlinear wave problems can be studied (Currò *et al.* 2013), here we gave an exact description of a soliton-like wave interactions described by the viscoelastic model under interest. We proved that simple waves interact without any distortion in time (soliton-like behaviour) although the nonlinear effects involved in the model under interest. We remark that exact description of nonlinear interaction processes is fully developed for 2×2 hyperbolic models. Unfortunately such an analysis cannot be in general applied to quasilinear hyperbolic systems involving more dependent and/or independent variables although special wave interaction problems were solved for multicomponent models of hydrodynamic type (Currò *et al.* 2015a,b, 2017). Therefore the reduction to 2×2 systems

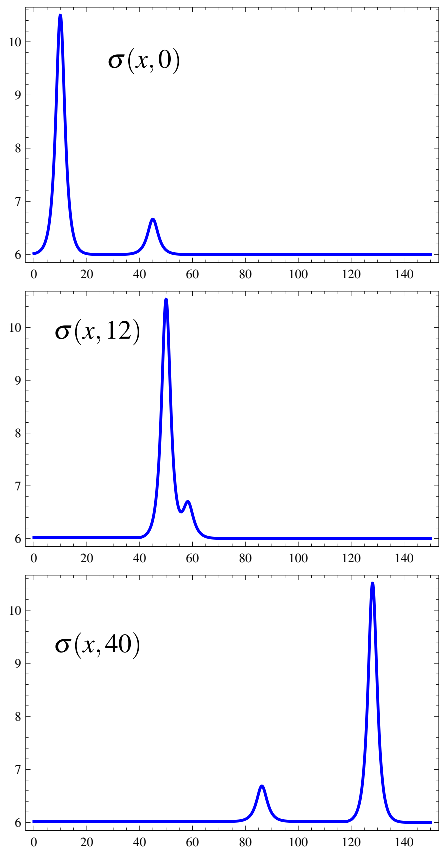

FIGURE 6. Numerical profile of the stress $\sigma(x,t)$ obtained from (41) with $v(x,t)$ and $\varepsilon(x,t)$ as in Fig. 5.

here considered can be useful also for studying wave interactions for multicomponent hyperbolic models.

Finally, we remark that the governing system of equations (1), (2) and (5) describes a viscoelastic medium in a mechanical framework (i. e. the temperature is assumed to be constant so that the heat flux vanishes). Therefore it should be of a certain interest to investigate the possible supplementary laws admitted by (1), (2) and (5). Such an analysis is under investigation and local as well as nonlocal conservation laws are determining. Our preliminary results show that the only local supplementary law admitted by (1), (2) and (5) is, of course, the energy balance law. This is in agreement to what happens in the purely elastic case where the entropy is constant so that the entropy principle reduces to the dissipation energy principle.

Appendix

For convenience here we describe briefly the main steps of the method of differential constraints. Let us consider the system

$$\mathbf{U}_t + A(\mathbf{U})\,\mathbf{U}_x = \mathbf{B}(\mathbf{U}), \tag{84}$$

where $\mathbf{U} \in R^{\mathrm{N}}$ is a column vector denoting the dependent field variables, $A(\mathbf{U})$ is the matrix coefficients while $\mathbf{B}(\mathbf{U})$ is a column vector denoting the source terms. The system (84) is assumed to be strictly hyperbolic.

It can be proved that the more general first order set of differential constraints which can be appended to system (84) must adopt the form

$$\mathbf{l}^{(i)} \cdot \mathbf{U}_x = q^{(i)}(x,t,\mathbf{U}), \qquad i = 1\ldots M \le N, \tag{85}$$

where $\mathbf{l}^{(i)}$ denote left eigenvectors of the matrix A corresponding to the eigenvalues $\lambda^{(i)}$, while the functions $q^{(i)}$ have to be determined during the process.

The first step in the procedure at hand is to require the differential compatibility between equations (84) and (85). This leads to an overdetermined set of conditions to be fulfilled by the functions $q^{(i)}$ which also impose restrictions on the structural form of the matrix coefficients A and $\mathbf{B}$. Once the consistency conditions have been satisfied, then a class of exact solutions to a given governing model can be obtained by integrating the overdetermined system (84), (85). Thus the differential constraints (85) select particular exact solutions to the original system (84).

In the case $M = N-1$ which is under interest in the present paper, it is straigtforward to ascertain that (84) and (85) lead to

$$\mathbf{U}_t + \lambda^{(\mathrm{N})}\mathbf{U}_x = \mathbf{B} + \sum_{i=1}^{N-1} q^{(i)}\left(\lambda^{(\mathrm{N})} - \lambda^{(i)}\right)\mathbf{d}^{(i)}, \tag{86}$$

so that the solution of an initial or boundary value problem of the governing equations can be obtained through integration along the characteristic curves associated to the eigenvalue $\lambda^{(\mathrm{N})}$, namely along the family of characteristics selected by the equations (85). Finally it can be proved that the initial (or boundary) data must satisfy the constraints (85) which therefore select the classes of initial (or boundary) value problems compatible with the procedure at hand. The resulting exact solution will be parameterized by one arbitrary function.

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