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COMPLETING SIMPLE PARTIAL k-LATIN SQUARES

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ABSTRACT. We study the completion problem for simple k-Latin rectangles, which are a special case of the generalized latin rectangles for which embedding theorems are given by Andersen and Hilton (1980) in "Generalized Latin rectangles II: Embedding", *Discrete Mathematics* **31**(3). Here an alternative proof of those theorems are given for k-Latin rectangles in the "simple" case. More precisely, generalizing two classic results on the completability of partial Latin squares, we prove the necessary and sufficient conditions for a completion of a simple $m \times n$ k-Latin rectangle to a simple k-Latin square of order n and we show that if $m \leq n/2$, any simple partial k-Latin square P of order m embeds in a simple k-Latin square L of order n.

1. Introduction

An $m \times n$ *k-Latin rectangle* is an $m \times n$ array of multisets, each of size k, such that each element of $N(n) = \{1, 2, \ldots, n\}$ occurs k times in each row and at most k times in each column. If there are no repeated elements within a cell we say that the k-Latin rectangle is *simple*. If $m = n$ then we obtain a *k-Latin square* of order n. A 1-Latin square is simply a Latin square. Observe that $k \leq n$ is necessary for the existence of a simple k-Latin square of order n. It is not hard to show that a simple k-Latin square of order n exists whenever $k \leq n$; see Lemma 1.2 of Cavenagh *et al.* (2011).

The above combinatorial structures are examples of a more general structure introduced by Andersen and Hilton (1980). Here a *partial (p, q, x)-Latin rectangle* is a a rectangular matrix in which each cell is filled with at most x symbols, each symbol occurs at most p times in each row and each symbol occurs at most q times in each column. Thus the above structures are particular types of (k, k, k)-Latin rectangles.

Using Hall's theorem, it was shown by Cavenagh *et al.* (2011) that any $m \times n$ k-Latin rectangle embeds in a k-Latin square of order n. Unbeknownst to the authors this is implied by Theorem 4.1.1 of Andersen and Hilton (1980).

Theorem 1.1. (Andersen and Hilton 1980; Cavenagh *et al.* 2011) *Let R be an $m \times n$ k-Latin rectangle of order n. Then R completes to a k-Latin square of order n.*

However, the above theorem does not hold in general if we require a completion to a simple k-Latin square, even when $k = 2$. For $n = 3$, we exhibit an example of a 2×3 simple 2-Latin rectangle which clearly does not complete to a simple 2-Latin square.

$1,2$	$2,3$	$3,1$
$1,2$	$2,3$	$3,1$

A partial *k-Latin square* of order n is an $n \times n$ array of multisets, each of size between 0 and k, such that each element of $N(n)$ occurs at most k times in each row and at most k times in each column. Again, if there are no repeated elements within a cell we say that the partial k-Latin square is *simple*.

In this paper we consider the completability of certain partial k-Latin squares to k-Latin squares, generalizing two classic results on the completability of partial Latin squares. We show that an $m \times n$ k-Latin rectangle R completes to a simple k-Latin square of order n whenever R is simple and $m \leq n - k$ (Theorem **2.4**), generalizing Evan's theorem on Latin rectangles (1960). Moreover we give necessary and sufficient conditions for a completion if $m > n - k$. In Theorem **2.9** we generalize the well-known result (a special case of Ryser's condition, 1951) that any partial Latin square of order m embeds in a Latin square of order n whenever $n \geq 2m$. It is easy to show that this is best possible as any k-Latin square of order m cannot embed in a k-Latin square of order n if $n < 2m$. Finally, some examples are given to explain the hypotheses of the results.

Theorems **2.4** and **2.9** are each implied by Theorem 4.1.2 of Andersen and Hilton (1980) which gives a much more general result, however the proofs in our paper were found independently.

2. Main Results

To obtain our main results we will apply a graph-theoretic result which will be given in terms of factors and factorizations and proved for the sake of completeness and self-sufficiency, although it is a particular case of a more general result on balanced edge-colorings of bipartite graphs (see De Werra 1970).

In the following any graph is allowed to have parallel edges unless it is specified to be simple.

Let f be some function from the vertices of a graph G to the non-negative integers. A subgraph H of G is said to be an f-factor if the degree of each vertex v in H is equal to $f(v)$. In the case where $f(v) = k$ for each vertex v where k is a positive integer we say that H is a k-factor. For any vertex v in a given graph G, $\delta_H(v)$ is defined to be the number of edges incident with v and a vertex from H, where H is any subset of the vertex set of G.

We make use of the following results, the first of which is a well-known corollary of Hall's theorem. The second is due to Ore (1957) and Folkman and Fulkerson (1970); see also Akiyama and Kano (2011) for a proof.

Theorem 2.1. *Let G be a regular bipartite graph. Then G has a* 1*-factorization, i.e. a partition of the edges into* 1*-factors.*

Theorem 2.2. (Ore 1957; Folkman and Fulkerson 1970) **The f-Factor Theorem for Bipartite Graphs:** *Let G be a bipartite graph with bipartition (U,V). Then G has an f-factor if and only if $\sum_{x\in U} f(x) = \sum_{x\in V} f(x)$ and*

$$\sum_{x\in S} f(x) + \sum_{x\in T} (\delta_{U\setminus S}(x) - f(x)) \geq 0$$

for all subsets $S \subset U$ and $T \subset V$.

A *simple k-factorization* of a graph G is a set of simple k-factors of G which include each edge of G exactly once. The following graph theoretic theorem allows us to prove the main results in this paper.

Theorem 2.3. *Let B be an mk-regular bipartite graph with maximum edge multiplicity at most m. Then B has a simple k-factorization.*

Proof. The result is trivially true for $m = 1$, so we assume that $m \geq 2$. Let E be the (possibly empty) set of edges of multiplicity m (here E is a set rather than a multiset, i.e. only one copy of each such edge is included). It suffices to show that B has a simple k-factor F which includes each edge from E. The removal of F then results in an $(m-1)k$-regular bipartite graph with maximum edge multiplicity at most $m-1$ and the result follows inductively.

Let B_1 be the simple graph obtained from B by replacing each edge of multiplicity at least 2 with a single edge. We need to show that the graph B_1 has a k-factor which includes all the edges from E.

Let B_2 be the graph obtained by removing the edges of E from B_1. For each vertex v in B_2, define $f(v) = k - d_v$, where d_v is the number of edges in E which are incident with v. We are done if we can show that B_2 has an f-factor.

Let B (and thus B_1 and B_2) have bipartition (U,V). It is clear that $\sum_{x\in U} f(x) = \sum_{x\in V} f(x)$, since each edge from E is incident with both U and V. Let $S \subset U$ and $T \subset V$. Let B_3 be the bipartite graph obtained by removing from B all edges of multiplicity m. Let e_1 (respectively, a_1) be the number of edges from E (respectively, B_3) between S and T. Let e_2 (respectively, a_2) be the number of edges from E (respectively, B_3) between S and $V \setminus T$. Let e_3 (respectively, a_3) be the number of edges from E (respectively, B_3) between T and $U \setminus S$.

Observe the following, where f and δ are defined with respect to the graph B_2.

$$\begin{aligned} |S|km &= a_1 + e_1 m + a_2 + e_2 m; \\ |T|km &= a_1 + e_1 m + a_3 + e_3 m; \\ \sum_{x\in S} f(x) &= k|S| - e_1 - e_2; \\ \sum_{x\in T} f(x) &= k|T| - e_1 - e_3; \\ \sum_{x\in T} \delta_{U\setminus S}(x) &\geq a_3/(m-1). \end{aligned}$$

Thus:

$$\sum_{x\in S} f(x) + \sum_{x\in T} (\delta_{U\setminus S}(x) - f(x)) \geq (a_2 - a_3)/m + a_3/(m-1) \geq 0.$$

The result follows by Theorem **2.2**. □

Theorem 2.4. *Let R be an $m \times n$ simple k-Latin rectangle of order n where $m \leq n-k$. Then R completes to a simple k-Latin square L of order n.*

Proof. Label the columns of R with $N(n)$. Construct a bipartite graph B with bipartition $(U = \{u_1, u_2, \ldots, u_n\}, V = \{v_1, v_2, \ldots, v_n\})$ where there are λ edges from u_i to v_j if and only if symbol j appears $k-\lambda$ times in column i of R. Observe that B is $k(n-m)$-regular. Thus by Theorem **2.1**, B has a 1-factorization.

Choose any k^2 1-factors from B to create a k^2-regular graph B'. Since B' is a subgraph of B, the maximum multiplicity of an edge in B' is at most k. Thus by Theorem **2.3**, the edges of B' may be partitioned into k simple k-factors $F_1, F_2, \ldots, F_k$.

Let $n-m = qk+r$, where $q \geq 0$ and $1 \leq r \leq k$. We extend R to an $(m+r) \times n$ simple k-Latin rectangle R' by placing symbol e in cell $(m+i, j)$ of R' whenever $\{u_j, v_e\}$ is an edge in the k-factor F_i, for each $i \in N(r)$.

Observe that either R' is a simple k-Latin square of order n (and we are done) or R' is a simple $m' \times n$ k-Latin rectangle such that $n-m'$ is a multiple of k. The result follows by recursion, adding k rows at a time. □

Theorem 2.5. *Let R be an $m \times n$ simple k-Latin rectangle of order n where $m > n-k$. Then R completes to a simple k-Latin square L of order n if and only if each symbol from $N(n)$ appears at least $k-n+m$ times in each column of R.*

Proof. If there exists a symbol e which appears less than $k-n+m$ times in a particular column, e must appear more than $n-m$ times within the final $n-m$ rows, making L non-simple. To see sufficiency, create a bipartite graph B as in the previous theorem. Observe that B is $k(n-m)$-regular with maximum edge multiplicity at most $n-m$. Thus the edges of B may be partitioned into $n-m$ simple k-factors by Theorem **2.3**. The extension to L then follows as in the previous theorem. □

In the following theorems, the operation $\oplus_\ell$ denotes arithmetic modulo ℓ with residues from $N(l)$ (performed on an element or a set). Formally, for any positive integer ℓ and subset $S \subseteq N(\ell)$, define the set $S \oplus_\ell 1$ by $\{s+1 \mid s \in S\}$ where $\ell+1$ is replaced by 1. We then recursively define $S \oplus_\ell a := (S \oplus_\ell (a-1)) \oplus_\ell 1$ for each $a > 1$.

Theorem 2.6. *Let $0 < n-k < m < n$. Then there exists an $m \times n$ simple k-Latin rectangle R of order n which does not complete to a simple k-Latin square L of order n.*

Proof. In order to construct the desired $m \times n$ k-Latin rectangle R, we distinguish two cases (denoting the contents of cell (i, j) of R by $R_{i,j}$):

(1) if $m \leq k < n$, then for each $1 \leq i \leq m$, let cell $R_{i,1} = N(k)$ and let cell $R_{i,j} = R_{i,1} \oplus_n (j-1)$ for each $2 \leq j \leq n$;
(2) if $m > k$, then for each $1 \leq i \leq m$, let $R_{i,1} = N(k) \oplus_m (i-1)$ and $R_{i,j} = R_{i,1} \oplus_n (j-1)$ for each $2 \leq j \leq n$.

Consider the first column of R. In Case 1, each element $1, 2, \ldots, k$ appears m times, while in Case 2 each element $1, 2, \ldots, m$ appears k times. In both cases there exists at least one element that must appear k times in the remaining $n-m$ rows of any k-Latin square of order n that completes our $m \times n$ simple k-Latin rectangle. Since $k > n-m$, a such k-Latin square of order n cannot be simple. □

We illustrate the previous theorem with the following examples.

Example 2.7 $n = 8$, $k = 5$, $m = 4$

12345	23456	34567	45678	56781	67812	78123	81234
23451	34562	45673	56784	67815	78126	81237	12348
34512	45623	56734	67845	78156	81267	12378	23481
45123	56234	67345	78456	81567	12678	23781	34812
66?							
6							
6							
6							

Example 2.8 $n = 8$, $k = 4$, $m = 5$

1234	2345	3456	4567	5678	6781	7812	8123
2345	3456	4567	5678	6781	7812	8123	1234
3451	4562	5673	6784	7815	8126	1237	2348
4512	5623	6734	7845	8156	1267	2378	3481
5123	6234	7345	8456	1567	2678	3781	4812
66?							
6							
6							

The case when L is not required to be simple in the theorem below is proved in Theorem 3.1 of Cavenagh *et al.* (2011). In the following we must be careful with multiset operations. Let the multiplicity of an element s in a multiset A be denoted by $\mu_A(s)$. Then, $\mu_{A\cup B}(s)$ =max$\{\mu_A(s), \mu_B(s)\}$, $\mu_{A\setminus B}(s)$ =max$\{0, \mu_A(s) - \mu_B(s)\}$ and $\mu_{A\uplus B}(s) = \mu_A(s) + \mu_B(s)$.

Theorem 2.9. *Let $m \leq n/2$. Then any simple partial k-Latin square P of order m embeds in a simple k-Latin square L of order n.*

Proof. Since P is simple, $k \leq m$. By definition the symbols in P belong to the set $N(m)$; for notational convenience we complete P to a simple k-Latin square L based on symbol set

$$S := N(m) \cup \{1', 2' \ldots, m'\} \cup \{1'', 2'', \ldots, (n-2m)''\}.$$

Let S^* be the multiset of size nk consisting of k copies of each element from S.

Let $P_{i,j}$ be the set of symbols in cell (i, j) of P. We first fill any empty or partially-filled in cells of P with elements from $\{1', 2', \ldots, m'\}$ by adding the elements

$$(i \oplus_m (j+1))', (i \oplus_m (j+2))', \ldots, (i \oplus_m (j+k-|P_{i,j}|))'$$

to cell (i, j) of P, where $i, j \in N(m)$. We also add $n-m$ empty rows and $n-m$ empty columns; the resultant structure P' is a simple partial k-Latin square with cell (i, j) filled if and only if $i \leq m$ and $j \leq m$.

Our aim is to fill the rest of the first m rows of P' to create an $m \times n$ simple k-Latin rectangle R; the result then follows by Theorem **2.4**. For each $i \in N(m)$, let $X(i)$ be the multiset $\uplus_{j=1}^{m} P'_{i,j}$. Define $A(i)$ to be the multiset containing $k - x_i(e)$ copies of each symbol $e \in S$, where $x_i(e)$ is the number of occurrences of e in $X(i)$.

We claim there exists a $k(n-m)$-regular bipartite graph B with bipartition

$$\{A(1),\ldots,A(m),C(1),\ldots,C(n-m)\}$$

(for some sets $C(1),C(2),\ldots,C(n-m)$) and S, where there are λ edges between a multiset and an element e if and only if e occurs λ times within that multiset. As part of our claim we also require that the maximum multiplicity of an edge in B is at most k.

To prove our claim we must specify the sets $C(1),C(2),\ldots,C(n-m)$. We construct these inductively. Let $C_0(i) = X(i)$ for each $i \in N(m)$. Our inductive hypothesis is that at stage $0 \leq j \leq n-2m$:

- For each $i \in N(m)$, $C_j(i)$ is a multiset of size $(m+j)k$ containing no element more than k times;
- The multisets $C(m+1),\ldots,C(m+j)$ each have size $(n-m)k$;
- For each $i \in N(j)$, $S^* = C(m+i) \uplus \biguplus_{j \in N(m)} (C_j(i) \setminus C_j(i-1))$.

Observe the hypothesis is true for $j = 0$. Assuming it holds for some fixed $j \geq 0$, observe that for each $i \in N(m)$, the number of elements in $C_j(i)$ which occur exactly k times is at most $m+j$. Let $D(1)$ be any subset of size k from the multiset $S^* \setminus C_j(1)$. For $1 < i \leq k$, we recursively define $D(i)$ to be any subset of size k from the multiset

$$S^* \setminus (C_j(k) \uplus \biguplus_{j=1}^{i-1} D(j)).$$

Since

$$|S^* \setminus (C_j(k) \uplus \biguplus_{j=1}^{i-1} D(j))| \geq nk - (m+j)k - (m-1)k = (n-2m-j+1)k \geq k,$$

such a multiset always exists. We emphasize that each $D(i)$ may be a multiset. We then define $C_{j+1}(i) = C_j(i) \uplus D(i)$ for each $i \in N(m)$ and we let

$$C(m+j+1) := S^* \setminus (\biguplus_{j=1}^{k} D(j)).$$

The induction then follows. Letting $C(i) = C_k(i)$ for each $i \in N(m)$, the claim above also follows, i.e. such a bipartite graph B exists.

Thus by Theorem **2.1** B has a 1-factorization; choose any k^2 of these 1-factors to form a bipartite graph B'. Since B' is a subgraph of B, the maximum multiplicity of an edge in B' is at most k. Thus by Theorem **2.3**, the edges of B' may be partitioned into k simple k-factors $F_1, F_2, \ldots, F_k$.

Let $n - m = qk + r$, where $q \geq 0$ and $1 \leq r \leq k$. We extend P' to an $m \times (m+r)$ simple partial k-Latin rectangle P'' by placing symbol e in cell $(i, m+j)$ of P'' whenever $\{A(i), e\}$ is an edge in the k-factor F_j, for each $i \in N(r)$.

Observe that either P'' is an $m \times n$ simple k-Latin rectangle (and we are done) or P'' is a simple $m \times m'$ k-Latin rectangle such that $n - m'$ is a multiple of k. The result follows by recursion, adding k columns at a time. □

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