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# Some 2D heat transfer problems in metals described by a linearized extended thermodynamics model

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**Abstract.** We describe qualitatively some 2D stationary problems for metal electrons in the presence of a temperature gradient and a magnetic field. To this aim we refer to a linearized Extended Thermodynamics model that allows a semianalytical construction of the solutions. The results are in agreement with the thermomagnetic effects already known in the literature.

## 1. Introduction

In this paper we consider a metallic plate in the presence of a temperature gradient and of a magnetic flux density. In order to describe the stationary physical phenomena related to this case, we refer to a Rational Extended Thermodynamics (RET) model for free electrons bounded within a metal, in agreement with the original idea by Sommerfeld [1, 2, 3, 4, 5]. The electrons in the metallic body are described as free fermion particles of mass  $m$ . Electrons can collide occasionally with a lattice ion (mass  $M$ ), but their energy is unchanged by a collision, due to the large ratio  $M/m$ . In fact, the ions are modeled as rigid spheres, at rest at their lattice point and their density is uniform and constant. Moreover, the collision between electrons are neglected.

Our analysis of the physical system will be developed within the the framework of RET [4, 6], a macroscopic theory based on a different strategy with respect to Classical Thermodynamics (CT). In fact, RET considers as field variables not only those of CT (mass density, momentum and energy) but also the stress tensor, the heat flux and others. The corresponding field equations are balance laws supplemented by local and instantaneous constitutive equations, that are determined by the requirement of validity of universal physical principles, like the entropy principle and the principle of relativity. During the last decades, RET proved to be a very powerful theory, capable of describing non-stationary physical phenomena through hyperbolic PDE systems, overcoming the paradox of infinite velocities due to parabolic PDE models. At the beginning, the theory was proposed by Müller, Ruggeri and other researchers for rarefied monatomic gases [4], but in the last years it has been generalized to rarefied polyatomic gases both in the classical [6, 7] and in the relativistic framework [8] and also to quantum systems [9], obtaining relevant results and very good agreement with experimental data (see for example [4, 6, 10, 11]). So, it is natural to turn to this theory also in the present case.

The physical phenomena that we will study here are basically associated to heat transfer and we will show the relevance of the boundary conditions assignment, in contrast to what happens



for a classical gas. In particular, a special attention will be devoted to the Righi-Leduc effect.

The structure of the paper is the following. Section 2 is devoted to the equation model, while the physical problems is presented and analysed in Section 3. Some concluding remarks are proposed in Section 4.

## 2. The model

In this section we refer to the RET 8-moment model obtained by Müller [3, 4] through a microscopic approach and recently revisited in [5] at a macroscopic level. In order to derive this set of balance equations, following [1, 2], it is assumed that the state of the electron gas is described by the phase density  $f(\mathbf{x}, \mathbf{c}, t)$  of electrons at the position  $\mathbf{x}$ , with the velocity  $\mathbf{c}$  and at the time  $t$ . The phase density must satisfy the Boltzmann equation, which assumes the form

$$\frac{\partial f}{\partial t} + c_i \frac{\partial f}{\partial x_i} + f_i \frac{\partial f}{\partial c_i} = S, \quad (1)$$

where  $S$  stands for the collision term, while  $f_i$  represents the specific external force acting on the particles which, in a metal gas, is the Lorentz force

$$f_i = -\frac{q}{m} [E_i + (\mathbf{c} \wedge \mathbf{B})_i], \quad (2)$$

if  $-q$  denotes the charge of an electron,  $m$  denotes its mass,  $\mathbf{E}$  is the electromotive intensity and  $\mathbf{B}$  represents the magnetic flux density, while ' $\wedge$ ' is the symbol of the vector product.

Multiplying equations (1) by a generic function  $\varphi(\mathbf{x}, \mathbf{c}, t)$  and integrating it over the whole range of  $\mathbf{c}$ , it is possible to construct the generic moment equation for the electron gas:

$$\frac{\partial \bar{\varphi}}{\partial t} + \frac{\partial \bar{c_i \varphi}}{\partial x_i} - \overline{\frac{\partial \varphi}{\partial t} + c_i \frac{\partial \varphi}{\partial x_i} + (f_i + i_i^c) \frac{\partial f}{\partial c_i}} = \int \varphi S d\mathbf{c}. \quad (3)$$

where the symbol  $\bar{\varphi}$  indicates the moment  $\int \varphi f d\mathbf{c} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{+\infty} \varphi f dc_1 dc_2 dc_3$ .

To derive the macroscopic equation model, setting  $\varphi = m, mc_i, mc^2, mc^2 c_i, \dots$  in equation (3), it is possible to construct an infinite hierarchy of moment equations [4, 6] that are truncated to the first eight moments:

$$\begin{aligned} \text{the mass density} & \quad \rho = \int m f d\mathbf{c}, \\ \text{the mass flux or the momentum density} & \quad J_i = \int m c_i f d\mathbf{c}, \\ \text{the energy density} & \quad e = \frac{1}{2} \int m c^2 f d\mathbf{c}, \\ \text{the heat flux} & \quad q_i = \frac{1}{2} \int m c^2 c_i f d\mathbf{c}, \end{aligned} \quad (4)$$

where the electric current density reads  $S_i = -\frac{q}{m} J_i$ .

From equation (3), the set of eight balance equations for the eight field variables  $\rho, J_i, e$  and  $q_i$  assumes the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial J_k}{\partial x_k} &= 0, \\ \frac{\partial J_i}{\partial t} + \frac{\partial P_{ik}}{\partial x_k} - \left(-\frac{q}{m} E_i\right) \rho - \left(-\frac{q}{m} \epsilon_{ijk} B_k\right) J_j &= G_i, \\ \frac{\partial e}{\partial t} + \frac{\partial q_k}{\partial x_k} - \left(-\frac{q}{m} E_k\right) J_k &= 0, \\ \frac{\partial q_i}{\partial t} + \frac{1}{2} \frac{\partial \rho_{lik}}{\partial x_k} - \left(-\frac{q}{m} E_k\right) \left(P_{<ik>} + \frac{5}{3} e \delta_{ik}\right) - \left(-\frac{q}{m} \epsilon_{ijk} B_k\right) q_j &= H_i, \end{aligned} \quad (5)$$

if  $\epsilon_{ijk}$  and  $\delta_{ik}$  denote, as usual, the Levi-Civita tensor and the Kronecker tensor,  $P_{ij} = \int m c_i c_j f d\mathbf{c}$  represents the momentum flux,  $\rho_{lik} = \int m c^2 c_i c_k f d\mathbf{c}$  is the trace of the fourth moment,  $G_i$  and

$H_i$  are the productions, while the square brackets stand for the traceless part of a symmetric tensor ( $P_{ij} = P_{kk}\delta_{ij}/3 + P_{<ij>}$ ).

In order to obtain a closed set of field equations from system (5), the fluxes  $P_{ik}$  and  $\rho_{lik}$  and the productions  $G_i$  and  $H_i$  must be expressed in terms of the eight field variables  $\rho$ ,  $J_i$ ,  $e$  and  $q_i$ , by material dependent relations. Following the procedure of RET at macroscopic level, the closure of the system is obtained starting from the validity requirement of the entropy principle:

$$\frac{\partial h}{\partial t} + \frac{\partial \Phi_i}{\partial x_i} = \Sigma \geq 0, \quad (6)$$

for an entropy density  $h$ , an entropy flux  $\Phi_i$  and an entropy production  $\Sigma$ . An approximation of the constitutive relations in the neighborhood of the equilibrium, the Gibbs relation and the Fermion statistics are the main ingredient used to reach the goal. In particular, we recall that at equilibrium the distribution function for fermion particles reads

$$f_E = \frac{y}{e^{-\frac{m}{k_B} \frac{g}{T} + \frac{mc^2}{2k_B} \frac{1}{T}} + 1}, \quad (7)$$

if  $k_B$  is the Boltzmann constant,  $g = \frac{e}{\rho} - \frac{Th_E}{\rho} + \frac{p_E}{\rho}$  is the specific free enthalpy,  $h_E$  denotes the entropy density at equilibrium, while  $p_E$  represents the equilibrium pressure, and  $\frac{1}{y} = \frac{h^3}{2m^3}$  [3, 4], where  $h$  denotes Plank's constant. To simplify the expressions of the final results, it is convenient to adopt as independent field variables the following ones:  $\vartheta = 2k_B T/m$ ,  $J_i$ ,  $\alpha = -\frac{mq}{k_B T}$  and  $q_i$ . In this way one gets

$$\begin{aligned} P_{<ij>} &= 0, \\ e(\vartheta, \alpha) &= 2\pi m y (\vartheta)^{\frac{5}{2}} i_4(\alpha), \\ \rho(\vartheta, \alpha) &= 4\pi m y (\vartheta)^{\frac{3}{2}} i_2(\alpha), \\ a(\vartheta, \alpha) &= \frac{4}{3}\pi m y (\vartheta)^{\frac{7}{2}} i_6(\alpha) + K(\vartheta), \end{aligned} \quad (8)$$

for an arbitrary function  $K(\vartheta)$  and the integral functions

$$i_n(\alpha) = \frac{I_n(\alpha)}{n+1} = \int_0^{+\infty} \frac{x^n}{e^{\alpha+x^2} + 1} dx. \quad (9)$$

The function  $i_n(\alpha)$  is well-known in the framework of degenerate gas theory and several numerical techniques were adopted to calculate it, for an exhaustive review see [12]. It presents some peculiar properties, in particular we recall the following recurrence relation

$$\frac{di_n(\alpha)}{d\alpha} = -\frac{n-1}{2} i_{n-2}(\alpha); \quad (10)$$

moreover, for completely degenerate gases (i.e. when  $\alpha \rightarrow -\infty$ ) it holds  $I_n(\alpha) \approx (-\alpha)^{(n+1)/2}$ .

With regard to the production terms, the macroscopic approach requires that the production terms are expressed as a linear combination of  $J_i$  and  $q_i$  through a positive definite matrix that could depend on  $\vartheta$  and  $\alpha$ .

Through the comparison with Müller's kinetic approach it is possible to fix the remaining arbitrary functions:  $K(\vartheta) = 0$  and

$$G_i = A_{11}(\vartheta, \alpha) J_i + A_{12}(\vartheta, \alpha) q_i, \quad H_i = A_{21}(\vartheta, \alpha) J_i + A_{22}(\vartheta, \alpha) q_i, \quad (11)$$

where  $l$  is the mean free path of an electron between two collisions and

$$\begin{aligned}
 A_{11}(\vartheta, \alpha) &= \frac{\vartheta^{1/2}}{l} \frac{I_4(\alpha)I_5(\alpha) - I_3(\alpha)I_6(\alpha)}{I_2(\alpha)I_6(\alpha) - I_4^2(\alpha)}, \\
 A_{12}(\vartheta, \alpha) &= \frac{2}{l\vartheta^{1/2}} \frac{I_3(\alpha)I_4(\alpha) - I_2(\alpha)I_5(\alpha)}{I_2(\alpha)I_6(\alpha) - I_4^2(\alpha)}, \\
 A_{21}(\vartheta, \alpha) &= \frac{\vartheta^{3/2}}{2l} \frac{I_4(\alpha)I_7(\alpha) - I_5(\alpha)I_6(\alpha)}{I_2(\alpha)I_6(\alpha) - I_4^2(\alpha)}, \\
 A_{22}(\vartheta, \alpha) &= \frac{\vartheta^{1/2}}{l} \frac{I_4(\alpha)I_5(\alpha) - I_2(\alpha)I_7(\alpha)}{I_2(\alpha)I_6(\alpha) - I_4^2(\alpha)}.
 \end{aligned} \tag{12}$$

It was proven that the PDE system (5) is hyperbolic and equipped with an entropy law with concave entropy density in the neighborhood of equilibrium, if the electron gas is not completely degenerate [5].

### 2.1. The dimensionless variables

Here we will deal with processes independent of the time that will be described by the stationary equations

$$\begin{aligned}
 \frac{\partial J_k}{\partial x_k} &= 0, \\
 \frac{2}{3} \frac{\partial e}{\partial x_i} &= \left( -\frac{q}{m} E_i \rho - \frac{q}{m} \epsilon_{ijk} B_k J_j \right) + A_{11} J_i + A_{12} q_i, \\
 \frac{\partial q_k}{\partial x_k} &= -\frac{q}{m} E_k J_k, \\
 \frac{1}{2} \frac{\partial a}{\partial x_i} &= \left( -\frac{5q}{3m} E_i e - \frac{q}{m} \epsilon_{ijk} B_k q_j \right) + A_{21} J_i + A_{22} q_i.
 \end{aligned} \tag{13}$$

In order to rewrite the system in dimensionless variables, we have to introduce preliminarily some suitable values of length, temperature and  $\alpha$  denoted by  $D$ ,  $T_0$  and  $\alpha_0$ , respectively.

Referring to relations (8) it is natural to propose the following reference quantities

$$\begin{aligned}
 \vartheta_0 &= \frac{2kT_0}{m}, \quad \rho_0 = \frac{4\pi m y \vartheta_0^{3/2} (-\alpha_0)^{3/2}}{3}, \\
 e_0 &= \frac{2\pi m y \vartheta_0^{5/2} (-\alpha_0)^{5/2}}{5}, \quad a_0 = \frac{4\pi m y \vartheta_0^{7/2} (-\alpha_0)^{7/2}}{7},
 \end{aligned} \tag{14}$$

under the assumption that  $\alpha \ll -1$  and to define the following dimensionless quantities

$$\begin{aligned}
 \hat{x}_k &= \frac{x_k}{D}, \quad \hat{i}_n(\alpha) = \frac{(n+1)i_n(\alpha)}{(-\alpha_0)^{(n+1)/2}}, \quad \hat{T} = \frac{T}{T_0}, \quad \hat{\vartheta} = \frac{\vartheta}{\vartheta_0}, \\
 \hat{v}_k &= \frac{v_k}{\sqrt{\vartheta_0}}, \quad \hat{J}_k = \frac{J_k}{\rho_0 \sqrt{\vartheta_0}}, \quad \hat{q}_k = \frac{q_k}{\rho_0 \vartheta_0^{3/2}}, \quad \hat{\alpha} = \frac{\alpha}{\alpha_0}, \\
 \hat{\rho} &= \frac{\rho}{\rho_0} = \hat{\vartheta}^{3/2} \hat{i}_2(\alpha), \quad \hat{e} = \frac{e}{e_0} = \hat{\vartheta}^{5/2} \hat{i}_4(\alpha), \quad \hat{a} = \frac{a}{a_0} = \hat{\vartheta}^{7/2} \hat{i}_6(\alpha).
 \end{aligned} \tag{15}$$

Moreover, taking into account the previous relations, we will use the following dimensionless

coefficients

$$\begin{aligned}
 \hat{A}_{11}(\alpha) &= \frac{\hat{i}_4(\alpha)\hat{i}_5(\alpha) - \hat{i}_3(\alpha)\hat{i}_6(\alpha)}{\hat{i}_2(\alpha)\hat{i}_6(\alpha) - \hat{i}_4^2(\alpha)}, & \hat{A}_{12}(\alpha) &= \frac{\hat{i}_3(\alpha)\hat{i}_4(\alpha) - \hat{i}_2(\alpha)\hat{i}_5(\alpha)}{\hat{i}_2(\alpha)\hat{i}_6(\alpha) - \hat{i}_4^2(\alpha)}, \\
 \hat{A}_{21}(\alpha) &= \frac{\hat{i}_4(\alpha)\hat{i}_7(\alpha) - \hat{i}_5(\alpha)\hat{i}_6(\alpha)}{\hat{i}_2(\alpha)\hat{i}_6(\alpha) - \hat{i}_4^2(\alpha)}, & \hat{A}_{22}(\alpha) &= \frac{\hat{i}_4(\alpha)\hat{i}_5(\alpha) - \hat{i}_2(\alpha)\hat{i}_7(\alpha)}{\hat{i}_2(\alpha)\hat{i}_6(\alpha) - \hat{i}_4^2(\alpha)}, \\
 \hat{F}_k^e &= \frac{DqE_k}{m\vartheta_0}, & \hat{F}_k^B &= \frac{DqB_k}{m\vartheta_0^{1/2}}, & \beta &= \frac{D}{l}.
 \end{aligned} \tag{16}$$

Hence, the stationary equations can be rewritten as

$$\begin{aligned}
 \frac{\partial \hat{J}_k}{\partial \hat{x}_k} &= 0, \\
 \hat{\vartheta}^{3/2} \hat{i}_4(\alpha) \frac{\partial \hat{\vartheta}}{\partial \hat{x}_i} - \frac{\hat{\vartheta}^{5/2} \hat{i}_2(\alpha)}{-\alpha_0} \frac{\partial \alpha}{\partial \hat{x}_i} &= \\
 \left[ -\frac{2\hat{F}_i^e}{(-\alpha_0)} \hat{\vartheta}^{3/2} \hat{i}_2(\alpha) - \frac{6}{(-\alpha_0)^2} \epsilon_{ijk} \hat{F}_k^B \hat{J}_j \right] + \beta \left[ \frac{2\hat{A}_{11}}{(-\alpha_0)^{1/2}} \hat{\vartheta}^{1/2} \hat{J}_i + \frac{4\hat{A}_{12}}{(-\alpha_0)^{3/2}} \frac{\hat{q}_i}{\hat{\vartheta}^{1/2}} \right], \\
 \frac{\partial \hat{q}_k}{\partial \hat{x}_k} &= -\hat{F}_k^e \hat{J}_k, \\
 \hat{\vartheta}^{5/2} \hat{i}_6(\alpha) \frac{\partial \hat{\vartheta}}{\partial \hat{x}_i} - \frac{\hat{\vartheta}^{7/2} \hat{i}_4(\alpha)}{-\alpha_0} \frac{\partial \alpha}{\partial \hat{x}_i} &= \\
 \left[ -\frac{2\hat{F}_i^e}{(-\alpha_0)} \hat{\vartheta}^{5/2} \hat{i}_4(\alpha) - \frac{4}{(-\alpha_0)^2} \epsilon_{ijk} \hat{F}_k^B \hat{q}_j \right] + \beta \left[ \frac{2\hat{A}_{21}}{(-\alpha_0)^{1/2}} \hat{\vartheta}^{3/2} \hat{J}_i + \frac{4\hat{A}_{22}}{(-\alpha_0)^{3/2}} \hat{\vartheta}^{1/2} \hat{q}_i \right].
 \end{aligned} \tag{17}$$

## 2.2. The 2D stationary set of equations

In this paper we will consider a metallic plate whose thickness is so small with respect to other dimensions that we can neglect it and describe the electron behavior inside the plate as a 2D problem in a bounded domain. Moreover, we will assume the possible presence of a magnetic flux density orthogonal to the plate, but the absence of any electromotive intensity. Hence the stationary 2D system is

$$\begin{aligned}
 B_{11} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_i} + B_{12} \frac{\partial \hat{\alpha}}{\partial \hat{x}_i} &= \epsilon_{ij3} \Omega \hat{J}_j + C_{11} \hat{J}_i + C_{12} \hat{q}_i, \\
 B_{21} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_i} + B_{22} \frac{\partial \hat{\alpha}}{\partial \hat{x}_i} &= \epsilon_{ij3} \Gamma \hat{q}_j + C_{21} \hat{J}_i + C_{22} \hat{q}_i, \\
 \frac{\partial \hat{q}_k}{\partial \hat{x}_k} &= 0,
 \end{aligned} \tag{18}$$

for  $i = 1, 2$  and if

$$\begin{aligned}
B_{11} &= \hat{\vartheta}^{3/2} \hat{i}_4(\alpha), & B_{12} &= -\hat{\vartheta}^{5/2} \hat{i}_2(\alpha), & B_{21} &= \hat{\vartheta}^{5/2} \hat{i}_6(\alpha), \\
B_{22} &= -\hat{\vartheta}^{7/2} \hat{i}_4(\alpha), & C_{11} &= \frac{2\beta \hat{A}_{11}(\alpha)}{(-\alpha)^{1/2}} \hat{\vartheta}^{1/2}, & C_{12} &= \frac{4\beta \hat{A}_{12}(\alpha)}{(-\alpha)^{3/2} \hat{\vartheta}^{1/2}}, \\
C_{21} &= \frac{2\beta \hat{A}_{21}(\alpha)}{(-\alpha)^{1/2}} \hat{\vartheta}^{3/2}, & C_{22} &= \frac{4\beta \hat{A}_{22}(\alpha)}{(-\alpha)^{3/2}} \hat{\vartheta}^{1/2}, \\
\Omega &= -\frac{6}{(-\alpha)^2} \hat{F}_3^B, & \Gamma &= -\frac{4}{(-\alpha)^2} \hat{F}_3^B.
\end{aligned} \tag{19}$$

Now, if we assume that there is no electric current, i.e.  $J_i = 0$  and we rewrite in a suitable way equations (18) for  $i = 1, 2$ , we get the following system:

$$\begin{aligned}
\frac{\partial \hat{\alpha}}{\partial \hat{x}_1} &= \nu \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} + \varsigma \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\
\frac{\partial \hat{\alpha}}{\partial \hat{x}_2} &= -\varsigma \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} + \nu \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\
\hat{q}_1 &= \mu \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2} + \eta \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\
\hat{q}_2 &= -\eta \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} + \mu \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\
\frac{\partial \hat{q}_1}{\partial \hat{x}_1} + \frac{\partial \hat{q}_2}{\partial \hat{x}_2} &= 0,
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\nu(\alpha, \hat{\vartheta}) &= \frac{(-B_{21}C_{12} + B_{11}C_{22})(B_{22}C_{12} - B_{12}C_{22}) - B_{11}B_{12}\Gamma^2}{\Lambda}, \\
\varsigma(\alpha, \hat{\vartheta}) &= \frac{(-B_{12}B_{21} + B_{11}B_{22})C_{12}\Gamma}{\Lambda}, \\
\mu(\alpha, \hat{\vartheta}) &= \frac{(B_{12}B_{21} - B_{11}B_{22})(-B_{22}C_{12} + B_{12}C_{22})}{\Lambda}, \\
\eta(\alpha, \hat{\vartheta}) &= -\frac{(B_{12}B_{21} - B_{11}B_{22})B_{12}\Gamma}{\Lambda}, \\
\Lambda(\alpha, \hat{\vartheta}) &= (B_{22}C_{12} - B_{12}C_{22})^2 + (B_{12}\Gamma)^2.
\end{aligned} \tag{21}$$

We underline that all the previous coefficients depend on  $B_3$  through  $\Gamma$  and, in particular,  $\varsigma$  and  $\eta$  vanish in the absence of a magnetic field. This fact will play a relevant role in the next sections.

### 2.3. The linearized equations

In order to simplify the integration of the previous equations, we linearize system (17) in the neighborhood of  $\alpha = \alpha_0$  and  $\vartheta = \vartheta_0$ , so that, setting

$$\nu_0 = \nu(\alpha_0, \hat{\vartheta}_0), \quad \varsigma_0 = \varsigma(\alpha_0, \hat{\vartheta}_0), \quad \mu_0 = \mu(\alpha_0, \hat{\vartheta}_0), \quad \eta_0 = \eta(\alpha_0, \hat{\vartheta}_0), \tag{22}$$

system (20) reduces to

$$\begin{aligned}\frac{\partial \hat{\alpha}}{\partial \hat{x}_1} &= \nu_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} + \varsigma_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\ \frac{\partial \hat{\alpha}}{\partial \hat{x}_2} &= -\varsigma_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} + \nu_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\ \nabla^2 \hat{\vartheta} &= 0,\end{aligned}\tag{23}$$

and is equipped with the following algebraic relations:

$$\begin{aligned}\hat{q}_1 &= \mu_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2} + \eta_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}, \\ \hat{q}_2 &= -\eta_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} + \mu_0 \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2}.\end{aligned}\tag{24}$$

For a problem in a bounded domain described by the previous set of equations it is necessary to prescribe the values of the temperature or of the heat flux along the boundaries (to solve the Laplace equation). In addition, a value of  $\alpha$  has to be fixed in a point of the boundary in order to determine completely the  $\alpha$  quantity. We imagine, for example, to set  $\alpha = \alpha_0$  in a vertex of the plate.

### 3. Heat transfer problems

We consider the application of the previous model to heat transfer problems. For a classical gas the stationary heat transfer problem in a bounded domain is described equivalently assigning a heat flux or different temperatures on the boundaries (see for example [13, 14, 15]). What we will verify here and in the following section is that for an electron gas the two assignments give rise to different phenomena. To present both cases, let us imagine to deal with a square plate whose side length is  $D$ , and introduce a reference system with the axis passing through two consecutive sides of the plate: in this way, we can assume that  $\hat{x}_1 \in [0, 1]$ ,  $\hat{x}_2 \in [0, 1]$ . Referring to reasonable values for the mean free path and for the mean electron density, accounting for a small sample at room temperature and a very high maximum magnetic flux density, we will consider here and in the next section the following parameter values:  $\beta \leq 10^6$ ,  $\alpha_0 = -100$ ,  $|\hat{F}_3^B| \leq 10^5$ . The value of  $\alpha_0$  will be prescribed in the vertex  $(\hat{x}_1, \hat{x}_2) = (0, 0)$ .

#### 3.1. Temperature assignment

First of all, let us study the assignment of the temperature values (or, it is the same, of  $\hat{\vartheta}$ ) on the boundaries of the plate. Just to analyse a simple case (although a bit rough) we prescribe the same temperature values on three sides of the plate and a different temperature on the remaining one. The behavior of the temperature corresponding to this case can be described analytically, referring to the classical solution of the Laplace equations, under the previous conditions.

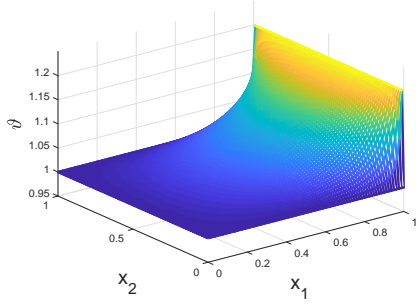
Concerning the temperature, let  $T_0$  be the temperature common to three sides  $x_1 = 0$ ,  $x_2 = 0$  and  $x_2 = 1$  so that  $\hat{\vartheta} = \hat{\vartheta}_0 = 1$  on these boundaries, while on  $x_1 = 1$ ,  $\hat{\vartheta} = \hat{\vartheta}_1$ . Then, the solution for  $\hat{\vartheta}$  reads:

$$\hat{\vartheta}(\hat{x}_1, \hat{x}_2) = \hat{\vartheta}_0 + \Delta \hat{\vartheta} \sum_{n=1}^{\infty} c_n \sinh(n\pi \hat{x}_1) \sin(n\pi \hat{x}_2),\tag{25}$$

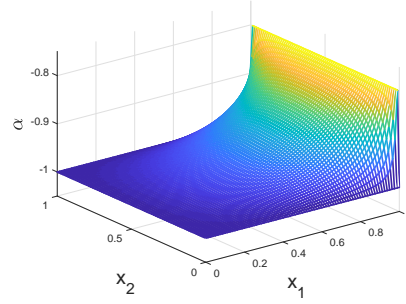
while the expression for  $\hat{\alpha}$  is

$$\hat{\alpha}(\hat{x}_1, \hat{x}_2) = K_0 + \Delta \hat{\vartheta} \sum_{n=1}^{\infty} c_n [\nu \sinh(n\pi \hat{x}_1) \sin(n\pi \hat{x}_2) + \varsigma \cosh(n\pi \hat{x}_1) \cos(n\pi \hat{x}_2)],\tag{26}$$

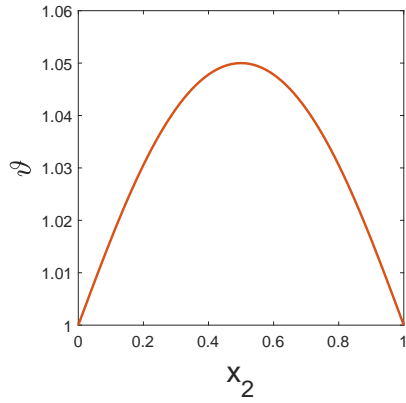




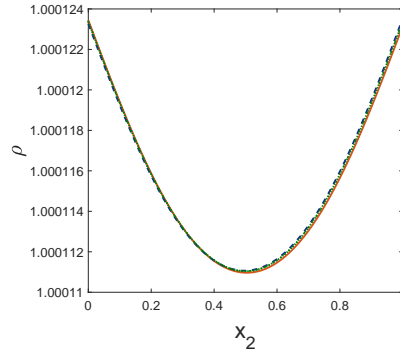
**Figure 1.** A 3D-plot of the analytical  $\hat{\vartheta}$  solution ( $\hat{\vartheta}_1 = 1.2$ ,  $\hat{F}_3^B = 10^5$ ,  $\beta = 10^6$ ).



**Figure 2.** A 3D-plot of the analytical  $\hat{\alpha}$  solution ( $\hat{\vartheta}_1 = 1.2$ ,  $\hat{F}_3^B = 10^5$ ,  $\beta = 10^6$ ).



**Figure 3.** The transversal profile of the analytical expression of  $\hat{\vartheta}$  independent of  $\hat{F}_3^B$  ( $\hat{\vartheta}_1 = 1.2$ ,  $\beta = 10^6$ ).



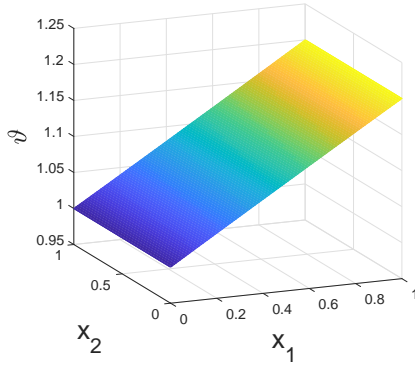
**Figure 4.** The transversal profile of the analytical expression of  $\hat{\rho}$  for  $\hat{F}_3^B = -10^5$  (continuous line),  $\hat{F}_3^B = 10^5$  (dashed line),  $\hat{F}_3^B = 0$  (dotted line), ( $\hat{\vartheta}_1 = 1.2$ ,  $\beta = 10^6$ ).

with

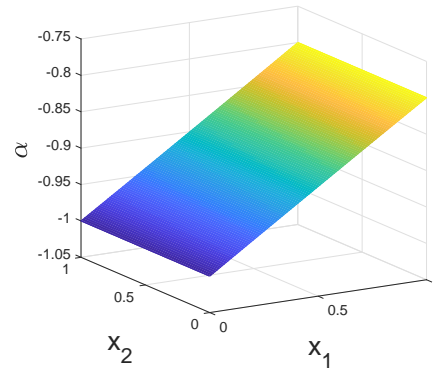
$$c_n = \frac{2[1 - (-1)^n]}{n\pi \sinh(n\pi)}, \quad \Delta\hat{\vartheta} = \hat{\vartheta}_1 - \hat{\vartheta}_0, \quad K_0 = \hat{\alpha}_0 - \varsigma \Delta\hat{\vartheta} \sum_{n=1}^{\infty} c_n. \quad (27)$$

Unfortunately, due to the discontinuities in  $(\hat{x}_1, \hat{x}_2) = (1, 0)$  and  $(\hat{x}_1, \hat{x}_2) = (1, 1)$ , the previous series are only pointwise and not uniformly convergent; anyway, in the present case it is possible to get an analytical expression of the solution.

Figures 1 and 2 show a 3D-plot of  $\hat{\vartheta}$  and  $\hat{\alpha}$  as functions of the two spatial variables. It is evident that in the present case the temperature behaviour does not depend on the magnetic flux density. In contrast,  $\alpha$  and also the electron mass density are related to  $B_3$  through the coefficient  $\mu$  and  $\eta$ . Nevertheless, very small differences are observable for  $\hat{\alpha}$  and  $\hat{\rho}$ , when the value of  $\hat{F}_3^B$  is varied. This fact is evident in Figures 3 and 4, where the profile of  $\hat{\vartheta}$  and  $\hat{\rho}$  along the  $\hat{x}_2$  axis ( $\hat{x}_2 = 0.5$ ) is presented for different values of  $\hat{F}_3^B$ . We remark that  $\hat{\rho}$  was defined as dimensionless variable before the linearization, but here it was determined through a linear approximation, so in  $(\hat{x}_1, \hat{x}_2) = (0, 0)$  in general  $\hat{\rho} \neq 1$ .



**Figure 5.** A 3D-plot of the  $\hat{\vartheta}$  behaviour, in the case of Section 3.2 ( $\hat{\vartheta}_1 = 1.2$ ,  $\hat{F}_3^B = 10^5$ ,  $\beta = 10^6$ ).



**Figure 6.** A 3D-plot of the  $\hat{\alpha}$  behaviour, in the case of Section 3.2 ( $\hat{\vartheta}_1 = 1.2$ ,  $\hat{F}_3^B = 10^5$ ,  $\beta = 10^6$ ).

### 3.2. Heat flux assignment and Righi-Leduc effect

The Righi-Leduc effect [16, 17] is a well-known effect in the literature. In the case of a metallic plate it predicts a transversal temperature gradient ( $\frac{\partial \hat{\vartheta}}{\partial \hat{x}_2} \neq 0$ ) in the presence of a longitudinal heat flux  $q_1$  and an orthogonal magnetic field  $B_3$  under the adiabatic condition  $q_2 = 0$ . This fact is easily verified for the previous equations (20). If we impose that  $\hat{q}_2 = 0$  we get

$$\begin{aligned} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_2} &= \frac{\eta}{\mu} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} = - \frac{B_{12}\Gamma}{(-B_{22}C_{12} + B_{12}C_{22})} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1}, \\ \hat{q}_1 &= \frac{\mu^2 + \eta^2}{\mu} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1} = \frac{(B_{12}B_{21} - B_{11}B_{22})}{(-B_{22}C_{12} + B_{12}C_{22})} \frac{\partial \hat{\vartheta}}{\partial \hat{x}_1}. \end{aligned} \quad (28)$$

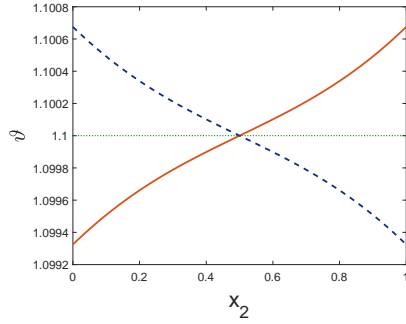
The coefficient for the Righi-Leduc effect is  $\eta/\mu$  and through a mathematical technique similar to the one used in the Appendices of [5], it is possible to show that if the electron gas is not completely degenerate, it holds

$$\text{sgn}(\eta/\mu) = -\text{sgn}(B_3), \quad (29)$$

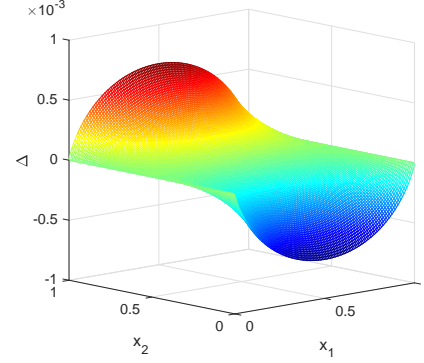
in agreement with previous studies [1, 2].

In the framework of a real experiment, in a metallic plate it is not possible to impose that  $\hat{q}_2 = 0$  everywhere. What is reasonable to prescribe at the boundaries is a temperature difference at two parallel sides and the vanishing orthogonal component of the heat flux at the other sides ( $\hat{q}_2 = 0$ ). Unlike [5], we neglect here the effect due to the interaction with the boundaries for two reasons: first of all we are interested to isolate the thermomagnetic phenomenon, secondly we are dealing here with a rough linearized model and the interaction effects could represent a more refined aspect. The previous conditions bring to a problem that cannot be easily solved analytically, even if we refer here to the linearized equations (23). The results obtained through a numerical integration of (23), based on finite difference schemes and carried on with Matlab®, are shown in the following Figures. In particular, Figure 5 and 6 show the profile of  $\hat{\vartheta}$  and  $\hat{\alpha}$  as functions of the spatial variables.

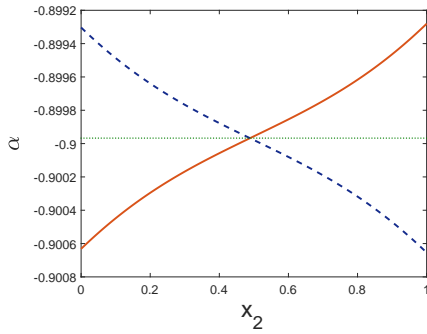
The details about the longitudinal and the transverse tendency of the field variables could not be observed through a 3D-plot. Therefore, in Figure 7 we present the  $\hat{\vartheta}$  profile for  $\hat{x}_1 = 0.5$ . In Figure 8 we resort again to a 3D-picture to get an overview of the temperature difference for different values of the magnetic field.



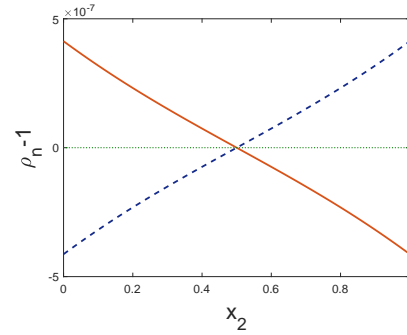
**Figure 7.**  $\hat{\vartheta}$  as a function of  $\hat{x}_2$  ( $\hat{\vartheta}_1 = 1.2$ ,  $\beta = 10^6$ ) for different values of  $\hat{F}_3^B$ :  $\hat{F}_3^B = -10^5$  continuous line,  $\hat{F}_3^B = 10^5$  dashed line,  $\hat{F}_3^B = 0$  dotted green line.



**Figure 8.** A 3D-plot of the difference  $\Delta = \hat{\vartheta}|_{\hat{F}_3^B = -10^5} - \hat{\vartheta}|_{\hat{F}_3^B = 0}$  ( $\hat{\vartheta}_1 = 1.2$ ,  $\beta = 10^6$ ).



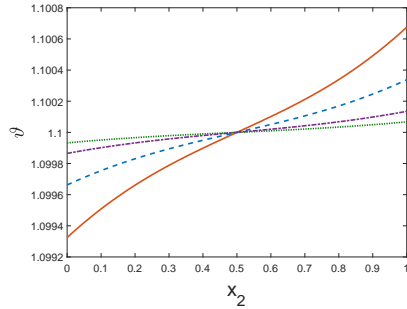
**Figure 9.** The behaviour of  $\hat{\alpha}$  as a function of  $\hat{x}_2$  ( $\hat{\vartheta}_1 = 1.2$ ,  $\beta = 10^6$ ) for different values of  $\hat{F}_3^B$ :  $\hat{F}_3^B = -10^5$  continuous line,  $\hat{F}_3^B = 10^5$  dashed line,  $\hat{F}_3^B = 0$  dotted green line.



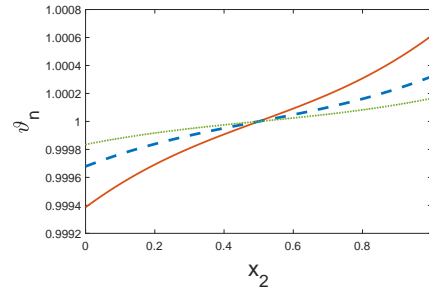
**Figure 10.** The behaviour of  $\hat{\rho}_n = \rho(x_1, x_2)/\rho(0.5, 0.5)$  as a function of  $\hat{x}_2$  ( $\hat{\vartheta}_1 = 1.2$ ,  $\beta = 10^6$ ) for different values of  $\hat{F}_3^B$ :  $\hat{F}_3^B = -10^5$  continuous line,  $\hat{F}_3^B = 10^5$  dashed line,  $\hat{F}_3^B = 0$  dotted green line.

The transversal behavior of  $\hat{\alpha}$  and  $\hat{\rho}$  along the  $\hat{x}_2$ -axis (when  $\hat{x}_1 = 0.5$ ) is illustrated in Figures 9 and 10. To better compare the different solutions of  $\hat{\rho}$  we have plotted the variable normalized with respect to the central value. Figure 11 presents a comparison of the transversal profile of  $\hat{\vartheta}$  for different values of  $\beta$ , that is to say for different mean free paths (i.e. different metals). For Figure 12 we introduce the quantity  $\vartheta_n(\hat{x}_1, \hat{x}_2) = \hat{\vartheta}(\hat{x}_1, \hat{x}_2)/\hat{\vartheta}(0.5, 0.5)$  in order to compare the transversal profile for different values of  $\Delta\hat{\vartheta}$ . From the previous analysis it is natural to conclude that the Righi-Leduc effect is enhanced for processes further from equilibrium: if the magnetic field, the difference between the temperature at the boundaries or the mean free path increases, the same happens also to the transversal temperature gradient.

These results confirm the idea that the choice of the boundary conditions in a heat transfer problem plays a crucial role to describe different phenomena related to metal electrons.



**Figure 11.** The profile of  $\hat{\vartheta}$  as a function of  $\hat{x}_2$  when  $\hat{x}_1 = 0.5$  ( $\hat{\vartheta}_1 = 1.2$ ,  $\hat{F}_3^B = -10^5$ ) for different values of  $\beta$ :  $\beta = 10^6$  continuous line,  $\beta = 2 \times 10^6$  dashed line,  $\beta = 5 \times 10^6$  dotted-dashed purple line,  $\beta = 10^7$  dotted green line.



**Figure 12.** The profile of  $\hat{\vartheta}_n = \theta(x_1, x_2)/\theta(0.5, 0.5)$  as a function of  $\hat{x}_2$  when  $\hat{x}_1 = 0.5$  ( $\beta = 10^6$ ,  $\hat{F}_3^B = -10^5$ ) for different values of  $\Delta\hat{\vartheta}$ :  $\Delta\hat{\vartheta} = 0.2$  continuous line,  $\Delta\hat{\vartheta} = 0.1$  dashed line,  $\Delta\hat{\vartheta} = 0.05$  dotted green line.

#### 4. Conclusions

This paper aims to a preliminary study of 2D heat transfer problem for metal electrons confined in a very thin plate. The role of the magnetic field is also investigated. Although the oversimplification of the equation system (few moments and linearization), it is possible to describe qualitatively well the main features of the phenomena and in particular to "capture" the Righi-Leduc effect, obtaining results also for the electron density behavior. To our knowledge, these cases are described here through a 2D integration for the first time in the literature. The linearized model allows a semi-analytical approach to the solution and constitute the first step to a more general treatment of the problem, that is already in progress.

#### Acknowledgments

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