University of Catania and University of Messina

Department of Economics and Business

Robustness Analysis for Interacting Criteria in Advanced Decision Support Systems

Sally Giuseppe Arcidiacono

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Supervisor

Prof. **Salvatore Greco**

PhD in Economics, Management and Statistics

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Introduction

Decisions inspire our reflections everyday. They are intrinsically related to a plurality of points of view and are often characterized by uncertainty, imperfect knowledge or ill-determination. So, it frequently occurs that decisions are faced in conditions of arbitrariness, incompleteness and imprecision. Decision aiding is the activity that, through the use of some formalized models, helps to obtain recommendations to the questions posed by a Decision Maker (DM) in a decision process.

A further source of complexity of real decision problems originates from the fact that they can be affected by many different issues at the same time which can not be neglected. To deal with this complexity, it seems natural to support the DM by providing integrated approaches which, trying to remaining intelligible, are able to manage the different features that decision problems can conjointly present. Since the uncertainty and the ambiguity which characterize decision problems can be treated by generalizing the deterministic cases, it appear natural to explore methods and procedures that analyze what happens when certain parameters of the models are not stable. According with this point of view, through this work we analyze some advantages to adopt in detail two families of methods which are frequently used to cope with a plurality of instances compatible with the preferences information provided by a DM. In particular, these methods provide results in terms of proportions or in terms of relations which are both inferred by considering a whole set of parameters that, in turn, represents all possible preferences which a DM can own.

The methods which deal with uncertainty and imprecision can be usefully applied also when the preferences can not be restored by using simple mathematical models. In the last part of this thesis, we analyze the advantages to adopt a decision support model that merges the capacity to represent a wide range of preferences together with the utility to support the recommendations in terms of their robustness.

The thesis is organized as follows. In Chapter 1, we shall recall some basic concepts and notions of Multiple Criteria Decision Aiding (MCDA, [38]). Chapter 2 contains contributions related to an extension of the PROMETHEE methods [20], while, in Chapter 3 we investigate some robustness concerns for the level dependent Choquet integral [53]. Some concluding remarks are contained in Chapter 4.

Contents

Chapter 1

Multiple Criteria Decision Aiding: basic concepts and notions

1.1 Aim of Multiple Criteria Decision Aiding

The Multiple Criteria Decision Aiding (MCDA, for some surveys on MCDA see [14, 38, 65]) is a decision support methodology and its aim is to provide some instruments which allow to obtain appropriate recommendations for decision problems in which a plurality of points of view, in general not all concordant, must be taken into account. Therefore, the multicriteria approach aims to help a decision maker (DM) in the analysis of a decision problem characterized by a set of eligible alternatives and a set of criteria which is supposed to satisfy some conditions[1].

So, in MCDA, a set of alternatives $A = \{a, b, c, \ldots\}$ is evaluated with respect to a set of criteria $G = \{g_1, \ldots, g_n\}$ which should be finite and consistent (see [15, 89]). A criterion is a real-valued function on the set A of the alternatives, i.e. $g : A \to I \subseteq \mathbb{R}$, reflecting a value of each alternative from a particular point of view, such that, in order to compare any two alternatives $a, b \in A$ with respect to the criterion i, it is sufficient to compare two values ($g_i(a)$, $g_i(b)$) which are also called performances. Without loss of the generality, we can suppose that the greater the evaluation $g_i(a)$ the better the performance of alternative $a \in A$ on criterion $g_i \in G$.

According with B. Roy [89], one can distinguish the following four decision problems:

- The *choice* decision problem consists in selecting a small number (as small as possible) of "good" alternatives in such a way that a single alternative may finally be chosen;

[1] A set of criteria is consistent if it does not contains criteria which violate the following: *exhaustive*: if $\forall a, b \in A$, $g_i(a)$ is indifferent to $g_i(b)$ for all $i = 1, \ldots, n$ then a must be indifferent to b; *monotonic*: if a is strictly preferred to b and there is an alternative c such that $g_i(c)$ is weekly preferred to $g_i(a)$ for all $i = 1, \ldots, n$ then c must be indifferent to b.

- The *sorting* decision problem consists in assigning each alternative to one of the predefined and ordered classes;

- The *ranking* decision problem consists in defining a complete or partial ranking on A; this ranking is the result of a procedure allowing to put together in equivalence classes alternatives which can be judged indifferent, and to order these classes.

- The *description* consists in elaborating an appropriate set of alternatives A, building a consistent family of criteria G and determining, for all or some $a \in A$, their performances on the considered criteria;

In the rest of this section on one hand, we briefly recall the dominance relation being a fundamental concept in decision theory and decision analysis, on the other hand, we discuss inference of the parameters used to model the DM preferences.

Dominance

One of the starting point and main concept in the field of MCDA is the dominance relation. Given $a, b \in A$ we say that a dominates b (aDb) iff $g_i(a) \geq g_i(b)$, $\forall g_i \in G$ with at least one strict inequality. In other terms, aDb if all the evaluations of a are better or equal to the corresponding evaluations of b with at least one evaluation of a strictly better than the corresponding evaluation of b.

The strength of the concept of dominance relation lies in its objectivity. This because of its independence from the subjective importance of each criterion $g_i \in G$. Indeed the dominance relation only depends on the evaluations of alternatives from A on criteria from G and it does not require the elicitation of any preferences information by the DM(s). In this sense, the dominance can be viewed as the unanimity of the points of view.

Since sometimes phenomena, quantities or qualities are imprecise, uncertain and, more generally, poorly understood or ill-determined, instead of the dominance it can be considered the stochastic dominance. The last one is a generalization of the dominance in the sense that the comparison between two alternatives is made by using probability distributions instead of single evaluations. We only remark that its use requires the assumption usually referred as "More is better"(for some details see [37]).

In general, the alternatives show discordant[2] evaluations when they are evaluated w.r.t. different points of view, so, in a lot of situations the dominance relation does not allow to represent overall

[2] In the most of the cases one alternatives is preferred to another one w.r.t. a proper subset of all considered criteria while the opposite direction of preference is showed in the complement of the whole set.

preferences between all alternatives. This is the main weakness of this concept.

To cope with this problems, practitioners use procedures that, on the basis of some assumptions, enrich the dominance relation to make comparable as many as alternatives in A taking into account all the criteria in G. So, it is important to remark that there is always a trade-off between objectivity and feasibility in MCDA.

Preference parameters

As we already argued, because of the poverty of the (objective) relation of dominance, each method used to deal with multicriteria decision problems needs some additional information that can be inferred, with some specific procedure, from the preferences information provided by the DM. The meaning and the values of the parameters induced from these preference information are closely linked to the specific definition of the criteria and to the adopted aggregation models [92].

Several methodologies have been proposed to infer the values of these preferential parameters. Two important strategies can be adopted to elicit this preference information:

- a *direct* procedure consists in asking the DM to provide exact values of the parameters;
- an *indirect* procedure consists in asking the DM to provide some preferences from which parameters compatible with these preferences can be inferred.

In other terms, in the indirect technique the DM provides some preference information on reference alternatives $A^* \subseteq A$ in terms of pairwise comparisons, or on considered criteria in terms of comparison between their importance, allowing also the possibility to express the intensity of these preferences.

Generally, MCDA methods based on indirect preference information and on disaggregation approaches [68, 35] are considered more interesting because they require a relatively smaller cognitive effort from the DM than methods based on direct preference information. Moreover, psychologists confirm that DMs are more confident exercising their decisions than explaining them [78, 95].

1.2 Aggregation operators

In order to cope with one of the four problematics mentioned in the previous Section 1.1, it arises the necessity to aggregate the evaluations of the alternatives taking into account the preferences of the DM.

Aggregation is the process of combining several numerical values into a single representative value. An aggregation function performs this operation and this may be specifically important in many problems related to the fusion of many information. In the field of MCDA, aggregation operators [48] are largely used and they are often chosen relying upon their simplicity. As we will see in the next paragraphs, unfortunately some features of multicriteria decision problems violate the assumption on which these procedures are founded.

The value computed by one aggregation function should synthesize all individual points of view but, on one hand, not all functions are good candidate to be an aggregation function, while, on the other hand, it is not easy to define a minimal set of properties that should be fulfilled by an aggregation function.

As a first requirement it can be considered the fact that, if inputs are supposed to lie in the interval $[a, b] \in \mathbb{R}$, the output should also lie in the same interval and this leads to define some boundary condition, while, a second natural requirement is the non-decreasing monotonicity, meaning that if some of the input values increase then the representative output value should reflect this increment, so, at least it should remain constant.

In general, if we assume that each variables of the aggregation function belongs to a common domain $I = [0, 1]$, which is very often considered in the literature, we have the following:

Definition 1.2.1. *An aggregation function in $I^n \subseteq \mathbb{R}^n$ is a function $f : I^n \rightarrow I$ that:*

- *is non-decreasing (in each variable), i.e. for each $\mathbf{x}, \mathbf{y} \in I^n$ if $\mathbf{x} \leq \mathbf{y} \Rightarrow f(\mathbf{x}) \leq f(\mathbf{y})$;*
- *fulfills the following boundary conditions:*

$$\inf_{\mathbf{z} \in I^n} f(\mathbf{z}) = \inf I, \quad \sup_{\mathbf{z} \in I^n} f(\mathbf{z}) = \sup I.$$

The most intuitive and simple aggregation function is the additive one:

$$f(g_1(a), \ldots, g_n(a)) = \sum\nolimits_{i=1}^{n} u_i(g_i(a)), \tag{1.1}$$

where we are considering n criteria and $u_i(\cdot)$ are marginal value functions non-decreasing w.r.t. the evaluation expressed by their argument.

1.2.1 Additive value functions

Among many possible additive aggregation functions, a large amount of literature is spend on the study of several means and their properties. Many of these means are applied to describe real decision problems for their simplicity and immediate understanding by the DM.

The weighted quasi-arithmetic means, also called quasi-linear means, are (weighted) aggregation function [48]. In what follow we briefly describe the class of the quasi-linear means that, as generalization of the class of the quasi-arithmetic means[3], comprise most of the algebraic means of common use such as the arithmetic mean and the geometric mean.

Definition 1.2.2. *Given $I \in \mathbb{R}$ and $f : I \to \mathbb{R}$, a function $F_f(\cdot) : I^n \to I$ is a quasi-linear mean if $\forall\, \mathbf{x} \in I^n$:*

$$F_f(\mathbf{x}) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right), \tag{1.2}$$

where $w_1, \ldots, w_n > 0$ are real numbers satisfying $\sum_i^n w_i = 1$ and $f : I \to \mathbb{R}$ is a continuous and strictly monotonic function.

In particular, one example of the quasi-linear mean is the class of weighted root-mean-power which can be expressed as follows:

$$F(\mathbf{x}) = \left[\sum_{j=1}^{n} w_j \left(x_j\right)^{\alpha}\right]^{1/\alpha}, \tag{1.3}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $w_i \in [0,1] : \sum_{i=1}^{n} w_i = 1$.

The analytical expressions related to these functions is showed in Table 1.1.

Table 1.1: Weighted root-mean-power

Analytical expression	Marginal value functions	α	Names
$\sum_{j=1}^{n} w_j x_j$	x	1	weighted arithmetic mean
$\sum_{j=1}^{n} (w_j x_j^2)^{1/2}$	x^2	2	weighted quadratic mean
$\sum_{j=1}^{n} (w_j x_j^{-1})^{-1}$	x^{-1}	-1	weighted harmonic mean
$\prod_{j=1}^{n} x_j^{w_j}$	$log(x)$	$\to \infty$	weighted geometric mean

Looking from the MCDA perspective, we note that most common preference models are based on the weighted sum:

$$f(\mathbf{a}, \mathbf{w}) = \sum_{i=1}^{n} w_i g_i(a), \tag{1.4}$$

where, as before, each $w_i \geq 0,\ i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i = 1$.

[3]Let $f : I \to \mathbb{R}$ be a continuous and strictly monotonic function. The n-ary quasi-arithmetic means generated by f is the function $M_f : I^n \to I$ defined as: $M_f(\mathbf{x}) := f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right)$.

In this case, the choice of the weights $\mathbf{w}$ establishes a trade-off between the performances of the criteria. Indeed, by using the Eq. (1.4) as preference model, it can be consider the following fact:

$$f(g_1(a), g_2(a), w_1, w_2) = f(g_1(a) + 1, g_2(a) - x, w_1, w_2);$$

$$\Rightarrow$$

$$g_1(a)w_1 + g_2(a)w_2 = (g_1(a) + 1)w_1 + (g_2(a) - x)w_2 \;\Rightarrow\; x = \frac{w_1}{w_2}.^4$$

Moreover, in this case the weights and thus the trade-offs are constant for the whole range of variation of criteria values and this, as we will see in Chapter 3, cannot always be allowable for every decision problem.

In general, for all additive aggregation functions, the great advantage of the simplicity is compensated by the fact that their use requires the strong assumption that the criteria must be independent in the sense of preferences, i.e. they require the assumption of the *mutual preference independence* between any subset of considered criteria. In this sense, the set of criteria $T \subseteq G$ is *preferentially independent* [105] of $G \setminus T$ if, for all $a_T, b_T \in \prod_{j \in T} I_j$, and for all $c_{G\setminus T}, d_{G\setminus T} \in \prod_{j \in G\setminus T} I_j$,

$$(a_T, c_{G\setminus T}) \succsim (b_T, c_{G\setminus T}) \Leftrightarrow (a_T, d_{G\setminus T}) \succsim (b_T, d_{G\setminus T})$$

that is, the preference of $(a_T, c_{G\setminus T})$ over $(b_T, c_{G\setminus T})$ does not depend on $c_{G\setminus T}$. The whole set of criteria G is said to be *mutually preferentially independent* if T is preferentially independent of $G \setminus T$ for every $T \subseteq G$.

1.2.2 Non-additive integral

In Section 1.2.1, we argued that to represent the preference of a DM by using an additive function we must consider the independence between the considered criteria and this means that if there is a certain degree of interaction between them others aggregation operators must by taken into account. More specifically, in some multicriteria decision problems it is necessary to consider positive interaction (synergy) or negative interaction (redundancy) between criteria, where we assume that two or more criteria are synergic (resp. redundant) when their joint weight is greater (resp. smaller) than the sum of the weights given to the criteria considered singularly. Therefore, it is natural the

[4] For example, if $w_1 = 0.6,\ w_2 = 0.4$ this means that the change on criterion g_2 able to compensate a change by 1.0 on criterion g_1 is $x = 1.5$.

interest to consider non-additive weights.

Now, whereas the weighted sum is the discrete form of the Lebesgue integral w.r.t. an additive measure, to deal with a more general framework we may refer to the Choquet integral [24] which, in turn, is the same integration [32, 40] w.r.t. a non-additive measure (otherwise called fuzzy measure, or capacity) expressed as a non-linear continuous aggregation function. So, the main reason to investigate this type of operators (also called non-additive integrals) is that the notion of capacity allows to consider a wider range of possibilities respect to the weighted sum and which includes this last one as a particular case.

Among many types of non-additive integrals, such as Sugeno integral [99] and the Shilkret integral [94], in the next sections we will focus on the Choquet integral, the bipolar Choquet integral and the level dependent Choquet integral.

The Choquet integral

The basic scope of the Choquet integral is to represent the importance of the criteria by considering not only weights $\lambda_i \geq 0$, $\forall g_i \in G$, but also by considering a measure for each pair $\{g_i, g_j\} \in 2^G$, a measure for each triple $\{g_i, g_j, g_k\} \in 2^G$ and so on until to assign a measure to each subset of G. Indeed, the Choquet integral is a non-additive integral based on the concept of fuzzy measure, or capacity [24], being a set function $\mu : 2^G \rightarrow [0,1]$, such that $\mu(T)$ represents the comprehensive importance of criteria in T for each $T \subseteq G$, satisfying the following boundary and monotonicity conditions

1a) $\mu(\emptyset) = 0$, $\mu(G) = 1$ (*boundary conditions*),

2a) $\forall R \subseteq T \subseteq G$, $\mu(R) \leq \mu(T)$ (*monotonicty condition*).

So, given $R, T \subseteq G$ such that $R \cap T = \emptyset$, three cases can occur:

case 1) $\mu(R \cup T) = \mu(R) + \mu(T)$ (additivity),

case 2) $\mu(R \cup T) > \mu(R) + \mu(T)$ (superadditivity),

case 3) $\mu(R \cup T) < \mu(R) + \mu(T)$ (subadditivity).

In **case 1)**, the sets of criteria R and T are not interacting since the importance assigned to the set of criteria $R \cup T$ ($\mu(R \cup T)$) is equal to the sum of the importance assigned to the two subsets of criteria considered separately ($\mu(R) + \mu(T)$); in **case 2)**, the sets of criteria R and T are positively interacting since the importance assigned to the set of criteria $R \cup T$ ($\mu(R \cup T)$) is greater than the sum of the importance assigned to the two subsets of criteria considered separately ($\mu(R) + \mu(T)$), while in **case 3)** the sets of criteria R and T are negatively interacting since the importance assigned to the set of criteria $R \cup T$ ($\mu(R \cup T)$) is lower than the sum of the importance assigned to the two subsets of criteria considered separately ($\mu(R) + \mu(T)$).

Given the capacity μ, the Choquet integral assigns to each alternative $a \in A$ the following value:

$$Ch(\mathbf{a}, \mu) = \sum_{i=1}^{n} \mu\left(A_i\right) \cdot \left(g_{(i)}(a) - g_{(i-1)}(a)\right), \tag{1.5}$$

where $\mathbf{a} = (g_1(a), \ldots, g_n(a))$ and $_{(\cdot)}$ stands for a permutation of the indices of criteria such that:

$$0 = g_{(0)}(a) \leq g_{(1)}(a) \leq \ldots \leq g_{(n)}\left(a\right), \ A_i = \{(i), \ldots, (n)\}.$$

A meaningful and useful reformulation of the capacity μ and of the Choquet integral can be obtained by means of the Möbius representation [87] of the capacity μ, which is a function $m : 2^G \rightarrow \mathbb{R}$ defined as follows:

$$\mu(R) = \sum_{T \subseteq R} m(T). \tag{1.6}$$

Now, if R is a singleton, i.e. $R = \{i\}$ with $i = 1, 2, \ldots, n$, then $\mu(\{i\}) = m(\{i\})$ while, if R is a couple (non-ordered pair) of criteria, i.e. $R = \{i, j\}$, then $\mu(\{i, j\}) = m(\{i\}) + m(\{j\}) + m(\{i, j\})$ and so on. Note also that, if $|R| = n$, then there exists a bijection $h : \{1, \ldots, 2^n\} \leftrightarrow 2^R$ such that it can be defined in the following way:

$$\sum\nolimits_{T \subseteq R} m(T) = \sum\nolimits_{i=1}^{2^n} m(h(i)) = \sum\nolimits_{i=0}^{2^n - 1} m(h(i)). \tag{1.7}$$

The Möbius representation $m(R)$ can be obtained from $\mu(R)$ as follows:

$$m(R) = \sum_{T \subseteq R} (-1)^{(|R| - |T|)} \mu(T). \tag{1.8}$$

In terms of the Möbius representation of the capacity μ, properties **1a)** and **2a)** can be restated as follows [22]:

1b) $m(\emptyset) = 0, \ \sum_{T \subseteq G} m(T) = 1,$

2b) $\forall\, g_i \in G$ and $\forall R \subseteq G \setminus \{g_i\}\,,\ m\left(\{g_i\}\right) + \sum\limits_{T \subseteq R} m(T) \geq 0,$

while, the Choquet integral may be written as [42]:

$$Ch(\mathbf{a}, \mu) = \sum_{T \subseteq G} m(T) \cdot \min_{g_i \in T} g_i\left(a\right). \tag{1.9}$$

Since the application of the Choquet integral preference model involves the knowledge of $2^{|G|} - 2$ parameters (because $\mu(\emptyset) = 0$ and $\mu(G) = 1$), it is advisable using k-additive measures [45]. A fuzzy measure is k-additive if $m(T) = 0\ \forall\, T \subseteq G$ such that $|T| > k$. In other words, considering k-additive measures we do not take into account the interactions between more than k criteria. In real world applications, 2-additive measures are considered. In this case, the Choquet integral can be applied knowing $n + \binom{n}{2}$ values that is, one value for each criterion g_i and one value for each pair of criteria $\{g_i, g_j\}$. Considering 2-additive measures, properties **1b)** and **2b)** become

1c) $m\left(\emptyset\right) = 0,\ \sum\limits_{g_i \in G} m\left(\{g_i\}\right) + \sum\limits_{\{g_i, g_j\} \subseteq G} m\left(\{g_i, g_j\}\right) = 1,$

2c) $\begin{cases} m\left(\{g_i\}\right) \geq 0,\ \forall g_i \in G, \\ m\left(\{g_i\}\right) + \sum\limits_{g_j \in T} m\left(\{g_i, g_j\}\right) \geq 0,\ \forall g_i \in G \text{ and } \forall\, T \subseteq G \setminus \{g_i\}\,,\, T \neq \emptyset. \end{cases}$

Finally we arrive at the Möbius representation of the Choquet integral with a 2-additive measure for the alternative a

$$Ch(\mathbf{a}, \mu) = \sum_{g_i \in G} m\left(\{g_i\}\right) g_i\left(a\right) + \sum_{\{g_i, g_j\} \subseteq G} m\left(\{g_i, g_j\}\right) \min\{g_i\left(a\right), g_j\left(a\right)\} \tag{1.10}$$

where $g(a) = (g_1(a), \ldots, g_n(a))$.

In this context, the importance of criterion $g_i \in G$ does not depend only on itself, but also on its contribution to all coalitions of criteria. For this reason we use some indices taken from the game theory.

The Shapley value [93] is used to represent the importance of criterion g_i, $i = 1, \ldots, n$, and it is given by

$$\varphi(i, \mu) = \sum_{T \subseteq G : i \notin T} \frac{|T|!(|G| - |T| - 1)!}{|G|!} \left(\mu(T \cup \{i\}) - \mu(T)\right), \tag{1.11}$$

or,

$$\varphi(i,\mu) = \sum_{T \subseteq G} \frac{(|G|-|T|)!(|T|-1)!}{|G|!} \left(\mu(T) - \mu(T \setminus \{i\})\right). \tag{1.12}$$

The main property of this index is that $\sum_{i \in G} \varphi(i,\mu) = \mu(G)$. The Shapley value can be considered a mean of the values $(\mu(T \cup \{i\}) - \mu(T))$ with weights $\frac{|T|!(|G|-|T|-1)!}{|G|!}$. Indeed, to prove that $\frac{|T|!(|G|-|T|-1)!}{|G|!}$ are non-negative and sum to one we note that:

1. $\frac{|T|!(|G|-|T|-1)!}{|G|!} \geq 0,\ \forall\, T \subseteq G$, which is trivial.

2. By using the Eq. (1.7) and Eq, (1.12), we also have that:

$$\sum_{T \subseteq G : i \in T} \frac{(|G|-|T|)!(|T|-1)!}{|G|!} = \sum_{i=1}^{2^{n-1}} \frac{(n-|h(i)|)!(|h(i)|-1)!}{n!}.$$

Now, since

$$\sum_{i=1}^{2^{n-1}} = 2^{n-1} = 2^n 2^{-1} = 2^n - 2^{n-1} = 2^n - 2^{n-1} + 1 - 1 = 2^n - 1 - (2^{n-1} - 1) =$$

$$= \sum_{i=1}^{n} \binom{n}{i} - \left[\left(\sum_{i=0}^{n-1} \binom{n-1}{i}\right) - 1\right] = \sum_{i=1}^{n} \binom{n}{i} - \left[\left(\sum_{i=1}^{n-1} \binom{n-1}{i}\right) + 1 - 1\right] =$$

$$= \sum_{i=1}^{n} \binom{n}{i} - \sum_{i=1}^{n-1} \binom{n-1}{i},$$

where we have considered the fact that $\binom{n-1}{0} = 1$. Noting that, if $k > n$ then $\binom{n}{k} = 0$ [49, 3], we have

$$\sum_{i=1}^{n} \binom{n}{i} - \sum_{i=1}^{n-1} \binom{n-1}{i} + 0 = \sum_{i=1}^{n} \binom{n}{i} - \sum_{i=1}^{n-1} \binom{n-1}{i} + \binom{n-1}{n} = \sum_{i=1}^{n} \binom{n}{i} - \sum_{i=1}^{n} \binom{n-1}{i} =$$

$$= \sum_{i=1}^{n} \left(\binom{n}{i} - \binom{n-1}{i}\right) = \sum_{i=1}^{n} \binom{n-1}{i-1}.$$

So, we can state the following

$$\sum_{i=1}^{2^{n-1}} \frac{(n-|h(i)|)!(|h(i)|-1)!}{n!} = \sum_{i=1}^{n} \binom{n-1}{|h(i)|-1} \frac{(n-|h(i)|)!(|h(i)|-1)!}{n!} =$$

$$= \sum_{i=1}^{n} \frac{(n-1)!}{(|h(i)|-1)!(n-|h(i)|)!} \frac{(n-|h(i)|)!(|h(i)|-1)!}{n!} = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

Now, by considering the 2-additive Möbius representation of a capacity μ, the Shapley value can be expressed as follows:

$$\varphi(i, m) = m(\{i\}) + \sum_{j \in G \setminus \{i\}} \frac{m(\{i, j\})}{2}. \tag{1.13}$$

Another index of the importance of a criterion that can be mentioned is the Banzhaf interaction index [11] which can be computed as follows:

$$\varphi(i, \mu) = \sum_{T \subseteq G: i \notin T} \frac{1}{2^{|G|-1}} \left(\mu(T \cup \{i\}) - \mu(T)\right). \tag{1.14}$$

Note that, in this last case all the subsets to which the criterion i can belong are equally weighted[5]. Analogously, by considering only couples of criteria, the Murofushi-Soneda index [84] expresses the importance of the pair $\{g_i, g_j\}$ as follow:

$$\varphi(\{i, j\}, \mu) = \sum_{T \subseteq G: i, j \notin T} \frac{|T|!(|G|-|T|-2)!}{(|G|-1)!} \left(\mu(T \cup \{i, j\}) - \mu(T \cup \{i\}) - \mu(T \cup \{j\}) + \mu(T)\right). \tag{1.15}$$

In case of the 2-additive Möbius representation of a capacity μ, the Murofushi-Soneda index can be computed as follows:

$$\varphi(\{i, j\}, m) = m(\{i, j\}). \tag{1.16}$$

The bipolar Choquet integrals

There are multicriteria decision problems in which the underlying scales on which the alternatives are assessed can be considered as being bipolar scales. In these cases it exists a neutral point (usually 0) on the evaluation scale associated to each criterion. Above this neutral point lie the evaluations corresponding to a good performance, while, below that point we have evaluations corresponding to bad performance. The motivation for this argument is that human decision makers do often distinguish gains and losses and behave differently according to this distinction. For example, if we

[5]Note that: $\sum_{T \subseteq G: i \notin T} \frac{1}{2^{|G|-1}} = \sum_{i=1}^{2^{n-1}} \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}} 2^{n-1} = 1.$

consider the maximum speed of a car, the reference neutral level can be 130 km/h, in this case, a maximum speed over 130 km/h is considered positive, while, a maximum speed under 130 km/h is considered negative. This has lead to models based on the bipolar Choquet integral, the bipolar Shilkret integral, bipolar Sugeno integral (for further details see [61]) and other models in decision under risk or uncertainty on Cumulative Prospect Theory ([102]).

In the bipolar case the importance of each subset of criteria depends on whether these criteria express a positive or negative evaluation with respect to the neutral point and it also depends on which criteria express an evaluation of the opposite sign. So in this context, for each action $a \in A$ we consider all possible pairs of sets of criteria (E^+) which express positive evaluations, and, in contrast, the set of criteria (E^-) which express negative evaluations, with $E^+, E^- \subset G$ such that $E^+ \cap E^- = \emptyset$. Then we assign a weight $\mu^+(E^+, E^-)$ to the set of criteria E^+ and a weight $\mu^-(E^+, E^-)$ to the set of criteria E^-, as a function of the particular pair (E^+, E^-). The difference $\mu(E^+, E^-) = \mu^+(E^+, E^-) - \mu^-(E^+, E^-)$ represents the net balance between weights of criteria in favor of good evaluations and weights of criteria opposing to them.

Formally, the bipolar Choquet integral is based on a bicapacity [46, 47, 61], being a function $\hat{\mu} : \mathcal{P}(G) \to [-1, 1]$, where $G = \{g_1, \ldots, g_n\}$[6] is the set of evaluation criteria and
$\mathcal{P}(G) = \{(C, D) : C, D \subseteq G \text{ and } C \cap D = \emptyset\}$, such that:

- $\hat{\mu}(\emptyset, G) = -1, \;\; \hat{\mu}(G, \emptyset) = 1, \;\; \hat{\mu}(\emptyset, \emptyset) = 0$ (boundary conditions),
- for all $(C, D), (E, F) \in \mathcal{P}(G)$, if $C \subseteq E$ and $D \supseteq F$, then $\hat{\mu}(C, D) \leq \hat{\mu}(E, F)$ (monotonicity condition).

As we noted, the value $\hat{\mu}(C, D)$, with $(C, D) \in \mathcal{P}(G)$, can be interpreted as the net importance of criteria from C when criteria from D are opposing to them so that, if $\hat{\mu}(C, D) > 0$ ($\hat{\mu}(C, D) < 0$ or $\hat{\mu}(C, D) = 0$), then criteria in C are more (less or equally) important than those in D. Given a vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and a bicapacity $\hat{\mu}$ on $\mathcal{P}(G)$, the bipolar Choquet integral of $\mathbf{x}$ with respect to the bicapacity $\hat{\mu}$ can be defined as

$$Ch^B(\mathbf{x}, \hat{\mu}) = \sum_{j=p}^{n} \left(|x_{(j)}| - |x_{(j-1)}|\right) \hat{\mu}\left(\left\{i \in G : x_i \geq |x_{(j)}|\right\}, \left\{i \in G : x_i \leq -|x_{(j)}|\right\}\right) \quad (1.17)$$

where $(1), (2), \ldots, (n)$ is a permutation of the indices such that $0 = |x_{(0)}| \leq |x_{(1)}| \leq \ldots \leq |x_{(n)}|$ and

[6]Note that, as we already mentioned, we shall sometimes write criterion $i \in G$ instead of criterion $g_i \in G$; that is we shall denote a criterion by its index.

$p = \min\limits_{\substack{j\in\{1,\dots,n\}:\\ |x_{(j)}|>0}} \{j\}$[7]; moreover, if $x_{(n)} = 0 \Rightarrow Ch^B(\mathbf{x},\hat{\mu}) = 0$.

$Ch^B(\mathbf{x},\hat{\mu})$ can be interpreted as the weighted mean of $\mathbf{x} \in \mathbb{R}^n$ when the importance of criteria is represented by the bicapacity $\hat{\mu}$.

In [50, 55] the following decomposition of the bicapacity $\hat{\mu}$ has been proposed

$$\hat{\mu}(C,D) = \mu^+(C,D) - \mu^-(C,D), \quad \text{for all } (C,D) \in \mathcal{P}(G) \tag{1.18}$$

where $\mu^+, \mu^- : \mathcal{P}(G) \to [0,1]$ are such that:

$$\left.\begin{array}{ll} \mu^+(G,\emptyset) = 1 \text{ and } \mu^+(\emptyset,B) = 0, & \forall B \subseteq G, \\ \mu^+(C,D) \leq \mu^+(E,D), & \forall C \subseteq E \subseteq G, \\ \mu^+(C,D) \geq \mu^+(C,F), & \forall D \subseteq F \subseteq G, \end{array}\right\} \quad \left.\begin{array}{ll} \mu^-(\emptyset,G) = 1 \text{ and } \mu^-(B,\emptyset) = 0, & \forall B \subseteq G, \\ \mu^-(C,D) \leq \mu^-(C,F), & \forall D \subseteq F \subseteq G, \\ \mu^-(C,D) \geq \mu^-(E,D), & \forall C \subseteq E \subseteq G. \end{array}\right\}$$

Given a set of alternatives $A = \{a_1,\dots,a_m\}$ and $a, b \in A$, $\mu^+(C,D)$ gives the degree of the importance of criteria in C in favor of the preference of a over b when criteria in D are opposing to it. Conversely, $\mu^-(C,D)$ represents the importance of criteria in D opposing to criteria in C. As a consequence, as above explained, $\hat{\mu}(C,D)$ is a balance of the importance of criteria in C (in favor of the preference) and criteria in D (opposing to the same preference).

Using the decomposition of the bicapacity $\hat{\mu}$ defined above, the positive and negative bipolar Choquet integrals of $\mathbf{x} \in \mathbb{R}^n$ can be defined:

$$Ch^{B+}(\mathbf{x},\hat{\mu}) = \sum_{j=p}^{n} \left(|x_{(j)}| - |x_{(j-1)}|\right) \mu^+ \left(\left\{i \in G : x_i \geq |x_{(j)}|\right\}, \left\{i \in G : x_i \leq -|x_{(j)}|\right\}\right) \tag{1.19}$$

$$Ch^{B-}(\mathbf{x},\hat{\mu}) = \sum_{j=p}^{n} \left(|x_{(j)}| - |x_{(j-1)}|\right) \mu^- \left(\left\{i \in G : x_i \geq |x_{(j)}|\right\}, \left\{i \in G : x_i \leq -|x_{(j)}|\right\}\right). \tag{1.20}$$

where $p = \min\limits_{\substack{j\in\{1,\dots,n\}:\\ |x_{(j)}|>0}} \{j\}$. Supposing that $\overline{x}$ represents the degree of preference of a over b, $Ch^{B+}(x,\hat{\mu})$ and $Ch^{B-}(x,\hat{\mu})$ give, respectively, how much a is comprehensively preferred to b (considering the reasons in favor of a) and how much b is preferred to a (considering the reasons against a). Consequently, the bipolar Choquet integral of $\mathbf{x} \in \mathbb{R}^n$ is obtained as the difference between the positive

[7] For example, if $x = (-1, 1, 0, 2, -2)$, then $|x_{(1)}| = |x_3| = 0$; $|x_{(2)}| = |x_1| = 1$; $|x_{(3)}| = |x_2| = 1$; $|x_{(4)}| = |x_4| = 2$; $|x_{(5)}| = |x_5| = 2$. Therefore, $p = 2$.

and the negative bipolar Choquet integrals.

$$Ch^B(\mathbf{x}, \hat{\mu}) = Ch^{B+}(\mathbf{x}, \hat{\mu}) - Ch^{B-}(\mathbf{x}, \hat{\mu}). \tag{1.21}$$

In the following we consider 2-additive decomposable bicapacities, such that the next decomposition of μ^+ and μ^-, proposed in [50], holds:

- $\mu^+(C, D) = \sum_{j \in C} a^+(\{j\}, \emptyset) + \sum_{\{j,k\} \subseteq C} a^+(\{j,k\}, \emptyset) + \sum_{j \in C,\, k \in D} a^+(\{j\}, \{k\}),$
- $\mu^-(C, D) = \sum_{j \in D} a^-(\emptyset, \{j\}) + \sum_{\{j,k\} \subseteq D} a^-(\emptyset, \{j,k\}) + \sum_{j \in C,\, k \in D} a^-(\{j\}, \{k\}).$

$a^+(\{j\}, \emptyset)$ represents the importance of criterion j when it is in favor of the preference; $a^+(\{j,k\}, \emptyset)$ represents the importance of the pair of criteria $\{j,k\}$ when they are in favor of the preference; finally, $a^+(\{j\}, \{k\})$ represents the antagonistic effect exercised by k over j when j is in favor of the preference and k is against it. The coefficients $a^-(\emptyset, \{j\})$, $a^-(\emptyset, \{j,k\})$ and $a^-(\{j\}, \{k\})$ can be interpreted analogously. $a^-(\emptyset, \{j\})$ represents the importance of criterion j when it is against the preference; $a^-(\emptyset, \{j,k\})$ represents the importance of the pair of criteria $\{j,k\}$ when they are against the preference; finally, $a^-(\{j\}, \{k\})$ represents the antagonistic effect exercised by j over k when k is opposing to the preference and j is in favor of it. Let us observe that $a^+_{j|k} \leq 0$ and $a^-_{j|k} \leq 0$ for all $j, k = 1, \ldots, n$, $j \neq k$, being representative of the antagonistic effect exercised by k over j in $a^+_{j|k}$ and of j over k in $a^-_{j|k}$.

Denoting by a^+_j, a^+_{jk} and $a^+_{j|k}$ the coefficients $a^+(\{j\}, \emptyset)$, $a^+(\{j,k\}, \emptyset)$ and $a^+(\{j\}, \{k\})$ and by a^-_j, a^-_{jk} and $a^-_{j|k}$ the coefficients $a^-(\emptyset, \{j\})$, $a^-(\emptyset, \{j,k\})$ and $a^-(\{j\}, \{k\})$, the following conditions have to be satisfied (see [25]):

- monotonicity: $\begin{cases} a^+_j + \sum_{k \in C} a^+_{jk} + \sum_{k \in D} a^+_{j|k} \geq 0, \quad \forall\, j \in G,\ \forall (C \cup \{j\}, D) \in \mathcal{P}(G) \\ a^-_j + \sum_{k \in D} a^-_{jk} + \sum_{k \in C} a^-_{k|j} \geq 0, \quad \forall\, j \in G,\ \forall (C, D \cup \{j\}) \in \mathcal{P}(G) \end{cases}$
- normalization: $\begin{cases} \mu^+(G, \emptyset) = 1, \text{ i.e., } \sum_{j \in G} a^+_j + \sum_{\{j,k\} \subseteq G} a^+_{jk} = 1 \\ \mu^-(\emptyset, G) = 1, \text{ i.e., } \sum_{j \in G} a^-_j + \sum_{\{j,k\} \subseteq G} a^-_{jk} = 1 \end{cases}$

Finally, by using the 2-additive decomposition of the bicapacity $\hat{\mu}$ defined above, the positive and negative bipolar Choquet integrals of $\mathbf{x} \in \mathbb{R}^n$ w.r.t. the bicapacity $\hat{\mu}$ can be redefined as follows [25]:

$$Ch^{B+}(\mathbf{x},\hat{\mu}) = \sum_{j\in G,\, x_j>0} a_j^+ x_j + \sum_{\substack{j,k\in G,\, j\neq k\\ x_j,\, x_k>0}} a_{jk}^+ \min\{x_j,x_k\} + \sum_{\substack{j,k\in G,\, j\neq k\\ x_j>0,\, x_k<0}} a_{j|k}^+ \min\{x_j,-x_k\} \tag{1.22}$$

$$Ch^{B-}(\mathbf{x},\hat{\mu}) = -\sum_{j\in G,\, x_j<0} a_j^- x_j - \sum_{\substack{j,k\in G,\, j\neq k\\ x_j,\, x_k<0}} a_{jk}^- \max\{x_j,x_k\} - \sum_{\substack{j,k\in G,\, j\neq k\\ x_j>0,\, x_k<0}} a_{j|k}^- \max\{-x_j,x_k\} \tag{1.23}$$

$$Ch^{B}(\mathbf{x},\hat{\mu}) = \sum_{j\in G,\, x_j>0} a_j^+ x_j + \sum_{j\in G,\, x_j<0} a_j^- x_j + \sum_{\substack{j,k\in G,\\ j\neq k,\, x_j,x_k>0}} a_{jk}^+ \min\{x_k,x_j\} + \tag{1.24}$$

$$+ \sum_{\substack{j,k\in G,\\ j\neq k,\, x_j,x_k<0}} a_{jk}^- \max\{x_k,x_j\} + \sum_{\substack{j,k\in G,\\ x_j>0,\, x_k<0}} a_{j|k}^+ \min\{x_j,-x_k\} + \sum_{\substack{j,k\in G,\\ x_j<0,\, x_k>0}} a_{j|k}^- \max\{x_j,-x_k\} \tag{1.25}$$

The level dependent Choquet integral

Let us consider a multicriteria decision problem of buying a car by taking into account three criteria: price, maximum speed and comfort which, for the sake of simplicity, we consider normalized on a common scale, let us, say to fix the idea, between 0, being the worst evaluation, and 1, being the best evaluation. Now, it can be argued that:

- above a certain threshold t of price, the maximum speed is preferred to comfort,
- for values of price between t and another threshold s, with $s < t$, the comfort is preferred to maximum speed,
- below the threshold s of price, the maximum speed is again preferred to comfort.

It is easy to note how the relations of importance between criteria changes according to the level of the price. Beyond their justification (which can be not so unreal or unreasonable), these sophisticated preferences are interesting because it can be showed [53] that they cannot be represented, for example, by using a bipolar Choquet integral or a Cumulative Prospect Theory functional. The reason is that, in this case, the capacity depends also on the level of the evaluations. These considerations leads to consider integration with respect to a capacity which is also function of the level of the evaluation, as shown by what follows.

A level dependent capacity μ^L [53] is defined as a function $\mu^L : 2^G \times (\alpha,\beta) \to [0,1],\ (\alpha,\beta) \subseteq \mathbb{R}$, such that:

1.a) for all $t \in (\alpha,\beta)$ and $E \subseteq B \subseteq G,\ \mu^L(E,t) \leq \mu^L(B,t)$,

2.a) for all $t \in (\alpha, \beta)$, $\mu^L(\emptyset, t) = 0$ and $\mu^L(G, t) = 1$,

3.a) for all $t \in A \subseteq G$, $\mu^L(E, t)$ considered as a function with respect to t is Lebesgue measurable.

The level dependent Choquet integral of $\mathbf{x} = (x_1, \ldots, x_n) \in (\alpha, \beta)^n$, $(\alpha, \beta) \subseteq \mathbb{R}$, w.r.t. the level dependent capacity μ^L, can be defined as follows:

$$Ch^L(\mathbf{x}, \mu^L) = \int_{\min\{x_1,\ldots,x_n\}}^{\max\{x_1,\ldots,x_n\}} \mu^L(E(\mathbf{x}, t), t)dt + \min\{x_1, \ldots, x_n\}$$

where $E(\mathbf{x}, t) = \{i \in G : x_i \geq t\}$.

Taking into account α and β, the level dependent Choquet integral can be formulated as:

$$Ch^L(\mathbf{x}, \mu^L) = \int_{\alpha}^{\beta} \mu^L(E(\mathbf{x}, t), t)dt + \alpha,$$

such that, if the value set of criteria is normalized with $\alpha = 0$, $\beta = 1$ and $\mathbf{x} \in [0, 1]^n$, we get

$$Ch^L(\mathbf{x}, \mu^L) = \int_{0}^{1} \mu^L(E(\mathbf{x}, t), t)dt.$$

Let us remark that the level dependent Choquet integral can also be written as

$$Ch^L(\mathbf{x}, \mu^L) = \sum_{i=2}^{n} \int_{x_{(i-1)}}^{x_{(i)}} \mu^L(E(\mathbf{x}, t), t)dt + x_{(1)}.$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that:

$$x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(i)} \leq \ldots \leq x_{(n)}.$$

The use of the level dependent Choquet integral can have some limitations in real world applications because it needs a definition of an arbitrary large number of capacities $\mu^L(A, t)$, one for each level $t \in (\alpha, \beta)$, $(\alpha, \beta) \subseteq \mathbb{R}$. To overcome this problem in [53] the authors suggest the use of an *interval* level dependent capacity which is defined with respect to a unique partition of (α, β) as follow:

(1) $a_0, a_1, \ldots, a_{p-1}, a_p \in (\alpha, \beta)$, $p > 0$, $p \in \mathbb{N}$, such that $\alpha = a_0 < a_1 < \ldots < a_{p-1} < a_p = \beta$,

(2) p capacities $\mu_1, \ldots, \mu_p$ on $G = \{1, \ldots, n\}$,

such that, for all $E \subseteq G$ and all $t \in (\alpha, \beta)$, $\mu^L(E, t) = \mu_j(E)$ if $t \in]a_{j-1}, a_j[$, $j = 1, \ldots, p$.

When the μ^L is an interval level dependent capacity, the following theorem [53] allows to calculate the level dependent Choquet integral Ch^L as the sum of a finite number of classical Choquet integrals, more precisely one of them for each interval $(a_{j-1}, a_j),\ j = 1, \ldots, p$.

Theorem 1.2.1. *If μ^L is the interval level dependent capacity on G relative to the breakpoints $a_0, a_1, \ldots, a_{p-1}, a_p \in [\alpha, \beta]$ and to the capacities $\mu_1, \ldots, \mu_p$, then for each $\mathbf{x} \in (\alpha, \beta)^n$*

$$Ch^L(\mathbf{x}, \mu^L) = \sum_{j=1}^{p} Ch(\mathbf{x}^j, \mu_j),$$

where $\mathbf{x}^j \in (0, a_j - a_{j-1})^n$ is the vector with elements $x_i^j,\ i = 1, \ldots, n$, defined as follows:

$$x_i^j = \begin{cases} 0 & \text{if} \quad x_i < a_{j-1}; \\ x_i - a_{j-1} & \text{if} \quad a_{j-1} \leq x_i < a_j; \\ a_j - a_{j-1} & \text{if} \quad x_i \geq a_j. \end{cases}$$

So, by using the vector with elements $x_i^j,\ i = 1, \ldots, n$, the level dependent Choquet integral can be formulated as:

$$Ch^L(\mathbf{x}, \mu^L) = \sum_{j=1}^{p} \sum_{i=1}^{n} \mu_j(N_i) \left(x_{(i)}^j - x_{(i-1)}^j \right),$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that $x_{(1)}^j \leq x_{(2)}^j \leq \ldots \leq x_{(i)}^j \leq \ldots \leq x_{(n)}^j$ and $N_i = \{(i), \ldots, (n)\}$ for $i = 1, \ldots, n$.

To make manageable the previous results, from now on, we will refer to the interval level dependent capacity assuming that only one partition of (α, β) is chosen. This also means that we fix the number p of the levels which define the level dependent capacity.

Given a level dependent capacity μ^L, the Möbius transform of a level dependent capacity, is a function $m^L : 2^G \times (\alpha, \beta) \rightarrow \mathbb{R}$, such that, for all $E \subseteq G$ and all $t \in (\alpha, \beta)$ we have that

$$\mu^L(E, t) = \sum_{B \subseteq E} m^L(B, t). \tag{1.26}$$

From the previous Eq. 1.26, we obtain that, for all $A \subseteq G$ and all $t \in (\alpha, \beta)$,

$$m^L(E, t) = \sum_{B \subseteq E} (-1)^{|E|-|B|} \mu^L(B, t). \tag{1.27}$$

In terms of Möbius representation, properties **1.a** and **2.a** are, respectively, restated as:

1.b) $m^L(\emptyset,t)=0,\ \sum_{E\subseteq G} m^L(E,t)=1,\ \forall t\in(\alpha,\beta),$

2.b) $\forall\, i\in G$ and $\forall\, E\subseteq G\setminus\{i\},\ \ m^L(\{i\},t)+\sum_{E\subseteq G} m^L(E\cup\{i\},t)\geq 0,\ \forall t\in(\alpha,\beta).$

Now, recalling the Theorem 1.2.1 in [53], for all $\mathbf{x}\in(\alpha,\beta)^n$ and for all level dependent capacities μ^L the following holds:

$$Ch^L(\mathbf{x},\mu^L)=\sum_{B\subseteq G}\int_{\min_{i\in G}\{x_i\}}^{\min_{i\in B}\{x_i\}} m^L(B,t)dt+\min_{i\in G}\{x_i\}. \tag{1.28}$$

In case of the interval level dependent capacity μ^L relative to the breakpoints $a_0,a_1,\ldots,a_{p-1},a_p\in(\alpha,\beta)$ and to the capacities $\mu_1,\ldots,\mu_p$, it is necessary to determine $p(2^n-2)$ values corresponding to the values $\mu_j(E),\ j=1,...,p$, for all $E\subseteq G$ with the exception of $\mu_j(\emptyset)$ and $\mu_j(G)$. Now, to be used in real applications, one need a more manageable interval level dependent capacity which can be obtained by restricting the analysis to the more significant parameters. So, considering an interval level dependent capacity where $a_0,a_1,\ldots,a_p$ are the elements decomposing the interval (α,β) and m_r the Möbius representation of the capacities μ_r, $r=1,\ldots,p$, the level dependent Choquet integral can be written in the following way

$$Ch^L(\mathbf{x},\mu^L)=\sum_{r=1}^{p} Ch(\mathbf{x}^r,\mu_r)=\sum_{r=1}^{p}\left(\sum_{E\subseteq G} m_r(E)\ \min_{i\in E} x_i^r\right), \tag{1.29}$$

where the $x_i^r,\ i=1,\ldots,n$ have been defined above.

A Möbius representation m^L of an interval level dependent capacity μ^L is k-order additive ($1\leq k\leq n-1$) if, for all $t\in[\alpha,\beta]$, $m^L(E,t)=0$ for all $E\subseteq G$ such that $|E|>k$. In particular, the value that a 2-additive interval level dependent capacity μ^L assigns to a set $E\subseteq G$ can be expressed in terms of the Möbius representation as follows:

$$\mu^L(E,t)=\sum_{i\in E} m^L(\{i\},t)+\sum_{\{i,j\}\subseteq E} m^L(\{i,j\},t),\ \forall E\subseteq G.$$

With regard to 2-additive capacities, properties **1.b** and **2.b** have, respectively, the following expressions:

1.c) $m^L(\emptyset,t)=0,\ \sum_{i\in G} m^L(\{i\},t)+\sum_{\{i,j\}\subseteq G} m^L(\{i,j\},t)=1,$

$$\textbf{1.c)}\quad \begin{cases} m^L\left(\{i\},t\right) \geq 0,\ \forall i \in G, \\ m^L\left(\{i\},t\right) + \sum\limits_{j\in E} m^L\left(\{i,j\},t\right) \geq 0,\ \forall i \in G \text{ and } \forall\, E \subseteq G \setminus \{i\}\,,\ E \neq \emptyset. \end{cases}$$

It is a straightforward consequence of the definitions of k-additive capacities and k-additive interval level dependent capacities that, given:

- an interval level dependent capacity μ^L over $2^G \times (\alpha, \beta)$, with $a_0, \ldots, a_p$ being a partition of the interval (α, β) and μ_r the corresponding capacities, $r = 1, \ldots, p$,
- m^L and m_r as Möbius representations of μ^L and μ_r, respectively,

then μ^L is a k-additive interval level dependent capacity if and only if all μ_r are k-additive, $r = 1, \ldots, p$.

Remembering the Eq. 1.28, we get that, if the level dependent capacity μ^L is k-additive, then for all $\mathbf{x} \in [\alpha, \beta]^n$

$$Ch^L(\mathbf{x}, \mu^L) = \sum_{E\subset G:|E|\leq k} \int_{\min_{i\in G}\{x_i\}}^{\min_{i\in E}\{x_i\}} m^L(E,t)dt + \min_{i\in G}\{x_i\}. \tag{1.30}$$

In terms of 2-additive level dependent capacity, the level dependent Choquet integral may be also reformulated as follows:

$$Ch^L(\mathbf{x}, \mu^L) = \sum_{t=1}^{p} \left(\sum_{i\in G} m^L(\{i\},t) x_i^j + \sum_{i,y\in G} m^L(\{i,y\},t) \min\{x_i^j, x_y^j\} \right). \tag{1.31}$$

Likewise to the indices we have remarked in Section 1.2.2, also in case of level dependent capacity it can be evaluated the importance and the interaction between criteria. So, given a generalized capacity μ^L, $i \in G$ and $t \in (\alpha, \beta)$, the Shapley index of importance of the i-th criterion $\phi(\mu^L, i, t)$, can be calculated as

$$\phi(i, \mu^L, t) = \sum_{E\subseteq G\setminus\{i\}} \frac{|E|!(|G|-|E|-1)!}{|G|!} \left(\mu^L(E\cup\{i\},t) - \mu^L(E,t) \right). \tag{1.32}$$

Considering the Möbius transformation m^L of μ^L, the importance index $\phi(\mu^L, i, t)$ can be rewritten as

$$\phi(i, \mu^L, t) = \sum_{E\subseteq G\setminus\{i\}} \frac{m^L(E\cup\{i\},t)}{|E\cup\{i\}|}. \tag{1.33}$$

Finally, if μ^L is an interval level dependent capacity written in terms of Möbius transformation, then

$$\phi(i,\mu^L,t) = \sum_{E\subseteq G\setminus\{i\}} \frac{m_r(E\cup\{i\})}{|E\cup\{i\}|}, \ \ \forall t\in[a_{r-1},a_r[. \tag{1.34}$$

From Eqs. (1.32)-(1.34), the comprehensive importance of criterion i can be obtained as [53]:

$$\phi(i,\mu^L) = \frac{\int_\alpha^\beta \phi(\mu^L,i,t)\; dt}{\beta-\alpha}.$$

In particular, for an interval level dependent capacity, the importance of criterion i on the interval $[a_{r-1},a_r[$ is given by

$$\phi(i,\mu^L,[a_{r-1},a_r[) = \sum_{E\subseteq G\setminus\{i\}} \frac{m_r(E\cup\{i\})}{|E\cup\{i\}|}\cdot\frac{a_r-a_{r-1}}{\beta-\alpha}$$

and, consequently,

$$\phi(i,\mu^L) = \sum_{r=1}^{p}\left(\sum_{E\subseteq G\setminus\{i\}} \frac{m_r(E\cup\{i\})}{|E\cup\{i\}|}\cdot\frac{a_r-a_{r-1}}{\beta-\alpha}\right).$$

For a 2-additive interval level dependent capacity, the previous index can be rewritten as

$$\phi(i,\mu^L) = \sum_{r=1}^{p}\left[\left(m_r(\{i\}) + \sum_{j\in G\setminus\{i\}} \frac{m_r(\{i,j\})}{2}\right)\cdot\frac{a_r-a_{r-1}}{\beta-\alpha}\right]. \tag{1.35}$$

Given a level dependent capacity μ^L, $E\subseteq G$ and $t\in[\alpha,\beta]$, the interaction index, which measures the interaction of criteria from E with respect to the generalized capacity μ^L at the level $t\in(\alpha,\beta)$, can be calculated as

$$I(E,\mu^L,t) = \sum_{B\subseteq G\setminus E} \frac{|B|!(|G|-|B|-|E|)!}{|G|!}\left(\sum_{C\subseteq E}(-1)^{E\setminus C}\mu^L(C\cup B,t)\right). \tag{1.36}$$

The interaction index $I(E,\mu^L,t)$ can be represented in terms of Möbius transform as follows:

$$I(E,\mu^L,t) = \sum_{B\subseteq G\setminus E} \frac{m^L(E\cup B,t)}{|B|+1}. \tag{1.37}$$

If μ^L is an interval level dependent capacity, then

$$I(E,\mu^L,t) = \sum_{B\subseteq G\setminus E} \frac{m_r(E\cup B)}{|B|+1}, \ \ \forall t\in[a_{r-1},a_r[. \tag{1.38}$$

As a consequence, the importance of criteria in E for the interval $[a_{r-1}, a_r[$ is

$$I(E, \mu^L, [a_{r-1}, a_r[) = \sum_{B \subseteq G \setminus E} \frac{m_r(E \cup B)}{|B|+1} \cdot \frac{a_r - a_{r-1}}{\beta - \alpha}$$

and, therefore,

$$I(E, \mu^L) = \sum_{r=1}^{p} \left(\sum_{B \subseteq G \setminus E} \frac{m_r(E \cup B)}{|B|+1} \cdot \frac{a_r - a_{r-1}}{\beta - \alpha} \right). \tag{1.39}$$

If $E = \{i, j\}$ and μ^L is a 2-additive interval level dependent capacity, then Eq. (1.39) can be rewritten as

$$I(\{i, j\}, \mu^L) = \sum_{r=1}^{p} \left(m_r(\{i, j\}) \cdot \frac{a_r - a_{r-1}}{\beta - \alpha} \right). \tag{1.40}$$

1.3 Reference approaches

What is usually called preference aggregation model is a well-defined model which produces results allowing comparison of any pair of potential alternatives, in crisp or fuzzy terms, which reflects the relations of preference, indifference, or incomparability.

The three main families of preference aggregation models are:

- *Multiple Attribute Utility Theory* (MAUT) [72]: the main hypothesis in the utility theory is that there exists a function $U(\cdot)$ mapped into reals, called utility function, which associates to each alternative a value (real number) that represents its degree of "goodness". So, the number assigned to $a \in A$ is independent from the evaluations of the other alternatives. It leads us to define a complete preorder on A and it does not allow any incomparability among the alternatives.

- *Decision Rule Approach* [54, 56, 57]: in this approach the DM's preferences are represented by a set of decision rules ("if..., then..."), which are expressed in a natural language and are easily understandable by the DM. Decision rule model is the most general of all three.

- *Outranking methods* [51, 88, 89]: first developed in France in the late sixties by B. Roy, they are based on a preference relation, usually called an outranking relation, which is a binary relation R on the set of alternatives A, such that aRb means that alternative a is at least as good as alternative b and it holds if a majority of the criteria supports this assertion and the opposition of the other criteria is not too strong. This outranking relation is neither complete (it is possible that not(aRb) and not(bRa)) nor transitive (it is possible that aRb and bRc but not(aRc)). The two most known families of outranking methods are ELECTRE [91] and PROMETHEE [18, 19, 20].

We conclude this paragraph by briefly summarizing the main features of ELECTRE methods, while, in Section 1.3.1 we show the PROMETHEE methods a little more in detail because we will deal with them it in Chapter 2.

A glimpse of ELECTRE methods

ELECTRE methods build one (or several) binary outranking relations which is based on two principal concepts:

1. the *concordance* verify if for an outranking aRb to be validated, a certain majority of criteria is in favor of this assertion;

2. the *non-discordance* test verifies that none of the criteria in the minority is so strongly in favor of action b against action a that is no more possible to conclude that aRb.

These two conditions must be both verified for validating the assertion aRb.

In the ELECTRE methods the relative importance w_i attached to each criterion reflects its voting power when it contributes to the majority which is in favor of an outranking relation. Note that these parameters cannot be interpreted in terms of substitution rates as in the weighted sum preference model (see Section 1.2.1).

1.3.1 PROMETHEE methods

PROMETHEE [20] is a family of MCDA methods developed to deal with ranking, choice and sorting problems (see [13] for a survey of PROMETHEE methods). In order to apply the PROMETHEE methods, one has to define for each criterion $g_j \in G$ a preference function $p_j(a, b)$ expressing, for each couple of alternatives $(a, b) \in A \times A$, the degree of preference of a over b on criterion g_j. Six different value functions have been proposed in [20]. Here we shall recall the V-shape function $P_j(a, b)$ being the most used in practice,

$$P_j(a,b) = \begin{cases} 0 & \text{if } \; d_j(a,b) \leq q_j, \\ \frac{d_j(a,b)-q_j}{p_j-q_j} & \text{if } \; q_j < d_j(a,b) < p_j, \\ 1 & \text{if } \; d_j(a,b) \geq p_j \end{cases} \tag{1.41}$$

where $d_j(a,b) = g_j(a) - g_j(b)$, and q_j, p_j are the indifference and preference thresholds for criterion g_j, respectively. Given the positive weights of n criteria $(w_1, \ldots, w_n)$ such that $\sum_{j=1}^{n} w_j = 1$, the comprehensive preference of a over b is given by $\pi(a,b) = \sum_{j=1}^{n} P_j(a,b) w_j$, while the positive, negative and net flows of $a \in A$ are consequently computed as follows:

$$\phi^+(a) = \frac{1}{m-1} \sum_{b \in A \setminus \{a\}} \pi(a,b), \qquad \phi^-(a) = \frac{1}{m-1} \sum_{b \in A \setminus \{a\}} \pi(b,a), \qquad \phi(a) = \phi^+(a) - \phi^-(a),$$

with m being the cardinality fo the set A of feasible alternatives. $\phi^+(a)$ represents how much, in average, a is preferred to the other alternatives in A; $\phi^-(a)$ represents how much, in average, the other alternatives in A are preferred to a, while $\phi(a)$ represents the comprehensive value of a. On one hand, PROMETHEE I provides a partial preorder on A building a preference (P^I), an indifference (I^I) and an incomparability (R^I) relations. In particular, aP^Ib if $\phi^+(a) \geq \phi^+(b)$, $\phi^-(a) \leq \phi^-(b)$ and at least one of the two inequalities is strict; aI^Ib if $\phi^+(a) = \phi^+(b)$ and $\phi^-(a) = \phi^-(b)$; aR^Ib otherwise. On the other hand, PROMETHEE II provides a complete ranking on A by comparing the alternatives with respect to their net flow. More in detail, it defines a preference P^{II} and an indifference I^{II} relations stating that $aP^{II}b$ iff $\phi(a) > \phi(b)$, while $aI^{II}b$ iff $\phi(a) = \phi(b)$.

In the following, we briefly recall the main characteristics of some others PROMETHEE methods.

PROMETHEE III

PROMETHEE III [19] defines a complete interval order by associating to each action $a \in A$ an interval $[x_a, y_a]$ which is given by

$$\begin{cases} x_a = \overline{\phi}(a) - \alpha\sigma_a, \\ y_a = \overline{\phi}(a) + \alpha\sigma_a; \end{cases} \tag{39}$$

where:

- $\overline{\phi}(a) = \frac{1}{m} \sum_{x \in A} [\pi(a,x) - \pi(x,a)]$,
- $\sigma_a^2 = \frac{1}{m} \sum_{x \in A} \left[\pi(a,x) - \pi(x,a) - \overline{\phi}(a)\right]^2$,

For $\alpha > 0$ this method takes into account the variability of the net flows. The complete interval order (P^{III}, I^{III}) is defined as follows:

$$aP^{III}b \text{ (a is preferred to b) iff } x_a > y_b;$$
$$aI^{III}b \text{ (a is indifferent to b) iff } x_a \leq y_b \text{ and } x_b \leq y_a.$$

PROMETHEE IV

PROMETHEE IV [19] handles multicriteria decision problems with a continuous infinity of alternatives. Therefore denoting by $X_j \subseteq \mathbb{R}$ the set of values that can be assumed by criterion $g_j \in G$, it is assumed that there exists a distribution ρ on $X = X_1 \times \dots X_n$ such that, for all $\mathbf{x} \in X$, $\rho(\mathbf{x}) \geq 0$ and $\int_X \rho(\mathbf{x})d\mathbf{x} = 1$. In this context the positive and the negative flows for each alternative a are defined as follows:

$$\phi^+(a) = \int_X \pi(a, \mathbf{x})\rho(\mathbf{x})d\mathbf{x} \tag{40}$$

$$\phi^-(a) = \int_X \pi(\mathbf{x}, a)\rho(\mathbf{x})d\mathbf{x} \tag{41}$$

PROMETHEE V

PROMETHEE V [16] deals with problems in which alternatives have to be grouped in clusters or segments. After defining different clusters of alternatives $C_1, \dots, C_k, \dots, C_R$, and after computing the net flow of each alternative $a \in A$, the following 0-1 problem has to be solved

$$\begin{aligned} &\max \sum_{a \in A} x_a \phi(a), \\ &\sum_{a \in A} \alpha_a x_a \geq [\leq, =]\beta, \ [BeCl] \\ &\sum_{a \in C_k} \gamma_{a,k} x_a \geq [\leq, =]\delta_k, \ [WiCl], \end{aligned} \tag{44}$$

where $x_a \in \{0, 1\}$ for all $a \in A$, while $\alpha_a, \gamma_{a,k} \in \mathbb{R}$ are coefficients attached to the single alternatives. Inequalities $[BeCl]$ are used to include constraints between the different clusters (for example, a maximum number of alternatives that should be selected), while inequalities $[WiCl]$ are used to include constraints within the same cluster (for example, the maximum investment in a particular region cannot be greater than a certain threshold δ_k). Of course, the chosen alternatives are those for which $x_a = 1$ after solving the 0-1 program.

PROMETHEE VI

In PROMETHEE VI [17], the DM does not provide exact values for the weights of criteria but intervals of possible values:

$$w = (w_1, \ldots, w_n) : \; w_j^- \leq w_j \leq w_j^+, \;\; \text{for all } j = 1, \ldots, n.$$

Then, all allowable weight vectors are considered and projected on a specific graphic representation of the multicriteria decision problem called GAIA plane [79] (for the detail see the following Section 2.2.2) in order to distinguish between hard and soft problems.

1.4 Robustness of the recommendations

In every model, the values assigned to the required parameters reflect in some sense the preferences expressed by the DM and they directly influence the recommendations obtained according to the procedure followed. So, it can be useful to verify which conclusions could be considered stable with respect to some perturbations of the values that can be assigned to that parameters or, at least, with respect to a subset of value clearly identified. Conclusions which satisfy this property are defined robust and the whole procedure to verify the stability of the results is called robustness analysis [90].

The robustness analysis takes into account a different point of view with respect to the sensitivity analysis. Indeed, the first one considers simultaneous variations of all parameters together, while, the last one is focused on the variation of only one parameter each time by leaving fixed the others (ceteris paribus analysis). Clearly, one of the two analysis does not exclude the other.

In the following two sections we remark two approaches which account for robustness concerns. One is based on the ordinal regression [35, 68] of the preference information and another which can be considered a volume based approach [104].

1.4.1 Robust Ordinal Regression

Usually, from among many sets of parameters of a preference model representing the preference information given by the DM, only one specific set is selected and used to work out a recommendation. In general, more than one set of parameters compatible with the preferences provided by the DM exists. Clearly, each of them provides the same results on the reference alternatives but they can differ w.r.t. the recommendations on the other alternatives that do not belong to the reference set

A^*.

For example, let us consider two alternatives $a, b \in A^*$ for which the DM is able to state that a^* is preferred to b^*. Then, all compatible models will be concordant with the preference of a over b but, considering other two alternatives $c, d \in A \setminus A*$ on which the DM did not provide any information, it is possible that for some compatible models, c is preferred to d, while for other compatible models, d is preferred to c. For this reason, the choice of only one among the many compatible models is, in some sense, arbitrary. From this information the parameters of a decision model can be induced by using a methodology called ordinal regression [58]. The ordinal regression paradigm which has been applied within the approaches using a value functin [21, 67, 68, 86, 98], and those using an outranking relation [82, 83].

Robust Ordinal Regression (ROR) [27] methods takes into account simultaneously all models compatible with the preferences information provided by the DM. The results of the ROR are given in terms of necessary and possible consequences of applying all the compatible preference models on the considered set of alternatives. More precisely, the couple (a, b) belongs to the necessary preference relation, denoted a $\succsim^N b$, iff a is at least as good as b for all compatible models, while the couple (a, b) belongs to the possible preference relation, denoted $a \succsim^P b$, iff a is at least as good as b for at least one model compatible.

ROR has been already applied to value functions in [28, 59, 60, 71], to the Choquet integral preference model in [6] and to the outranking methods in [29, 52, 71].

Non Additive Robust Ordinal Regression (NAROR)

The NAROR [6] belongs to the family of ROR methods and, as any ROR method, it is based on the indirect preference information.

In general, the parameters of the decision model can be inferred by some preference information supplied by the DM in the following form:

- possibly incomplete binary preference relation $\succsim^*$ on an improper subset of alternatives $A^* \subseteq A$ and/or a possibly incomplete preference relation one on an improper subset $G^* \subseteq G$ of criteria,
- the assignment of positive or negative interactions between criteria, and the comparison of their strenght.

Such inputs of the analysis at hand induce some restrictions on the parameters $\theta \in \boldsymbol{\Theta}$ of the function $U(a, \theta)$ chosen to evaluate the alternatives, defining a set of parameters P, consistent with the provided preferences information, defined as follow

$$P := \{\theta \in \Theta \mid U(a,\theta) \geq U(b,\theta) \text{ iff } a \succsim^* b, \;\; \forall a, b \in A^*\}.$$

Since the aim is to extend the relation $\succsim^*$ to all A, as output, the NAROR builds two very informative preference relations, one necessary preference relation $\succsim^N$ and one possible preference relation $\succsim^P$, which are returned to DM in the following terms:

$$a \succsim^N b \;\text{ iff }\; U(a,\theta) \geq U(b,\theta) \;\text{ for all }\; \theta \in P,$$

$$a \succsim^P b \;\text{ iff }\; U(a,\theta) \geq U(b,\theta) \;\text{ for at least one }\; \theta \in P.$$

1.4.2 Stochastic Multicriteria Acceptability Analysis

Stochastic Multicriteria Acceptability Analysis (SMAA [75]) is a family of MCDA methods which take into account uncertainty or imprecision on the evaluations and preference model parameters. In this section we describe SMAA-2 [76], since our proposed methodology also regards ranking problems.

In SMAA methods, once chosen a value function, for example the linear $U(\mathbf{a},\mathbf{w}) = \sum_{i=1}^{n} w_i g_i(a)$, where $\mathbf{w} \in W = \{(w_1,\ldots,w_n) \in \mathbb{R}^n \colon w_i \geq 0 \text{ and } \sum_{i=1}^{n} w_i = 1\}$ and $\mathbf{a} = (g_1(a),\ldots,g_n(a))$, the indirect preference information is composed of two probability distributions, f_χ and f_W, defined on the evaluation space χ and on the weight space W, respectively. Defining the rank function $rank(\cdot)$ as

$$rank(a,\xi,\mathbf{w}) = 1 + \sum_{x \in A\setminus\{a\}} \rho\left(U(\xi_x,\mathbf{w}) > U(\xi_a,\mathbf{w})\right),$$

(where $\rho(false) = 0$ and $\rho(true) = 1$) that, for all $x \in A$, $\xi \in \chi$ and $w \in W$ gives the rank position of alternative a, SMAA-2 computes the set of weights of criteria for which alternative a assumes rank $r = 1, 2, \ldots, n$, as follows:

$$W^r(a,\xi) = \{\mathbf{w} \in \mathbf{W} : rank(a,\xi,\mathbf{w}) = r\}.$$

The following further indices are computed in SMAA-2:

- *The rank acceptability index*:

$$b_a^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{\mathbf{w} \in W^r(a,\xi)} f_W(\mathbf{w})\, d\mathbf{w}\, d\xi; \tag{1.42}$$

that measures the variety of different parameters compatible with the DM's preference information giving to the alternative a the rank r. Clearly, the best alternatives are those having rank acceptability index greater than zero for the first positions and rank acceptability index close to zero for the lower positions and it is within the range $[0,1]$.

- *The central weight vector*:

$$\mathbf{w}_d^c = \frac{1}{b_d^1}\int_{\xi\in\chi} f_\chi(\xi)\int_{\mathbf{w}\in W^1(d,\xi)} f_W(\mathbf{w})\mathbf{w}\, d\mathbf{w}\, d\xi; \tag{1.43}$$

describes in terms of a representative weight vectors the preferences of a typical DM that make alternative d the most preferred;

- *The pairwise winning index*:

$$p_{de} = \int_{\mathbf{w}\in W} f_W(\mathbf{w})\int_{\xi\in\chi: u(\xi_d,\mathbf{w})>u(\xi_e,\mathbf{w})} f_\chi(\xi)d\xi\, d\mathbf{w}. \tag{1.44}$$

considers the frequency with which an alternative d is weakly preferred to an alternative e in the space of the preference parameters and it is within the range $[0,1]$.

From a computational point of view, the multidimensional integrals defining the considered indices are estimated by using the Monte Carlo method.

The following is an example of situation in which SMAA methodology can help the DM.

Example 1.4.1. *Let us consider three alternatives $A=\{a_1,a_2,a_3\}$ and three criteria $G=\{g_1,g_2,g_3\}$. The evaluation matrix is showed in Table 1.2.*

Table 1.2: The evaluation table computed for the example at hand

a_i	$g_1(\cdot)$	$g_2(\cdot)$	$g_3(\cdot)$
a_1	6	8	7
a_2	10	6	5
a_3	8	4	9

The three alternatives share the same arithmetic mean and it is easy to show that their rank strictly depends on the choice of the weights. For example if we choose the three different vectors of weights below

$$w^1 = (w_1^1,\ w_2^1,\ w_3^1) \;=\; (0,5; 0,25; 0,25);$$

$w^2 = (w_1^2,\ w_2^2,\ w_3^2) \ = \ (0,25;0,5;0,25);$

$w^3 = (w_1^3,\ w_2^3,\ w_3^3) \ = \ (0,25;0,25;0,5);$

we get that the utilities of the alternatives at hand and, consequently, their rankings are the following:

$U(a_1, w^1) = 6,75;\ U(a_2, w^1) = 7,75;\ U(a_3, w^1) = 7,25 \ \Rightarrow a_2 \succ a_3 \succ a_1;$

$U(a_1, w^2) = 7,25;\ U(a_2, w^2) = 6,75;\ U(a_3, w^2) = 6,25 \ \Rightarrow a_1 \succ a_2 \succ a_3;$

$U(a_1, w^3) = 7,00;\ U(a_2, w^3) = 6,50;\ U(a_3, w^3) = 7,50 \ \Rightarrow a_3 \succ a_1 \succ a_2.$

By using the SMAA methodology, after that we sampled a certain number of vectors of weights and, consequently, we obtained the rankings of the alternatives for each sampled vector, we can compute the rank acceptability indices and the pairwise winning indices to evaluated the goodness of all considered alternatives with respect to the problem at hand.

The SMAA for the Choquet integral preference model

Assuming as preference model the Choquet integral, in the computation of the indices in Eqs. (1.42) and (1.44) we have to consider two probability distributions $f_{\mathcal{M}}(\mu)$ and $f_{\chi}(\xi)$ on $\mathcal{M}$ and χ, respectively, where

$$\mathcal{M} = \{\mu : 2^G \rightarrow [0,1] : \mu(\emptyset) = 0,\ \mu(G) = 1, \text{ and } \mu(R) \leq \mu(T) \text{ for all } R \subseteq T \subseteq G\}$$

is the set of admissible capacities and $\chi \subseteq \mathbb{R}^{m\times n}$ is the evaluation space, that is the set of all the performance matrices $\chi = [g_i(a),\ g_i \in G,\ a \in A]$. For each $\mu \in \mathcal{M}$ and for each $\xi \in \chi$, a complete ranking of the alternatives at hand can be computed. SMAA-2 introduces a ranking function relative to the alternative a,

$$rank(a, \xi, w) = 1 + \sum_{x \in A\setminus\{a\}} \rho\left(Ch\left(\xi_x, \mu\right) > Ch\left(\xi_a, \mu\right)\right), \tag{1.45}$$

where $\rho(false) = 0$, $\rho(true) = 1$ and $\xi_a = (\xi_{a1}, \ldots, \xi_{an}) \in \mathbb{R}^{1\times n}$ is the vector of the performances of alternative $a \in A$. Consequently, since the position filled by an alternative depends on the capacity μ, for each alternative $a \in A$, for each performance matrix $\xi \in \chi$ and for each rank $r = 1, \ldots, m$, SMAA-2 computes the set of capacities for which alternative a assumes rank r:

$$\mathcal{M}_a^r(\xi) = \{\mu \in \mathcal{M} : rank(a, \xi, w) = r\}. \tag{1.46}$$

1.5 Introductive example

Sport provides a natural setting for interesting decision making problems and the first applications of the Operation Research tools on this issue have occurred since the 50's [41, 80, 107]. Even if the greatest part of contributions in literature are related to the scheduling of different sport championships (for a survey of scheduling in sports see [73]), many proposals regarding the application of MCDA to sport can be found as well. In [97], the authors apply the MACBETH method [10] to rank the performances of Brazilian athletes in 2008 Olympic Games; in [33] the AHP method is applied to compare cricket bowlers in the Indian premier league, while [30] use the TOPSIS method [64] to evaluate the performances of basketball players of the Lithuanian league; [85] measures the accuracy of PROMETHEE [20], SMART [36] and a centroid method [12] in describing the abilities of baseball teams in the American major league. Finally, [66] proposes several models to aggregate the results obtained by the athletes in the 10 decathlon events analyzing, at the same time, the aggregated results obtained by the PROMETHEE II method.

In the following, we propose to apply the SMAA-Choquet method [5] to evaluate the performances of different sailboats in sailing regattas. This method is taken into account, instead of other possible methods, mainly because it permits to deal with interactions between criteria, expressed in different scales, also considering robustness concerns. More precisely:

- through the use of the Choquet integral, it permits to handle positive and negative interactions between criteria,
- through the construction of a common scale for the evaluations on the considered criteria, it permits to aggregate criteria of heterogeneous nature,
- through the consideration of a plurality of preference parameters for the Choquet integral (common scales and sets of non-additive weights called capacities), it permits to handle robustness concerns in multicriteria aggregation.

In general, sailboats of different design (shape, size, etc.) compete in worldwide regattas. Obviously, the design of the sailboats strongly influences their performances, so that comparing them taking into account the time necessary to complete the course is not fair. Indeed, for example, long sailboats have an advantage over short ones in case of strong wind, while short sailboats are favored in case of low wind. The scoring options used to rank the participants to the regattas, assign a "corrected time" to each sailboat taking into account the time spent to complete the race and the specific design of each sailboat. Nevertheless, the way in which the scoring options compute this corrected time are based on different assumptions and, consequently, the complete ranking of the participants varies with respect to the considered scoring option.

Here, we propose to aggregate the corrected times computed by the different scoring options to assign a single comprehensive corrected time to each sailboat taking into account the good characteristics of each scoring option. To this aim, we can restate this problem as an MCDA problem, by considering the sailboats as alternatives, the different scoring options as the evaluation criteria on which the different alternatives have to be evaluated and, consequently, the corrected time assigned to each sailboat by the considered scoring option as the evaluation of the alternative on the criterion at hand. Since the corrected time values computed by the scoring options are not expressed on the same scale and some type of interaction can be observed among the scoring options, we shall evaluate the comprehensive performance of the different sailboats by using the SMAA-Choquet method.

Sailboats handicapping rules: the ORC

Longer sailboats are inherently faster than shorter ones, and so, in the interest of fairness, since the 1820s in the sailboat races, the larger sailboat started to be imprecisely handicapped. Unfortunately, the results showed to have many problems with the allowance system. Modern rating systems are based on a precise analysis of very specific physical parameters (such as: length, sail area, sail material, etc.) and this analysis is done in a unique way, differently from the calculation of the corrected time on which the final ranking of the sailboats depends. The most prevalent handicap rating systems today are the ORC, the Offshore Racing Rule (ORR)[8], the IRC [9] and the Performance Handicap Racing Fleet (PHRF)[10].

The ORC is an international body for the sport of competitive sailing and it is responsible for fixing and maintaining the rating and classification standards used to define offshore, sailboat racing handicap categories. In the ORC handicap system, the final ranking can be established by using

[8]http://offshoreracingrule.org/

[9]https://www.rorcrating.com/

[10]http://www.ussailing.org/racing/offshore-big

a certain number of scoring options but, usually, none of them is universally accepted or always adopted. This disagreement exists because each of these scoring options takes into account different conditions of the race and, therefore, it is meaningful according to some aspects of the race such as course distance, wind direction, wind strength, etc. In order to make comparable the performances of the different sailboats, each of these scoring options transforms the real time spent by a sailboat to finish the race in another corrected time, taking into account also the different physical characteristics of the sailboat.

The ORC gives a certificate to each sailboat providing the parameters necessary to compute the corrected time in each of the considered scoring options. For any of the simple scoring options, parameters are given for the offshore (coastal/long distance) and for the inshore (windward/leeward) races. In the ORC, there are four main scoring options through which it is possible to build the final ranking and these are the Time on Distance (ToD), the Time on Time (ToT), the Performance Line (PL), and the Triple Number (ToT:low, Medium, Hight). Moreover, in our application, we shall take into account also another scoring option, the General Purpose Handicap method, being a scoring method universally recognized in this sector. The selection of the scoring option usually depends on the size of the sailboats, type and level of the fleet, type of the race and local racing conditions. Moreover, the choice of the scoring option is at the discretion of National Authorities or local event organizers. Figure 1.1 shows an extract of a sailboat's ORC certificate where we can find the parameters necessary to compute the corrected time, while Table 1.3 shows the formulas used to obtain the corrected time in each of the five scoring options. The term *Distance* appearing in the formulas in Table 1.3 is the distance between the start and the end points of the course expressed in miles.

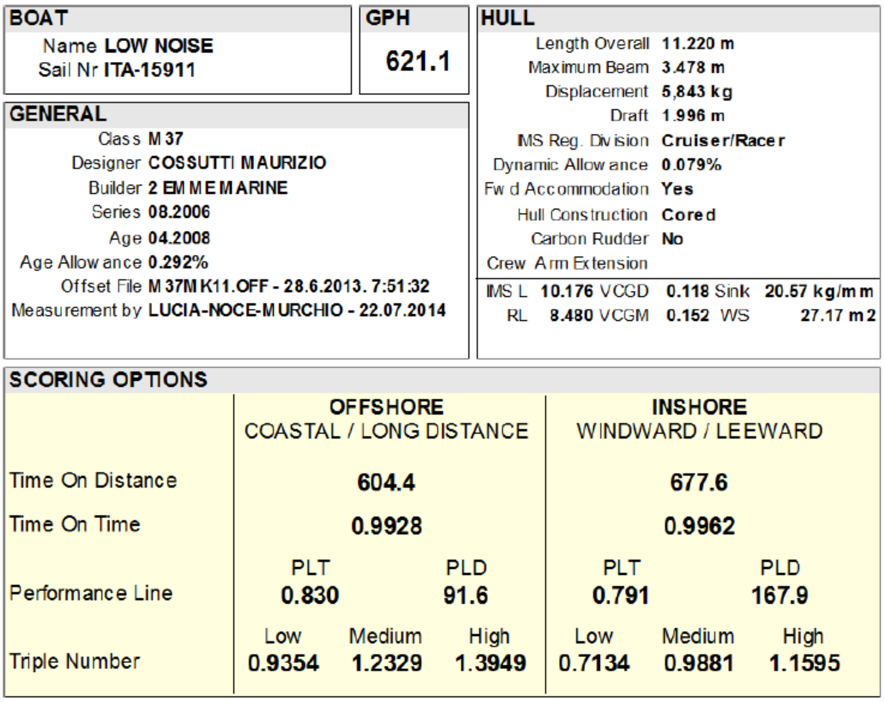
Table 1.3: Formulas for Corrected Times through ORC simple scoring option

Scoring option	Formula
$Time\ on\ Distance$	Elapsed Time - (ToD × Distance)
$Time\ on\ Time$	Elapsed Time × ToT
$Performance\ Line$	Elapsed Time × PLT - PLD × Distance
$Triple\ Number$	Elapsed Time × ToT (Low, Medium, High)
$General\ Purpouse\ Handicap$	Elapsed Time - (GPH × Distance)

Example 1.5.1. *Let us consider the certificate shown in Figure 1.1 and the five scoring options described in Table 1.3 supposing that the time (expressed in seconds) necessary to the sailboat for*

[11] http://www.orc.org/rules/ORC%20Rating%20Systems%202015.pdf

Figure 1.1: Example of an ORC certificate[11]



BOAT		GPH	HULL	
Name	LOW NOISE	621.1	Length Overall	11.220 m
Sail Nr	ITA-15911		Maximum Beam	3.478 m
			Displacement	5,843 kg
			Draft	1.996 m

GENERAL	
Class	M 37
Designer	COSSUTTI MAURIZIO
Builder	2 EMMEMARINE
Series	08.2006
Age	04.2008
Age Allowance	0.292%
Offset File	M37MK11.OFF - 28.6.2013. 7:51:32
Measurement by	LUCIA-NOCE-MURCHIO - 22.07.2014

HULL (cont.)	
IMS Reg. Division	Cruiser/Racer
Dynamic Allowance	0.079%
Fwd Accommodation	Yes
Hull Construction	Cored
Carbon Rudder	No
Crew Arm Extension	

IMS L	10.176	VCGD	0.118	Sink	20.57 kg/mm
RL	8.480	VCGM	0.152	WS	27.17 m2

SCORING OPTIONS	OFFSHORE COASTAL / LONG DISTANCE			INSHORE WINDWARD / LEEWARD		
Time On Distance	604.4			677.6		
Time On Time	0.9928			0.9962		
Performance Line	PLT 0.830		PLD 91.6	PLT 0.791		PLD 167.9
Triple Number	Low 0.9354	Medium 1.2329	High 1.3949	Low 0.7134	Medium 0.9881	High 1.1595

concluding the offshore course is t and the covered distance is d, then the corrected time in the five scoring options is computed as follows:

- *Time on Distance:* $t - 604.4d$,
- *Time on Time:* $t \times 0.9928$,
- *Performance Line:* $t \times 0.83 - 91.6 \times d$,
- *Triple Number:* $t \times c$*. The parameter* c *for which the elapsed time has to be multiplied is chosen in the end of the course from the authorities or organizers depending on the speed of the wind. If the speed of the wind is Low, then* t *has to be multiplied by* $c = 0.9354$*, if it is Medium, by* $c = 1.2329$ *and, if it is High, by* $c = 1.3949$,
- *General Purpose Handicap:* $t - 621.1d$.

Now, let us show how the application of the different scoring options affects the final ranking of the alternatives. Let us consider three sailboats (XP-ACT, PROFILO, MAYDA) whose parameters necessary to compute the corrected time are shown in Table 1.4.

Using each of the five scoring options, we computed the corrected time for the three sailboats shown in Table 1.5.

Table 1.4: Parameters necessary for the computation of the corrected time for the five considered scoring options; t is expressed in seconds

Sailboat / Paramaters / Real Time	ToD	ToT	PLT	PLD	TN: Low; Medium; High	GPH	t
XP-ACT	542.50	1.11	0.972	116.70	1.0593; 1.3851; 1.5609	624.70	76,873
PROFILO	605.80	0.99	0.881	124.30	0.9503; 1.2368; 1.3952	615.20	84,321
MAYDA	592.80	1.01	0.882	114.40	0.9629; 1.2638; 1.4246	598.50	83,188

Table 1.5: Corrected time of the three sailboats with respect to the five scoring options; the considered distance is $d = 82.4$ miles, while t is expressed in seconds

Sailboat / Scoring Option	ToD	ToT	PL	TN	GPH
XP-ACT	32,171	85,014	65,104	102,633	31,504
PROFILO	34,403	83,520	64,044	100,688	33,629
MAYDA	34,341	84,195	63,945	101,248	33,872

Analyzing the values of the corrected time, it is possible to compute the relative ranking of the three sailboats for each scoring option as shown in Table 1.6.

Table 1.6: Ranking of the three sailboats with respect to the five scoring options

Sailboat / Scoring Option	ToD	ToT	PL	TN	GPH
XP-ACT	1^{st}	3^{rd}	3^{rd}	3^{rd}	1^{st}
PROFILO	3^{rd}	1^{st}	2^{nd}	1^{st}	2^{nd}
MAYDA	2^{nd}	2^{nd}	1^{st}	2^{nd}	3^{rd}

As one can see, each of the three sailboats can fill the first position, depending on the considered scoring option. Moreover, we can also underline that:

- *XP-ACT fills the first position in two of the considered scoring options, while in the remaining three, it is in the third position,*
- *PROFILO fills the first position in two of the considered scoring options, the second position in other two scoring options and the third position in the remaining one,*
- *MAYDA fills the first and the third position for one score option only, while it is in the second position for the other three score options.*

SMAA approach applied to the ORC simple scoring

In what follow we apply some MCDA methods combined with the SMAA approach to the "53rd Syracuse - Malta" regatta[12] which length is 82.4 miles. At this end, we shall consider 13 sailboats participating to the regatta that will be the considered alternatives, while the 5 scoring options will be the

[12] Official website of the Siracusa-Malta regatta: http://www.rmyc.org/races/archive/?id=115&page=4

evaluation criteria. In the following, we shall denote the five scoring options by g_{ToD}, g_{ToT}, g_{PL}, g_{TN} and g_{GPH} and, consequently, the set of criteria will be denoted by $G = \{ToD, ToT, PL, TN, GPH\}$. The corrected time computed by the five considered scoring options will be the evaluations of the alternatives (the different sailboats) with respect to the considered criteria and they are provided in Table 1.7.

Table 1.7: Table of the evaluations of the different sailboats expressed in seconds for each simple scoring option. For the sake of simplicity, with respect to Triple Number scoring option, we averaged the three evaluations with respect to Low, Medium and High wind speed

	Scoring Option	g_{ToD}	g_{ToT}	g_{PL}	g_{TN}	g_{GPH}
a_1	OTTOVOLANTE	23,530	73,144	54,225	88,191	22,549
a_2	JUNO OILTANKING	25,193	76,546	57,768	92,471	24,773
a_3	WOLVERINE	26,744	74,856	55,853	89,827	25,533
a_4	NAUTILUS QQ7	26,518	77,374	57,958	93,400	25,760
a_5	UNICA	30,135	81,139	61,187	97,610	29,047
a_6	ARTIE RTFX	30,194	81,358	62,087	98,029	29,683
a_7	RICOMINCIO DA TRE	30,954	78,812	59,338	94,642	29,859
a_8	DREAMER TECH	31,212	87,667	64,689	105,588	30,339
a_9	XP-ACT	32,171	85,014	65,104	102,633	31,504
a_{10}	SQUALO BIANCO	33,599	82,026	61,998	98,540	32,495
a_{11}	EMILE GALLE'	33,672	83,070	63,999	100,020	32,815
a_{12}	PROFILO	34,403	83,520	64,044	100,688	33,629
a_{13}	MAYDA	34,341	84,195	63,945	101,248	33,872

The SMAA-PROMETHEE II method: the weighted sum is often referred as compensatory operator by meaning that a decrease in one evaluation of an action w.r.t. a certain criterion can be compensated, i.e. balanced, by the improvement of its evaluation with respect to one (or more) different criteria among those considered. In other words, low scores on one criterion may be compensated by high scores on another one. The concept of compensation is based on the intensity of preference expressed by each criterion in the sense that the global comparison between two alternatives depends on how much one action is preferred to the other compared to each criterion considered rather than compared to which criteria is preferred.

Instead, the basic idea of a non-compensatory aggregation procedure is that only ordinal information on individual criteria is taken into account. According with [39], an aggregation procedure in non-compensatory if $\forall a, b, c, d \in A$ we have that

$$\{g_j \in G : aP_jb\} \subseteq \{g_j \in G : cP_jd\}\,,\ \{g_j \in G : bP_ja\} \supseteq \{g_j \in G : dP_jc\} \quad \Rightarrow (aPb \Rightarrow cPd)\,,$$

where cP_jd is a preference relation of c over d w.r.t. criterion j while, cPd is the preference relation of c over d considered as comprehensive.

In order to use a locally non-compensatory approach, we implement the PROMETHEE II method supported by the SMAA method.

Let us start from the following reformulation of the PROMETHEE net flow:

$$\phi(a) = \phi^+(a) - \phi^-(a) = \frac{1}{m-1}\sum_{j=1}^{n}\sum_{x\in A}\left[P_j(a,x) - P_j(x,a)\right] =$$

$$= \sum_{j=1}^{n}\frac{1}{m-1}\sum_{x\in A}\left[P_j(a,x) - P_j(x,a)\right], \tag{1.47}$$

where, if $\phi_j(a) = \sum_{x\in A}\left[P_j(a,x) - P_j(x,a)\right]$ is the single criterion net flow [20], then

$$\phi(a) = \sum_{j=1}^{n}\phi_j(a)w_j. \tag{1.48}$$

So, the Eq. (1.48) clearly shows how the final rank is directly influenced by the choice of the weight of each criterion.

Now, we can apply SMAA-2 in order to explore the weight space, describing for example:

- the preferences (weight vector) that make an alternative the most preferred one,
- the preferences that would give a certain rank to a specific alternative,
- the preferences that make an alternative preferred to another one.

These results are presented in the form of the typical indices of SMAA. In particular, the rank acceptability indices are showed in Table 1.8 and the pairwise winning indices are showed in Table 1.9.

Table 1.8: The table of the rank acceptability indices

	Sailboat	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}
a_1	OTTOVOLANTE	100	0	0	0	0	0	0	0	0	0	0	0	0
a_2	JUNO OILTANKING	0	46,91	53,09	0	0	0	0	0	0	0	0	0	0
a_3	WOLVERINE	0	53,09	44,79	2,12	0	0	0	0	0	0	0	0	0
a_4	NAUTILUS QQ7	0	0	2,12	97,88	0	0	0	0	0	0	0	0	0
a_5	UNICA	0	0	0	0	59,41	40,59	0	0	0	0	0	0	0
a_6	ARTIE RTFX	0	0	0	0	0	9,08	89,82	1,1	0	0	0	0	0
a_7	RICOMINCIO DA TRE	0	0	0	0	40,59	50,33	9,08	0	0	0	0	0	0
a_8	DREAMER TECH	0	0	0	0	0	0	0	7,68	20,89	20,41	9,7	17,53	23,79
a_9	XP-ACT	0	0	0	0	0	0	0	0	2,09	21,66	31,29	26,76	18,2
a_{10}	SQUALO BIANCO	0	0	0	0	0	0	1,1	91,22	5,59	2,09	0	0	0
a_{11}	EMILE GALLE'	0	0	0	0	0	0	0	0	69,73	11,65	18,62	0	0
a_{12}	PROFILO	0	0	0	0	0	0	0	0	0	25,79	16,72	30,94	26,55
a_{13}	MAYDA	0	0	0	0	0	0	0	0	1,7	18,4	23,67	24,77	31,46

In particular we observe that OTTOVOLANTE is the best alternative and it is ranked first. Instead, for the last position we can consider the sailboats PROFILO and MAYDA but on the other

Table 1.9: Table of the pairwise winning indices

Sailboat	Sailboat	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
a_1	OTTOVOLANTE	0	100	100	100	100	100	100	100	100	100	100	100	100
a_2	JUNO OILTANKING	0	0	46,91	100	100	100	100	100	100	100	100	100	100
a_3	WOLVERINE	0	53,09	0	97,88	100	100	100	100	100	100	100	100	100
a_4	NAUTILUS QQ7	0	0	2,12	0	100	100	100	100	100	100	100	100	100
a_5	UNICA	0	0	0	0	0	100	59,41	100	100	100	100	100	100
a_6	ARTIE RTFX	0	0	0	0	0	0	9,08	100	100	98,9	100	100	100
a_7	RICOMINCIO DA TRE	0	0	0	0	40,59	90,92	0	100	100	100	100	100	100
a_8	DREAMER TECH	0	0	0	0	0	0	0	0	67,66	7,68	28,57	58,14	58,07
a_9	XP-ACT	0	0	0	0	0	0	0	32,34	0	2,09	18,61	53,72	55,92
a_{10}	SQUALO BIANCO	0	0	0	0	0	1,1	0	92,32	97,91	0	100	100	100
a_{11}	EMILE GALLE'	0	0	0	0	0	0	0	71,43	81,39	0	0	100	98,29
a_{12}	PROFILO	0	0	0	0	0	0	0	41,86	46,28	0	0	0	53,61
a_{13}	MAYDA	0	0	0	0	0	0	0	41,93	44,08	0	1,71	46,39	0

side, looking at the pairwise winning index we can say that PROFILO is preferred to MAYDA in the 53,61% of the times so the worst alternative is MAYDA. Looking at the second best alternative we have to chose between JUNO OILTANKING (46.91%) and WOLVERINE (53,09%). Again, looking at the pairwise winning index we can observe that WOLVERINE is preferred to JUNO OILTANKING in the 53,09% of the times.

The SMAA-Choquet method: the application of the Choquet integral preference model requires, on one hand, that the evaluations of the alternatives on the considered criteria are given on the same scale and, on the other hand, that a capacity μ is given.

Regarding the first point, in order to build a scale common to all the evaluations of the alternatives, in [5] an heuristic, that we shall briefly recall in the following, has been proposed.

For each criterion $g_i \in G$, α different values $x_1, \ldots, x_\alpha$ are uniformly sampled in the interval [0,1], where α is the number of different evaluations got on criterion g_j by the alternatives in A, that is $|\{x \in \mathbb{R} : g_i(a_k) = x, a_k \in A\}| = \alpha$. Then, after ordering the α real numbers in an increasing way, that is $x_{i(1)} < x_{i(2)} < \ldots < x_{i(\alpha)}$, one has to assign the value $x_{i(h)}$, $h = 1, \ldots, \alpha$, to the alternative(s) having the h-th evaluation in an increasing way with respect to the preferences of the DM on criterion g_i. For example, let us consider five alternatives evaluated on criterion g_i as shown in the first line of Table 1.10.

Table 1.10: An example of the scale construction

Criterion/Alternative	a_1	a_2	a_3	a_4	a_5
g_i	3	5	3	4	6
Constructed value	0.1270	0.9058	0.1270	0.8147	0.9134

After sampling five different real numbers in $[0, 1]$ such as, $x_1 = 0.8147$, $x_2 = 0.9058$, $x_3 = 0.1270$, $x_4 = 0.9134$ and $x_5 = 0.9134$ and after ordering them in an increasing way, $x_{i(1)} = 0.1270 <$

$x_{i(2)} = 0.1270 < x_{i(3)} = 0.8147 < x_{i(4)} = 0.9058 < x_{i(5)} = 0.9134$, these values are assigned to the alternatives $a_1, \ldots, a_5$ as shown in the second line of Table 1.10. Applying the same procedure for each of the n criteria, a common scale to the evaluations of the alternatives on the considered criteria is built.

Regarding the second point, in order to get a capacity μ necessary for the application of the Choquet integral preference model, the DM has to provide a direct or an indirect preference information. In the direct case, the DM has to provide a value for each subset of criteria $T \subseteq G$ ($\mu(T)$) representing the importance of the set of criteria T. Since the DM, in general, is not able to provide this information, the indirect preference information is more used in practice [68]. In the indirect case, the DM has to provide some preferences in terms of comparisons between alternatives (for example, a_h is preferred to a_k or a_h and a_k are indifferent) or comparison between importance of criteria and interaction between criteria (for example, g_i is more important than g_j or criteria g_i and g_j are positively interacting) from which a capacity compatible with these preferences can be inferred. After that these preferences are translated to constraints, the space of the 2-additive Möbius representations of capacities μ compatible with the preferences provided by the DM is defined by the following set of constraints,

$$
\left.\begin{array}{l}
\left.\begin{array}{ll}
\varphi(\{g_i\}) \geq \varphi(\{g_j\}) + \varepsilon, & \text{if criterion } i \text{ is more important than criterion } j, \text{ with } i, j \in G, \\
\varphi(\{g_i, g_j\}) \geq \varepsilon, & \text{if criteria } i \text{ and } j \text{ are positively interacting, with } i, j \in G, \\
\varphi(\{g_i, g_j\}) \leq -\varepsilon, & \text{if criteria } i \text{ and } j \text{ are negatively interacting, with } i, j \in G,
\end{array}\right\} E^C \\
\left.\begin{array}{ll}
Ch(a, \mu) \geq Ch(b, \mu) + \varepsilon, & \text{if } a \text{ is preferred to } b \\
Ch(a, \mu) = Ch(b, \mu), & \text{if } a \text{ is indifferent to } b
\end{array}\right\} E^A \\
\left.\begin{array}{l}
m(\{\emptyset\}) = 0, \ \sum\limits_{g_i \in G} m(\{g_i\}) + \sum\limits_{\{g_i, g_j\} \subseteq G} m(\{g_i, g_j\}) = 1, \\
m(\{g_i\}) \geq 0, \ \forall g_i \in G, \\
m(\{g_i\}) + \sum\limits_{g_j \in T} m(\{g_i, g_j\}) \geq 0, \forall g_i \in G \text{ and } \forall\, T \subseteq G \setminus \{g_i\}
\end{array}\right\} E^{MB}
\end{array}\right\} E^{DM}
$$

where ε is an auxiliary variable used to convert the strict inequalities into weak inequalities. For example, the constraint $\varphi(\{g_i\}) > \varphi(\{g_j\})$, translating the preference of criterion g_i over criterion g_j, is converted into the weak constraint $\varphi(\{g_i\}) \geq \varphi(\{g_j\}) + \varepsilon$. If E^{DM} is feasible and $\varepsilon^* > 0$, where $\varepsilon^* = \max \varepsilon$ subject to E^{DM}, then there exists at least one capacity compatible with the preferences provided by the DM.

Considering the two aspects simultaneously, many common scales can be built by using the

proposed heuristic and many capacities compatible with the preferences provided by the DM can be inferred. For this reason, the SMAA-Choquet method proposes to take into account at the same time a plurality of common scales and a plurality of compatible capacities by means of an iterative procedure. At each iteration, a common scale is built by using the procedure described above and, therefore, the existence of a capacity compatible with the preferences provided by the DM has to be checked by solving the previous LP problem.
We have to distinguish the case in which the DM is able to provide preference information in terms of comparisons between alternatives ($E^A \neq \emptyset$) from the case in which (s)he is not able to do it ($E^A = \emptyset$), since the sampling procedure depends on it:

- In the first case, since constraints translating the preferences of the DM on the considered alternatives are dependent on the evaluations of the same alternatives on the considered criteria, that are replaced by the values obtained in the common scale, the set of constraints E^{DM} will vary depending on the built common scale. Consequently, it is possible that for a certain common scale no capacity compatible with the preferences provided by the DM there exists. If there exists a compatible capacity, then it is stored. The procedure of building the common scale ends when a satisfying number of scales, for example 10,000, and the corresponding 10,000 capacities compatible with the preferences provided by the DM have been stored.

- In the second case, the set of constraints E^{DM} will be independent on the built common scale. Consequently, the procedure used to build the common scale and the procedure used to sample a capacity compatible with the preferences provided by the DM are independent. Therefore, 10,000 common scales are built by using the suggested heuristic, while 10,000 compatible capacities can be sampled by using the Hit-And-Run method [96, 101] since the set of compatible capacities defined by the constraints in E^{DM} is convex.

In both cases, 10,000 pairs (scale, capacity) have been stored and for each of them the Choquet integral of the alternatives at hand and their ranking can be obtained. After computing the corresponding 10,000 rankings of the alternatives, the rank acceptability indices and pairwise winning indices described in Section 1.4.2 can be computed. More details on the SMAA-Choquet method can be found in [5].

To apply the SMAA-Choquet method to this particular problem, the preference information was collected by a DM being an expert on the Syracuse-Malta regatta. We asked him to provide us some preference information of the nature shown in Section 1.5. On one hand he was confident enough in providing preference information on the considered criteria, while he was not sure in the comparisons between the alternatives at hand since he did not know very well their design characteristics. Consequently, $E^A = \emptyset$. In the following, we shall give in detail the statements provided by the expert:

- Information in terms of *importance of criteria*:

    1. the Performance Line is considered the most important scoring option since it gives the best approximation[13] of the time necessary to each sailboat to complete the race. This preference is translated to the constraints $\varphi(\{g_{PL}\}) > \varphi(\{g_i\})$ for all $i \in G \setminus \{PL\}$;
    2. the Time on Distance and the Time on Time scoring options are the most adopted in long and short races, respectively. In a comparison between the two criteria, the DM thinks that they have the same importance. This piece of information is translated to the constraint $\varphi(\{g_{ToD}\}) = \varphi(\{g_{ToT}\})$;
    3. as previously observed, the corrected time computed by the Triple Number scoring option depends only on the speed of the wind that in a long race, such as the Syracuse - Malta is, cannot be assumed as constant, while it does not consider the length of the course of the race. Since, instead, the corrected time computed by the Time on Distance scoring option depends on the covered distance, the DM considers the Time on Distance scoring option more important than the Triple Number one. The related constraint is $\varphi(\{g_{ToD}\}) > \varphi(\{g_{TN}\})$;
    4. a DM can argue that the General Purpouse Handicap scoring option gives less information on the race with respect to the others and, therefore, we can add the following constraints: $\varphi(\{g_{GPH}\}) < \varphi(\{g_i\})$ for all $i \in G \setminus \{GPH\}$.

- Information in terms of *interaction between criteria*:

    1. since, as already stated above, the Time on Distance scoring option and the Time on Time scoring option take into account different aspects of the considered race, a sailboat presenting good values of the corrected time for both scoring options will be particularly appreciated. Therefore the two criteria g_{ToD} and g_{ToT} present a positive interaction. The constraint translating this piece of preference information is $\varphi(\{g_{ToD}, g_{ToT}\}) > 0$;
    2. the formulas used in the Time on Distance and in the General Purpouse Handicap scoring options differ only in the coefficients used to compute the corrected time. For this reason,

[13]The Performance Line is a linearization of the Performance Curve which is a function that, given the wind speed, gives back the speed of the sailboat.

the two criteria can be considered as negatively interacting because, in general, a good performance on one of the two criteria can be expected when there is a good performance on the other criterion. The constraint translating this piece of preference information is $\varphi(\{g_{ToD}, g_{GPH}\}) < 0$;

3. for the same reasons expressed in the previous item, the Time on Time scoring option and the Triple Number one present a negative interaction. The constraint $\varphi(\{g_{ToT}, g_{TN}\}) < 0$ is used to translate this piece of preference information;

4. the Performance Line scoring option takes into account aspects already considered both in the Time on Distance and Time on Time scoring options. Consequently, the DM thinks that the two pairs of criteria $\{g_{PL}, g_{ToD}\}$ and $\{g_{PL}, g_{ToT}\}$ are negatively interacting. The constraints $\varphi(\{g_{ToD}, g_{PL}\}) < 0$ and $\varphi(\{g_{ToT}, g_{PL}\}) < 0$ are therefore used to translate this piece of preference information.

Summing up, the set of constraints translating the preferences of the DM is the following:

$$\left.\begin{array}{l}
\varphi(\{g_{PL}\}) \geq \varphi(\{g_{ToD}\}) + \varepsilon, \\
\varphi(\{g_{ToD}\}) = \varphi(\{g_{ToT}\}), \\
\varphi(\{g_{ToT}\}) \geq \varphi(\{g_{TN}\}) + \varepsilon, \\
\varphi(\{g_{TN}\}) \geq \varphi(\{g_{GPH}\}) + \varepsilon, \\
\varphi(g_{ToT}, g_{PL}) \leq -\varepsilon, \\
\varphi(\{g_{ToD}, g_{ToT}\}) \geq \varepsilon, \\
\varphi(\{g_{ToD}, g_{GPH}\}) \leq -\varepsilon, \\
\varphi(\{g_{ToT}, g_{TN}\}) \leq -\varepsilon, \\
\varphi(\{g_{ToD}, g_{PL}\}) \leq -\varepsilon.
\end{array}\right\} E^C$$

Based on the preference information provided by the DM, and observing that she did not provide any preference in terms of comparisons between alternatives, we apply the SMAA-Choquet method by sampling 10,000 pairs (common scale, capacity) computing for each of them the Choquet integral of the 13 alternatives[14]. Consequently, the rank acceptability indices and the pairwise winning indices described in Section 1.4.2 have been computed and reported in Tables 1.11 and 1.12, respectively.

Looking at Table 1.11 one can see that, apart from OTTOVOLANTE that is the best among the considered sailboats in each case, each sailboat can fill different positions depending on the built common scale and on the sampled capacity compatible with the preferences of the DM. For example,

[14]From the computational point of view, the method has been implemented in Matlab environment and executed on a computer with an Intel Core 2 Quad 2.0 GHz and 4 Gb of RAM.

Table 1.11: Rank Acceptability Indices of the considered sailboats expressed in percentage

	Sailboat	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}
a_1	OTTOVOLANTE	100	0	0	0	0	0	0	0	0	0	0	0	0
a_2	JUNO OILTANKING	0	39.64	60.36	0	0	0	0	0	0	0	0	0	0
a_3	WOLVERINE	0	60.36	38.94	0.70	0	0	0	0	0	0	0	0	0
a_4	NAUTILUS QQ7	0	0	0.70	99.30	0	0	0	0	0	0	0	0	0
a_5	UNICA	0	0	0	0	48.66	51.34	0	0	0	0	0	0	0
a_6	ARTIE RTFX	0	0	0	0	0	1.17	97.53	1.30	0	0	0	0	0
a_7	RICOMINCIO DA TRE	0	0	0	0	51.34	47.49	1.17	0	0	0	0	0	0
a_8	DREAMER TECH	0	0	0	0	0	0	0	0	2.95	17.98	25.95	32.04	21.08
a_9	XP-ACT	0	0	0	0	0	0	0	0	1.29	11.06	20.58	28.73	38.34
a_{10}	SQUALO BIANCO	0	0	0	0	0	0	1.30	98.7	0	0	0	0	0
a_{11}	EMILE GALLÉ	0	0	0	0	0	0	0	0	89.39	8.99	1.43	0.19	0
a_{12}	PROFILO	0	0	0	0	0	0	0	0	0	12.51	31.91	22.68	32.90
a_{13}	MAYDA	0	0	0	0	0	0	0	0	6.37	49.46	20.13	16.36	7.68

Table 1.12: Pairwise Winning Indices for each pair of alternatives expressed in percentage

	Sailboat	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
a_1	OTTOVOLANTE	0	100	100	100	100	100	100	100	100	100	100	100	100
a_2	JUNO OILTANKING	0	0	39.64	100	100	100	100	100	100	100	100	100	100
a_3	WOLVERINE	0	60.36	0	99.3	100	100	100	100	100	100	100	100	100
a_4	NAUTILUS QQ7	0	0	0.70	0	100	100	100	100	100	100	100	100	100
a_5	UNICA	0	0	0	0	0	100	48.66	100	100	100	100	100	100
a_6	ARTIE RTFX	0	0	0	0	0	0	1.17	100	100	98.70	100	100	100
a_7	RICOMINCIO DA TRE	0	0	0	0	51.34	98.83	0	100	100	100	100	100	100
a_8	DREAMER TECH	0	0	0	0	0	0	0	0	62.92	0	3.58	52.94	30.24
a_9	XP-ACT	0	0	0	0	0	0	0	37.08	0	0	2.18	44.78	24.19
a_{10}	SQUALO BIANCO	0	0	0	0	0	1.30	0	100	100	0	100	100	100
a_{11}	EMILE GALLÉ	0	0	0	0	0	0	0	96.42	97.82	0	0	100	93.34
a_{12}	PROFILO	0	0	0	0	0	0	0	47.06	55.22	0	0	0	21.75
a_{13}	MAYDA	0	0	0	0	0	0	0	69.76	75.81	0	6.66	78.25	0

JUNO OILTANKING can fill the second and the third positions with a frequency of the 39.64% and 60.36%, respectively. WOLVERINE is the sailboat filling the second position more frequently since $b_3^2 = 60.36\%$, NAUTILUS QQ7 is almost always in the fourth position ($b_4^4 = 99.3\%$) while one among UNICA and RICOMINCIO DA TRE is the fifth among the considered sailboats since $b_5^5 = 48.66\%$ and $b_7^5 = 51.34\%$. Looking at the last positions in the ranking, one can observe that the worst is one between DREAMER TECH, XP-ACT, PROFILO and MAYDA. Indeed, only these four sailboats can fill the last position in the ranking and, moreover, their best possible rank position is the 9^{th}.

Analogously, looking at Table 1.12 one can observe that MAYDA is the best among the ones that can fill the last position since it is preferred to the other three with a frequency at least equal to the 69.76%; PROFILO is preferred with a frequency greater than the 50% only to XP-ACT ($p(a_{12,9}) = 55.22\%$) while DREAMER TECH is preferred to XP-ACT and PROFILO with a frequency of the 62.92% and 52.94%. The worst among the four considered sailboats appears to be XP-ACT since the other three sailboats are preferred to it with a frequency at least equal to the 55.22%.

Since the DM is interested in getting a unique ranking of the alternatives at hand, we need to consider only one among the built common scales and only one among the capacities compatible with the preferences provided by the DM. As explained in section 1.5, to each built common scale

S_k corresponds a value ε_k^* obtained by solving the corresponding LP problem. Therefore, among the built common scales, we consider the most discriminant one, that is the common scale S_{DM} such that $\varepsilon_{DM}^* = \max_k \varepsilon_k^*$. The most discriminant common scale is shown in Table 1.13.

Table 1.13: Most discriminant built common scale

Sailboat / Criterion	ToD	ToT	PL	TN	GPH
OTTOVOLANTE	0.9016	0.8564	0.9218	0.9102	0.9028
JUNO OILTANKING	0.8089	0.6022	0.5966	0.7698	0.5542
WOLVERINE	0.6412	0.7925	0.6205	0.7961	0.5457
NAUTILUS QQ7	0.7864	0.5979	0.5555	0.7647	0.5422
UNICA	0.6393	0.2760	0.2098	0.6315	0.4841
ARTIE RTFX	0.5703	0.1967	0.1719	0.4595	0.4436
RICOMINCIO DA TRE	0.5583	0.3777	0.5085	0.7245	0.3698
DREAMER TECH	0.4457	0.0479	0.0540	0.0367	0.3545
XP-ACT	0.4260	0.0557	0.0055	0.0523	0.2145
SQUALO BIANCO	0.3793	0.1898	0.2017	0.2828	0.1668
EMILE GALLE'	0.3322	0.1217	0.1646	0.2346	0.1560
PROFILO	0.0285	0.0658	0.1158	0.1788	0.1367
MAYDA	0.1405	0.0641	0.1672	0.1736	0.0001

At the same time, among the capacities compatible with the preferences provided by the DM, we shall consider their barycenter that is their average component by component (see Table 1.14).

Table 1.14: The Möbius representation of the barycenter of the capacities compatible with the preferences provided by the DM

$m(g_{ToD})$	$m(g_{ToT})$	$m(g_{PL})$	$m(g_{TN})$	$m(g_{GPH})$	$m(g_{ToD}, g_{ToT})$	$m(g_{ToD}, g_{PL})$	$m(g_{ToD}, g_{GPH})$	$m(g_{ToT}, g_{PL})$	$m(g_{ToT}, g_{TN})$	$m(g_{PL}, g_{TN})$
0.2626	0.2803	0.3836	0.2034	0.1290	0.078	-0.0963	0.0093	-0.0645	-0.0952	0.0017

By using the common scale and the Möbius representation of the barycenter of the compatible capacities shown in Tables 1.13 and 1.14, we can compute the Choquet integral of the 13 sailboats obtaining, therefore, their ranking shown in Table 1.15.

One can observe that the recommendations obtained by the rank acceptability indices shown in Table 1.11 are almost confirmed. Indeed, the first nine positions are filled by the sailboats presenting the highest rank acceptability index for that position (for example WOLVERINE is second with a frequency of the 60.36% while EMILE GALLÉ fills the ninth position with a frequency equal to the 89.39%). The last four positions (from the 10^{th} to the 13^{th}) are filled by DREAMER TECH, XP-ACT, MAYDA and PROFILO being the four sailboats presenting the highest rank acceptability indices for these positions.

Table 1.15: Ranking of the sailboats considerig the most discriminat scale and the Möbius representation of the barycenter of the compatible capacities

Position	Sailboat
1^{st}	OTTOVOLANTE
2^{nd}	WOLVERINE
3^{rd}	JUNO OILTANKING
4^{th}	NAUTILUS QQ7
5^{th}	RICOMINCIO DA TRE
6^{th}	UNICA
7^{th}	ARTIE RTFX
8^{th}	SQUALO BIANCO
9^{th}	EMILE GALLÉ
10^{th}	DREAMER TECH
11^{th}	XP-ACT
12^{th}	MAYDA
13^{th}	PROFILO

Chapter 2

GAIA-SMAA-PROMETHEE for a hierarchy of interacting criteria

Real world decision problems generally present the following features: criteria structured in a hierarchical way, positive, negative and antagonist effects between considered criteria and uncertainty or imprecision in the definition of the parameters as well as on the evaluations of the alternatives at hand. Many multiple criteria methods have been proposed to deal with these complex aspects of a decision problem and among them some of the most appreciated are PROMETHEE methods.

Basic PROMETHEE methods make the following assumptions:

- *there is a finite set of criteria:* in general the considered criteria are not too numerous; moreover, it is also supposed that the criteria are not hierarchically organized;
- *the criteria are not interacting:* there are no phenomena of mutual strengthening or weakening, or antagonistic effects between them;
- *it is possible to define precisely the parameters of the decision model:* weights of considered criteria are supposed to be clearly fixed.

Real world decision problems are very often violating the above assumptions, so that the application of PROMETHEE methods in these cases is affected by a simplified reformulation which neglects several aspects that can be indeed quite salient for the definition of the final recommendation. For example, in risk-return evaluation of industrial companies, it is quite natural to cope with the following situations:

- taking into account economic indices such as:

  - debt ratio, current ratio and quick ratio defining solvency and liquidity aspects,
  - return on equity, return on investment and gross profit ratio defining the profitability aspects,
  - interest expense ratio, account receivable ratio and administration cost per unit defining managerial aspects.

  In these cases, it is therefore natural to compare industrial companies not only at the global level, but also at the intermediate level of profitability, solvency and liquidity and managerial aspects. Anyway, these intermediate comparisons are not supplied by basic PROMETHEE methods;

- managerial aspects and profitability aspects are both quite important, therefore both of them receive a relevant weight. However, if a company is profitable, in general, it is also well managed. Consequently, there is the risk of an overevaluation of companies that are both profitable and well managed. In this case, we can say that there is a mutual weakening effect between managerial aspects and profitability aspects. Basic PROMETHEE methods have no possibility to avoid such type of overevaluation. Analogously, other interactions between criteria, such as mutual strengthening and antagonistic effects, cannot be properly handled with basic PROMETHEE methods;

- the financial analysts have not a clear-cut idea about the weights of considered criteria. Again, basic PROMETHEE methods cannot handle in a systematic way these zones of imprecision and hesitation.

Even if there are some extensions of PROMETHEE methods able to deal with the above points singularly (see [29, 26, 25, 71]), to the best of our knowledge, there is not any PROMETHEE method taking into consideration the above points all together. Moreover, in so complex problems there is not the possibility to use the very useful visualization support of the GAIA plane [79]. In fact, the representation on a two dimensional plane supplied by the GAIA plane handles only non-interacting criteria without any consideration of hierarchy of criteria.

In the last years many extensions of PROMETHEE methods have been proposed. They regard weights determination [43], integration with other methods such as Data Envelopment Analysis (DEA [8, 9]), Analytic Network Process (ANP) [74], Analytic Hierarchy Process (AHP) [31] or fuzzy approaches [23, 100], integration of quantitative and qualitative information [62], robustness of the recommendations as in [34] and, finally, sorting methods [70]. Looking at the appeal of the

PROMETHEE methods, witnessed by the many recent papers in the topic mentioned above, we think that it could be interesting an extension of PROMETHEE methods permitting to conjointly handle hierarchy of criteria, interaction between criteria and imprecise definition of weights. Methodologically, the construction of such an extension of the PROMETHEE methods is based on the following tools:

- the Multiple Criteria Hierarchy Process (MCHP, [28]), which permits to deal with a set of criteria hierarchically structured,
- the bipolar Choquet integral which permits to deal with interacting criteria,
- the Robust Ordinal Regression and the Stochastic Multicriteria Acceptability Analysis, which permit to deal with imprecision and hesitation in the definition of weights.
- an extension of the GAIA plane [79] which permits, taking into account the plurality of instances of the preference model considered by SMAA (with a procedure different from that one proposed by [63] for the basic PROMETHEE methods), to obtain a graphical visualization of alternatives and criteria, as well as interactions between criteria organized in a hierarchy. In this way, some relevant features (such as clusters, agreements and conflicts between criteria) of the decision problem at hand can be observed.

Of course, MCHP, bipolar Choquet integral, ROR and SMAA, and GAIA plane are not simply put together, but they have to be harmonized within PROMETHEE methods, taking care that the interrelations properly answer to the requirements of the most complex real world decision problems.

2.1 The Bipolar PROMETHEE methods

To extend the PROMETHEE methods to the bipolar case let consider that for each criterion $g_j \in G$, in [25] a bipolar preference function $P_j^B : A \times A \rightarrow [-1, 1]$ has been defined such that:

$$P_j^B(a,b) = P_j(a,b) - P_j(b,a) = \begin{cases} P_j(a,b) & \text{if} \quad P_j(a,b) > 0 \\ -P_j(b,a) & \text{if} \quad P_j(a,b) = 0 \end{cases} \tag{2.1}$$

Of course a positive value of $P_j^B(a,b)$ involves the preference of a over b on j, while a negative value represents the preference of b over a on the same criterion.

In order to deal with interactions and antagonistic effects, the bipolar Choquet integral (presented in the previous Section 1.2.2) is applied to aggregate the bipolar vector $\left[P_j^B(a,b)\right]_j = \mathbf{P^B}(\mathbf{a},\mathbf{b})$. By using the 2-additive decomposition of the bicapacity $\hat{\mu}$ (Eqs. (1.22)-(1.25)) we have the following reformulation:

$$\begin{aligned} Ch^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}) &= \sum_{j\in G:\ P_j^B(a,b)>0} a_j^+ {P_j}^B(a,b) + \sum_{\substack{j,k\in G,\\ P_j^B(a,b)>0,\ P_k^B(a,b)>0}} a_{jk}^+ \min\{P_j^B(a,b), P_k^B(a,b)\} + \\ &+ \sum_{\substack{j,k\in G,\\ P_j^B(a,b)>0,\ P_k^B(a,b)<0}} a_{j|k}^+ \min\{P_j^B(a,b), -P_k^B(a,b)\}, \end{aligned} \tag{2.2}$$

$$\begin{aligned} Ch^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}) &= -\sum_{j\in G:\ P_j^B(a,b)<0} a_j^- {P_j}^B(a,b) - \sum_{\substack{j,k\in G,\\ P_j^B(a,b)<0,\ P_k^B(a,b)<0}} a_{jk}^- \min\{P_j^B(a,b), P_k^B(a,b)\} - \\ &- \sum_{\substack{j,k\in G,\\ P_j^B(a,b)>0,\ P_k^B(a,b)<0}} a_{j|k}^- \max\{-P_j^B(a,b), P_k^B(a,b)\}. \end{aligned} \tag{2.3}$$

and

$$Ch^B(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}) = Ch^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}) - Ch^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}). \tag{2.4}$$

where $Ch^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu})$ represents the magnitude of the reasons for which a is preferred to b, while $Ch^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu})$ represents the magnitude of the reasons against this preference. Therefore, $Ch^B(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu})$ gives an estimate of the overall preference of a over b.

In order to impose that the amount of the reasons in favor of the preference of a over b is the same of the amount of the reasons against the preference of b over a, technically $Ch^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}) = Ch^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu})$, the following symmetry conditions must be fulfilled (see [25]):

$$\begin{cases} a_j^+ = a_j^-, & \text{for each } j\in G, \\ a_{jk}^+ = a_{jk}^-, & \text{for each } \{j,k\}\subseteq G, \\ a_{j|k}^+ = a_{k|j}^-, & \text{for each } j,k\in G. \end{cases} \tag{2.5}$$

Now, for each alternative $a\in A$, we can define the bipolar positive flow, the bipolar negative flow and the bipolar net flow needed by the bipolar PROMETHEE methods to build the relations of preference, indifference and incomparability

$$\phi^{B+}(a) = \frac{1}{m-1}\sum_{x\in A\setminus\{a\}} Ch^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{x}),\hat{\mu}), \quad \phi^{B-}(a) = \frac{1}{m-1}\sum_{x\in A\setminus\{a\}} Ch^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{x}),\hat{\mu}),$$

where $|A| = m$. Reminding Eq. (2.4) and using the two equations above, the bipolar net flow $\phi^B(a) = \phi^{B+}(a) - \phi^{B-}(a)$ can be written as $\phi^B(a) = \frac{1}{m-1}\sum_{x\in A\setminus\{a\}} Ch^B(\mathbf{P^B}(\mathbf{a},\mathbf{x}),\hat{\mu})$.

2.1.1 The bipolar Choquet integral for criteria hierarchically structured

When the alternatives are evaluated with respect to criteria which are not all located at the same level but which are hierarchically structured, the MCHP [28] approach can be applied. Thus, there is a *root criterion* (the "comprehensive objective") at level zero, a set of subcriteria of the root criterion at level one, etc. The criteria at the lower level of the hierarchy (which are the "leaves" of the associated tree of criteria) are called elementary criteria. In what follow we will use the following notation:

- G is the comprehensive set of criteria (at all levels of the hierarchy), and $g_{\mathbf{0}}$ is the root criterion;
- $\mathsf{I_G}$ is the set of indices of the criteria in G;
- $\mathsf{G_E} \subseteq \mathsf{G}$ is the set of all elementary criteria in G, that are the criteria located at the bottom of the hierarchy;
- $\mathsf{E_G} \subseteq \mathsf{I_G}$ is the set of indices of elementary criteria;
- $g_{\mathbf{r}}$ is a non-elementary criterion if $g_{\mathbf{r}} \in \mathsf{G} \setminus \mathsf{G_E}$;
- $\mathsf{E}(g_{\mathbf{r}}) \subseteq \mathsf{E_G}$ is the set of indices of all the elementary criteria descending from $g_{\mathbf{r}}$ (it follows that $\mathsf{E}(g_{\mathbf{0}}) = \mathsf{E_G}$);
- given $\mathsf{F} \subseteq \mathsf{G}$, $\mathsf{E}(\mathsf{F}) = \cup_{g_{\mathbf{r}}\in\mathsf{F}}\mathsf{E}(g_{\mathbf{r}})$;
- $\mathsf{G}^k_{\mathbf{r}} \subseteq \mathsf{G}$ is the set of subcriteria of $g_{\mathbf{r}}$ located at level k.

Example 2.1.1. *In order to better explain the proposed extensions, we shall introduce a decision problem that we will analyze through this chapter and that will be discussed in detail in Section 2.4. Let us consider the case of financial decision problem of ranking some companies for possible investment. From a risk-return tradeoff point of view, an investment in a company can be measured according to the following main aspects: Profitability (P), Solvency and Liquidity (S&L) and Managerial Performance (MP). Each of these three aspects can be defined by using some financial ratios which can be considered as elementary criteria since they are located at the bottom of the hierarchy (see Figure 2.1): Gross Profit Ratio (GP/S) and Return on Total Assets (EBIT/S) are subcriteria of*

Profitability; Debt Ratio (TL/TA) and Current Ratio (CA/CL) are elementary criteria descending from Solvency and Liquidity, while Account Receivable Ratio (AR/S) and Interest Expense Ratio (IE/S) are subcriteria of Managerial Performance. The description of the elementary criteria is provided in Table 2.1.

Figure 2.1: Hierarchical structure of criteria considered in the example

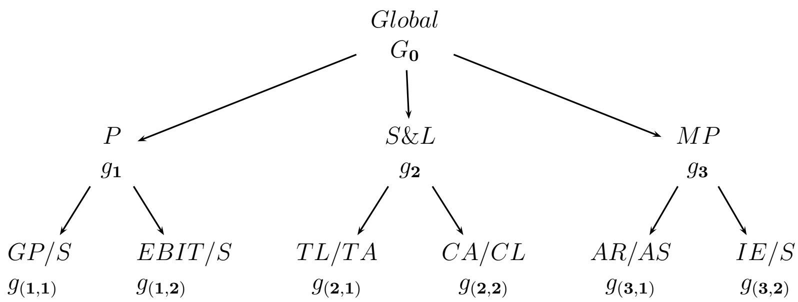


Table 2.1: Criteria Description

Notation	Criterion	Description
$g_{\mathbf{1}}$	Profitability (P)	It assesses the ability of a firm to generate earnings.
$g_{\mathbf{2}}$	Solvency and Liquidity (S&L)	It assesses the dependency of the firms on debt financing and their overall level of leverage.
$g_{\mathbf{3}}$	Managerial Performance (MP)	It focuses on the efficiency of a firm's policy towards its creditors and clients as well as its financial efficiency.
Notation	Elementary Criterion	Description
$g_{(\mathbf{1},\mathbf{1})}$	Gross Profit Ratio (GP/S)	It shows the proportion of profits generated by the sales, before selling and administrative expenses.
$g_{(\mathbf{1},\mathbf{2})}$	Return On Total Assets (EBIT/S)	It is an indicator of how effectively a company is using its assets to generate earnings before contractual obligations must be paid.
$g_{(\mathbf{2},\mathbf{1})}$	Debt Ratio (TL/TA)	It shows a company's ability to pay off its liabilities with its assets.
$g_{(\mathbf{2},\mathbf{2})}$	Current Ratio (CA/CL)	It measures the company's ability to meet short-term commitments.
$g_{(\mathbf{3},\mathbf{1})}$	Account Receivable Ratio (AR/S)	It shows the relationship between unpaid sales and the total sales revenue.
$g_{(\mathbf{3},\mathbf{2})}$	Interest Expense Ratio (IE/S)	It computes the amount of gross income that is being spent to pay the interest on borrowed money.

Now, we will apply the bipolar Choquet integral to the hierarchical case to compute, in the next section 2.2, the bipolar PROMETHEE preference of an alternative a over an alternative b with respect to a particular criterion $g_{\mathbf{r}}$. To do this, we need to introduce a small change in the notation used until now. Indeed, while in case of a flat structure of criteria the bicapacity $\hat{\mu}$ was defined on $\mathcal{P}(G)$, where G was the set of criteria $\{g_1, \ldots, g_m\}$, from here on, by $\hat{\mu}$ we shall denote a bicapacity defined on $\mathcal{P}(\mathsf{G_E}) = \{(C, D) : C, D \subseteq \mathsf{G_E} \text{ and } C \cap D = \emptyset\}$ that is on the set composed of the couples of disjoint subsets of elementary criteria. The same applies to μ^+ and μ^- which will be defined on $\mathcal{P}(\mathsf{G_E})$ too.

From a formal point of view, to apply the bipolar Choquet integral of $\mathbf{x} \in \mathbb{R}^{|\mathsf{G_E}|}$ to the hierarchical case, for each non-elementary criterion $g_{\mathbf{r}}$ in the hierarchy, we need to define a bicapacity $\hat{\mu}_{\mathbf{r}}^k$ on $\mathcal{P}(\mathsf{G}_{\mathbf{r}}^k) = \big\{(C, D) : C, D \subseteq \mathsf{G}_{\mathbf{r}}^k \text{ and } C \cap D = \emptyset\big\}$, such that, for all $(A, B) \in \mathcal{P}(\mathsf{G}_{\mathbf{r}}^k)$,

$$\hat{\mu}^k_{\mathbf{r}}(A,B) = \frac{\hat{\mu}\left(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(A)\}, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(B)\}\right)}{\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(\mathsf{G}^k_{\mathbf{r}})\}, \emptyset)} = \frac{\hat{\mu}\left(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(A)\}, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(B)\}\right)}{\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset)} \tag{2.6}$$

where $\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset) \neq 0$, because, on the contrary, criteria from $\mathsf{G}^k_{\mathbf{r}}$ would have null importance and would not be meaningful for the decision problem.

Proposition 2.1.1. *If $\hat{\mu}$ is 2-additive and it satisfies the symmetry conditions in (2.5), for each non-elementary criterion $g_{\mathbf{r}}$ the following holds:*

$$\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\}, \emptyset) = -\hat{\mu}(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\}).$$

Proof. By Eq. (1.18),

$$\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset) = \mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset) - \mu^-(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset).$$

Now, considering a 2-additive decomposition of μ^+ and μ^- and observing that $\mu^-(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset) = 0$, we have that

$$\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset) - \mu^-(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset) = \sum_{\mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})} a^+(\{g_{\mathbf{t}}\}, \emptyset) + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\} \subseteq \mathsf{E}(g_{\mathbf{r}})} a^+(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}, \emptyset);$$

reminding the symmetry conditions (13), and that $\mu^+(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}) = 0$, we obtain

$$\sum_{\mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})} a^+(\{g_{\mathbf{t}}\}, \emptyset) + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\} \subseteq \mathsf{E}(g_{\mathbf{r}})} a^+(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}, \emptyset) = \sum_{\mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})} a^-(\emptyset, \{g_{\mathbf{t}}\}) + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\} \subseteq \mathsf{E}(g_{\mathbf{r}})} a^-(\emptyset, \{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}) =$$

$$= \mu^-(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}) - \mu^+(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}) = -\hat{\mu}(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\})).$$

□

Note 2.1.1. *As a consequence of Proposition 2.1.1, the normalization conditions for the bicapacity $\hat{\mu}^k_{\mathbf{r}}$ are satisfied. Indeed:*

- $\hat{\mu}^k_{\mathbf{r}}(\emptyset, \emptyset) = \frac{\hat{\mu}(\emptyset,\emptyset)}{\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\},\emptyset)} = 0$ *because* $\hat{\mu}(\emptyset, \emptyset) = 0$,
- $\hat{\mu}^k_{\mathbf{r}}(G^k_{\mathbf{r}}, \emptyset) = \frac{\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in E(G^k_{\mathbf{r}})\},\emptyset)}{\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\},\emptyset)} = 1$ *because* $E(G^k_{\mathbf{r}}) = E(g_{\mathbf{r}})$,
- $\hat{\mu}^k_{\mathbf{r}}(\emptyset, G^k_{\mathbf{r}}) = \frac{\hat{\mu}(\emptyset,\{g_{\mathbf{t}} : \mathbf{t} \in E(G^k_{\mathbf{r}})\})}{\hat{\mu}(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\},\emptyset)} = \frac{\hat{\mu}(\emptyset,\{g_{\mathbf{t}} : \mathbf{t} \in E(G^k_{\mathbf{r}})\})}{-\hat{\mu}(\emptyset,\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\})} = -1$ *because* $E(G^k_{\mathbf{r}}) = E(g_{\mathbf{r}})$ *and by Proposition 2.1.1,*

As the bicapacity $\hat{\mu}$ has been decomposed by means of μ^+ and μ^-, also the bicapacity $\hat{\mu}_{\mathbf{r}}^k$ can be decomposed in a similar way,

$$\hat{\mu}_{\mathbf{r}}^k(A,B) = \mu_{\mathbf{r}}^{k,+}(A,B) - \mu_{\mathbf{r}}^{k,-}(A,B)$$

where $\mu_{\mathbf{r}}^{k,+}, \mu_{\mathbf{r}}^{k,-} : \mathcal{P}(\mathsf{G}_{\mathbf{r}}^k) \rightarrow [0,1]$ and

$$\mu_{\mathbf{r}}^{k,+}(A,B) = \frac{\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(A)\}, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(B)\})}{\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset)} \quad \text{for all } (A,B) \in \mathcal{P}(\mathsf{G}_{\mathbf{r}}^k), \tag{2.7}$$

$$\mu_{\mathbf{r}}^{k,-}(A,B) = \frac{\mu^-(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(A)\}, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(B)\})}{\mu^-(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\})} \quad \text{for all } (A,B) \in \mathcal{P}(\mathsf{G}_{\mathbf{r}}^k). \tag{2.8}$$

with $\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset),\ \mu^-(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}) \neq 0$ because, on the contrary, criteria from $\mathsf{E}(g_{\mathbf{r}})$ would have null importance and would not be meaningful for the decision problem at hand.

Example 2.1.2. *Let us continue the example introduced above, and let us suppose we wish to compute the bicapacities $\mu_{\mathbf{0}}^{1,+}$, $\mu_{\mathbf{0}}^{1,-}$, $\hat{\mu}_{\mathbf{0}}^1$ on $G_{\mathbf{0}}^1 = \{g_{\mathbf{1}}, g_{\mathbf{2}}, g_{\mathbf{3}}\}$. As shown in Eqs. (2.7) and (2.8), the three bicapacities are dependent on the bicapacities μ^+ and μ^- defined on $\mathcal{P}(G_E)$. For illustrative reasons, let us consider the 2-additive decomposition of μ^+ and μ^- shown in Table 2.2; we considered the coefficients a^+ only because the analogous coefficients a^- can be obtained by a^+ due to the symmetry conditions in 2.5. Moreover, all coefficients a^+ not specified in Table 2.2 are fixed equal to zero.*

In the following, we shall show how to compute $\mu_{\mathbf{0}}^{1,+}(\{g_{\mathbf{1}}\},\{g_{\mathbf{2}}\})$ and $\mu_{\mathbf{0}}^{1,-}(\{g_{\mathbf{1}}\},\{g_{\mathbf{2}}\})$. From Eqs.

Table 2.2: Example of the 2-additive decomposition of μ^+

	$a^+(\{\cdot\},\{\cdot\})$		$a^+(\{\cdot\},\{\cdot\})$		$a^+(\{\cdot\},\{\cdot\})$
$(\{g_{(\mathbf{1},\mathbf{1})}\},\emptyset)$	0.281	$(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{3},\mathbf{1})}\},\emptyset)$	0.043	$(\{g_{(\mathbf{2},\mathbf{2})}, g_{(\mathbf{3},\mathbf{1})}\},\emptyset)$	-0.028
$(\{g_{(\mathbf{1},\mathbf{2})}\},\emptyset)$	0.282	$(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{3},\mathbf{2})}\},\emptyset)$	-0.054	$(\{g_{(\mathbf{2},\mathbf{2})}, g_{(\mathbf{3},\mathbf{2})}\},\emptyset)$	-0.077
$(\{g_{(\mathbf{2},\mathbf{1})}\},\emptyset)$	0.218	$(\{g_{(\mathbf{1},\mathbf{2})}, g_{(\mathbf{3},\mathbf{1})}\},\emptyset)$	-0.076	$(\{g_{(\mathbf{1},\mathbf{1})}\},\{g_{(\mathbf{2},\mathbf{1})}\})$	-0.015
$(\{g_{(\mathbf{2},\mathbf{2})}\},\emptyset)$	0.173	$(\{g_{(\mathbf{1},\mathbf{2})}, g_{(\mathbf{3},\mathbf{2})}\},\emptyset)$	0.083	$(\{g_{(\mathbf{1},\mathbf{1})}\},\{g_{(\mathbf{2},\mathbf{2})}\})$	-0.0700
$(\{g_{(\mathbf{3},\mathbf{1})}\},\emptyset)$	0.121	$(\{g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{2},\mathbf{2})}\},\emptyset)$	-0.053	$(\{g_{(\mathbf{1},\mathbf{1})}\},\{g_{(\mathbf{3},\mathbf{1})}\})$	-0.036
$(\{g_{(\mathbf{3},\mathbf{2})}\},\emptyset)$	0.234	$(\{g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{3},\mathbf{1})}\},\emptyset)$	-0.015	$(\{g_{(\mathbf{1},\mathbf{2})}\},\{g_{(\mathbf{2},\mathbf{1})}\})$	-0.049
$(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}\},\emptyset)$	-0.032	$(\{g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{3},\mathbf{2})}\},\emptyset)$	-0.102	$(\{g_{(\mathbf{1},\mathbf{2})}\},\{g_{(\mathbf{2},\mathbf{2})}\})$	-0.051

(2.7) and (2.8), we have

$$\mu_{\mathbf{0}}^{1,+}(\{g_{\mathbf{1}}\},\{g_{\mathbf{2}}\}) = \frac{\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{1}})\}, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{2}})\})}{\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{0}})\}, \emptyset)}$$

and

$$\mu_{\mathbf{0}}^{1,-}(\{g_{\mathbf{1}}\},\{g_{\mathbf{2}}\}) = \frac{\mu^-(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{1}})\}, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{2}})\})}{\mu^-(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{0}})\})}.$$

In consequence of the 2-additive decomposition of μ^+ *and* μ^- *provided in Section 1.2.2, we have that:*

- $\mu^+\left(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{1}})\}, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{2}})\}\right) =$

  $= \sum_{\mathbf{t}\in E(g_1)} a^+\left(\{g_{\mathbf{t}}\}, \emptyset\right) + \sum_{\mathbf{t_1},\mathbf{t_2}\in E(g_1)} a^+\left(\{g_{\mathbf{t_1}}, g_{\mathbf{t_2}}\}, \emptyset\right) + \sum_{\substack{\mathbf{t_1}\in E(g_1)\\ \mathbf{t_2}\in E(g_2)}} a^+\left(\{g_{\mathbf{t}_1}\}, \{g_{\mathbf{t}_2}\}\right) = a^+\left(\{g_{(\mathbf{1},\mathbf{1})}\}, \emptyset\right) +$

  $+a^+\left(\{g_{(\mathbf{1},\mathbf{2})}\}, \emptyset\right) + a^+\left(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}\}, \emptyset\right) + a^+\left(\{g_{(\mathbf{1},\mathbf{1})}\}, \{g_{(\mathbf{2},\mathbf{1})}\}\right) + a^+\left(\{g_{(\mathbf{1},\mathbf{1})}\}, \{g_{(\mathbf{2},\mathbf{2})}\}\right) +$

  $+a^+\left(\{g_{(\mathbf{1},\mathbf{2})}\}, \{g_{(\mathbf{2},\mathbf{1})}\}\right) + a^+\left(\{g_{(\mathbf{1},\mathbf{2})}\}, \{g_{(\mathbf{2},\mathbf{2})}\}\right) = 0.281 + 0.282 - 0.032 - 0.015 - 0.070 - 0.049-$

  $-0.051 = 0.344.$

- $\mu^+\left(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{0}})\}, \emptyset\right) = \mu^+\left(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}, g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{2},\mathbf{2})}, g_{(\mathbf{3},\mathbf{1})}, g_{(\mathbf{3},\mathbf{2})}\}, \emptyset\right) = 1$ *in consequence of the normalization constraints on* μ^+,

- $\mu^-\left(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{1}})\}, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{2}})\}\right) =$

  $= \sum_{\mathbf{t}\in E(g_2)} a^-\left(\emptyset, \{g_{\mathbf{t}}\}\right) + \sum_{\mathbf{t_1},\mathbf{t_2}\in E(g_2)} a^-\left(\emptyset, \{g_{\mathbf{t_1}}, g_{\mathbf{t_2}}\}\right) + \sum_{\substack{\mathbf{t_1}\in E(g_1)\\ \mathbf{t_2}\in E(g_2)}} a^-\left(\{g_{\mathbf{t}_1}\}, \{g_{\mathbf{t}_2}\}\right) = a^-\left(\emptyset, \{g_{(\mathbf{2},\mathbf{1})}\}\right) +$

  $+a^-\left(\emptyset, \{g_{(\mathbf{2},\mathbf{2})}\}\right) + a^-\left(\emptyset, \{g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{2},\mathbf{2})}\}\right) + a^-\left(\{g_{(\mathbf{1},\mathbf{1})}\}, \{g_{(\mathbf{2},\mathbf{1})}\}\right) + a^-\left(\{g_{(\mathbf{1},\mathbf{1})}\}, \{g_{(\mathbf{2},\mathbf{2})}\}\right) +$

  $+a^-\left(\{g_{(\mathbf{1},\mathbf{2})}\}, \{g_{(\mathbf{2},\mathbf{1})}\}\right) + a^-\left(\{g_{(\mathbf{1},\mathbf{2})}\}, \{g_{(\mathbf{2},\mathbf{2})}\}\right) =$ [1] $a^+\left(\{g_{(\mathbf{2},\mathbf{1})}\}, \emptyset\right) + a^+\left(\{g_{(\mathbf{2},\mathbf{2})}\}, \emptyset\right) +$

  $+a^+\left(\{g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{2},\mathbf{2})}\}, \emptyset\right) + a^+\left(\{g_{(\mathbf{2},\mathbf{1})}\}, \{g_{(\mathbf{1},\mathbf{1})}\}\right) + a^+\left(\{g_{(\mathbf{2},\mathbf{2})}\}, \{g_{(\mathbf{1},\mathbf{1})}\}\right) + a^+\left(\{g_{(\mathbf{2},\mathbf{1})}\}, \{g_{(\mathbf{1},\mathbf{2})}\}\right) +$

  $+a^+\left(\{g_{(\mathbf{2},\mathbf{2})}\}, \{g_{(\mathbf{1},\mathbf{2})}\}\right) = 0.218 + 0.173 - 0.053 = 0.338.$

- $\mu^-\left(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{0}})\}\right) = \mu^-\left(\emptyset, \{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}, g_{(\mathbf{2},\mathbf{1})}, g_{(\mathbf{2},\mathbf{2})}, g_{(\mathbf{3},\mathbf{1})}, g_{(\mathbf{3},\mathbf{2})}\}\right) = 1$ *in consequence of the normalization constraints on* μ^-.

Analogously, one can compute the values of the bicapacity for each pair of subsets of criteria $(A, B) \in \mathcal{P}(G^1_{\mathbf{0}})$ *as shown in Table 2.3.*

Considering Eqs. (2.7) and (2.8), the positive and negative bipolar Choquet integral of $\mathbf{x} \in \mathbb{R}^{|\mathsf{E}_G|}$ on a particular node $g_{\mathbf{r}}$ can be computed as follows:

$$Ch^{B+}_{\mathbf{r}}(\mathbf{x}, \hat{\mu}^k_{\mathbf{r}}) = \frac{Ch^{B+}(\mathbf{x_r}, \hat{\mu})}{\mu^+\left(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset\right)} \tag{2.9}$$

$$Ch^{B-}_{\mathbf{r}}(\mathbf{x}, \hat{\mu}^k_{\mathbf{r}}) = \frac{Ch^{B-}(\mathbf{x_r}, \hat{\mu})}{\mu^-\left(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}\right)} \tag{2.10}$$

[1] Remember that $a^+(\{g_{\mathbf{t}}\}, \emptyset) = a^-(\{g_{\mathbf{t}}\}, \emptyset)$ for all $g_{\mathbf{t}} \in \mathsf{G_E}$; $a^+\left(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}, \emptyset\right) = a^-\left(\emptyset, \{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}\right)$ and $a^+\left(\{g_{\mathbf{t}_1}\}, \{g_{\mathbf{t}_2}\}\right) = a^-\left(\{g_{\mathbf{t}_2}\}, \{g_{\mathbf{t}_1}\}\right)$ for all $g_{\mathbf{t_1}}, g_{\mathbf{t_2}} \in \mathsf{G_E}$

Table 2.3: Values of the bicapacities $\mu_{\mathbf{0}}^{1,+}$, $\mu_{\mathbf{0}}^{1,-}$ and $\hat{\mu}_{\mathbf{0}}^{1}$ considering the 2-additive decomposition of μ^+ given in Table 2.2

$(\cdot,\cdot)$	$\mu_{\mathbf{0}}^{1,+}(\cdot,\cdot)$	$\mu_{\mathbf{0}}^{1,-}(\cdot,\cdot)$	$\hat{\mu}_{\mathbf{0}}^{1}(\cdot,\cdot)$	$(\cdot,\cdot)$	$\mu_{\mathbf{0}}^{1,+}(\cdot,\cdot)$	$\mu_{\mathbf{0}}^{1,-}(\cdot,\cdot)$	$\hat{\mu}_{\mathbf{0}}^{1}(\cdot,\cdot)$	$(\cdot,\cdot)$	$\mu_{\mathbf{0}}^{1,+}(\cdot,\cdot)$	$\mu_{\mathbf{0}}^{1,-}(\cdot,\cdot)$	$\hat{\mu}_{\mathbf{0}}^{1}(\cdot,\cdot)$
$(g_1,\emptyset)$	0.531	0	0.531	$(\{g_2\},\{g_1\})$	0.338	0.344	-0.005	$(\{g_1,g_3\},\{g_2\})$	0.698	0.338	0.359
$(g_2,\emptyset)$	0.338	0	0.338	$(\{g_2\},\{g_3\})$	0.338	0.356	0.018	$(\{g_2,g_3\},\{g_1\})$	0.471	0.307	0.163
$(g_3,\emptyset)$	0.356	0	0.356	$(\{g_3\},\{g_1\})$	0.356	0.494	-0.138	$(\emptyset,\{g_1\})$	0	0.531	-0.531
$(\{g_1,g_2\},\emptyset)$	0.870	0	0.870	$(\{g_3\},\{g_2\})$	0.356	0.338	0.017	$(\emptyset,\{g_2\})$	0	0.338	-0.338
$(\{g_1,g_3\},\emptyset)$	0.885	0	0.885	$(\{g_1\},\{g_2,g_3\})$	0.307	0.471	-0.163	$(\emptyset,\{g_3\})$	0	0.356	-0.356
$(\{g_2,g_3\},\emptyset)$	0.471	0	0.471	$(\{g_2\},\{g_1,g_3\})$	0.338	0.698	-0.359	$(\emptyset,\{g_1,g_2\})$	0	0.870	-0.870
$(\{g_1\},\{g_2\})$	0.344	0.338	0.006	$(\{g_3\},\{g_1,g_2\})$	0.356	0.833	-0.477	$(\emptyset,\{g_1,g_3\})$	0	0.885	-0.885
$(\{g_1\},\{g_3\})$	0.494	0.356	0.138	$(\{g_1,g_2\},\{g_3\})$	0.833	0.356	0.477	$(\emptyset,\{g_2,g_3\})$	0	0.471	-0.471
$(\emptyset,\emptyset)$	0	0	0	$(\{g_1,g_2,g_3\},\emptyset)$	1	0	1	$(\emptyset,\{g_1,g_2,g_3\})$	0	1	-1

where $\mathbf{x_r}$ is a fictitious vector having the same components of $\mathbf{x}$ on the indices corresponding to the elementary criteria descending from $g_{\mathbf{r}}$ and zero on the others.

Proposition 2.1.2. *If $\hat{\mu}$ is 2-additive and it satisfies the symmetry conditions in 2.5, for each non-elementary criterion $g_{\mathbf{r}}$, the following holds:*

$$\mu^{+}\left(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\}, \emptyset\right) = \mu^{-}\left(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{r}})\}\right).$$

Proof. Considering a 2-additive bicapacity and in consequence of the symmetry conditions in Eq. (2.5), we have that:

$$\mu^{+}\left(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}, \emptyset\right) = \sum_{\mathbf{t}\in \mathsf{E}(g_{\mathbf{r}})} a^{+}(\{g_{\mathbf{t}}\}, \emptyset) + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\}\subseteq \mathsf{E}(g_{\mathbf{r}})} a^{+}(\{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}, \emptyset) =$$

$$= \sum_{\mathbf{t}\in \mathsf{E}(g_{\mathbf{r}})} a^{-}(\emptyset, \{g_{\mathbf{t}}\}) + \sum_{\{\mathbf{t}_1,\mathbf{t}_2\}\subseteq \mathsf{E}(g_{\mathbf{r}})} a^{-}(\emptyset, \{g_{\mathbf{t}_1}, g_{\mathbf{t}_2}\}) = \mu^{-}\left(\emptyset, \{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}\right).$$

□

The bipolar Choquet integral of $\mathbf{x}$ on a node $g_{\mathbf{r}}$ can be defined as

$$Ch^{B}_{\mathbf{r}}(\mathbf{x}, \hat{\mu}^{k}_{\mathbf{r}}) = Ch^{B+}_{\mathbf{r}}(\mathbf{x}, \hat{\mu}^{k}_{\mathbf{r}}) - Ch^{B-}_{\mathbf{r}}(\mathbf{x}, \hat{\mu}^{k}_{\mathbf{r}}) \tag{2.11}$$

that, on the basis of the positive and negative bipolar Choquet integral of $\mathbf{x}$ w.r.t. $g_{\mathbf{r}}$ given in Eqs. (2.9) and (2.10) and considering Proposition 2.1.2, can be written as

$$Ch^B_{\mathbf{r}}(\mathbf{x},\hat{\mu}^k_{\mathbf{r}}) = \frac{Ch^B(\mathbf{x_r},\hat{\mu})}{\hat{\mu}\left(\{g_{\mathbf{t}}:\mathbf{t}\in \mathsf{E}(g_{\mathbf{r}})\},\emptyset\right)}. \tag{2.12}$$

2.2 The Bipolar PROMETHEE methods for the MCHP

While in the previous section we proposed to apply the bipolar Choquet integral to the case of a set of criteria hierarchically structured, in this section we will consider the bipolar PROMETHEE method to the case of interacting criteria organized in a hierarchy but also interacting between them. On the basis of the comprehensive positive and negative preference of a over b given in Eqs. (2.2) and (2.3) and by considering the definitions of the bipolar positive, negative and comprehensive Choquet integral with respect to a criterion $g_{\mathbf{r}}$ in the hierarchy, we can give the following definition.

Definition 2.2.1. *Given $a,b\in A$ and a non-elementary criterion $g_{\mathbf{r}}$ in the hierarchy, we can define:*

- *The comprehensive positive preference of a over b on $g_{\mathbf{r}}$:*

$$Ch^{B+}_{\mathbf{r}}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}^k_{\mathbf{r}}) = \frac{Ch^{B+}(\mathbf{P^B_r}(\mathbf{a},\mathbf{b}),\hat{\mu})}{\mu^+\left(\{g_{\mathbf{t}}:\ \mathbf{t}\in E(g_{\mathbf{r}})\},\emptyset\right)}, \tag{2.13}$$

- *The comprehensive negative preference of a over b on $g_{\mathbf{r}}$:*

$$Ch^{B-}_{\mathbf{r}}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}^k_{\mathbf{r}}) = \frac{Ch^{B-}(\mathbf{P^B_r}(\mathbf{a},\mathbf{b}),\hat{\mu})}{\mu^-\left(\emptyset,\{g_{\mathbf{t}}:\ \mathbf{t}\in E(g_{\mathbf{r}})\}\right)} \tag{2.14}$$

- *The comprehensive preference of a over b on $g_{\mathbf{r}}$:*

$$Ch^{B}_{\mathbf{r}}(\mathbf{P^B}(\mathbf{a},\mathbf{b}),\hat{\mu}^k_{\mathbf{r}}) = \frac{Ch^{B}(\mathbf{P^B_r}(\mathbf{a},\mathbf{b}),\hat{\mu})}{\hat{\mu}\left(\{g_{\mathbf{t}}:\ \mathbf{t}\in E(g_{\mathbf{r}})\},\emptyset\right)}, \tag{2.15}$$

In all three definitions, $\mathbf{P^B_r}(\mathbf{a},\mathbf{b})$ is the vector belonging to $\mathbb{R}^{|E(g_{\mathbf{r}})|}$ such that

$$\left(\mathbf{P^B_r}(\mathbf{a},\mathbf{b})\right)_{\mathbf{t}} = \begin{cases} P^B_{\mathbf{t}}(a,b) & \textit{if}\ \ \mathbf{t}\in E(g_{\mathbf{r}}),\\ 0 & \textit{if}\ \ \mathbf{t}\notin E(g_{\mathbf{r}}). \end{cases}$$

Example 2.2.1. *Let us back to the example introduced in Section 2.1.1. The evaluations of the eight most capitalized European pharmaceutical companies on the six elementary criteria are shown in Table 2.4, while the indifference and preference thresholds for all elementary criteria are given in Table 2.5.*

Table 2.4: Evaluations of the eight pharmaceutical companies on the considered elementary criteria

Pharma Companies	GP/S	EBIT/S	TL/TA	CA/CL	AR/S	IE/S
Glaxo	0.664	0.209	0.904	1.236	0.160	0.027
Sanofi	0.687	0.054	0.430	1.482	0.212	0.007
Novartis	0.660	0.067	0.414	0.964	0.162	0.013
Roche Holding	0.708	0.177	0.723	1.186	0.170	0.018
AstraZeneca	0.818	0.068	0.692	1.077	0.188	0.016
Bayer	0.543	0.085	0.672	1.403	0.214	0.016
Merck	0.683	0.048	0.664	0.783	0.213	0.024
Novo Nordisk	0.850	0.539	0.488	1.349	0.143	0.001
Preference direction	Incr.	Incr.	Decr.	Incr.	Decr.	Decr.

Table 2.5: Indifference and preference thresholds for the elementary criteria

$q_{(\mathbf{1},\mathbf{1})}$	$p_{(\mathbf{1},\mathbf{1})}$	$q_{(\mathbf{1},\mathbf{2})}$	$p_{(\mathbf{1},\mathbf{2})}$	$q_{(\mathbf{2},\mathbf{1})}$	$p_{(\mathbf{2},\mathbf{1})}$	$q_{(\mathbf{2},\mathbf{2})}$	$p_{(\mathbf{2},\mathbf{2})}$	$q_{(\mathbf{3},\mathbf{1})}$	$p_{(\mathbf{3},\mathbf{1})}$	$q_{(\mathbf{3},\mathbf{2})}$	$p_{(\mathbf{3},\mathbf{2})}$
0.02	0.06	0.01	0.03	0.05	0.15	0.10	0.20	0.01	0.02	0.003	0.006

Considering Sanofi and Novartis we can compute the bipolar vector $\mathbf{P^B}(\mathbf{Sanofi},\mathbf{Novartis})$ *at the global level as well as on all three macrocriteria:*

- $\mathbf{P^B}(\mathbf{Sanofi},\mathbf{Novartis}) = (0.1711, -0.1379, 0, 1, -1, 1)$,
- $\mathbf{P^B_1}(\mathbf{Sanofi},\mathbf{Novartis}) = (0.1711, -0.1379, 0, 0, 0, 0)$,
- $\mathbf{P^B_2}(\mathbf{Sanofi},\mathbf{Novartis}) = (0, 0, 0, 1, 0, 0)$,
- $\mathbf{P^B_3}(\mathbf{Sanofi},\mathbf{Novartis}) = (0, 0, 0, 0, -1, 1)$.

Now, let us consider the 2-additive decomposition of a bicapacity $\hat{\mu}$ *given in Table 2.2. To compute the comprehensive positive preference of Sanofi over Novartis on Profitability, by Eq. (2.13), we have that*

$$Ch^{B+}_{\mathbf{1}}(\mathbf{P^B}(\mathbf{Sanofi},\mathbf{Novartis}),\hat{\mu}^2_{\mathbf{1}}) = \frac{Ch^{B+}(\mathbf{P^B_1}(\mathbf{Sanofi},\mathbf{Novartis}),\hat{\mu})}{\mu^+(\{g_{\mathbf{t}} : \mathbf{t} \in E(g_{\mathbf{1}})\},\emptyset)} =$$

$$= \frac{Ch^{B+}((0.1711, -0.1379, 0, 0, 0, 0)\,,\hat{\mu})}{\mu^+(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}\},\emptyset)}.$$

By Eq. (2.2) and considering the 2-additive decomposition of μ^+ *we get*

- $Ch^{B+}((0.1711, -0.1379, 0, 0, 0, 0)\,,\hat{\mu}) =$

  $= 0.1711 \cdot a^+\left(\{g_{(\mathbf{1},\mathbf{1})}\},\emptyset\right) + \min\{0.1711, 0.1379\} \cdot a^+\left(\{g_{(\mathbf{1},\mathbf{1})}\},\{g_{(\mathbf{1},\mathbf{2})}\}\right) = 0.1711 \cdot 0.281 = 0.048.$

- $\mu^+(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}\},\emptyset) =$

  $= a^+\left(\{g_{(\mathbf{1},\mathbf{1})}\},\emptyset\right) + a^+\left(\{g_{(\mathbf{1},\mathbf{2})}\},\emptyset\right) + a^+\left(\{g_{(\mathbf{1},\mathbf{1})}, g_{(\mathbf{1},\mathbf{2})}\},\emptyset\right) = 0.281 + 0.282 - 0.032 = 0.531.$

Consequently, $Ch_{\mathbf{1}}^{B+}(\mathbf{P^B}(\mathbf{Sanofi}, \mathbf{Novartis}), \hat{\mu}_{\mathbf{1}}^2) = \dfrac{0.048}{0.531} = 0.090.$

By using the definitions above, the comprehensive negative preference and the comprehensive preference of $Sanofi$ over $Novartis$ at a global level as well as at partial level can be computed (see Table 2.6).

Table 2.6: Comprehensive positive and negative preference of $Sanofi$ over $Novartis$ at global level as well as at partial level

	$Ch_{\mathbf{r}}^{B+}(\mathbf{P^B}(\mathbf{Sanofi}, \mathbf{Novartis}), \hat{\mu}_{\mathbf{r}}^2)$	$Ch_{\mathbf{r}}^{B-}(\mathbf{P^B}(\mathbf{Sanofi}, \mathbf{Novartis}), \hat{\mu}_{\mathbf{r}}^2)$	$Ch_{\mathbf{r}}^{B}(\mathbf{P^B}(\mathbf{Sanofi}, \mathbf{Novartis}), \hat{\mu}_{\mathbf{r}}^2)$
$\mathbf{r} = (\mathbf{1})$	0.0906	0.0734	0.0172
$\mathbf{r} = (\mathbf{2})$	0.5117	0	0.5117
$\mathbf{r} = (\mathbf{3})$	0.6581	0.3419	0.3163

On the basis of the previous definitions, the positive, negative and net bipolar flows w.r.t. a criterion $g_{\mathbf{r}}$ can be computed.

Definition 2.2.2. *Given $a \in A$ and a non-elementary criterion $g_{\mathbf{r}}$, we can define:*

- *The bipolar positive flow of a on $g_{\mathbf{r}}$:*

$$\phi_{\mathbf{r}}^{B+}(a) = \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} Ch_{\mathbf{r}}^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{x}), \hat{\mu}_{\mathbf{r}}^k) = \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} \frac{Ch^{B+}(\mathbf{P_r^B}(\mathbf{a},\mathbf{x}), \hat{\mu})}{\mu^+(\{g_{\mathbf{t}}:\ \mathbf{t} \in E(g_{\mathbf{r}})\}, \emptyset)}$$

- *The bipolar negative flow of a on $g_{\mathbf{r}}$:*

$$\phi_{\mathbf{r}}^{B-}(a) = \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} Ch_{\mathbf{r}}^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{x}), \hat{\mu}_{\mathbf{r}}^k) = \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} \frac{Ch^{B-}(\mathbf{P_r^B}(\mathbf{a},\mathbf{x}), \hat{\mu})}{\mu^-(\emptyset, \{g_{\mathbf{t}}:\ \mathbf{t} \in E(g_{\mathbf{r}})\})}$$

- *The bipolar net flow of a on $g_{\mathbf{r}}$:*

$$\begin{aligned}\phi_{\mathbf{r}}^{B}(a) &= \phi_{\mathbf{r}}^{B+}(a) - \phi_{\mathbf{r}}^{B-}(a) = \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} \left[Ch_{\mathbf{r}}^{B+}(\mathbf{P^B}(\mathbf{a},\mathbf{x}), \hat{\mu}_{\mathbf{r}}^k) - Ch_{\mathbf{r}}^{B-}(\mathbf{P^B}(\mathbf{a},\mathbf{x}), \hat{\mu}_{\mathbf{r}}^k)\right] = \\ &= \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} Ch_{\mathbf{r}}^{B}(\mathbf{P^B}(\mathbf{a},\mathbf{x}), \hat{\mu}_{\mathbf{r}}^k) = \frac{1}{m-1}\sum_{x \in A\setminus\{a\}} \frac{Ch^{B}(\mathbf{P_r^B}(\mathbf{a},\mathbf{x}), \hat{\mu})}{\hat{\mu}(\{g_{\mathbf{t}}: \mathbf{t} \in E(g_{\mathbf{r}})\}, \emptyset)}.\end{aligned}$$

Using definitions 2.2.1 and 2.2.2, the extension of the bipolar PROMETHEE I and II methods to the case of criteria organized in a hierarchical way (that we shall call hierarchical bipolar PROMETHEE I and II methods) can be provided. In particular, given $a, b \in A$ and a non-elementary criterion $g_{\mathbf{r}}$,

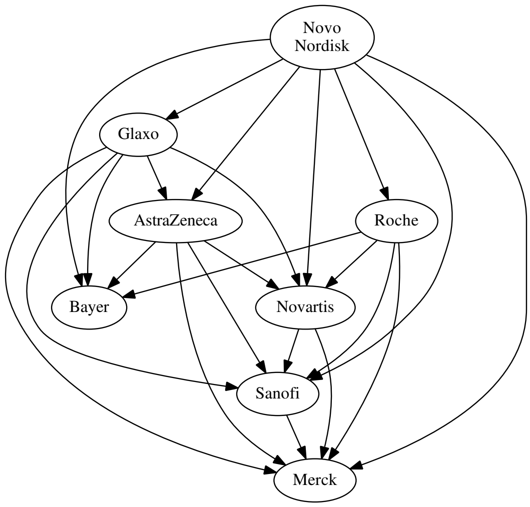
- the hierarchical bipolar PROMETHEE I method defines a preference ($\mathcal{P}^{I}_{\mathbf{r},B}$), an indifference ($\mathcal{I}^{I}_{\mathbf{r},B}$) and an incomparability ($\mathcal{R}^{I}_{\mathbf{r},B}$) relation as follows:
  - $a\mathcal{P}^{I}_{\mathbf{r},B}b$ iff $\phi^{B+}_{\mathbf{r}}(a) \geq \phi^{B+}_{\mathbf{r}}(b)$, $\phi^{B-}_{\mathbf{r}}(a) \leq \phi^{B-}_{\mathbf{r}}(b)$ and at least one of the two inequalities is strict,
  - $b\mathcal{P}^{I}_{\mathbf{r},B}a$ iff $\phi^{B+}_{\mathbf{r}}(b) \geq \phi^{B+}_{\mathbf{r}}(a)$, $\phi^{B-}_{\mathbf{r}}(b) \leq \phi^{B-}_{\mathbf{r}}(a)$ and at least one of the two inequalities is strict,
  - $a\mathcal{I}^{I}_{\mathbf{r},B}b$ iff $\phi^{B+}_{\mathbf{r}}(a) = \phi^{B+}_{\mathbf{r}}(b)$ and $\phi^{B-}_{\mathbf{r}}(a) = \phi^{B-}_{\mathbf{r}}(b)$,
  - $a\mathcal{R}^{I}_{\mathbf{r},B}b$, otherwise.
- the hierarchical bipolar PROMETHEE II method defines a preference relation ($\mathcal{P}^{II}_{\mathbf{r},B}$) and an indifference relation ($\mathcal{I}^{II}_{\mathbf{r},B}$), only:
  - $a\mathcal{P}^{II}_{\mathbf{r},B}b$ iff $\phi^{B}_{\mathbf{r}}(a) > \phi^{B}_{\mathbf{r}}(b)$,
  - $a\mathcal{I}^{II}_{\mathbf{r},B}b$ iff $\phi^{B}_{\mathbf{r}}(a) = \phi^{B}_{\mathbf{r}}(b)$.

Example 2.2.2. *Continuing the previous example, we shall show how to compute the relations defined above on profitability for the hierarchical bipolar PROMETHEE I and II methods. Let us consider the 2-additive decomposition of the bicapacity $\hat{\mu}$ given in Table 2.2 and the evaluations of the eight companies on the eight elementary subcriteria shown in Table 2.4. The bipolar positive, negative and net flow of each alternative on criterion $g_{\mathbf{1}}$ shown in Table 2.7 can be computed. One*

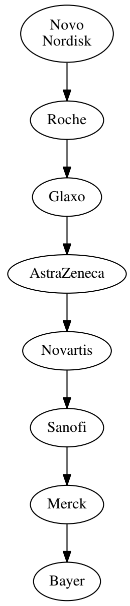
Table 2.7: Bipolar positive, negative and net flows on Profitability

	$\phi^{B+}_{\mathbf{1}}(\cdot)$	$\phi^{B-}_{\mathbf{1}}(\cdot)$	$\phi^{B}_{\mathbf{1}}(\cdot)$
Glaxo	0.5230	0.2693	0.2537
Sanofi	0.0942	0.4791	-0.3849
Novartis	0.1194	0.4629	-0.3435
Roche Holding	0.5486	0.2946	0.2540
AstraZeneca	0.5007	0.2708	0.2299
Bayer	0.2032	0.7313	-0.5218
Merck	0.0808	0.5217	-0.4409
Novo Nordisk	0.9256	0	0.9256

can observe that Roche is preferred to Glaxo on profitability considering the hierarchical bipolar PROMETHEE II method (Roche$\mathcal{P}^{II}_{\mathbf{1},B}$Glaxo), while the two companies are incomparable considering the bipolar PROMETHEE I method (Roche$\mathcal{R}^{I}_{\mathbf{1},B}$Glaxo). The preferences between all pairs of companies obtained by the new hierarchical bipolar PROMETHEE methods are shown in Figure 2.2.



(a) Hierarchical bipolar PROMETHEE I method



(b) Hierarchical bipolar PROMETHEE II method

Figure 2.2: Preference relations obtained by the hierarchical bipolar PROMETHEE methods on Profitability

2.2.1 The Bipolar PROMETHEE III, IV, V, VI

PROMETHEE III

We observe that PROMETHEE III can be easily extended to the hierarchical bipolar case by considering an adaptation of the formulas provided in Section 1.3.1. Indeed, defining

$$\overline{\phi}^B_{\mathbf{r}}(a) = \frac{1}{m}\sum_{x\in A}\frac{Ch^B(\mathbf{P}^{\mathbf{B}}_{\mathbf{r}}(\mathbf{a},\mathbf{x}),\hat{\mu})}{\hat{\mu}\left(\{g_{\mathbf{t}}:\mathbf{t}\in \mathsf{E}(g_{\mathbf{r}})\},\emptyset\right)},$$

$$\sigma^2_{a,\mathbf{r}} = \frac{1}{m}\sum_{x\in A}\left[\frac{Ch^B(\mathbf{P}^{\mathbf{B}}_{\mathbf{r}}(\mathbf{a},\mathbf{x}),\hat{\mu})}{\hat{\mu}\left(\{g_{\mathbf{t}}:\mathbf{t}\in \mathsf{E}(g_{\mathbf{r}})\},\emptyset\right)} - \overline{\phi}^B_{\mathbf{r}}(a)\right]^2$$

and replacing $\overline{\phi}(a)$ and σ_a with $\overline{\phi}^B_{\mathbf{r}}(a)$ and $\sigma_{a,\mathbf{r}}$ in Eq. (39), we obtain a complete interval order on the set of alternatives A. Note that it is possible to obtain the complete interval order w.r.t. each non-elementary criterion $g_{\mathbf{r}}$ and w.r.t. each compatible bicapacity $\hat{\mu}$.

PROMETHEE IV

PROMETHEE IV can be extended to a hierarchy of interacting criteria by defining $\phi_{\mathbf{r}}^{B+}(a)$ and $\phi_{\mathbf{r}}^{B-}(a)$ as follows:

$$\phi_{\mathbf{r}}^{B+}(a) = \int_X \frac{Ch^{B+}(\mathbf{P}_{\mathbf{r}}^{\mathbf{B}+}(\mathbf{a},\mathbf{x}),\hat{\mu})}{\mu^{+}\left(\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\},\emptyset\right)}\rho(\mathbf{x})\, d\mathbf{x} \tag{42}$$

$$\phi_{\mathbf{r}}^{B-}(a) = \int_X \frac{Ch^{B-}(\mathbf{P}_{\mathbf{r}}^{\mathbf{B}-}(\mathbf{a},\mathbf{x}),\hat{\mu})}{\mu^{-}\left(\emptyset,\{g_{\mathbf{t}} : \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})\}\right)}\rho(\mathbf{x})\, d\mathbf{x} \tag{43}$$

PROMETHEE V

By considering PROMETHEE V, for each alternative $a \in A$ and for each criterion $g_{\mathbf{r}}$, one can compute the bipolar net flow of a on $g_{\mathbf{r}}$, $\phi_{\mathbf{r}}^{B}(a)$, being the equivalent of $\phi(a)$ in our case. As a consequence, the 0-1 programming problem in Eq. (44) can be easily extended to our methodology. Indeed, for each criterion $g_{\mathbf{r}}$ in the hierarchy, after defining the clusters $C_1^{\mathbf{r}},\ldots,C_k^{\mathbf{r}},\ldots,C_R^{\mathbf{r}}$, one has to solve the following 0-1 program

$$\begin{array}{l} \max \displaystyle\sum_{a\in A} x_a^{\mathbf{r}}\phi_{\mathbf{r}}^{B}(a), \\ \displaystyle\sum_{a\in A} \alpha_a^{\mathbf{r}} x_a^{\mathbf{r}} \geq [\leq,=]\beta^{\mathbf{r}},\ [BeCl_{\mathbf{r}}] \\ \displaystyle\sum_{a\in C_k^{\mathbf{r}}} \gamma_{a,k}^{\mathbf{r}} x_a^{\mathbf{r}} \geq [\leq,=]\delta_k^{\mathbf{r}},\ [WiCl_{\mathbf{r}}], \end{array} \tag{45}$$

where constraints $[BeCl_{\mathbf{r}}]$ and $[WiCl_{\mathbf{r}}]$ are the equivalent of constraints $[BeCl]$ and $[WiCl]$. The difference is related to the fact that they will take into account a criterion only. In this way, one can look at the best portfolio of alternatives when considering a particular criterion $g_{\mathbf{r}}$ in the hierarchy and, therefore, different subsets of alternatives can be chosen for different aspects of the problem at hand. For example, in a project selection problem in which economic, environmental and social aspects are considered, one can look at the best subset of alternatives that should be chosen when all three aspects are taken into account simultaneously but also the sets of alternatives that should be selected when the three different aspects are considered separately.

PROMETHEE VI

We have already taken into account this preference information in a direct or an indirect way by using the SMAA methodology as explained in Section 4. In order to consider a similar case, we added the possibility that the DM provides some preferences in terms of intervals of possible values for the weights of elementary criteria $g_{\mathbf{t}}$. For example, he can state that the importance of elementary criterion $g_{\mathbf{t}}$ varies between a lower ($l_{\mathbf{t}}$) and an upper ($u_{\mathbf{t}}$) bound:

$$a_{\mathbf{t}} \in [l_{\mathbf{t}}, u_{\mathbf{t}}] \, .$$

Consequently, the application of SMAA permits to study the different ranking of the alternatives varying the weights in the considered intervals.

2.2.2 The GAIA plane for the bipolar PROMETHEE methods

Since PROMETHEE methods are often implemented along with the GAIA plane [79], we propose an extension of this visualization tool to the case of bipolar PROMETHEE methods with criteria organized in a hierarchy taking also into account the plurality of instances of the preference model considered by SMAA.

As known, the purpose of this technique is to obtain a graphical representation of the matrix of the partial criteria net flows by using the Principal Component Analysis (PCA) [2, 69, 106] to reduce the dimensionality. In particular, in the classical case, the GAIA plane permits to visualize in a two dimensional space the projections of criteria and alternatives which look as vectors and points, respectively. In many applications of the GAIA plane, the PCA is based on the covariance matrix. In this case, if there are large differences between the variances of the single criteria net flows then, those single criteria net flow which have the largest variance tend to dominate the first principal component. For this reason, here we apply the PCA to the correlation matrix[2] in order to avoid an unreasonably large impact on the principal components by few variables treating all variables on an equal footing.

To apply this visualization technique to the bipolar case, we start by decomposing the net flow w.r.t. the bipolar PROMETHEE method defined in Section 2.1 in terms of parameters useful for the visual tool, as follows:

$$\phi^B(a) = \frac{1}{m-1} \sum_{x \in A \setminus \{a\}} \left[\sum_{j \in G} a_j^+ P_j^B(a,x) + \sum_{(j,k) \in Syn} a_{jk}^+ \alpha_{jk}(a,x) + \sum_{(j,k) \in Ant} a_{j|k}^+ \beta_{j|k}(a,x) \right] \quad (2.16)$$

where:

- $Syn \subseteq \{(j,k) : j,k \in G, j < k\} = \{(j_1^s, k_1^s), \ldots, (j_q^s, k_q^s)\}$ is the set of non-ordered pairs of criteria for which the DM declared that there is interaction and therefore $a_{jk}^+ \neq 0$,

[2]In this case the vectors of loadings (principal component coefficients) consist of the eigenvectors of the correlation matrix.

- $Ant \subseteq \{(j,k) : j,k \in G, j \neq k\} = \{(j_1^a, k_1^a), \ldots, (j_t^a, k_t^a)\}$ is the set of ordered pairs of criteria for which the DM declared that there is antagonism and therefore $a^{+}_{j|k} < 0$,

-
$$\alpha_{jk}(a,x) = \begin{cases} \min\{P_j^B(a,x), P_k^B(a,x)\} & \text{if } P_j^B(a,x) > 0,\ P_k^B(a,x) > 0; \\ \max\{P_j^B(a,x), P_k^B(a,x)\} & \text{if } P_j^B(a,x) < 0,\ P_k^B(a,x) < 0; \\ 0 & \text{otherwise} \end{cases} \tag{2.17}$$

-
$$\beta_{j|k}(a,x) = \begin{cases} \min\{P_j^B(a,x), -P_k^B(a,x)\} & \text{if } P_j^B(a,x) > 0,\ P_k^B(a,x) < 0; \\ \max\{-P_j^B(a,x), P_k^B(a,x)\} & \text{if } P_k^B(a,x) > 0,\ P_j^B(a,x) < 0; \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Through this decomposition, we are considering interactions and antagonistic effects as criteria for which the evaluations of alternatives are given by the functions defined in Eqs. (2.17) and (2.18), respectively.

Now, instead of the classical matrix containing the single criterion net flows, we have to consider a matrix $\mathbf{\Phi}$ which, as in the classic case, takes into account the preferences given by the generalized criteria but with the additional information given by the interactions between criteria as well as the antagonistic effects between them. The size of the matrix $\mathbf{\Phi}$ depends on the information provided by the DM in the sense that it contains the columns related to interactions and antagonistic effects only if the DM provides them. This means that the matrix $\mathbf{\Phi}$ does not contain the columns for which the related information is not provided[3]. We consider the matrix $\mathbf{\Phi}$ as input data to perform the PCA. The general form of the matrix $\mathbf{\Phi}$ is shown in Table 2.8, where

$$\phi_j^B(a) = \frac{1}{m-1} \sum_{x \in A} P_j^B(a,x) \tag{2.19}$$

$$\phi_{jk}^B(a) = \frac{1}{m-1} \sum_{x \in A} \alpha_{jk}(a,x) \tag{2.20}$$

$$\phi_{j|k}^B(a) = \frac{1}{m-1} \sum_{x \in A} \beta_{j|k}(a,x) \tag{2.21}$$

Note that the vector $\phi^B = [\phi^B(a_1), \ldots, \phi^B(a_m)]$ containing the net flow of alternatives from A can be written in terms of the product between the matrix $\mathbf{\Phi}$ and the vector $\hat{\mathbf{a}}$ containing the

[3] The maximum size of the matrix is $m \times [n + \binom{n}{2} + D_{n,2}]$, where n is the number of considered criteria and m is the number of alternatives. Note that each element of $\mathbf{\Phi}$ belongs to $[-1, 1]$.

Table 2.8: Matrix of the parameters involved in the extension of the GAIA method. Partial bipolar preference index for each criterion and parameters representing interaction and antagonistic effects between criteria.

	$\phi_1^B(\cdot)$	$\ldots$	$\phi_n^B(\cdot)$	$\phi_{j_1^s k_1^s}^B(\cdot)$	$\ldots$	$\phi_{j_q^s k_q^s}^B(\cdot)$	$\phi_{j_1^a \vert k_1^a}^B(\cdot)$	$\ldots$	$\phi_{j_t^a \vert k_t^a}^B(\cdot)$
a_1	$\phi_1^B(a_1)$	$\ldots$	$\phi_n^B(a_1)$	$\phi_{j_1^s k_1^s}^B(a_1)$	$\ldots$	$\phi_{j_q^s k_q^s}^B(a_1)$	$\phi_{j_1^a \vert k_1^a}^B(a_1)$	$\ldots$	$\phi_{j_t^a \vert k_t^a}^B(a_1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_m	$\phi_1^B(a_m)$	$\ldots$	$\phi_n^B(a_m)$	$\phi_{j_1^s k_1^s}^B(a_m)$	$\ldots$	$\phi_{j_q^s k_q^s}^B(a_m)$	$\phi_{j_1^a \vert k_1^a}^B(a_m)$	$\ldots$	$\phi_{j_t^a \vert k_t^a}^B(a_m)$

decomposition of the bicapacity $\hat{\mu}$ as shown in Eq. (2.22).

$$\hat{\mathbf{a}} = \left(a_1^+, \ldots, a_n^+, a_{j_1^s k_1^s}^+, \ldots, a_{j_q^s k_q^s}^+, a_{j_1^a|k_1^a}^+, \ldots, a_{j_t^a|k_t^a}^+\right). \tag{2.22}$$

We apply the principal component analysis solving the following problem:

$$\begin{aligned} &max\ \mathbf{u}'C\mathbf{u}\ \text{ subject to} \\ &\mathbf{u}'\mathbf{u} = 1. \end{aligned} \tag{2.23}$$

where $\mathbf{u}$ is a unit vector, while

$$C = \frac{\Phi^{Std'}\Phi^{Std}}{n+q+t}$$

is the variance-covariance matrix of centered and standardized data

$$\Phi^{Std} = [\phi_r^{std}(a_i), r = 1, \ldots, n, j_1^s k_1^s, \ldots, j_q^s k_q^s, j_1^a k_1^a, \ldots, j_t^a k_t^a] \tag{2.24}$$

which generic element is obtained as

$$\phi_r^{Std}(a_i) = \frac{\phi_r^B(a_i) - M_r}{\sigma_r} \tag{2.25}$$

with M_r and σ_r denoting the the mean and the standard deviation of the elements in the r-th column of matrix Φ, respectively, that is

$$M_r = \frac{\sum_{i=1}^{m} \phi_r^B(a_i)}{m}, \quad \sigma_r = \sqrt{\frac{\sum_{i=1}^{m} (\phi_r^B(a_i) - M_r)^2}{m}}. \tag{2.26}$$

Using the Lagrange theorem this amounts to solve the system

$$\begin{cases} C\mathbf{u} = \lambda \mathbf{u} \\ \mathbf{u}'\mathbf{u} = 1. \end{cases} \tag{2.27}$$

After this, the two largest eigenvalues λ_1 and λ_2 of C and the corresponding eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ have to be calculated. Then each point

$$\mathbf{x} = \left(x_1^+, \ldots, x_n^+, x^+_{j_1^s k_1^s}, \ldots, x^+_{j_q^s k_q^s}, x^+_{j_1^a|k_1^a}, \ldots, x^+_{j_t^a|k_t^a}\right), \tag{2.28}$$

is projected on the GAIA plane to the point having coordinates $(\mathbf{u}_1'\mathbf{x}, \mathbf{u}_2'\mathbf{x})$. The explained variance of the GAIA plane, that is the percentage of information kept by the projection, is given by

$$\delta = \frac{\lambda_1 + \lambda_2}{\sum\limits_{r=1}^{n+q+t} \lambda_r} \tag{2.29}$$

where $\lambda_r, r = 1, \ldots, n+q+t$, is the r-th eigenvalue of C. The value of δ is in general larger than 60% and most times larger than 85% [17].

In this framework, each alternative $a \in A$ is represented by the projection on the GAIA plane of the following vector

$$\phi^{\mathbf{B}}(\mathbf{a}) = \left[\phi_1^B(a), \ldots, \phi_n^B(a), \phi^B_{j_1^s k_1^s}(a), \ldots, \phi^B_{j_q^s k_q^s}(a), \phi^B_{j_1^a|k_1^a}(a), \ldots, \phi^B_{j_t^a k_t^a}(a)\right]. \tag{2.30}$$

Analogously, importance of criteria, positive and negative interactions and antagonism between pairs of criteria are projected on the GAIA plane as follows:

- criterion $g_j \in G$ is represented by the axis linking the origin of the plane with the projection of the vector $\mathbf{x}$ whose components are all null with the exception of x_j being equal to 1,
- the negative or positive interaction between criteria $(j_e^s, k_e^s), e = 1, \ldots, q$, is represented by the axis linking the origin of the plane with the projection of the vector $\mathbf{x}$ which components are all null with the exception of $x_{j_e^s, k_e^s}$ being equal to 1,
- the antagonism between criteria $(j_f^a, k_f^a), f = 1, \ldots, t$ is represented by the axis linking the origin of the plane with the projection of the vector $\mathbf{x}$ which components are all null with the exception of $x_{j_f^a, k_f^a}$ being equal to 1.

We briefly recall some properties of the GAIA plane:

- the length of the projection of a criterion, the interaction or the antagonism between a pair of criteria, is representative of its capacity to discriminate between the alternatives;
- projections of criteria, positive or negative interactions or antagonisms between pairs of criteria oriented in similar directions express similar preferences. Analogously, projections of criteria, positive or negative interactions or antagonisms having opposite directions represent opposite preferences;
- criteria, synergies or antagonisms between pairs of criteria which projections appear orthogonal are uncorrelated in terms of preferences;
- the closer the points representing two alternatives on the GAIA plane, the more similar the alternatives are;
- an alternative $a \in A$ has a good evaluation with respect to criterion $g_j \in G$ if it is located in the same direction of g_j and, instead, it has a bad evaluation if it is located in the opposite direction of g_j. More precisely, in case of location in the same (opposite) direction of g_j, the longer the projection of the point representing an alternative $a \in A$ on the axis of a criterion $g_j \in G$, the better (worse) its evaluation on g_j; similar interpretations hold with respect to interaction and antagonism of pairs of criteria.

With respect to the overall evaluation of an alternative from A, the projection of the vector $\hat{\mathbf{a}}$ on the GAIA plane constitutes the *PROMETHEE decision axis* corresponding to the projection of the weight vector for the classical PROMETHEE methods. In general, the longer the projection of the point representing an alternative $a \in A$ on the decision axis, the greater the net flow $\phi^B(a)$.

The above general principles of the GAIA plane suffer of some exceptions due to the approximation of the projection of all the variables related to single criteria, interactions and antagonistic effects on the 2-dimensional plane. Anyway, the greater the variance δ explained by the first two components of PCA, the greater the information preserved and, consequently, the less these exceptions are.

In case of a hierarchy of criteria, one can get a GAIA plane for each non elementary criterion $g_{\mathbf{r}} \in \mathsf{G} \setminus \mathsf{G_E}$ by narrowing the above analysis to the elementary criteria $g_{\mathbf{t}}, \mathbf{t} \in \mathsf{E}(g_{\mathbf{r}})$, descending from $g_{\mathbf{r}}$.

Example 2.2.3. *Let us continue the example introduced above. Let us consider the 2-additive decomposable bicapacity whose decomposition vector* $\hat{\mathbf{a}}$ *is presented in Table 2.9. In this case, the PCA* *is applied to 21 variables (6 criteria, 10 interactions and 5 antagonistic effects). Figure 2.3 (a),(b) and (c) show axis representing single criteria $g_j \in G$, pairs of criteria $(j, k) \in Syn$ for which there is a positive or negative interaction, and pairs of criteria $(j, k) \in Ant$ for which there is an antagonistic effect, respectively. For reasons of space, in Figure 2.3, numbers 1,2,3,4,5 and 6 represent elementary criteria GP/S, EBIT/S, TL/TA, CA/CL, AR/S and IE/S, respectively. The points representing the alternatives as well as the PROMETHEE decision axis, indicated with the label "Capacity", are shown in all the three Figure 2.3 (a),(b) and (c). One can observe the following points:*

- *Figure 2.3(a), having an explained variance $\delta = 68.72\%$ (as well as Figure 2.3(b) and (c) that show different elements, but on the same GAIA plane), shows that all criteria are quite discriminating because their axis are quite long. Moreover, criteria CA/CL, TL/TA, IE/S, on one hand, and GP/S, EBIT/S and AR/S, on the other hand, express similar preferences. Novo Nordisk is in the same direction of all criteria and has long projections on their axis, so that it has good evaluations on all criteria. On the contrary, Merck is in the opposite direction of all criteria and has relatively long projections on their axis, so that it has bad evaluations on all criteria. AstraZeneca and Roche Holding are relatively close on the GAIA plane and this shows that they have similar evaluations on all the criteria.*

- *Figure 2.3(b) displays a certain proximity of the axis representing interaction between the pairs of criteria (TL/TA, CA/CL), (CA/CL, IE/S) and (TL/TA, IE/S), showing that they express similar aspects in the decision problem. Analogously, there is a certain similarity between the interaction of the pairs (CA/CL, AR/S), (TL/TA, AR/S), (GP/GS, IE/S), (EBIT/S, IE/S), on one hand, and the pairs (GP/S,EBIT/S), (GP/S,AR/S), (EBIT/S,AR/S), on the other hand.*

- *Figure 2.3(c) shows a certain similarity of the antagonism for the pairs $(GP/S \mid TL/TA)$ and $(GP/S \mid CA/CL)$, on one hand, and the pairs $(EBIT/S \mid TL/TA)$ and $(EBIT/S \mid CA/CL)$. Instead the antagonism of the pair $(GP/S \mid AR/S)$ has its own specific features as shown by the location of its axis which is quite isolated with respect to the other axis representing pairs of antagonistic criteria.*

- *Comparing the points representing alternatives with the PROMETHEE decision axis "Capacity" one can see that Novo Nordisk is by far the best alternative, while Bayer and Meck are the worst ones. Indeed, all these alternatives have a long projection on the "Capacity" axis. However, Novo Nordisk is in the same direction of "Capacity" axis, while Bayer and Merck are in the* *opposite direction.*

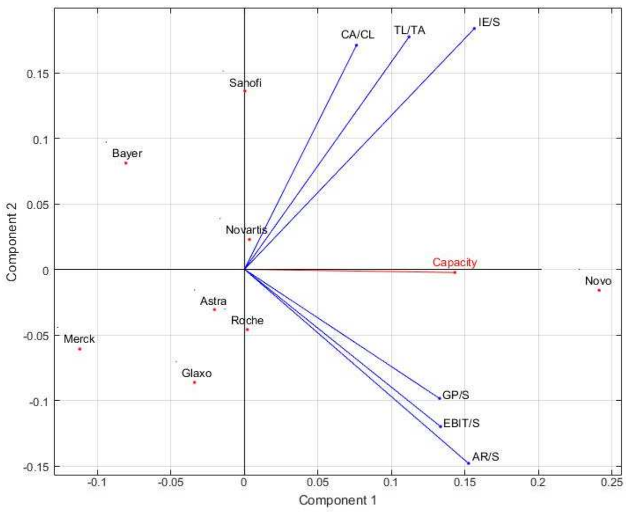
- *Figure 2.3(d), having an explained variance* $\delta = 99.47\%$*, shows that with respect to the non-elementary criterion Solvency and Liquidity, the two elementary criteria CA/CL and TL/TA are both discriminant (because of their long axis) and independent (because of the orthogonality of their axis), while the interaction between the two criteria is less discriminant (because of the relatively short axis). Alternatives are quite different between them (because of their quite scattered location), with the exception of a relative similarity between Novo Nordisk and Sanofi, on one hand, and Roche Holding and AstraZeneca, on the other hand. With respect to the overall evaluation related to Solvency and Liquidity, one can see that the best alternatives are Novo Nordisk and Sanofi while the worst alternatives are Merck and Glaxo. All these four alternatives have long projections on the PROMETHEE decision axis "Capacity", but Novo Nordisk and Sanofi are in the same direction of the "Capacity" axis, while Merck and Glaxo are in the opposite direction.*

Table 2.9: 2-additive decomposable bicapacity considered in the example.

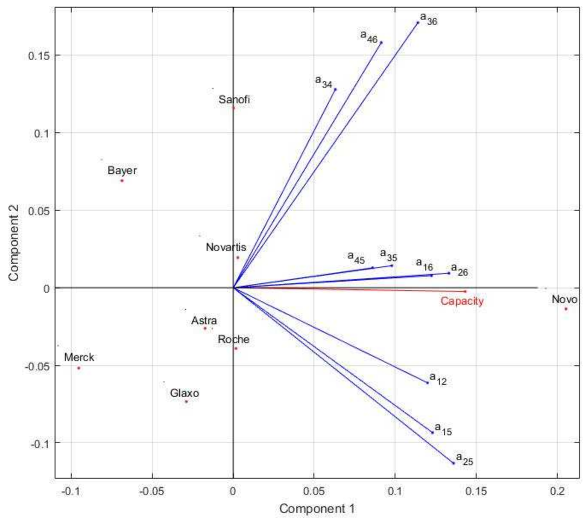
a_1^+	a_2^+	a_3^+	a_4^+	a_5^+	a_6^+	a_{12}^+	a_{15}^+	a_{16}^+	a_{25}^+	a_{26}^+
0.259	0.296	0.185	0.222	0.148	0.222	-0.037	0.037	-0.074	-0.037	0.037
a_{34}^+	a_{35}^+	a_{36}^+	a_{45}^+	a_{46}^+	$a_{1\|3}^+$	$a_{1\|4}^+$	$a_{1\|5}^+$	$a_{2\|3}^+$	$a_{2\|4}^+$	
-0.074	-0.037	-0.037	-0.037	-0.074	-0.037	-0.037	-0.037	-0.055	-0.055	

2.3 Exploring the results through SMAA and ROR

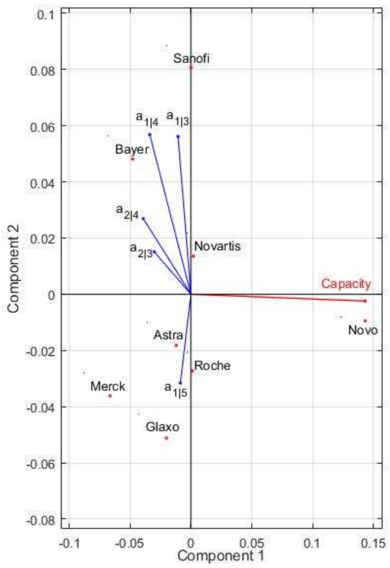
The proposed hierarchical bipolar PROMETHEE methods can be easily considered as a generalization of the classical PROMETHEE methods. Indeed, they are able to take into account several aspects that are not considered in the classical PROMETHEE methods such as the hierarchical structure of criteria, the interactions between criteria as well as the antagonistic effects between them. Anyway, the advantages coming from the capabilities of the new methodology are in part balanced from its complexity related, in particular, to the elicitation of the involved parameters. As already explained, the hierarchical bipolar PROMETHEE methods are based on a bicapacity defined on $\mathcal{P}(\mathsf{G_E})$. This means that values for the importance of each elementary criterion $a_{\mathbf{t}}$, the interaction coefficients $a_{\mathbf{t}_1\mathbf{t}_2}$ for each couple of elementary criteria $\{\mathbf{t}_1, \mathbf{t}_2\}$ and the antagonistic effects $a_{\mathbf{t}_1|\mathbf{t}_2}$ for each pair of elementary criteria $(\mathbf{t}_1, \mathbf{t}_2)$ $(\mathbf{t}_1, \mathbf{t}_2 \in \mathsf{E_G})$ have to be provided. Therefore, the total number of parameters necessary for the application of the methodology is equal to $|\mathsf{E_G}| + \binom{|\mathsf{E_G}|}{2} + 2D_{|\mathsf{E_G}|,2}$.



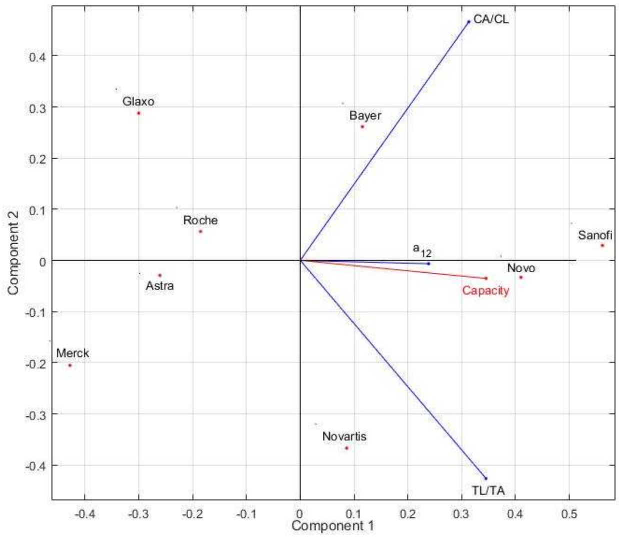
(a) Criteria



(b) Interactions



(c) Antagonistic effects



(d) Solvency and Liquidity

Figure 2.3: Extension of the GAIA plane to take into account interactions and antagonistic effects between criteria

For example, in the previously considered case, being $|\mathsf{E_G}| = 6$, the number of parameters is equal to $6 + \binom{6}{2} + D_{6,2} = 6 + 15 + 30 = 51$. Of course, asking the DM to provide values for all these parameters is unreasonable.

To solve this issue the use of an indirect technique is required [68], meaning that the DM is asked to provide some preference information in terms of comparisons between alternatives or criteria, interaction between criteria and antagonistic effects between them from which parameters compatible with this information can be inferred. In the following, according to [71], we distinguish between local and global preferences at different levels of the hierarchy. A preference, as well as an indifference or an incomparability between alternatives $a, b \in A$ is called local if it is based on considerations of the

relation between a and b in terms of $Ch_{\mathbf{r}}^{B}(\mathbf{P^B(a,b)},\hat{\mu})$, $Ch_{\mathbf{r}}^{B+}(\mathbf{P^B(a,b)},\hat{\mu})$ and $Ch_{\mathbf{r}}^{B-}(P^B(a,b),\hat{\mu})$ for $g_{\mathbf{r}} \in G \setminus \mathsf{G_E}$. Instead, a preference, as well as an indifference or an incomparability between a and b is called global if it is based on consideration of the relation between a and all the other alternatives as well as b and all the other alternatives in terms of $\phi_{\mathbf{r}}^{B}(a)$, $\phi_{\mathbf{r}}^{B+}(a)$, $\phi_{\mathbf{r}}^{B-}(a)$, $\phi_{\mathbf{r}}^{B}(b)$, $\phi_{\mathbf{r}}^{B+}(b)$ and $\phi_{\mathbf{r}}^{B-}(b)$ for $g_{\mathbf{r}} \in G \setminus \mathsf{G_E}$.
The DM can therefore express:

- Preferences on alternatives:

  1. a is locally preferred to b, translated into $Ch^B(\mathbf{P^B(a,b)},\hat{\mu}) > 0$,
  2. a is locally preferred to b on $g_{\mathbf{r}}$, translated into $Ch^B(\mathbf{P_r^B(a,b)},\hat{\mu}) > 0$,
  3. a is comprehensively preferred to b, translated into
     - $\phi^B(a) > \phi^B(b)$ if the bipolar PROMETHEE II is considered,
     - $\begin{cases} \phi^{B+}(a) \geq \phi^{B+}(b), \\ \phi^{B-}(a) \leq \phi^{B-}(b), \\ \phi^{B}(a) > \phi^{B}(b) \end{cases}$ if the bipolar PROMETHEE I is considered,
  4. a is comprehensively preferred to b on $g_{\mathbf{r}}$, translated into
     - $\phi_{\mathbf{r}}^B(a) > \phi_{\mathbf{r}}^B(b)$ if the hierarchical bipolar PROMETHEE II is considered,
     - $\begin{cases} \phi_{\mathbf{r}}^{B+}(a) \geq \phi_{\mathbf{r}}^{B+}(b), \\ \phi_{\mathbf{r}}^{B-}(a) \leq \phi_{\mathbf{r}}^{B-}(b), \\ \phi_{\mathbf{r}}^{B}(a) > \phi_{\mathbf{r}}^{B}(b) \end{cases}$ if the hierarchical bipolar PROMETHEE I is considered,
  5. a is locally indifferent to b, translated into the constraint $Ch^B(\mathbf{P^B(a,b)},\hat{\mu}) = 0$,
  6. a is locally indifferent to b on $g_{\mathbf{r}}$, translated into the constraint $Ch^B(\mathbf{P_r^B(a,b)},\hat{\mu}) = 0$,
  7. a is comprehensively indifferent to b, translated into the constraints
     - $\phi_{\mathbf{r}}^B(a) = \phi_{\mathbf{r}}^B(b)$ if the hierarchical bipolar PROMETHEE II is considered,
     - $\begin{cases} \phi_{\mathbf{r}}^{B+}(a) = \phi_{\mathbf{r}}^{B+}(b), \\ \phi_{\mathbf{r}}^{B-}(a) = \phi_{\mathbf{r}}^{B-}(b) \end{cases}$ if the hierarchical bipolar PROMETHEE I is considered,

- Preferences on importance of and interactions between elementary criteria:

  - the importance of criterion $g_{\mathbf{t}}$ varies between $l_{\mathbf{t}}$ and $u_{\mathbf{t}}$, translated into the constraint $l_{\mathbf{t}} \leq a_{\mathbf{t}} \leq u_{\mathbf{t}}$,
  - $g_{\mathbf{t}_1}$ is more important than $g_{\mathbf{t}_2}$, translated into the constraint $a_{\mathbf{t_1}} > a_{\mathbf{t_2}}$,

- $g_{\mathbf{t}_1}$ and $g_{\mathbf{t}_2}$ are equally important, translated into the constraint $a_{\mathbf{t_1}} = a_{\mathbf{t_2}}$,
- $g_{\mathbf{t}_1}$ and $g_{\mathbf{t}_2}$ are positively interacting, translated into the constraint $a_{\mathbf{t}_1\mathbf{t}_2} > 0$,
- $g_{\mathbf{t}_1}$ and $g_{\mathbf{t}_2}$ are negatively interacting, translated into the constraint $a_{\mathbf{t}_1\mathbf{t}_2} < 0$,
- $g_{\mathbf{t}_2}$ exercises an antagonistic effect over $g_{\mathbf{t}_1}$, translated into the constraint $a^{+}_{\mathbf{t}_1|\mathbf{t}_2} < 0$.

- Preferences on importance of and interactions between non-elementary criteria:
  - $g_{\mathbf{r}_1}$ is more important than $g_{\mathbf{r}_2}$, translated into the constraint

$$\sum_{\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1})} a_{\mathbf{t}_1} > \sum_{\mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})} a_{\mathbf{t}_2},$$

  - $g_{\mathbf{r}_1}$ and $g_{\mathbf{r}_2}$ are positively (negatively) interacting, translated into the constraint

$$\sum_{\substack{\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1}) \\ \mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})}} a_{\mathbf{t}_1\mathbf{t}_2} > 0 \quad (< 0)\,.$$

    In this case, the DM could be willing to be more precise in his preferences, specifying the nature of the interactions between some $g_{\mathbf{t}_1}$, $\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1})$ and some $g_{\mathbf{t}_2}$, $\mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})$; this means that $a_{\mathbf{t}_1\mathbf{t}_2}$ can be greater than zero for some $\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1})$ and $\mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})$, while $a_{\mathbf{t}_1\mathbf{t}_2}$ can be lower than zero for some $\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1})$ and $\mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})$;
  - criterion $g_{\mathbf{r}_2}$ exercises an antagonistic effect over $g_{\mathbf{r}_1}$, translated into the constraint:

$$\sum_{\substack{\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1}) \\ \mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})}} a_{\mathbf{t}_1|\mathbf{t}_2} < 0.$$

    As in the previous case, the DM can be willing to specify if there is some antagonist effect exercised from elementary criteria descending from $g_{\mathbf{r}_2}$ over elementary criteria descending from $g_{\mathbf{r}_1}$. This means that $a_{\mathbf{t}_1|\mathbf{t}_2} < 0$ for some $\mathbf{t}_1 \in \mathsf{E}(g_{\mathbf{r}_1})$ and $\mathbf{t}_2 \in \mathsf{E}(g_{\mathbf{r}_2})$.

Let us observe that because of the symmetry conditions (2.5), $a^{+}_{\mathbf{t}} = a^{-}_{\mathbf{t}} = a_{\mathbf{t}}$ and $a^{+}_{\mathbf{t}_1\mathbf{t}_2} = a^{-}_{\mathbf{t}_1\mathbf{t}_2} = a_{\mathbf{t}_1\mathbf{t}_2}$ for all $\mathbf{t}, \mathbf{t}_1, \mathbf{t}_2 \in \mathsf{E}_{\mathsf{G}}$. Consequently, in the above constraints, we used $a_{\mathbf{t}}$ and $a_{\mathbf{t}_1\mathbf{t}_2}$ only, without distinguishing between the positive and the negative importance of criteria and interaction coefficients. Moreover, monotonicity $[MC]$ and normalization $[NC]$ constraints can be written as follows:

$$[MC]\quad \begin{cases} a_{\mathbf{t}} + \sum_{\mathbf{t}_1 \in C} a_{\mathbf{t}\mathbf{t}_1} + \sum_{\mathbf{t}_1 \in D} a^{+}_{\mathbf{t}|\mathbf{t}_1} \geq 0, & \forall \mathbf{t} \in \mathsf{E_G}, \forall (C \cup \{\mathbf{t}\}, D) \in \mathcal{P}(\mathsf{G_E}), \\ a_{\mathbf{t}} + \sum_{\mathbf{t}_1 \in D} a_{\mathbf{t}\mathbf{t}_1} + \sum_{\mathbf{t}_1 \in C} a^{+}_{\mathbf{t}_1|\mathbf{t}} \geq 0, & \forall \mathbf{t} \in \mathsf{E_G}, \forall (C, D \cup \{\mathbf{t}\}) \in \mathcal{P}(\mathsf{G_E}), \end{cases}$$

$$[NC]\quad \sum_{\mathbf{t} \in \mathsf{E_G}} a_{\mathbf{t}} + \sum_{\mathbf{t}_1\mathbf{t}_2 \in \mathsf{E_G}} a_{\mathbf{t}_1\mathbf{t}_2} = 1.$$

In the following, by E^{DM} we shall denote the set of constraints translating the preferences provided by the DM, together with the normalization $[NC]$ and monotonicity $[MC]$ constraints, where strict inequalities are converted into weak inequalities by using an auxiliary variable ε assumed to be a small value greater than zero (for example, $Ch^B(P^B(a,b),\hat{\mu}) > 0$ is converted into $Ch^B(P^B(a,b),\hat{\mu}) \geq \varepsilon$). If E^{DM} is feasible and $\varepsilon^* > 0$, where $\varepsilon^* = \max \varepsilon$ subject to E^{DM}, then there is at least one set of parameters compatible with the preferences provided by the DM. In the opposite case, in which E^{DM} is infeasible or $\varepsilon^* \leq 0$, one can check the cause of the infeasibility by using some of the methodologies presented in [81].

In general, if there exists one set of compatible parameters, then there exists an infinity of them. Consequently, the choice of only one of these sets of parameters can be considered arbitrary. The SMAA and the ROR avoid this choice by considering simultaneously all compatible sets of parameters.

2.3.1 The SMAA for bipolar PROMETHEE with hierarchy of criteria

In Section 1.5 we have seen that SMAA methods explore the space of parameters compatible with the preferences provided by the DM and defined by the set of constraints E^{DM} as well as the space of the evaluations of the alternatives on the considered criteria in an iterative way [4]. In this case, we shall assume that the evaluations of the alternatives are fixed. Several compatible sets of parameters can be sampled from the corresponding space and, since the set of constraints E^{DM} is convex, one can use the Hit-And-Run algorithm to sample them [96, 101]. For each of these compatible sets of parameters, a different ranking of the considered alternatives can be obtained by the hierarchical bipolar PROMETHEE II method. Analogously, one preference, one indifference and one incomparability relation between the same alternatives can be obtained in case the hierarchical bipolar PROMETHEE I method is applied. For this reason, the application of SMAA in this context

[4]Let us observe that the DM is not obliged to provide any preference information. Just in case he is not willing to provide any preferences in terms of comparison between alternatives or in terms of comparison between criteria and interaction or antagonistic effects between them, then the set of constraints E^{DM} is composed of the monotonicity [MC] and normalization [NC] constraints only.

can provide the following indices:

- the *rank acceptability index* $b_{\mathbf{r}}^{k}(a)$ (in case one applies the hierarchical bipolar PROMETHEE II method) giving the frequency with which an alternative a gets the k-th position in the final ranking computed for criterion $g_{\mathbf{r}}$,

- the *preference index* $Pref_{\mathbf{r}}(a,b)$ (both in case of the hierarchical bipolar PROMETHEE I and II methods), called also pairwise winning index in [77], giving the frequency with which a is preferred over b on criterion $g_{\mathbf{r}}$,

- the indifference index $Ind_{\mathbf{r}}(a,b)$ (both in case of the hierarchical bipolar PROMETHEE I and II methods) giving the frequency with which a and b are indifferent on criterion $g_{\mathbf{r}}$,

- the incomparability index $Inc_{\mathbf{r}}(a,b)$ (in case of the hierarchical bipolar PROMETHEE I method) giving the frequency with which a and b are incomparable on criterion $g_{\mathbf{r}}$.

2.3.2 GAIA plane for SMAA

From the previous Section 2.3, it becomes clear that there is not only one bicapacity $\hat{\mu}$, represented by its corresponding vector $\hat{\mathbf{a}}$, of parameters compatible with the preferences provided by the DM, but a whole set of them. Consequently, the GAIA plane can be extended to permit to convey information about the plurality of bicapacities compatible with the preferences provided by the DM. The idea of considering the whole set of bicapacities corresponds to the imprecise determination of the weight vector for the classical PROMETHEE methods considered in the PROMETHEE VI Sensitivity Tool [17], in which the weight w_j of each criterion $g_j \in G$ is supposed to be in an interval, that is $w_j^- \leq w_j \leq w_j^+$. In this way, taking into account the constraint related to the normalization of the weights, that is $w_1 + \ldots + w_n = 1$, a polyhedron of feasible weight vector is determined and it is projected on the GAIA plane. This projection, denoted by Δ, has the same interpretation of the PROMETHEE decision axis, but, more realistically, it takes into account ill-determination and imprecision of the weights. The projection Δ permits to distinguish between:

- soft problems, when the projection of the set of feasible weight vectors does not include the origin of the axis in the GAIA plane and, then, the direction of the PROMETHEE decision axis is similar for all feasible weight vectors,

- hard problems, when the projection of the set of feasible weight vectors includes the origin of the axis in the GAIA plane and, then, the PROMETHEE decision axis corresponding to feasible weight vectors take all directions.

The plurality of feasible weight vectors is also considered by [63] that proposes to couple SMAA method with the GAIA plane. With this aim, on the basis of the results obtained by the random sampling of the preferential parameters, they estimate their distribution that is consequently projected on the GAIA plane.

In order to better explain this approach, we propose the Example 2.3.1 in which this methodology is applied to the decision problem at hand.

Example 2.3.1. *Provided the evaluations in Table 2.4 and the indifference and preference thresholds in Table 2.5, it can be calculated a matrix* **N** *which has the same size of the evaluation matrix and that has as elements the single criterion net flows* $\phi_j(a)$ *defined as in [20], which expresses how much an alternative* a *is outranking (*$\phi_j(a) > 0$*) or it is outranked (*$\phi_j(a) < 0$*) by all the other alternatives in* A *with respect to a single criterion only. The matrix* **N** *is showed in Table 2.10.*

Table 2.10: Matrix **N** of the single net flow

	$\phi_1(\cdot)$	$\phi_2(\cdot)$	$\phi_3(\cdot)$	$\phi_4(\cdot)$	$\phi_5(\cdot)$	$\phi_6(\cdot)$
Glaxo	-0,23866	0,71428	-1	0,11475	0,48778	-0,89476
Sanofi	-0,11122	-0,61934	0,72238	0,76078	-0,71429	0,71429
Novartis	-0,27615	-0,39875	0,74857	-0,61813	0,44347	0,09987
Roche Holding	0,06304	0,42857	-0,30131	-0,07738	0,38051	-0,13761
AstraZeneca	0,81553	-0,37322	-0,28571	-0,36568	-0,10214	0
Bayer	-1	-0,04679	-0,28376	0,66660	-0,71429	0
Merck	-0,1513	-0,70476	-0,27208	-0,97178	-0,71429	-0,78179
Novo Nordisk	0,89875	1	0,67189	0,49084	0,93324	1

We also suppose that the DM provides the following preference information about the comparisons between the importance of (elementary) criteria[5]:

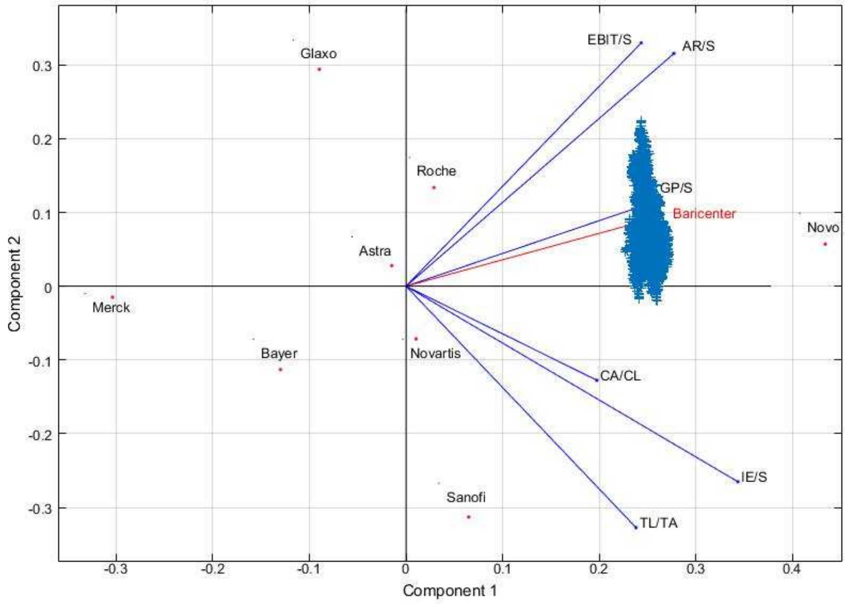
- *EBIT/S is more important than GP/S;*
- *GP/S is more important than IE/S;*
- *IE/S is more important than TL/TA;*
- *TL/TA is more important than AR/S.*

[5] Note that these preferences about comparison between criteria are the same of those in Section 2.4 but without considering interactions between criteria.

Now, according with the SMAA approach, we may sample a certain number of compatible weights vectors that can be projected in the GAIA plane appearing as a cloud of points. This visualization is showed in Figure 2.4 where it can also be noted that:

- *in blue, we have the sampling of 10,000 decision axes,*
- *in red, we have the decision axis corresponding to the barycenter of the sampling.*

Figure 2.4: GAIA plane with the 10,000 sampled decision axes (without interactions between criteria) with $\delta = 70,77\%$



Keeping in mind the previous Example 2.3.1 and with the aim of including the information given by the interactions and the antagonistic effects, we note that the vectors $\hat{\mathbf{a}}$ which represent the sampled feasible bicapacities $\hat{\mu}$, can be directly projected instead of representing an estimation of their distribution. In this way, one can obtain a useful visualization of the decision problem at hand, showing the main results obtained through this extension of the SMAA-PROMETHEE methods. More in detail, on the GAIA plane can be projected:

- the whole set of feasible bicapacities $\hat{\mu}$, represented by the corresponding vectors $\hat{\mathbf{a}}$, so that the DM can visualize the set of PROMETHEE decision axis as well as importance, interaction and antagonistic effects, compatible with the preference information;
- the set of feasible bicapacities $\hat{\mu}$ for which an alternative attains a given ranking position (first, second, etc...);

- the set of feasible bicapacities $\hat{\mu}$ for which one alternative is preferred, indifferent or incomparable to another one.

Clearly, the GAIA plane technique can be also applied to single non-elementary criteria in order to obtain a graphical representation of the SMAA results in case of hierarchical structure of the set of criteria. In Section 2.4, considering the problem of evaluating pharmaceutical companies, we show an application of the GAIA plane technique coupled with SMAA in case of hierarchy of interacting criteria.

2.3.3 The ROR for bipolar PROMETHEE with hierarchy of criteria

In the case of bipolar PROMETHEE with criteria hierarchically structured, the typical results of the ROR in terms of necessary and possible relations can be obtained as follows:

- a is necessarily preferred to b on $g_{\mathbf{r}}$:
  - locally, iff $E^N = E^{DM} \cup \{Ch^B(\mathbf{P}^{\mathbf{B}}_{\mathbf{r}}(\mathbf{a},\mathbf{b}),\hat{\mu}) + \varepsilon \leq 0\}$ is infeasible or $\varepsilon^*_N \leq 0$, where $\varepsilon^*_N = \max \varepsilon$ subject to E^N,
  - globally and considering the hierarchical bipolar PROMETHEE II method, if $E^N = E^{DM} \cup \{\phi^B_{\mathbf{r}}(a) + \varepsilon \leq \phi^B_{\mathbf{r}}(b)\}$ is infeasible or $\varepsilon^*_N \leq 0$, where $\varepsilon^*_N = \max \varepsilon$ subject to E^N,
  - globally and considering the hierarchical bipolar PROMETHEE I method, if E^N is infeasible or $\varepsilon^*_N \leq 0$, where $\varepsilon^*_N = \max \varepsilon$ subject to E^N and

$$E^N = \begin{cases} E^{DM}, \\ \phi^{B+}_{\mathbf{r}}(a) + \varepsilon \leq \phi^{B+}_{\mathbf{r}}(b) + 2M_1 \ \text{ and } \ \phi^{B-}_{\mathbf{r}}(a) + 2M_2 \geq \phi^{B-}_{\mathbf{r}}(b) + \varepsilon \\ \quad \text{where } \ M_i \in \{0,1\}, i = 1,2, \ \text{ and } \ \sum_{i=1}^{2} M_i \leq 1, \end{cases}$$

- a is possibly preferred to b on $g_{\mathbf{r}}$:
  - locally, iff $E^P = E^{DM} \cup \{Ch^B(\mathbf{P}^{\mathbf{B}}_{\mathbf{r}}(\mathbf{a},\mathbf{b}),\hat{\mu}) \geq 0\}$ is feasible and $\varepsilon^*_P > 0$, where $\varepsilon^*_P = \max \varepsilon$ subject to E^P,
  - globally and considering the hierarchical bipolar PROMETHEE II method, if $E^P = E^{DM} \cup \{\phi^B_{\mathbf{r}}(a) \geq \phi^B_{\mathbf{r}}(b)\}$ is feasible and $\varepsilon^*_P > 0$, where $\varepsilon^*_P = \max \varepsilon$ subject to E^P,

– globally and considering the hierarchical bipolar PROMETHEE I method, if E^P is feasible and $\varepsilon_P^* > 0$, where $\varepsilon_P^* = \max \varepsilon$ subject to E^P where

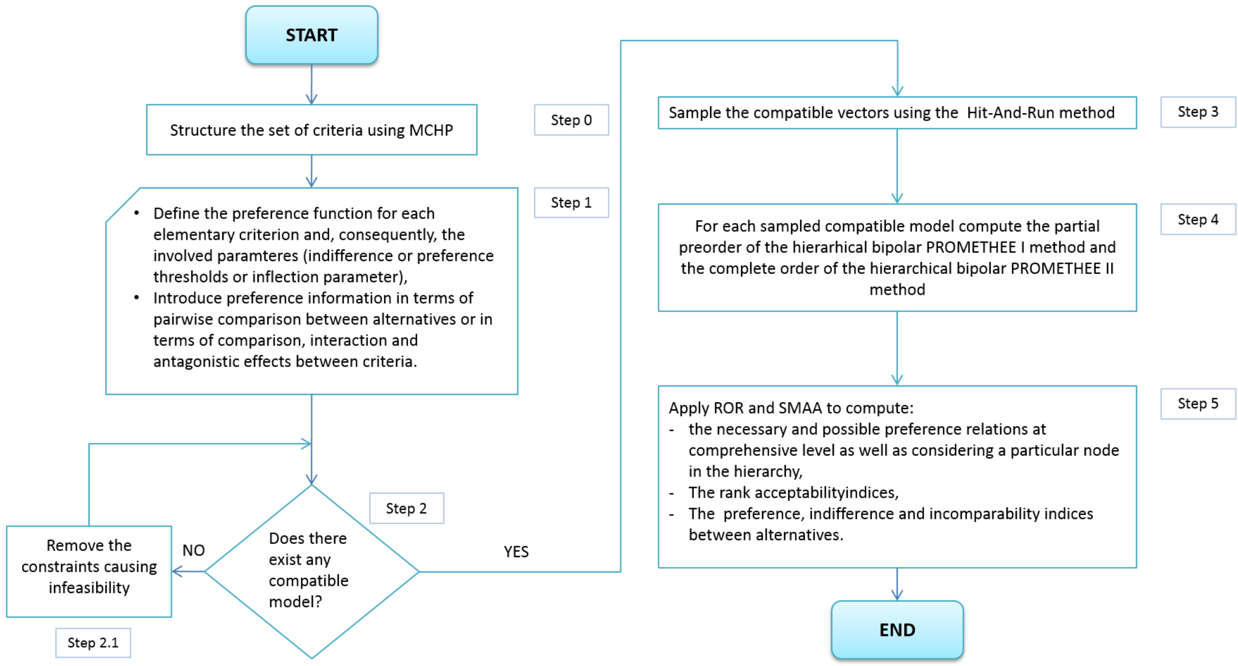
$$E^P = \begin{cases} E^{DM}, \\ \phi_{\mathbf{r}}^{B+}(a) \geq \phi_{\mathbf{r}}^{B+}(b), \\ \phi_{\mathbf{r}}^{B-}(a) \leq \phi_{\mathbf{r}}^{B-}(b). \end{cases}$$

All the parameters representing the interactions between criteria which are not expressed by the DM are set equal to zero.

The flow chart and main steps of the proposed method

In this section we shall briefly summarize the main steps involved in the application of the hierarchical bipolar PROMETHEE method (see the flow chart in Fig. 2.5).

Figure 2.5: Flow chart of the hierarchical bipolar PROMETHEE method



Step 0) The problem is structured in a hierarchical way by defining elementary criteria and all other criteria placed at intermediate level between the root node and the bottom of the hierarchy (Section 2.1.1);

Step 1) For each elementary criterion one of the six preference functions proposed in [20] has to be chosen. Each of them involves the definition of at most 2 parameters being indifference and preference threshold or the inflection parameter (Section 2.1);

The DM is invited to provide some preference in terms of pairwise comparison between alternatives (preference or indifference between them) at comprehensive level (that is considering all alternatives) or at a partial level (considering only two alternatives). Moreover, could be provided at the root level (considering all elementary criteria at the same time), or at a certain node (considering the elementary criteria descending from a certain criterion only). If he is willing to, the DM can also provide some preference information in terms of importance comparison between criteria, positive or negative interaction or antagonistic effect between them (Section 2.3);

Step 2) The analyst checks if there exists at least one model compatible with the preferences provided by the DM. If this is not the case, the cause of the infeasibility has to be investigated (Section 2.3);

Step 3) Several sets of parameters compatible with the preferences provided by the DM (compatible models) are randomly sampled by using the Hit-And-Run method (Section 2.3.1);

Step 4) For each compatible model the bipolar positive, negative and net flows of each alternative are computed at root level as well as at partial level. On the basis of this flow the preference and indifference relations can be computed considering the hierarchical bipolar PROMETHEE I and II methods, while the incomparability relation can be computed in considering the hierarchical bipolar PROMETHEE I method (Section 2.2);

Step 5) The SMAA (Section 2.3.1) and ROR (Section 2.3.3) methodologies are applied to get robust recommendations on the problem at hand and are graphically represented on the GAIA plane to supply an easily understandable intuition.

This procedure, that has been built for the extension of the PROMETHEE I and II methods to take into account hierarchy and interaction between criteria as well as robustness concerns, can be easily adapted to other PROMETHEE methods (PROMETHEE III, IV, V and VI) which extensions have been briefly described in the Section 2.2.1.

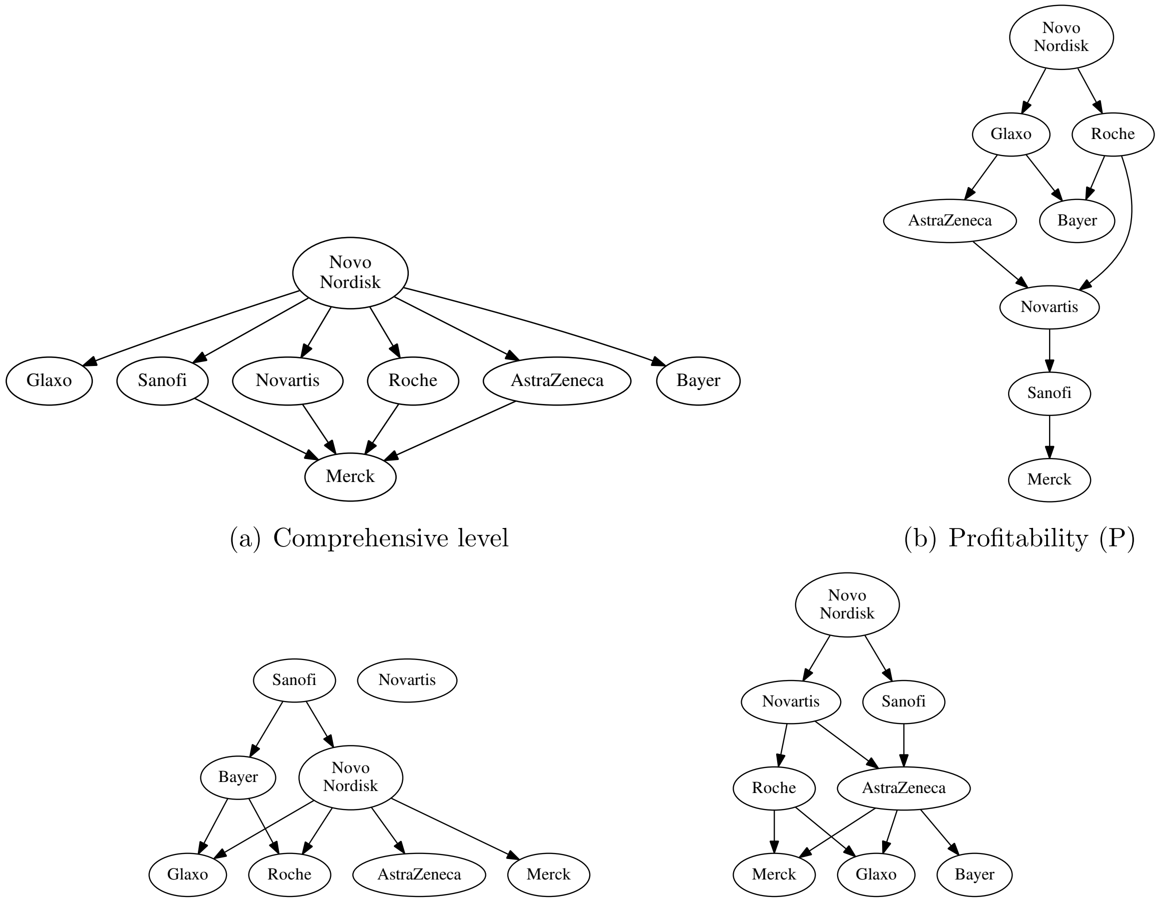
2.4 Illustrative example

In this section we show how to apply the proposed methodology to the financial decision problem handled starting from section 2.1.1. The eight most capitalized European pharmaceutical companies are evaluated on three aspects: Profitability (P), Solvency and Liquidity (S&L) and Managerial Performance (MP). The evaluations of the eight considered companies on the six elementary criteria descending from the three macro-aspects have been extracted from [1].

At first, for each pair of alternatives $(a, b) \in A \times A$ the bipolar vector $P^B(a, b)$ has been computed as described in section 2.1. After that, since we would like to apply the hierarchical extension of the bipolar PROMETHEE methods, we asked the DM to provide some preference information. He is confident in giving us the following information:

1. comparisons between the importance of criteria:
   - P is more important than $S\&L$ that, in turn, is more important than MP,
   - $\begin{cases} EBIT/S \text{ is more important than } GP/S; \\ GP/S \text{ is more important than } IE/S; \\ IE/S \text{ is more important than } TL/TA; \\ TL/TA \text{ is more important than } AR/S; \end{cases}$
2. positive and negative interactions between criteria:
   - P and MP are negatively interacting,
   - $S\&L$ and MP are negatively interacting; in particular, all elementary criteria descending from $S\&L$ are negatively interacting with all elementary criteria descending from MP; this means that the pairs of elementary criteria $\{TL/TA, AR/S\}$, $\{TL/TA, IE/S\}$, $\{CA/CL, AR/S\}$ and $\{CA/CL, IE/S\}$ are negatively interacting,
   - $\begin{cases} GP/S \text{ and } EBIT/S \text{ are negatively interacting;} \\ GP/S \text{ and } AR/S \text{ are positively interacting;} \\ EBIT/S \text{ and } IE/S \text{ are positively interacting;} \\ TL/TA \text{ and } CA/CL \text{ are negatively interacting;} \end{cases}$
3. antagonistic effect between criteria:
   - $S\&L$ has an antagonistic effect over P; in particular, all elementary criteria descending from $S\&L$ have an antagonistic effect over all elementary criteria descending from P; this means that an antagonistic effect is exercised from TL/TA over GP/S and $EBIT/S$ and from CA/CL over GP/S and $EBIT/S$;
   - AR/S has an antagonistic effect over GP/S.

At this point, the ROR and the SMAA methodologies are applied to get robust recommendations on the problem at hand. By applying the ROR to the hierarchical bipolar PROMETHEE II method, we get the necessary preference relation at global level as well as on the three considered aspects as shown in Figure 2.6.



(a) Comprehensive level

(b) Profitability (P)

(c) Solvency and Liquidity (S&L)

(d) Managerial Performance (MP)

Figure 2.6: Necessary preference relation at the comprehensive level, as well as with respect to macro-criteria P, S&L and MP considering the hierarchical bipolar PROMETHEE II method.

As one can see in Fig. 2.6(a), Novo Nordisk is comprehensively necessarily preferred to all other companies, while all companies apart from Bayer are comprehensively necessarily preferred to Merck. Considering the remaining six companies, nothing can be concluded looking at this figure since they are incomparable.

Something different can be observed on the three single main aspects:

- with respect to profitability (Fig. 2.6(b)), again Novo Nordisk is the best, while Merck is the worst one. Anyway, only the incomparabilities between AstraZeneca and Bayer and between Roche and AstraZeneca can be highlighted,

- on solvency and liquidity (Fig. 2.6(c)) Novo Nordisk is no longer the best company. Instead, Sanofi is necessarily preferred to all other companies but Novartis; looking at the lowest level of the diagram some companies are necessarily preferred to Glaxo, Roche, AstraZeneca and Merck, while each of them is not necessarily preferred to anyone else,
- considering managerial performance (Fig. 2.6(d)), Novo Nordisk is, again, the best company, while Merck, Glaxo and Bayer are at the bottom of the ranking.

Table 2.11: Results obtained by applying SMAA to the hierarchical bipolar PROMETHEE II method

(a) Rank acceptability indices (in percentage) at global level

Pharma Company	b^1	b^2	b^3	b^4	b^5	b^6	b^7	b^8
Glaxo	0	0	1.82	5.7	12.57	57.75	22.16	0
Sanofi	0	66.75	31.02	2.23	0	0	0	0
Novartis	0	0	3.79	17.07	57.88	15.85	5.41	0
Roche Holding	0	33.25	62.84	3.29	0.62	0	0	0
AstraZeneca	0	0	0.53	71.71	27.2	0.56	0	0
Bayer	0	0	0	0	1.73	25.84	72.43	0
Merck	0	0	0	0	0	0	0	100
Novo Nordisk	100	0	0	0	0	0	0	0

(b) Pairwise winning indices (in percentage) at global level

Pharma Company	Gl	Sa	No	Ro	AsZe	Ba	Me	NoNo
Glaxo (Gl)	0	1.82	19.94	0	8.08	77.43	100	0
Sanofi (Sa)	98.18	0	100	66.75	99.59	100	100	0
Novartis (No)	80.06	0	0	3.79	20.86	93.27	100	0
Roche (Ro)	100	33.25	96.21	0	99.26	100	100	0
AstraZeneca (AsZe)	91.92	0.41	79.14	0.74	0	100	100	0
Bayer (Ba)	22.57	0	6.73	0	0	0	100	0
Merck (Me)	0	0	0	0	0	0	0	0
Novo Nordisk (NoNo)	100	100	100	100	100	100	100	0

(c) Rank acceptability indices (in percentage) with respect to S&L

Pharma Company	b^1	b^2	b^3	b^4	b^5	b^6	b^7	b^8
Glaxo	0	0	0	0	1.75	13.17	77.68	7.4
Sanofi	100	0	0	0	0	0	0	0
Novartis	0	0	13.92	78.62	5.71	1.75	0	0
Roche Holding	0	0	0	7.46	92.54	0	0	0
AstraZeneca	0	0	0	0	0	85.08	14.92	0
Bayer	0	0	86.08	13.92	0	0	0	0
Merck	0	0	0	0	0	0	7.4	92.6
Novo Nordisk	0	100	0	0	0	0	0	0

(d) Pairwise winning indices (in percentage) on S&L

Pharma Company	Gl	Sa	No	Ro	AsZe	Ba	Me	NoNo
Glaxo (Gl)	0	0	1.75	0	14.92	0	92.6	0
Sanofi (Sa)	100	0	100	100	100	100	100	100
Novartis (No)	98.25	0	0	92.54	100	13.92	100	0
Roche (Ro)	100	0	7.46	0	100	0	100	0
AstraZeneca (AsZe)	85.08	0	0	0	0	0	100	0
Bayer (Ba)	100	0	86.08	100	100	0	100	0
Merck (Me)	7.4	0	0	0	0	0	0	0
Novo Nordisk (NoNo)	100	0	100	100	100	100	100	0

As previously observed, the necessary preference relations computed at the global level as well as with respect to the three considered aspects, leave many pairs of companies incomparable. For example, looking at Fig. 2.6(c), we can observe that, on one hand, Novartis is not necessarily preferred to any other company and, on the other hand, no company is necessarily preferred to it. For this reason, we decided to apply also the SMAA methodology to the hierarchical bipolar PROMETHEE methods to get some estimate of the frequency with which a company gets a certain rank position or the frequency with which one company is preferred to another not only at global level but also with respect to a particular aspect.

As shown in Tables 2.11(a) and 2.11(b), Novo Nordisk is undoubtedly the best among the eight companies since it achieves always the first position in the ranking and, consequently, it is always preferred to all other companies. On the opposite side, the Merck is always in the last position and all companies are always preferred to it. Other information can be gathered:

- apart from Novo Nordisk and Merck filling always the first and the last position, respectively, all other companies can reach different positions. For instance, Glaxo and Novartis show a non-zero percentage from the third to the seventh position, while the Bayer fills always a position from the fifth to the seventh. Moreover, Roche achieves the second position in 33.25% of the cases;

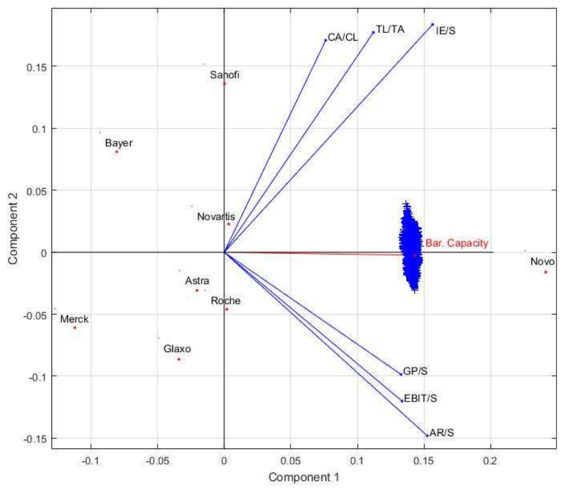
- looking at Table 2.11(b), Sanofi can be considered the second best company since it is preferred to all other companies, apart from Novo Nordisk, with a percentage no lower than 66.75%; conversely, the Bayer is the second worst company since all other ones, apart from Merck, are preferred to it with a percentage no lower than 77.43%.

Similar considerations can be done also on single aspects. Here we report those related to $S\&L$:

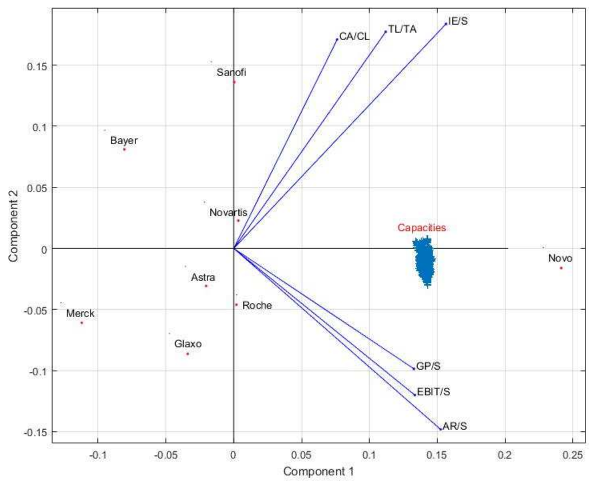
- while Novo Nordisk was always in the first position at a global level, a different situation can be observed on $S\&L$. Indeed, Sanofi ranks always first (see Table 2.11(c)) and, consequently, it is always preferred to all other companies (see Table 2.11(d));

- looking at Table 2.11(d), most of the pairwise comparisons are certain, while others are not true in all cases. For example, AstraZeneca is preferred to Glaxo with the 85.05%, while Glaxo is preferred to AstraZeneca with a percentage equal to 14.92%; analogously, Glaxo is preferred to Merck with a percentage of the 92.6%, while the viceversa is true in the 7.4% of the cases.

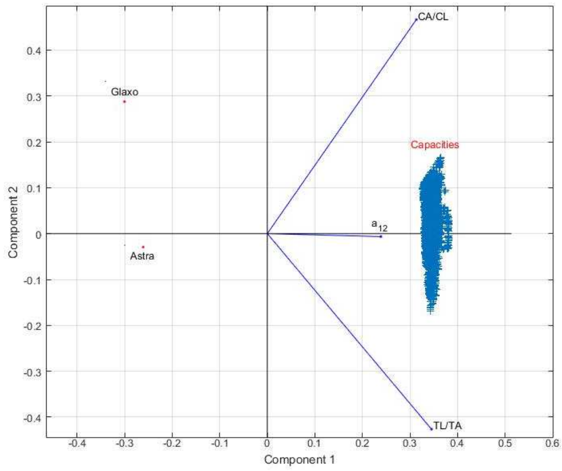
Let us show how the GAIA plane technique can support the DM in the interpretation of all the results supplied by the SMAA-PROMETHEE method. With this aim we present and comment Figures 2.7(a)-2.7(c). Let us start by pointing out that Figures 2.7(a) and 2.7(b) have the same explained variance $\delta = 78.72\%$ of Figures 2.3(a),2.3(b) and 2.3(c), because they refer to the same



(a) Sampling of 10,000 compatible capacities



(b) Compatible capacities for which Roche Holding achieves the second position at global level



(c) A sampling of compatible capacities for which AstraZeneca is preferred to Glaxo with respect to Solvency and Liquidity

Figure 2.7: GAIA plane for SMAA-PROMETHEE applied to the hierarchy of interacting criteria

matrix Φ. Analogously, Figure 2.7(c) has the same explained variance $\delta = 92.47\%$ of Figure 2.3(d). Figure 2.7(a) shows the projection on the GAIA plane of a random sample of 10,000 compatible bicapacities. It is presented also the PROMETHEE decision axis corresponding to the barycenter of the sampled compatible bicapacities. Looking at Figure 2.7(a) it is evident how much robust is the best position of Novo Nordisk, that is in the same direction of the projections of all the compatible bicapacities, as well as the last position of Merck, which is in the opposite direction. Figure 2.7(b), instead, permits to have a visualization of the 33.25% of compatible bicapacities for which, according to Table 2.11(a), Roche Holding gets the second rank position. Comparing them with the whole set of bicapacities represented in Figure 2.7(a), one can see that Roche Holding is the second for those bicapacities giving more importance to criteria GP/S, EBIT/S and AR/S. Observe that, again on the basis of Table 2.11(a), for the other compatible bicapacities Sanofi attains the second rank position. Consequently, the compatible bicapacities shown in 2.7(a) but not in 2.7(b) are those for which the second best company is Sanofi, which is therefore preferred to Roche Holding for those bicapacities giving more importance to criteria CA/CL, TL/TA and IE/S. Finally, Figure 2.7(c) presents the projection of the 85.08% compatible bicapacities for which, according to Table 2.11(d), AstraZeneca is preferred to Glaxo with respect to Solvency and Liquidity. Of course, all the rich and varied information supplied by PROMETHEE method with respect to all criteria in the hierarchy can be visualized analogously, permitting to the DM to gain a better insight of the decision problem and of the recommendation obtained through the decision process.

To conclude this section, we would like to show the usefulness of considering the MCHP in the problem at hand. To this aim, in Fig. 2.8 we have shown the comparison between Sanofi and Novartis in terms of frequency of preference of one over the other. In particular, Sanofi is always preferred to Novartis both at global level as well as on $S\&L$, while Novartis is always preferred to Sanofi on P. A not so certain comparison can be done on MP. Indeed, on one hand, Sanofi is preferred to Novartis with a frequency of 58.39%, while the viceversa is true in the remaining 41.61%. Let us observe that so detailed considerations can be possible only using the extension of the bipolar PROMETHEE method to the case of criteria structured in a hierarchical way.

Figure 2.8: Frequency of preference of Novartis and Sanofi and viceversa at global level as well as on the three main aspects considering the hierarhical bipolar PROMETHEE II method

Chapter 3

Robustness concerns for the level dependent Choquet integral

In this chapter we show a quite "rich" decision support model built by the ROR and SMAA methodologies applied to the level dependent Choquet integral.

On one hand, the level dependent Choquet integral permits to handle criteria for which the importance and the interaction change from one level to the other of the evaluations. Indeed, the Choquet integral permits to aggregate the evaluations taking into account interaction of criteria by using capacities that assign a weight to each subset of the considered criteria and not only to single criteria as in the usual weighted sum. The level dependent Choquet integral goes a further step forward taking into account a set of level dependent capacities, that is, a capacity for each level of the evaluation criteria. This permits to represent different attitudes, that is, different importance and interaction assigned to considered criteria, for different evaluation levels.

On the other hand, the robustness concerns are handled by an exploration of the space of the feasible parameters of the decision model, that are, the level dependent capacities compatible with the preference information supplied by the DM. This is done by applying:

- ROR, that, for each pair of alternatives a and b, tells us if a is preferred to b for all compatible level dependent capacities (in which case, we say that a is necessarily preferred to b) or for at least one compatible level dependent capacity (in which case we say that a is possibly preferred to b);

- SMAA, that, for each alternative a and for each rank order position r, tells us which is the probability that, randomly extracting a compatible level dependent capacity, a attains the position r, as well as, for each pair of alternatives a and b, the probability that a is preferred to b.

3.1 An introductive explanatory example

Let us explain the richness and usefulness of the level dependent Choquet integral coupled with methodologies taking into account robustness concerns, as ROR and SMAA, and representing quite articulated preferences.

Inspired by [44], let us suppose that the dean of a scientifically oriented high school wants to provide an overall evaluation of its students. For the sake of simplicity, the dean wants to take into account only notes on Mathematics and Physics, expressed on a thirty point scale. The dean expresses the following convictions:

- considering the required skills, Physics is more important than Mathematics in evaluating students with good notes on both subjects, that are those having a mark of at least 26 in these subjects;
- considering students needing to improve their background, that are students having a mark at most equal to 25 on the two subjects, Mathematics is more important than Physics;
- a student with notes at least equal to 26 in both subjects is excellent if he has very good notes both in Mathematics and in Physics;
- among students with notes at most equal to 25 in both subjects, the dean prefers students with a relatively good note in at least one of the subjects, because this can be seen as a starting point to reduce the gap from better students.

On the basis of the above considerations, taking into account the students whose evaluations on the two subjects are presented in Table 3.1, the dean expresses the following preferences

$$A \succ C \succ B \succ E \succ F \succ D$$

where $X \succ Y$ means that student X is preferred to student Y, with $X, Y \in \{A, B, C, D, E, F\}$.

One can see that the dean is giving a rich preference information, but one can also wonder if it is possible to represent his preferences using a model as simple as possible. The simplest model one can consider is the weighted sum, that is

$$U(X) = w_M \cdot M(X) + w_{Ph} \cdot Ph(X) \tag{3.1}$$

Table 3.1: Students' notes on Mathematics and Physics

	Mathematics (M)	Physics (Ph)
A	28	28
B	30	26
C	26	30
D	23	23
E	25	21
F	21	25

where

- $U(X)$ is the overall evaluation of X,
- $M(X)$ and $Ph(X)$ are the notes of X in Mathematics and Physics, respectively,
- w_M and w_{Ph} are the weights of Mathematics and Physics and they are such that, $w_M \geq 0$, $w_{Ph} \geq 0$ and $w_M + w_{Ph} = 1$.

Apparently a so simple model cannot model the preference information supplied by the dean. On one hand, taking into account students B and C and remembering that the dean prefers C over B, one would get,

$$U(B) = w_M \cdot 30 + w_{Ph} \cdot 26 < w_M \cdot 26 + w_{Ph} \cdot 30 = U(C) \tag{3.2}$$

implying that $w_M < w_{Ph}$. On the other hand, considering E and F and remembering that the dean prefers E over F, one could get that

$$U(E) = w_M \cdot 25 + w_{Ph} \cdot 21 > w_M \cdot 21 + w_{Ph} \cdot 25 = U(F) \tag{3.3}$$

that implies $w_M > w_{Ph}$. Of course the two inequalities translating the preferences of the dean are incompatible and, consequently, the weighted sum is not able to represent these preferences.

Looking for an amendment of the weighted sum able to represent preferences that cannot be expressed by using (3.1), one could imagine that weights depend on the level of the notes. For example, one can imagine that Mathematics has a weight w_M^1 for notes up to 25, that is $M^1(X) = min\{M(X), 25\}$, and weight w_M^2 for the possible excess to 25, that is, $M^2(X) = \max\{M(X) - 25, 0\}$. Analogously, one could assume that Physics has weight w_{Ph}^1 for notes up to 25, that is, $Ph^1(X) = \min\{Ph(X), 25\}$, and weight w_{Ph}^2 for the possible excess to 25, that is, $Ph^2(X) = \max\{Ph(X) - 25, 0\}$. Nonnegativity and normalization of the weights in this case would require that

$$w_M^1, w_M^2, w_{Ph}^1, w_{Ph}^2 \geq 0,\ w_M^1 + w_{Ph}^1 = 1,\ w_M^2 + w_{Ph}^2 = 1. \tag{3.4}$$

In this way, we could have a new model, the *level dependent weighted sum* ,(in fact, the two additive level dependent weighted sum) having the following form:

$$U_{LDWS}(X) = w_M^1 \cdot M^1(X) + w_M^2 \cdot M^2(X) + w_{Ph}^1 \cdot Ph^1(X) + w_{Ph}^2 \cdot Ph^2(X) \tag{3.5}$$

that is

$$U_{LDWS}(X) = w_M^1 \cdot \min\{M(X), 25\} + w_M^2 \cdot \max\{Ph(X) - 25, 0\} + w_{Ph}^1 \cdot \min\{Ph(X), 25\} + \tag{3.6}$$

$$+ w_{Ph}^2 \cdot \max\{Ph(X) - 25, 0\}.$$

The level dependent weighted sum permits to take into account that, for notes at least equal to 26, Physics is more important than Mathematics ($w_{Ph}^2 > w_M^2$), while for notes at most equal to 25, it is exactly the opposite ($w_M^1 > w_{Ph}^1$).

The level dependent weighted sum is therefore able to represent some of the preferences provided by the dean. Indeed, considering students B, C, E and F, the preference of C over B is translated to the constraint

$$U_{LDWS}(B) = w_M^1 \cdot \min\{30, 25\} + w_M^2 \cdot \max\{30 - 25, 0\} + w_{Ph}^1 \cdot \min\{26, 25\} + w_{Ph}^2 \cdot \max\{26 - 25, 0\} \tag{3.7}$$

$$<$$

$$w_M^1 \cdot \min\{26, 25\} + w_M^2 \cdot \max\{26 - 25, 0\} + w_{Ph}^1 \cdot \min\{30, 25\} + w_{Ph}^2 \cdot \max\{30 - 25, 0\} = U_{LDWS}(C)$$

while, the preference of E over F is translated to the constraint

$$U_{LDWS}(E) = w_M^1 \cdot \min\{25, 25\} + w_M^2 \cdot \max\{25 - 25, 0\} + w_{Ph}^1 \cdot \min\{21, 25\} + w_{Ph}^2 \cdot \max\{21 - 25, 0\} \tag{3.8}$$

$$>$$

$$w_M^1 \cdot \min\{21, 25\} + w_M^2 \cdot \max\{21 - 25, 0\} + w_{Ph}^1 \cdot \min\{25, 25\} + w_{Ph}^2 \cdot \max\{25 - 25, 0\} = U_{LDWS}(F).$$

On one hand, by (3.7) one gets

$$w_M^1 \cdot 25 + w_M^2 \cdot 5 + w_{Ph}^1 \cdot 25 + w_{Ph}^2 \cdot 1 < w_M^1 \cdot 25 + w_M^2 \cdot 1 + w_{Ph}^1 \cdot 25 + w_{Ph}^2 \cdot 5 \tag{3.9}$$

and, consequently, $w_M^2 < w_{Ph}^2$.

On the other hand, by (3.1) we get

$$w_M^1 \cdot 25 + w_M^2 \cdot 0 + w_{Ph}^1 \cdot 21 + w_{Ph}^2 \cdot 0 > w_M^1 \cdot 21 + w_M^2 \cdot 0 + w_{Ph}^1 \cdot 25 + w_{Ph}^2 \cdot 0, \tag{3.10}$$

and, consequently, $w_M^1 > w_{Ph}^1$.

Therefore, we have seen that the level dependent weighted sum permits to represent importance of criteria changing from a level to another. However, we have still to check if U_{LDWS} allows to consider also the conjunctive or disjunctive nature of the overall evaluation that changes from one level to another. In fact, according with the preference information supplied by the dean, the aggregation procedure has to be

- conjunctive for notes at least equal to 26 in both subjects: in this case, a student good in both subjects is more appreciated than a student very good in one subject and not so good in the other,
- disjunctive for notes at most equal to 25 in both subjects: in this case, a student relatively good in at least one of the two subjects is more appreciated than a student presenting medium marks on both subjects.

In consequence of the previous observations, in evaluating students A, B and C the dean stated that A is preferred to C that, in turn, is preferred to B. Using the U_{LDWS}, the previous preferences are translated into the constraints

$$U_{LDWS}(A) > U_{LDWS}(C) > U_{LDWS}(B).$$

The first inequality, that is, $U_{LDWS}(A) > U_{LDWS}(C)$, implies that

$$w_M^1 \cdot \min\{28, 25\} + w_M^2 \cdot \max\{28 - 25, 0\} + w_{Ph}^1 \cdot \min\{28, 25\} + w_{Ph}^2 \cdot \max\{28 - 25, 0\}$$

$$>$$

$$w_M^1 \cdot \min\{26, 25\} + w_M^2 \cdot \max\{26 - 25, 0\} + w_{Ph}^1 \cdot \min\{30, 25\} + w_{Ph}^2 \cdot \max\{30 - 25, 0\}$$

from which, $w_M^2 > w_{Ph}^2$.

The second inequality, that is $U_{LDWS}(C) > U_{LDWS}(B)$, instead, implies that

$$w_M^1 \cdot \min\{26,25\} + w_M^2 \cdot \max\{26-25,0\} + w_{Ph}^1 \cdot \min\{30,25\} + w_{Ph}^2 \cdot \max\{30-25,0\}$$

$$>$$

$$w_M^1 \cdot \min\{30,25\} + w_M^2 \cdot \max\{30-25,0\} + w_{Ph}^1 \cdot \min\{26,25\} + w_{Ph}^2 \cdot \max\{26-25,0\}$$

and, consequently, $w_{Ph}^2 > w_M^2$. Of course, the two obtained inequalities are incompatible since w_{Ph}^2 cannot be simultaneously lower and greater than w_M^2.

Analogously, taking into consideration students D, E and F, and reminding that the dean stated that E is preferred to F that, in turn, is preferred to D, the application of the level dependent weighted sum would translate these preferences into the following chain of constraints

$$U_{LDWS}(E) > U_{LDWS}(F) > U_{LDWS}(D).$$

On one hand, the first inequality, that is $U_{LDWS}(E) > U_{LDWS}(F)$, implies $w_M^1 > w_{Ph}^1$, while, on the other hand, the second inequality, that is $U_{LDWS}(F) > U_{LDWS}(D)$, implies that $w_M^1 < w_{Ph}^1$. Again, the two inequalities are incompatible and, consequently, even the level dependent weighted sum is not able to represent the preferences given by the DM.

To represent the more or less conjunctive character of the aggregation procedure one can use the Choquet integral that has been successfully applied in multiple criteria decision analysis to handle interaction between criteria. The Choquet integral of the evaluations of student X on Mathematics and Physics can be defined as follows:

$$U_{Ch}(X) = \begin{cases} M(X) + w_{Ph}^{Ch} \cdot (Ph(X) - M(X)) & \text{iff} \quad M(X) \leq Ph(x), \\ Ph(X) + w_M^{Ch} \cdot (M(X) - Ph(X)) & \text{iff} \quad M(X) \geq Ph(x), \end{cases} \tag{3.11}$$

with $0 \leq w_M^{Ch}, w_{Ph}^{Ch} \leq 1$ and not necessarily $w_M^{Ch} + w_{Ph}^{Ch} = 1$. Let us see if the Choquet integral permits to represent the preferences expressed by the dean on students A, B and C. Since $A \succ C \succ B$ we should have

$$U_{Ch}(A) > U_{Ch}(C) > U_{Ch}(B),$$

and, consequently,

$$28 + w_{Ph}^{Ch} \times 0 > 26 + 4 \cdot w_{Ph}^{Ch} > 26 + 4 \cdot w_M^{Ch}$$

from which

$$0.5 > w_{Ph}^{Ch} > w_M^{Ch} \tag{3.12}$$

which is in agreement with the greater importance assigned by the dean to Physics over Mathematics in case of notes not smaller than 26 in both the two subjects. Thus we proved that the Choquet integral can represent the preferences about students A, B and C.

With analogous considerations, we can show that the Choquet integral can represent also the preferences over students D, E and F. In fact, from $E \succ F \succ D$ we obtain that

$$U_{Ch}(E) > U_{Ch}(F) > U_{Ch}(D),$$

and, consequently,

$$21 + 4 \cdot w_M^{Ch} > 21 + 4 \cdot w_{Ph}^{Ch} > 23 + 0 \cdot w_{Ph}^{Ch}.$$

The inequalities above imply that

$$w_M^{Ch} > w_{Ph}^{Ch} > 0.5 \tag{3.13}$$

which are in agreement with the greater importance assigned by the dean to Mathematics over Physics in case of notes not greater than 25 in both subjects. Thus also the preferences between the students E, F and D can be represented by the Choquet integral U_{Ch}.

However let us remark that conditions (3.12) and (3.13) are incompatible, which means that the Choquet integral is not able to represent, simultaneously, all the preferences expressed by the dean over the six students. The problem is that the importance of criteria as well as the conjunctive or disjunctive nature of the aggregation procedure are different for students with notes at least equal to 26 and students with notes at most equal to 25. Thus, one has to further generalize the model considering the level dependent Choquet integral [53] that permits to take into account importance of criteria and interaction between criteria related to conjunctive or disjunctive nature of the aggregation procedure that can change from one level to the others of the evaluations on considered criteria. With respect to the overall evaluation of students on Mathematics and Physics, the level dependent Choquet integral can be formulated as follows:

$$U_{LDCh}(X) = \begin{cases} \overline{U}_M(X) & \text{iff} \quad M(X) \leq Ph(X), \\ \overline{U}_{Ph}(X) & \text{iff} \quad M(X) \geq Ph(X), \end{cases} \tag{3.14}$$

where

$$\overline{U}_M(X) = M(X)+w_{Ph}^{Ch,1}\cdot(\min\{Ph(X),25\}-\min\{M(X),25\})+w_{Ph}^{Ch,2}\cdot(\max\{Ph(X),25\}-\max\{M(X),25\}),$$

and

$$\overline{U}_{Ph}(X) = Ph(X)+w_{M}^{Ch,1}\cdot(\min\{M(X),25\}-\min\{Ph(X),25\})+w_{M}^{Ch,2}\cdot(\max\{M(X),25\}-\max\{Ph(X),25\}).$$

Let us check if the level dependent Choquet integral is able to represent the preferences over the six students, that is if there exist $w_M^{Ch,1}$, $w_M^{Ch,2}$, $w_{Ph}^{Ch,1}$ and $w_{Ph}^{Ch,2}$ such that

$$U_{LDCh}(A) > U_{LDCh}(C) > U_{LDCh}(B) > U_{LDCh}(E) > U_{LDCh}(F) > U_{LDCh}(D).$$

Using Eq. (3.14), the previous inequalities are translated into the following chain of constraints:

$$U_{LDCh}(A) = 28 + w_{Ph}^{Ch,1}\cdot(\min\{28,25\}-\min\{28,25\}) + w_{Ph}^{Ch,2}\cdot(\max\{28,25\}-\max\{28,25\})$$

$$>$$

$$U_{LDCh}(C) = 26 + w_{Ph}^{Ch,1}\cdot(\min\{30,25\}-\min\{26,25\}) + w_{Ph}^{Ch,2}\cdot(\max\{30,25\}-\max\{26,25\})$$

$$>$$

$$U_{LDCh}(B) = 26 + w_{M}^{Ch,1}\cdot(\min\{30,25\}-\min\{26,25\}) + w_{M}^{Ch,2}\cdot(\max\{30,25\}-\max\{26,25\})$$

$$>$$

$$U_{LDCh}(E) = 21 + w_{M}^{Ch,1}\cdot(\min\{25,25\}-\min\{21,25\}) + w_{M}^{Ch,2}\cdot(\max\{25,25\}-\max\{21,25\})$$

$$>$$

$$U_{LDCh}(F) = 21 + w_{Ph}^{Ch,1}\cdot(\min\{25,25\}-\min\{21,25\}) + w_{Ph}^{Ch,2}\cdot(\max\{25,25\}-\max\{21,25\})$$

$$>$$

$$U_{LDCh}(D) = 23 + w_{Ph}^{Ch,1}\cdot(\min\{23,25\}-\min\{23,25\}) + w_{Ph}^{Ch,2}\cdot(\max\{23,25\}-\max\{23,25\}).$$

We obtain the following conditions:

1. $U_{LDCh}(A) > U_{LDCh}(C) \Rightarrow 2 > w_{Ph}^{Ch,2}\cdot 4 \Rightarrow 0.5 > w_{Ph}^{Ch,2}$,

2. $U_{LDCh}(C) > U_{LDCh}(B) \Rightarrow w_{Ph}^{Ch,2} \cdot 4 > w_M^{Ch,2} \cdot 4 \Rightarrow w_{Ph}^{Ch,2} > w_M^{Ch,2}$,

3. $U_{LDCh}(E) > U_{LDCh}(F) \Rightarrow w_M^{Ch,1} \cdot 4 > w_{Ph}^{Ch,1} \cdot 4 \Rightarrow w_M^{Ch,1} > w_{Ph}^{Ch,1}$,

4. $U_{LDCh}(F) > U_{LDCh}(D) \Rightarrow w_{Ph}^{Ch,1} \cdot 4 > 2 \Rightarrow w_{Ph}^{Ch,1} > 0.5$.

The four inequalities above are not in contradiction and, therefore, the level dependent Choquet integral permits to represent the preferences expressed by the dean. Moreover, the conditions perfectly translate the preferences of the dean in terms of importance of criteria and the desired conjunctive and disjunctive nature of the aggregation. However, there is still another aspect that deserves to be investigated. Let us consider the new students G, H and I whose notes in Mathematics and Physics are presented in Table 3.2 trying to rank them on the basis of conditions 1.-4. above.

Table 3.2: Notes on Mathematics and Physics of the new students

	Mathematics (M)	Physics (Ph)
G	26	29
H	29	26
I	30	27

On one hand, comparing G and H, we get that

$$U_{LDCh}(G) = 26 + w_{Ph}^{Ch,1} \cdot (\min\{29,25\} - \min\{26,25\}) + w_{Ph}^{Ch,2} \cdot (\max\{29,25\} - \max\{26,25\})$$

$$>$$

$$26 + w_M^{Ch,1} \cdot (\min\{29,25\} - \min\{26,25\}) + w_M^{Ch,2} \cdot (\max\{29,25\} - \max\{26,25\}) = U_{LDCh}(H)$$

being always true since it coincides with condition 2.
On the other hand, comparing G and I, we can have $U_{LDCh}(G) > U_{LDCh}(I)$ as well as $U_{LDCh}(I) > U_{LDCh}(G)$, depending on the values of $w_M^{Ch,1}$, $w_M^{Ch,2}$, $w_{Ph}^{Ch,1}$ and $w_{Ph}^{Ch,2}$. Indeed, we get

$$U_{LDCh}(G) = 26 + w_{Ph}^{Ch,1} \cdot (\min\{29,25\} - \min\{26,25\}) + w_{Ph}^{Ch,2} \cdot (\max\{29,25\} - \max\{26,25\})$$

$$>$$

$$27 + w_M^{Ch,1} \cdot (\min\{30,25\} - \min\{27,25\}) + w_M^{Ch,2} \cdot (\max\{30,25\} - \max\{27,25\}) = U_{LDCh}(I),$$

iff

$$26 + w_{Ph}^{Ch,2} \cdot 3 > 27 + w_M^{Ch,2} \cdot 3, \tag{3.15}$$

as well as $U_{LDCh}(G) < U_{LDCh}(I)$ when

$$26 + w_{Ph}^{Ch,2} \cdot 3 < 27 + w_{M}^{Ch,2} \cdot 3. \tag{3.16}$$

None of the two inequalities is in contradiction with conditions 1.-4. Therefore, we have to conclude that the preference information given by the DM, and formally expressed by conditions 1.-4., does not permit to rank order unambiguously all students since different values of parameters can provide a different comparison of G and I. For example, taking $w_{M}^{Ch,2} = \frac{1}{18}$ and $w_{Ph}^{Ch,2} = \frac{4}{9}$ we get

$$U_{LDCh}(G) = 27.333 > 27.166 = U_{LDCh}(I) \tag{3.17}$$

while, taking $w_{M}^{Ch,2} = \frac{1}{6}$ and $w_{Ph}^{Ch,2} = \frac{1}{3}$ we get

$$U_{LDCh}(G) = 27 < 27.5 = U_{LDCh}(I). \tag{3.18}$$

With a more level of detail, we can say that student G will be overall evaluated at least as good as student I, that is, $U_{LDCh}(G) \geq U_{LDCh}(I)$, in case $w_{M}^{Ch,2}$ and $w_{Ph}^{Ch,2}$ satisfy conditions 1.-4. on one hand, and condition

$$U_{LDCh}(G) \geq= U_{LDCh}(I) \Leftrightarrow 26 + w_{Ph}^{Ch,2} \cdot 3 \geq 27 + w_{M}^{Ch,2} \cdot 3 \Leftrightarrow w_{Ph}^{Ch,2} \cdot 3 \geq 1 + w_{M}^{Ch,2} \cdot 3, \tag{3.19}$$

on the other hand.

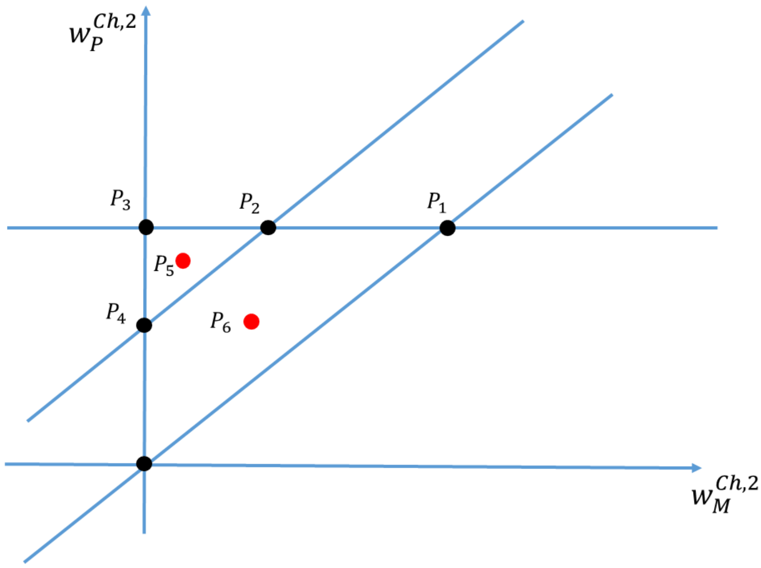
In the space $w_{M}^{Ch,2} - w_{Ph}^{Ch,2}$, this amounts to the set of weights in the triangle $W_{G \succsim I}$ having vertices $P_2 = (\frac{1}{6}, \frac{1}{2})$, $P_3 = (0, \frac{1}{2})$ and $P_4 = (0, \frac{1}{3})$ in Figure 3.1. Instead, student I will get an overall evaluation not worse than G, that is, $U_{LDCh}(I) \geq U_{LDCh}(G)$, in case $w_{M}^{Ch,2}$ and $w_{Ph}^{Ch,2}$ satisfy conditions 1.-4. and condition

$$U_{LDCh}(G) \leq= U_{LDCh}(I) \Leftrightarrow 26 + w_{Ph}^{Ch,2} \cdot 3 \leq 27 + w_{M}^{Ch,2} \cdot 3 \Leftrightarrow w_{Ph}^{Ch,2} \cdot 3 \leq 1 + w_{M}^{Ch,2} \cdot 3, \tag{3.20}$$

that is, the set of weights in the trapezoid $W_{I \succsim G}$ having vertices $P_1 = (\frac{1}{2}, \frac{1}{2})$, $P_2 = (\frac{1}{6}, \frac{1}{2})$, $P_4 = (0, \frac{1}{3})$ and $O = (0, 0)$ in Figure 3.1. The centroids of $W_{G \succsim I}$ and $W_{I \succsim G}$ are the pairs of weights $(w_{M}^{Ch,2}, w_{Ph}^{Ch,2})$ $P_5 = (\frac{1}{18}, \frac{4}{9})$, and $P_6 = (\frac{1}{6}, \frac{1}{3})$, respectively, that have been used as examples of parameters for which $U_{LDCh}(G) > U_{LDCh}(I)$ and $U_{LDCh}(G) < U_{LDCh}(I)$.

The above considerations show that a special attention has to be given to robustness concerns

Figure 3.1: Representation of the subset of weights in the $w_M^{Ch,2}$-$w_{Ph}^{Ch,2}$ space for which G is necessarily preferred to I (the triangle $W_{G \succsim I} \equiv P_2P_3P_4$) or I is necessarily preferred to G (the trapezoid $W_{I \succsim G} \equiv P_1P_2P_4O$)



related to the application of the decision model. In general, it is always advisable to avoid to consider only one instance of the decision model compatible with the preference information provided by the DM (in our example the dean) because its choice is arbitrary to some extent. One possible answer to this type of situations is to distinguish preferences that hold for all compatible decision models and preferences that hold for at least one compatible decision model, called respectively necessary and possible preferences ([27, 58]). With respect to students G, H and I we have to conclude that we have a necessary preference of G over H and, also, due to dominance with respect to notes in Mathematics and Physics, of H over I. Instead, with respect to students G and I we have only a possible preference both for G over I and for I over G.

One can try to give a more precise evaluation of the range of compatible decision models for which student G is more appreciated than student I and viceversa by estimating the probability that picking randomly a vector of compatible parameters one gets G preferred over I or viceversa. Let us observe that the overall evaluations of students G, H and I involve only the weights $w_M^{Ch,2}$ and $w_{Ph}^{Ch,2}$, while the weights $w_M^{Ch,1}$ and $w_{Ph}^{Ch,1}$ have no role. Consequently, to compare students G, H and I, the set of the compatible decision models can be identified with the set W of pairs of weights $\left(w_M^{Ch,2}, w_{Ph}^{Ch,2}\right)$ satisfying conditions 1.-4. (in fact, only conditions 1. and 2., because conditions 3. and 4. are related to the weights $w_M^{Ch,1}$ and $w_{Ph}^{Ch,1}$). The set W of compatible weights $\left(w_M^{Ch,2}, w_{Ph}^{Ch,2}\right)$ are represented by the triangle having vertices O, P_1 and P_3 in figure 3.1. We have already seen that the the triangle $W_{G \succsim I}$ represents the set of pairs of weights $(w_M^{Ch,2}, w_{Ph}^{Ch,2})$ for which G is at least as good as I, while the trapezoid $W_{I \succsim G}$ represents the set of pairs of weights for which I is at least as good as G. One can assume a uniform distribution on the set of compatible pairs of weights W. Under this hypothesis, the probability of random picking a decision model for which G is preferred to I is given by the ratio between the area of $W_{G \succsim I}$ and the area of W, while the analogous probability for I being preferred to G is given by the ratio between the area of $W_{I \succsim G}$ and the area of W. Since the area of W, $W_{G \succsim I}$ and $W_{I \succsim G}$ are $\frac{1}{8}$, $\frac{1}{72}$ and $\frac{8}{72}$, respectively, we have that the probability of obtaining G preferred over I is $\frac{1}{9}$, while the probability of having I preferred over G is $\frac{8}{9}$. Instead, we have already seen that for all compatible decision models G is preferred to H as well as I is preferred to H, so that the corresponding probabilities are equal to 1. One could also look for the probability for which each student G, H and I attains the first, the second and the third ranking position. Since we have already seen both G and I are always preferred to H, we have to conclude that H takes always the third ranking position, so that the probability of being the third is 1 while the probability of being the first or the second is null. Instead G takes the first ranking position if he is preferred to I, otherwise he is the second. Consequently, the probability that G attains the first ranking position is $\frac{1}{9}$, while the probability of being the second is $\frac{8}{9}$. Analogously, we have that for I the probabilities of being the first and the second are $\frac{8}{9}$ and $\frac{1}{9}$, respectively. Both for G and I the probability of being the third is null. In the following we shall show how to apply systematically the decision analysis approach proposed in this introductive example.

3.2 NAROR and SMAA applied to the level dependent Choquet integral preference model

In this section we shall describe the extension of the NAROR and the SMAA to the level dependent Choquet integral.

3.2.1 The NAROR for the level dependent Choquet integral

The extension of the NAROR analysis to the level dependent Choquet integral preference model can be implemented by considering that, given the preferences information provided by the DM and fixed a partition of the evaluation scale, it computes the necessary relations and the possible relations for each interval of the level dependent capacity and it also compute the same relations considering the global level dependent capacity μ^L.

So, the DM can provide (among parenthesis the constraints translating this piece of preference information):

- a possibly incomplete binary relation $\succsim^{DM}$ on a subset of alternatives $A^* \subseteq A$ for which $a \succsim^{DM} b$ iff a is at least as good as b $\left(Ch^L(\mathbf{a}, \mu^L) \geq Ch^L(\mathbf{b}, \mu^L)\right)$,

- a possibly incomplete binary relation $\succsim^{DM}$ on a subset of criteria $G^* \subseteq G$, for which $g_i \succsim^{DM} g_j$ iff g_i is at least as important as g_j $\left(\phi(i, \mu^L) \geq \phi(j, \mu^L)\right)$,

- a possibly incomplete binary relation $\succsim^{DM}_{[a_{r-1}, a_r[}$ on a subset of criteria $G^* \subseteq G$, for which $g_i \succsim^{DM}_{[a_{r-1}, a_r[} g_j$ iff, considering the evaluations in the interval $[a_{r-1}, a_r[$, g_i is at least as important as g_j $\left(\phi(i, \mu^L, [a_{r-1}, a_r[) \geq \phi(j, \mu^L, [a_{r-1}, a_r[)\right)$,

- positive or negative interactions between criteria g_i and g_j in a comprehensive way $(I(\mu^L, \{i, l\}) > 0$ if g_i and g_j are positively interacting, while $I(\{i, j\}, \mu^L) < 0$ if g_i and g_j are negatively interacting) or considering a particular interval of evaluations $\big(I(\{i, j\}, \mu^L, [a_{r-1}, a_r[) > 0$ if g_i and g_j are positively interacting, while $I(\{i, j\}, \mu^L, [a_{r-1}, a_r[) < 0$ if g_i and g_j are negatively interacting).

Considering a 2-additive interval level dependent capacity μ^L relative to the breakpoints $a_0, a_1, \ldots, a_p$ and m_r the Möbius representation of the capacities μ_r, $r = 1, \ldots, p$, to check if there exists at least one interval level dependent Choquet integral compatible with the preferences provided by the DM, the following LP problem has to be solved

$$
\left.\begin{array}{l}
\varepsilon^* = max\ \varepsilon,\ \text{subject to,} \\
Ch^L(\mathbf{a},\mu^L) \geq Ch^L(\mathbf{b},\mu^L) + \varepsilon,\ \text{if } a \succ^{DM} b, \\
Ch^L(\mathbf{a},\mu^L) = Ch^L(\mathbf{b},\mu^L),\ \text{if } a \sim^{DM} b, \\
\phi(i,\mu^L) \geq \phi(j,\mu^L) + \varepsilon,\ \text{if } g_i \succ^{DM} g_j, \\
\phi(i,\mu^L) = \phi(j,\mu^L),\ \text{if } g_i \sim^{DM} g_j, \\
\phi(i,\mu^L,[a_{r-1},a_r[) \geq \phi(j,\mu^L,[a_{r-1},a_r[) + \varepsilon,\ \text{if } g_i \succ^{DM}_{[a_{r-1},a_r[} g_j, \\
\phi(i,\mu^L,[a_{r-1},a_r[) = \phi(j,\mu^L,[a_{r-1},a_r[),\ \text{if } g_i \sim^{DM}_{[a_{r-1},a_r[} g_j, \\
I(\{i,j\},\mu^L) \geq \varepsilon\ \text{ if } g_i \text{ and } g_j \text{ are comprehensively positively interacting,} \\
I(\{i,j\},\mu^L,[a_{r-1},a_r[) \geq \varepsilon\ \text{ if } g_i \text{ and } g_j \text{ are positively interacting considering} \\
\quad \text{evaluations in the interval } [a_{r-1},a_r[, \\
I(\{i,j\},\mu^L) \leq -\varepsilon\ \text{ if } g_i \text{ and } g_j \text{ are comprehensively negatively interacting,} \\
I(\{i,j\},\mu^L,[a_{r-1},a_r[) \leq -\varepsilon\ \text{ if } g_i \text{ and } g_j \text{ are negatively interacting considering} \\
\quad \text{evaluations in the interval } [a_{r-1},a_r[, \\
\left.\begin{array}{l}
m_r(\emptyset) = 0,\ \sum\limits_{i\in G} m_r(\{i\}) + \sum\limits_{\{i,j\}\subseteq G} m_r(\{i,j\}) = 1, \\
m_r(\{i\}) \geq 0,\ \forall i \in G, \\
m_r(\{i\}) + \sum\limits_{j\in E} m_r(\{i,j\}) \geq 0,\ \forall i \in G \text{ and } \forall\, E \subseteq G \setminus \{i\},\ E \neq \emptyset.
\end{array}\right\} \forall r = 1,\ldots,p,
\end{array}\right\} E^{DM}
\tag{3.21}
$$

If E^{DM} is feasible and $\varepsilon^* > 0$, then there exists at least one interval level dependent capacity compatible with the preferences provided by the DM, otherwise, there is not any interval level depedent capacity compatible and therefore, the constraints causing this incompatibility can be checked by using one of the methods presented in [81].

In general, if there exists one interval level dependent capacity compatible with the preferences provided by the DM, more than one there exists and, therefore, the choice of only one of them can be considered arbitrary to some extent. For this reason, ROR takes into account all of them defining a necessary $\succsim^N$ and a possible $\succsim^P$ preference relation. The necessary preference relation holds between two alternatives a and b if a is at least as good as b for all compatible models, while the possible preference relation holds between a and b if a is at least as good as b for at least one compatible model.

From a computational point of view, the two preference relations can be computed as follows:

- $a \succsim^N b$ iff $E^N(a,b)$ is infeasible or $\varepsilon^N \leq 0$ where $= \varepsilon^N \max \varepsilon$ subject to $E^N(a,b)$ and

$$E^N(a,b) = \{Ch^L(\mathbf{b},\mu^L) \geq Ch^L(\mathbf{a},\mu^L) + \varepsilon\} \cup E^{DM},$$

- $a \succsim^P b$ iff $E^P(a,b)$ is feasible and $\varepsilon^P > 0$ where $= \varepsilon^P \max \varepsilon$ subject to $E^P(a,b)$ and

$$E^P(a,b) = \{Ch^L(\mathbf{a},\mu^L) \geq Ch^L(\mathbf{b},\mu^L)\} \cup E^{DM}.$$

3.2.2 The SMAA for the level dependent Choquet integral

In order to apply SMAA to the level dependent Choquet integral, we shall denote by χ the evaluation space and by $\mathcal{M}$ the set of all interval level dependent capacities satisfying the constraints in E^{DM}.

As we remarked in Section 1.4.2, SMAA takes into account all the models compatible with the preferences provided by the DM even if in a different way. The indirect preference information is composed of two probability distributions, f_χ and $f_\mathcal{M}$, defined on χ and $\mathcal{M}$, respectively.

So in the case of level dependent Choquet integral, given $\xi \in \chi$ and $\mu^L \in \mathcal{M}$, SMAA methods define the rank function as:

$$rank(a,\xi,\mu^L) = 1 + \sum_{b \in A \setminus \{a\}} \rho\left(Ch^L(\xi_b,\mu^L) > Ch^L(\xi_a,\mu^L)\right),$$

(where $\rho(false) = 0$ and $\rho(true) = 1$ and ξ_a are the evaluations of a in the matrix ξ) that gives the rank position of alternative a. On the basis of this rank function, SMAA-2 computes the set of capacities $\mathcal{M}^s(a,\xi) \subseteq \mathcal{M}$ for which alternative a assumes rank $s = 1,2,\ldots,|A|$, as follows:

$$\mathcal{M}^s(a,\xi) = \left\{\mu^L \in \mathcal{M} : rank(a,\xi,\mu^L) = s\right\}.$$

The following indices are therefore computed in SMAA-2:

- *The rank acceptability index* that measures the variety of different parameters compatible with the DM's preference information giving to a the rank s:

$$b^s(a) = \int_{\xi \in \chi} f_\chi(\xi) \int_{\mu^L \in \mathcal{M}^s(a,\xi)} f_\mathcal{M}(\mu^L)\, d\mu^L d\xi;$$

$b^s(a)$ gives the probability that a has rank s, and it is within the range $[0,1]$.

- *The pairwise winning index* [77] that is defined as the frequency that an alternative a is preferred to an alternative c in the space of the interval level dependent capacities:

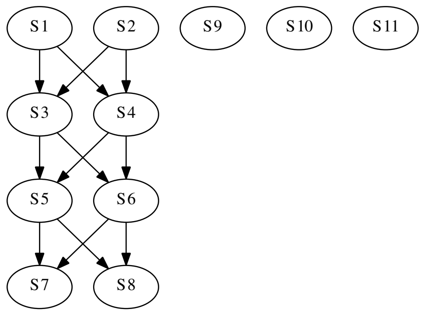
$$p(a,c) = \int_{\mu^L \in \mathcal{M}} f_{\mathcal{M}}(d\mu^L) \int_{\xi \in \chi:\, Ch^L(\xi_a,\mu^L) > Ch^L(\xi_c,\mu^L)} f_{\chi}(\xi)\; d\xi d\mu^L.$$

From a computational point of view, the multidimensional integrals defining the considered indices are estimated by using the Monte Carlo method. In particular, since the preference information provided by the DM and translated by constraints in E^{DM} defines a convex space of parameters (interval level dependent capacities), the Hit-And-Run (HAR) method can be used to sample several of these sets of parameters [96, 101, 103]. The application of the level dependent Choquet integral to each of these sets of parameters produces a ranking of the alternatives at hand. Consequently, considering all these rankings simultaneously, the rank acceptability index, as well as the pairwise winning indices described above, can be computed for each alternative and for each pair of alternatives, respectively.

3.3 Illustrative example

In this section we shall present an example of the application of the proposed methodology putting together with the interval level dependent Choquet integral, the NAROR and the SMAA methdologies. For this reason, we will recall the example provided in [53]. 11 students have to be evaluated on a $[18,30]$ scale on Mathematics (M), Physics (P) and Literature (L) as shown in Table 3.3. Let us observe that w.r.t. the original example, three more students (S_9, S_{10}, S_{11}) have been taken into account.

Figure 3.2: Dominance relation between the 11 students



Looking at the evaluations of the considered students, one can get the dominance relation shown in Figure 3.2. Anyway, many pairs of students are still incomparable and, therefore, the dean decides

Table 3.3: Students' evaluations on the three considered subjects

	Mathematics (M)	Physics (P)	Literature (L)
S_1	28	28	27
S_2	27	28	28
S_3	26	26	25
S_4	25	26	26
S_5	23	23	22
S_6	22	23	23
S_7	19	19	18
S_8	18	19	19
S_9	18	30	18
S_{10}	30	18	19
S_{11}	19	18	30

to provide the following preference information:

$$S_1 \succ S_2; \quad S_4 \succ S_3; \quad S_5 \succ S_6; \quad S_8 \succ S_7.$$

As already observed in [53], the Choquet integral is not able to represent these preferences. For this reason, we apply the interval level dependent Choquet integral considering the partition $a_0 = 18$, $a_1 = 22$, $a_2 = 25$, $a_3 = 27$, $a_4 = 30$ of the interval $[18, 30]$. Solving the LP problem (3.21), we get $\varepsilon^* > 0$. This means that there exists at least one interval level dependent capacity compatible with the preference information provided by the DM.

At this point, to have more insight on the considered ranking, we apply the NAROR. In particular, Figure 3.3 shows the necessary preference relation. This provides a complete ranking of the first 8 students. Anyway, the other three added students result incomparable with the remaining ones. For this reason, we decided to apply the SMAA methodology in order to investigate how the three added students can be compared with the other 8 students.

In Tables 3.4(a) and 3.4(b), we show the Rank Acceptability Indices and the Pairwise Winning Indices relative to the 11 students at hand.

As one can see in Table 3.4(a), S_1 and S_2 fill always the first and the second position in the considered ranking, while S_4 and S_3 are in the third and in the fourth positions, respectively, with a percentage very close to the 100%. Analogously, the last two positions are filled almost surely by S_8 and S_7 since $b^{10}(S_8)$ and $b^{11}(S_7)$ are equal to the 99.99%. Regarding the three students S_9, S_{10} and S_{11}, looking at the positions for which they present the highest rank acceptability index, these are the

Figure 3.3: Necessary preference relation between the 11 students. Red arrows represent the preferences provided by the DM.

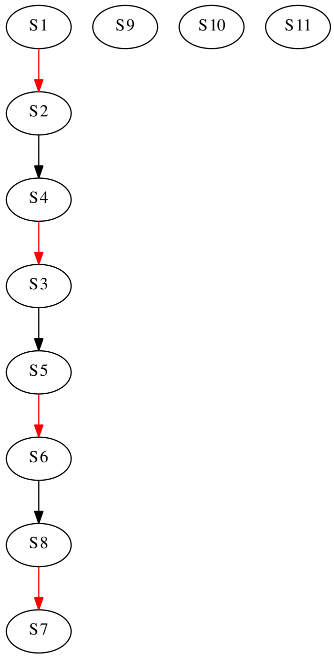


Table 3.4: Results obtained by applying SMAA to the interval level dependent Choquet integral

(a) Rank acceptability indices (in percentage)

	$b^1(\cdot)$	$b^2(\cdot)$	$b^3(\cdot)$	$b^4(\cdot)$	$b^5(\cdot)$	$b^6(\cdot)$	$b^7(\cdot)$	$b^8(\cdot)$	$b^9(\cdot)$	$b^{10}(\cdot)$	$b^{11}(\cdot)$
S_1	100	0	0	0	0	0	0	0	0	0	0
S_2	0	100	0	0	0	0	0	0	0	0	0
S_3	0	0	0	94.597	5.403	0	0	0	0	0	0
S_4	0	0	96.048	3.952	0	0	0	0	0	0	0
S_5	0	0	0	0	20.119	40.988	32.924	5.969	0	0	0
S_6	0	0	0	0	0	13.492	37.716	38.571	10.221	0	0
S_7	0	0	0	0	0	0	0	0	0	0.004	99.996
S_8	0	0	0	0	0	0	0	0	0.01	99.99	0
S_9	0	0	2.286	0.29	24.546	11.462	8.91	20.956	31.549	0.001	0
S_{10}	0	0	0.004	0.025	28.944	18.962	13.009	17.998	21.058	0	0
S_{11}	0	0	1.662	1.136	20.988	15.096	7.441	16.506	37.162	0.005	0.004

(b) Pairwise winning indices (in percentage)

	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}
S_1	0	100	100	100	100	100	100	100	100	100	100
S_2	0	0	100	100	100	100	100	100	100	100	100
S_3	0	0	0	0	100	100	100	100	97.424	99.971	97.202
S_4	0	0	100	0	100	100	100	100	97.714	99.996	98.338
S_5	0	0	0	0	0	100	100	100	61.66	52.807	60.79
S_6	0	0	0	0	0	0	100	100	55.903	43.035	55.541
S_7	0	0	0	0	0	0	0	0	0	0	0.004
S_8	0	0	0	0	0	0	100	0	0.001	0	0.009
S_9	0	0	2.576	2.286	38.34	44.097	100	99.999	0	44.905	54.308
S_{10}	0	0	0.029	0.004	47.193	56.965	100	100	55.095	0	57.541
S_{11}	0	0	2.798	1.662	39.21	44.459	99.996	99.991	45.692	42.459	0

9^{th} ($b^9(S_9) = 31.549\%$), the 5^{th} ($b^5(S_{10}) = 28.944\%$) and the 9^{th} again ($b^9(S_{11}) = 37.162\%$). Anyway, these three students are more or less similar among them if one looks at Table 3.4(b). Indeed, on one hand, S_{10} is preferred to S_9 and S_{11} more than the viceversa but with frequencies very close to the 50% ($p(S_{10}, S_9) = 55.095\%$ and $p(S_{10}, S_{11}) = 57.541\%$); on the other hand, S_9 and S_{10} are preferred to S_{11} more frequently than the viceversa ($p(S_{11}, S_9) = 45.692\%$ and $p(S_{11}, S_{10}) = 42.459\%$).

Chapter 4

Concluding remarks

In this thesis we have mainly dealt with two important issues of MCDA, which are the interaction between criteria and the robustness concerns of the results.

In Chapter 1 we have seen a procedure which integrated two preference models, such as the PROMETHEE II and the Choquet integral, with the Stochastic Multicriteria Acceptability Analysis (SMAA) methodologies to compare the performances of different sailboats in sailing regattas accounting also for the different scales in which each criterion is expressed [4]. In particular, the application of the SMAA-Choquet method permits, on one hand, to compute a scale common to all the evaluations and, on the other hand, it permits to take into account the possible positive and negative interactions among the five considered criteria highlighted by the Decision Maker (DM).

Dealing with real world decision problems, we generally face a lot of features such as: hierarchical structure of criteria, interaction between criteria and uncertainty or imprecision in the evaluations of the alternatives. Moreover, we have also observed that PROMETHEEE methods are widely used in Multiple Criteria Decision Aiding [13] to deal with such type of problems.

In Chapter 2 we have showed an extension of the PROMETHEE methods presented in [7], where the hierarchical bipolar PROMETHEE methods combine the Multiple Criteria Hierarchy Process (MCHP), the PROMETHEE methods and the bipolar Choquet integral together with the Robust Ordinal Regression (ROR) and the SMAA.

In particular, we showed an extension of the bipolar Choquet integral to problems presenting criteria organized in an hierarchy by considering indirect preference information provided by the DM. ROR and SMAA methodologies provide robust recommendations with respect to the problem at hand by exploring the whole set of parameters compatible with the provided preferences. All this decision procedure is supported by the graphical visualization supplied by an extension of the GAIA plane technique that permits to take into account interaction and antagonistic effects of criteria organized in a hierarchy as well as the plurality of feasible parameters compatible with the provided preference information.

We would like to underline that the proposed method is not a simple aggregation of methods already existing. The hierarchical bipolar PROMETHEE method is the result of a harmonization of all of them involving a fine adaptation. Moreover we believe that the points we considered are crucially relevant in many real world decision problems that can benefit from the methodology we have proposed.

Parsimony of models in multiple criteria decision analysis has to be tempered by the possibility to permit a rich interaction with the DM and the consideration of robustness concerns. In this perspective, in Chapter 3, we showed a decision model that has the following interesting properties:

- interaction between criteria is taken into consideration,
- importance and interaction of criteria can change from one level to the other of the evaluations taken by considered alternatives on criteria at hand,
- the parameters of the model are obtained by starting from some preference information supplied by the decision makers in terms that are relatively easily understandable by them such as
  - pairwise comparisons of some alternatives,
  - comparisons of criteria in terms of their importance,
  - nature of interaction between criteria (synergy or redundancy) and comparisons related to intensity of the interaction between pairs of criteria;
- consideration of the plurality of instances of the decision model compatible with the preferences provided by the DM so that, not only necessary preferences (valid for all compatible models) are distinguished from possible preferences (valid for at least one compatible model), but even a probabilistic evaluation of a preference of an alternative over another or the ranking position of the alternatives is given to the DM.

All this is obtained by coupling:

- the level dependent Choquet integral, which permits to take into account importance and interaction of criteria that can change from one level to the other of the evaluations of criteria,

- the Robust Ordinal Regression and the Stochastic Multiobjective Acceptability Analysis methodologies, which permit to take into account the plurality of instances of the preference model compatible with the preference information supplied by the decision makers.

We believe that the good properties of the methodology we have proposed can be very useful in many real life complex decision problems related to many domains of interest.

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