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Posner's Second Theorem and some Related Annihilating Conditions on Lie Ideals

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Abstract. Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, L a non-central Lie ideal of R, F and G two non-zero generalized derivations of R. If $[F(u), u]G(u) = 0$ for all $u \in L$, then one of the following holds: (a) there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$; (b) $R \subseteq M_2(\mathcal{F})$, the ring of 2×2 matrices over a field $\mathcal{F}$, and there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$, for all $x \in R$.

1. Introduction

Let R be a prime ring of characteristic different from 2. Throughout this paper $Z(R)$ always denotes the center of R, U the Utumi quotient ring of R and $C = Z(U)$, the center of U (C is usually called the extended centroid of R). Many results in literature indicate that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. A well known result of Posner [31] states that if d is a derivation of R such that $[d(x), x] \in Z(R)$, for any $x \in R$, then either $d = 0$ or R is commutative. Later in [24] Lanski proves that if d is a nonzero derivation of R so that $[d(x), x] \in Z(R)$ for all $x \in L$, a non-central Lie ideal of R, then $char(R) = 2$ and $R \subseteq M_2(\mathcal{F})$, the ring of 2×2 matrices over a field $\mathcal{F}$. More recently Chebotar, Lee and Wong [8] generalize the previous results in case the characteristic of R is different from 2 or 3. More precisely they prove that if L is a non central Lie ideal of R, then the additive subgroup S generated by $\{[d(x), x] : x \in L\}$ contains a non central Lie ideal W of R. In particular S is not contained in $Z(R)$, unless $d = 0$. Moreover, since both the left (right) annihilator $\mathcal{A}_W$ and the centralizer $\mathcal{C}_W$ of a Lie ideal W of a prime ring are trivial, that is $\mathcal{A}_W = (0)$ and $\mathcal{C}_W = Z(R)$, then both the left (right) annihilator and centralizer of S are trivial and these facts in a prime ring are natural tests which evidence that the set $\{[d(x), x] : x \in L\}$ is rather large in R.

In [11] De Filippis considers the problems concerning the annihilator of the commutators with derivations on Lie ideals and he shows that the left annihilator of the set $\{[d(x), x] : x \in L\}$ in R is zero if L is a non-central Lie ideal of R. Following this study Shiue [34] prove that if d is a non-zero derivation of R, L a non-central Lie ideal of R, $a \in R$ and $k \geq 1$ a fixed integer such that $a[d(u), u]_k = 0$ for all $u \in L$ then either

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$a = 0$ or R satisfies the standart identity s_4 and $char(R) = 2$.

This paper follows the line of investigation of the previous ones, by replacing the derivation d with some additive maps which generalize the concept of usual derivation on R.

An additive map $G : R \to R$ is called a generalized derivation of R if there exists a derivation d of R such that $G(xy) = G(x)y + xd(y)$, for all $x, y \in R$. The simplest example of generalized derivation is a map of the form $g(x) = ax + xb$, for some $a, b \in R$ and for all $x \in R$: such generalized derivations are called inner. Generalized inner derivations have been primarily studied on operator algebras. Therefore any investigation from the algebraic point of view might be interesting (see for example [21], [28], [29]). Here we will consider some related problems concerning identities with generalized derivations in prime rings. More precisely, let F be a generalized derivation of R and define the following subset of R:

$$T = \{[F(x), x] : x \in L\},$$

where L is a non-central Lie ideal of R.
A first approach to the study of T is contained in [12], [13] and [33]. More precisely, the following facts hold:

- Let $T \neq (0)$ and $a \in R$ be such that $aT = (0)$ (respectively $Ta = (0)$), then $a = 0$;
- Let $T \neq (0)$ and $a \in R$ be such that $[a, T] = (0)$, then $a \in Z(R)$,

that is both the annihilator and the centralizer of T are trivial, unless when $T = (0)$. Thus T is rather large in R.

It seems natural to investigate what happens when the annihilating element is not fixed, but it is depending on the choice of the element $x \in L$. In other words, what about the case when, for all $x \in L$ there exists $a_x \in R$ such that $[F(x), x]a_x = 0$ (or similarly $a_x[F(x), x] = 0$).

More recently in [15] a first answer to this question is given:

Theorem 1. *Let R be a prime ring of characteristic different from 2, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, F be a non-zero generalized derivations of R. Suppose that $[F(u), u]F(u) = 0$, for all $u \in L$, then one of the following holds:*

(a) *there exists $\alpha \in C$ such that $F(x) = \alpha x$, for all $x \in R$;*
(b) *$R \subseteq M_2(\mathcal{F})$ for some field $\mathcal{F}$ and there exist $a \in U$ and $\alpha \in C$, such that $F(x) = ax + xa + \alpha x$, for all $x \in R$.*

In this article we would like to give an answer to a more general question, considering such annihilating condition, when two different generalized derivations act on the evaluations of a non-central Lie ideal of R. More precisely we will prove the following:

Theorem 2. *Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, L a non-central Lie ideal of R, F and G two non-zero generalized derivations of R. If $[F(u), u]G(u) = 0$ for all $u \in L$, then one of the following holds:*

(a) *there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$;*
(b) *$R \subseteq M_2(\mathcal{F})$, the ring of 2×2 matrices over a field $\mathcal{F}$, and there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax+xa+\lambda x$, for all $x \in R$.*

2. The case of inner generalized derivations

In this section we will consider the generalized derivations $F(x) = ax + xb$ and $G(x) = cx + xq$, induced by suitable fixed elements $a, b, c, q \in R$.

We premit the following result (for the proof see Proposition 1 in [14]):

Lemma 2.1. *Let $\mathcal{F}$ be a field of $char(\mathcal{F}) \neq 2$, $R = M_t(\mathcal{F})$ the matrix ring over $\mathcal{F}$ and $t \geq 3$. Denote by e_{ij} the usual matrix unit, with 1 in the (i, j)-entry and zero elsewhere. Let a, b be elements of R, with $a = \sum_{r,s=1}^{t} a_{rs}e_{rs}$ and $b = \sum_{r,s=1}^{t} b_{rs}e_{rs}$, for $a_{rs}, b_{rs} \in \mathcal{F}$ and suppose that $a_{ij}b_{ij} = 0$ for all $i \neq j$. Assume that, for any inner automorphism φ of R, the following hold:*

$$\varphi(a) = \sum_{r,s=1}^{t} a'_{rs}e_{rs}, \quad \varphi(b) = \sum_{r,s=1}^{t} b'_{rs}e_{rs} \quad \text{and} \quad a'_{ij}b'_{ij} = 0.$$

Then either $a \in Z(R)$ or $b \in Z(R)$.

We begin with:

Lemma 2.2. *Let $\mathcal{F}$ be a field of $char(\mathcal{F}) \neq 2$, $R = M_t(\mathcal{F})$ the algebra of $t \times t$ matrices over $\mathcal{F}$ with $t \geq 3$, $Z(R)$ the center of R, $L = [R, R]$, a, b, c, q elements of R. Assume that $c \in Z(R)$. If $(au^2 + u(b - a)u - u^2b)(cu + uq) = 0$ for all $u \in L$, then one of the following holds:*

(a) $c = -q \in Z(R)$;
(b) $a, b \in Z(R)$.

Proof. Since $c \in Z(R)$, by the assumption we have that

$$(au^2 + u(b - a)u - u^2b)u(c + q) = 0 \tag{2.1}$$

for all $u \in [R, R]$. Here we denote $p = c + q = \sum p_{rs}e_{rs}$, $a = \sum a_{rs}e_{rs}$ and $b = \sum b_{rs}e_{rs}$, for $a_{rs}, b_{rs}, p_{rs} \in \mathcal{F}$. Let i, j, k three different indices and choose $u = e_{ii} - e_{jj} + e_{ik} \in [R, R]$ in (2.1). Left multiplying (2.1) by e_{kk} we get

$$e_{kk}a(e_{ii} + e_{ik} - e_{jj})p = 0. \tag{2.2}$$

On the other hand, for $u = e_{ii} - e_{jj} - e_{ik} \in [R, R]$ in (2.1) and left multiplying (2.1) by e_{kk} we also get

$$e_{kk}a(e_{ii} - e_{ik} - e_{jj})p = 0. \tag{2.3}$$

Comparing (2.2) with (2.3) and right and since $char(\mathcal{F}) \neq 2$, it follows $a_{ki}p_{ki} = 0$. Moreover, for any inner automorphism φ of R, the elements $\varphi(a)$ and $\varphi(p)$ satisfy the same algebraic condition as a and p. Therefore, by Lemma 2.1, one has that either $a \in Z(R)$ or $p \in Z(R)$.
In case $0 \neq p \in Z(R)$, the relation (2.1) reduces to $(au^2 + u(b - a)u - u^2b)u = 0$, for all $u \in [R, R]$. Thus by [6, Theorem 4.7] we have that $a, b \in Z(R)$, as required.
Assume now that $a \in Z(R)$. In this case (2.1) reduces to

$$(ubu - u^2b)u(c + q) = 0 \tag{2.4}$$

for all $u \in [R, R]$. For $u = e_{ii} - e_{jj} + e_{ik} \in [R, R]$ in (2.4) and left multiplying (2.4) by e_{jj}, since $char(\mathcal{F}) \neq 2$, we get

$$(-e_{jj}be_{ii} - e_{jj}be_{ik})p = 0. \tag{2.5}$$

Analogously, for $u = e_{ii} - e_{jj} - e_{ik} \in [R, R]$ in (2.4) and left multiplying (2.1) by e_{jj} we also get

$$-e_{jj}b(e_{ii} - e_{ik})p = 0. \tag{2.6}$$

Hence, by comparing (2.5) with (2.6) and using $char(\mathcal{F}) \neq 2$ it follows

$$b_{ji}p_{ir} = 0, \quad \forall r, \quad \forall i \neq j. \tag{2.7}$$

Let χ be the inner automorphism of R defined as $\chi(x) = (1 + e_{ij})x(1 - e_{ij}) = x + e_{ij}x - xe_{ij} - e_{ij}xe_{ij}$, and denote $\chi(b) = \sum b'_{rs}e_{rs}$, $\chi(p) = \sum p'_{rs}e_{rs}$, for $b'_{rs}, p'_{rs} \in \mathcal{F}$. Since $\chi(b)$ and $\chi(p)$ satisfy the same algebraic condition as b and p, then by (2.7) one has $b'_{ji}p'_{ir} = 0$, for all $r \in \{1, \ldots, t\}$ and for any $i \neq j$. By computations it follows that $b_{ji}p_{ji} = 0$, for all $i \neq j$. Hence, by Lemma 2.1, either $b \in Z(R)$ or $p \in Z(R)$. If $b \in Z(R)$, we are done. On the other hand, in case $p \in Z(R)$, as above the conclusion follows from [6, Theorem 4.7]. □

Lemma 2.3. *Let $\mathcal{F}$ be a field of $char(\mathcal{F}) \neq 2$, $R = M_t(\mathcal{F})$ the algebra of $t \times t$ matrices over $\mathcal{F}$ with $t \geq 3$, $Z(R)$ the center of R, $L = [R, R]$, a, b, c, q elements of R. Assume that $b - a \in Z(R)$. If $(au^2 + u(b - a)u - u^2 b)(cu + uq) = 0$ for all $u \in L$, then one of the following holds:*

(a) $a, b \in Z(R)$;
(b) $c = -q \in Z(R)$.

Proof. Since $b - a \in Z(R)$, by the assumption we have that

$$[a, u^2](cu + uq) = 0 \tag{2.8}$$

for all $u \in [R, R]$. Let $i \neq j$, choose $u = e_{ij} - e_{ji} \in [R, R]$ in (2.8) and multiply both on the left and on the right (2.8) by e_{kk}, for any $k \neq i, j$. One has

$$e_{kk}(-ae_{ij} + ae_{ji})qe_{kk} = 0. \tag{2.9}$$

In the same way, for $u = e_{ij} + e_{ji} \in [R, R]$ in (2.8) and multiplying both on the left and on the right (2.8) by e_{kk}, for any $k \neq i, j$, it follows

$$e_{kk}(ae_{ij} + ae_{ji})qe_{kk} = 0. \tag{2.10}$$

Comparing (2.9) with (2.10), since $char(\mathcal{F}) \neq 2$, we get

$$a_{kj}q_{ik} = 0, \quad \forall i \neq j, \quad \forall k \neq i, j. \tag{2.11}$$

Let φ' and φ'' be the inner automorphisms of R defined as

$$\varphi'(x) = (1 + e_{ik})x(1 - e_{ik}) = x + e_{ik}x - xe_{ik} - e_{ik}xe_{ik}$$

$$\varphi''(x) = (1 - e_{ik})x(1 + e_{ik}) = x - e_{ik}x + xe_{ik} - e_{ik}xe_{ik}$$

and denote $\varphi'(a) = \sum a'_{rs}e_{rs}$, $\varphi''(a) = \sum a''_{rs}e_{rs}$, $\varphi'(q) = \sum q'_{rs}e_{rs}$, $\varphi''(q) = \sum q''_{rs}e_{rs}$, for $a'_{rs}, a''_{rs}, q'_{rs}, q''_{rs} \in \mathcal{F}$. Since $\varphi'(a), \varphi''(a), \varphi'(q), \varphi''(q)$, satisfy the same algebraic condition as a and q, then by (2.11) one has $a'_{kj}q'_{ik} = 0$ and $a''_{kj}q''_{ik} = 0$, for all $i \neq j$ and for any $k \neq i, j$. By computations, the following hold simultaneously:

$$a_{kj}(q_{kk} - q_{ii} - q_{ki}) = 0 \tag{2.12}$$

$$a_{kj}(-q_{kk} + q_{ii} - q_{ki}) = 0. \tag{2.13}$$

Therefore, by comparing (2.12) with (2.13) and since one has

$$a_{kj}q_{ki} = 0, \quad \forall i \neq j, \quad \forall k \neq i, j. \tag{2.14}$$

Let now χ be the inner automorphism of R defined as

$$\chi(x) = (1 + e_{ji})x(1 - e_{ji}) = x + e_{ji}x - xe_{ji} - e_{ji}xe_{ji}$$

and denote $\chi(a) = \sum a'''_{rs}e_{rs}$, $\chi(q) = \sum q'''_{rs}e_{rs}$, for $a'''_{rs}, q'''_{rs} \in \mathcal{F}$. As above, $\chi(a)$ and $\chi(q)$ must satisfy relation (2.14), that is $a'''_{kj}q'''_{ki} = 0$, for all $i \neq j$ and $k \neq i, j$. It is easy to see that this implies $a_{kj}q_{kj} = 0$, for all $k \neq j$. Hence, by Lemma 2.1, either $a \in Z(R)$ or $q \in Z(R)$. If $a \in Z(R)$ then we are done, thus we assume here that $q \in Z(R)$. Let $c + q = p = \sum p_{rs}e_{rs}$, with $p_{rs} \in \mathcal{F}$. Then (2.8) reduces to

$$[a, u^2]pu = 0 \tag{2.15}$$

for all $u \in [R, R]$. Again we subsitute in (2.15) u with $e_{ij} - e_{ji}$ and multiply on the left by e_{kk}, for any $i \neq j$ and $k \neq i, j$. Then

$$e_{kk}(-ae_{ii}pe_{ij} + ae_{ii}pe_{ji} - ae_{jj}pe_{ij} + ae_{jj}pe_{ji}) = 0. \tag{2.16}$$

Similarly, for $u = e_{ij} + e_{ji}$ in (2.15) and left multiplying by e_{kk}, we also have

$$e_{kk}(ae_{ii}pe_{ij} + ae_{ii}pe_{ji} + ae_{jj}pe_{ij} + ae_{jj}pe_{ji}) = 0. \tag{2.17}$$

Comparing (2.16) with (2.17) we get

$$a_{ki}p_{ij} + a_{kj}p_{jj} = 0. \tag{2.18}$$

We finally choose the following automorphism of R:

$$\chi'(x) = (1 + e_{jk})x(1 - e_{jk}) = x + e_{jk}x - xe_{jk} - e_{jk}xe_{jk}$$

and denote $\chi'(a) = \sum a^{iv}_{rs}e_{rs}$, $\chi'(p) = \sum p^{iv}_{rs}e_{rs}$, for $a^{iv}_{rs}, p^{iv}_{rs} \in \mathcal{F}$. Thus, by (2.18), $a^{iv}_{ki}p^{iv}_{ij} + a^{iv}_{kj}p^{iv}_{jj} = 0$ and by computations it follows that $a_{kj}p_{kj} = 0$, for any $k \neq j$.

Once again, by Lemma 2.1, either $a \in Z(R)$ or $p \in Z(R)$. The first case implies that $b \in\in Z(R)$. Let $p = c + q \in Z(R)$. Since $q \in Z(R)$ we have $c \in Z(R)$. Using these facts in (2.8) we have either $[a, u^2]u = 0$ or $p = 0$. By [6, Theorem 4.7] the first case implies that $a \in Z(R)$. If $p = 0$ we have $c = -q \in Z(R)$, as required. □

Lemma 2.4. *Let $\mathcal{F}$ be a field of char($\mathcal{F}$) $\neq 2$, $R = M_t(\mathcal{F})$ the algebra of $t \times t$ matrices over $\mathcal{F}$ with $t \geq 3$, $Z(R)$ the center of R, $L = [R, R]$, a, b, c, q elements of R. If*

$$(au^2 + u(b - a)u - u^2b)(cu + uq) = 0 \tag{2.19}$$

for all $u \in L$, then one of the following holds:

(a) $a, b \in Z(R)$;
(b) $c = -q \in Z(R)$.

Proof. Denote $b - a = w = \sum w_{rs}e_{rs}$ and $c = \sum c_{rs}e_{rs}$, with $w_{rs}, c_{rs} \in \mathcal{F}$. Let $u = e_{ij} \in [R, R]$ in (2.19). Therefore, by our assumption we get $e_{ij}(b - a)e_{ij}ce_{ij} = 0$, that is $w_{ji}c_{ji} = 0$, for all $i \neq j$. By Lemma 2.1, either $c \in Z(R)$ or $b - a \in Z(R)$ and the conclusion follows respectively by Lemmas 2.2 and 2.3. □

Lemma 2.5. *Let $\mathcal{F}$ be a field of char($\mathcal{F}$) $\neq 2$, $R = M_t(\mathcal{F})$ the algebra of $t \times t$ matrices over $\mathcal{F}$, $Z(R)$ the center of R, $L = [R, R]$, a, b, c, q elements of R. If*

$$(au^2 + u(b - a)u - u^2b)(cu + uq) = 0 \tag{2.20}$$

for all $u \in L$, then one of the following holds:

(a) $b - a \in Z(R)$;
(b) $c = -q \in Z(R)$.

Proof. Let $c = \sum c_{rs}e_{rs}$ and $q = \sum q_{rs}e_{rs}$ and denote $p = b - a = \sum p_{rs}e_{rs}$, with $c_{rs}, q_{rs}, p_{rs} \in \mathcal{F}$.
For $i \neq j$ and $u = e_{ij}$ in (2.20), we have $e_{ij}pe_{ij}ce_{ij} = 0$, that is

$$p_{ji}c_{ji} = 0, \quad \forall i \neq j, \quad i, j \in \{1, 2\}. \tag{2.21}$$

Let now $u = e_{12} + e_{21}$ in (2.20), then the $(1, 1)$-entry of the matrix (2.20) is

$$(p_{11} - p_{22})(c_{12} + q_{21}) + (p_{12} - p_{21})(c_{22} + q_{11}) = 0. \tag{2.22}$$

On the other hand, for $u = -e_{12} + e_{21}$ in (2.20), the $(1, 1)$-entry of the matrix (2.20) is

$$(p_{22} - p_{11})(c_{12} - q_{21}) + (-p_{12} - p_{21})(c_{22} + q_{11}) = 0. \tag{2.23}$$

Subtracting (2.23) from (2.22), since $char(R) \neq 2$, it follows that

$$(p_{11} - p_{22})c_{12} + p_{12}(c_{22} + q_{11}) = 0. \tag{2.24}$$

Similarly one may obtain the following (we omit the computations for brevity):

$$(p_{22} - p_{11})c_{21} + p_{21}(c_{11} + q_{22}) = 0. \tag{2.25}$$

Assume that c is not a diagonal matrix, without loss of generality we suppose $c_{12} \neq 0$. Thus, by (2.21), $p_{12} = 0$ and, by (2.24), $p_{11} = p_{22}$.
Let φ' and φ'' be the inner automorphisms of R defined as

$$\varphi'(x) = (1 + e_{12})x(1 - e_{12}) = x + e_{12}x - xe_{12} - e_{12}xe_{12} \tag{2.26}$$

and

$$\varphi''(x) = (1 - e_{12})x(1 + e_{12}) = x - e_{12}x + xe_{12} - e_{12}xe_{12} \tag{2.27}$$

and denote $\varphi'(p) = \sum p'_{rs}e_{rs}$, $\varphi''(p) = \sum p''_{rs}e_{rs}$, $\varphi'(c) = \sum c'_{rs}e_{rs}$, $\varphi''(c) = \sum c''_{rs}e_{rs}$, for $p'_{rs}, p''_{rs}, c'_{rs}, c''_{rs} \in \mathcal{F}$. Clearly $\varphi'(p), \varphi''(p), \varphi'(c), \varphi''(c)$ satisfy the same algebraic condition as p and c.
If $c'_{12} \neq 0$, then, by the previous argument, both $0 = p'_{12} = p_{22} - p_{11} - p_{21}$ and $0 = p'_{11} - p'_{22} = p_{11} + 2p_{21} - p_{22}$, which imply $p_{21} = 0$. In this case $p \in Z(R)$ and so we are done. Analogously, if $c''_{12} \neq 0$ then p is a central matrix.
Hence we assume that both $c'_{12} = 0$ and $c''_{12} = 0$, that is:

$$c_{12} + c_{22} - c_{11} - c_{21} = 0 \tag{2.28}$$

and

$$c_{12} - c_{22} + c_{11} - c_{21} = 0. \tag{2.29}$$

By the last two relations it follows that $c_{12} - c_{21} = 0$, i.e. $c_{21} = c_{12} \neq 0$. Therefore, by (2.21), $p_{21} = 0$ and $p \in Z(R)$.Hence the following holds: either c is diagonal or p is central.

In the sequel we assume that p is not a central matrix. Hence $c = c_{11}e_{11} + c_{22}e_{22}$, moreover $\varphi(p) \notin Z(R)$, for all $\varphi \in Aut(R)$. In particular, let $\varphi(x) = (1 + e_{21})x(1 - e_{21}) = x + e_{21}x - xe_{21} - e_{21}xe_{21}$. Since $\varphi(c)$ must be a diagonal matrix, then easy computations show that $c_{11} = c_{22}$, that is $c \in Z(R)$. Hence, if denote $w = c + q$, (2.20) reduces to:

$$[au + ub, u]uw = 0 \tag{2.30}$$

for all $u \in [R, R]$. Recall that any commutator $[X, Y] \in [M_2(\mathcal{F}), M_2(\mathcal{F})]$ is either invertible or nilpotent. In particular we choose $u = e_{11} - e_{22}$ in (2.30).

If $[a(e_{11} - e_{22}) + (e_{11} - e_{22})b, e_{11} - e_{22}]$ is invertible, then $(e_{11} - e_{22})w = 0$, which implies easily that $w = 0$, as required. Suppose now that $w \neq 0$ and $[a(e_{11} - e_{22}) + (e_{11} - e_{22})b, e_{11} - e_{22}]$ is not invertible. We prove that a contradiction follows. Since

$$M = [a(e_{11} - e_{22}) + (e_{11} - e_{22})b, e_{11} - e_{22}]^2 = 0$$

we firstly notice that the $(1, 1)$-entry of the matrix M is $4p_{12}p_{21} = 0$. This means that, if c is a central matrix then $p_{12}p_{21} = 0$. By using the same above automorphisms (2.26) and (2.27), we have that $\varphi'(c) \in Z(R)$ and $\varphi''(c) \in Z(R)$, so that

$$0 = p'_{12}p'_{21} = (p_{12} + p_{22} - p_{11} - p_{21})p_{21} = (p_{22} - p_{11} - p_{21})p_{21} \tag{2.31}$$

and

$$0 = p''_{12}p''_{21} = (p_{12} - p_{22} + p_{11} - p_{21})p_{21} = (-p_{22} + p_{11} - p_{21})p_{21}. \tag{2.32}$$

By comparing (2.31) and (2.32) we get $p_{21}^2 = 0$ that is $p_{21} = 0$.
Finally we consider the following automorphisms of R:

$$\chi'(x) = (1 + e_{21})x(1 - e_{21}) = x + e_{21}x - xe_{21} - e_{21}xe_{21}$$

$$\chi''(x) = (1 - e_{21})x(1 + e_{21}) = x - e_{21}x + xe_{21} - e_{21}xe_{21}$$

and denote $\chi'(p) = \sum q'_{rs}e_{rs}$, $\chi''(p) = \sum q''_{rs}e_{rs}$, for $q'_{rs}, q''_{rs} \in \mathcal{F}$. Since $\chi'(c) \in Z(R)$ and $\chi''(c) \in Z(R)$, then

$$0 = q'_{12}q'_{21} = p_{12}(p_{11} - p_{22} - p_{12}) \tag{2.33}$$

and

$$0 = q''_{12}q''_{21} = p_{12}(-p_{11} + p_{22} - p_{12}). \tag{2.34}$$

Comparing (2.33) and (2.34) one has $p_{12}^2 = 0$ that is $p_{12} = 0$, which means that p is a diagonal matrix. This argument also shows that $\chi'(p)$ must be a diagonal matrix, in particular $0 = q'_{21} = p_{11} - p_{22}$, that is $p \in Z(R)$, which is a contradiction. $\square$

The following is an easy consequence of Lemmas 2.4 and 2.5:

Corollary 2.6. *Let $R = M_t(\mathcal{F})$ be the algebra of $t \times t$ matrices over a field $\mathcal{F}$ with $t \geq 2$, $Z(R)$ the center of R, $L = [R, R]$, a, c elements of R. If*

$$[a, u]_2[c, u] = 0$$

for all $u \in L$, then either $a \in Z(R)$ or $c \in Z(R)$.

Remark 2.7. If B is a basis of U over C then any element of $T = U *_C C\{x_1, \ldots, x_n\}$, the free product over C of the C-algebra U and the free C-algebra $C\{x_1, \ldots, x_n\}$, can be written in the form $g = \sum_i \alpha_i m_i$. In this decomposition the coefficients α_i are in C and the elements m_i are B-monomials, that is $m_i = q_0 y_1 q_1 \cdots y_h q_h$, with $q_i \in B$ and $y_i \in \{x_1, \ldots, x_n\}$. In [9] it is shown that a generalized polynomial $g = \sum_i \alpha_i m_i$ is the zero element of T if and only if all α_i are zero. Let $a_1, \ldots, a_k \in U$ be linearly independent over C and

$$a_1 g_1(x_1, \ldots, x_n) + \ldots + a_k g_k(x_1, \ldots, x_n) = 0 \in T,$$

for some $g_1, \ldots, g_k \in T$. If, for any i, $g_i(x_1, \ldots, x_n) = \sum_{j=1}^n x_j h_j(x_1, \ldots, x_n)$ and $h_j(x_1, \ldots, x_n) \in T$, then $g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n)$ are the zero element of T. The same conclusion holds if

$$g_1(x_1, \ldots, x_n)a_1 + \ldots + g_k(x_1, \ldots, x_n)a_k = 0 \in T,$$

and $g_i(x_1, \ldots, x_n) = \sum_{j=1}^n h_j(x_1, \ldots, x_n)x_j$ for some $h_j(x_1, \ldots, x_n) \in T$. (We refer the reader to [2] and [9] for more details on generalized polynomial identities).

Lemma 2.8. *Let R be a prime ring of characteristic different from 2, a, b, c, q elements of R such that*

$$\Phi(x_1, x_2) = (a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b)(c[x_1, x_2] + [x_1, x_2]q) \tag{2.35}$$

is satisfied by R. If R does not satisfy any non-trivial generalized polynomial identity, then either $a, b \in C$ or $c = -q \in C$.

Proof. Since R and U satisfy the same generalized polynomial identities (Theorem 2 in [2]), we have that $\Phi(x_1, x_2)$ is satisfied by U.

Assume first that $\{1, q\}$ is linearly C-independent. By the previous Remark 2.7 and since $\Phi(x_1, x_2)$ is a trivial generalized polynomial identity for U, it follows that

$$(a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b)[x_1, x_2]q = 0 \in T$$

which implies

$$(a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b)[x_1, x_2] = 0 \in T. \tag{2.36}$$

If $\{1, a\}$ is linearly C-independent, and since (2.36) is a trivial generalized identity for U, we have $a[x_1, x_2]^3 = 0 \in T$, which gives the contradiction $a = 0$. On the other hand, if $\{1, a\}$ is linearly C-dependent, then $a \in C$ and (2.36) reduces to

$$([x_1, x_2]b[x_1, x_2] - [x_1, x_2]^2 b)[x_1, x_2] = 0 \in T. \tag{2.37}$$

Moreover, since (2.37) is a trivial generalized identity for U, then $\{1, b\}$ is linearly C-dependent, that is $b \in C$, and we are done.

Let now $\{1, q\}$ be linearly C-dependent, i.e. $q \in C$, and denote $u = c + q$. Therefore by (2.35) one has that

$$(a[x_1, x_2]^2 + [x_1, x_2](b - a)[x_1, x_2] - [x_1, x_2]^2 b)u[x_1, x_2] = 0 \in T. \tag{2.38}$$

As above, If $\{1, a\}$ is linearly C-independent, and since (2.38) is a trivial generalized identity for U, we have

$$a[x_1, x_2]^2 u[x_1, x_2] = 0 \in T \tag{2.39}$$

implying $u = 0$, as required.

Finally, if $\{1, a\}$ is linearly C-dependent, then $a \in C$ and (2.38) reduces to

$$[x_1, x_2]\Big[b, [x_1, x_2]\Big]u[x_1, x_2] = 0 \in T. \tag{2.40}$$

Since (2.40) is a trivial generalized identity for R, it is easy to see that either $b \in C$ or $u = 0$, in any case we get the required conclusion. □

For the proof of the next result, we premit the following:

Fact 2.9. *Let R be a non-commutative prime ring of characteristic different from 2, d a derivation of R, $a \in R$ a non-zero element of R. If $d(r)a = 0$, for all $r \in R$, then $d = 0$ (the proof is a classical result, for instance contained in [4, Lemma 7]).*

Fact 2.10. *Let R be a non-commutative prime ring of characteristic different from 2, d a derivation of R, $a \in R$ a non-zero element of R. If $[d(r), r]a = 0$, for all $r \in R$, then $d = 0$ (it is an easy consequence of the result in [5]).*

Fact 2.11. *Let R be a prime ring of characteristic different from 2, d and g derivations of R, such that $d(r)g(r) = 0$, for all $r \in R$. Then either $d = 0$ or $g = 0$. (It is a reduced version of [35, Theorem 3]).*

Proposition 2.12. *Let R be a non-commutative prime ring of characteristic different from 2, a, b, c, q elements of R such that*

$$\Phi(x_1, x_2) = \Big[a[x_1, x_2] + [x_1, x_2]b, [x_1, x_2]\Big](c[x_1, x_2] + [x_1, x_2]q) \tag{2.41}$$

is satisfied by R. Then one of the following holds:

(a) $a, b \in C$;
(b) $R \subseteq M_2(\mathcal{F})$, *the ring of* 2×2 *over a field* $\mathcal{F}$ *and* $b - a \in C$;
(c) $c = -q \in C$.

Proof. By Lemma 2.8 we may assume that $\Phi(x_1, x_2)$ is a non trivial generalized polynomial identity for R. By a theorem due to Beidar (Theorem 2 in [2]) this generalized polynomial identity is also satisfied by U. Let $\mathcal{F}$ be the algebraic closure of C if C is infinite and set $\mathcal{F} = C$ for C finite. Clearly, the map $r \in U \mapsto 1 \in U \bigotimes_C \mathcal{F}$ gives a ring embedding. So we may assume U is a subring of $U \bigotimes_C \mathcal{F}$. By (Proposition in [26]), $\Phi(r_1, r_2)$ is also a nonzero *GPI* of $U \bigotimes_C \mathcal{F}$. Moreover, in view of (Theorems 2.5 and 3.5 in [18]), $U \bigotimes_C \mathcal{F}$ is a prime ring with $\mathcal{F}$ as its extended centroid and both U and $U \bigotimes_C \mathcal{F}$ are centrally closed. So we may replace R by either U or $U \bigotimes_C \mathcal{F}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over its extended centroid which is either finite or algebraically closed and $\Phi(r_1, r_2) = 0$, for all $r_1, r_2 \in R$. By Martindale's theorem [30], R is a primitive ring having a non-zero socle with $\mathcal{F}$ as the associated division ring. In light of Jacobson's theorem ([20], page 75) R is isomorphic to a dense ring of linear transformations on some vector space V over $\mathcal{F}$.

Assume first that V is finite-dimensional over $\mathcal{F}$. Then the density of R on V implies that $R \cong M_m(\mathcal{F})$, the ring of all $m \times m$ matrices over $\mathcal{F}$. In this case the conclusion follows by Lemmas 2.4 and 2.5.

Assume next that V is infinite-dimensional over $\mathcal{F}$. Since V is infinite dimensional over $\mathcal{F}$ then, as in Lemma 2 in [36], the set $[R, R]$ is dense on R and so from

$$\Big[a[r_1, r_2] + [r_1, r_2]b, [r_1, r_2]\Big](c[r_1, r_2] + [r_1, r_2]q) = 0$$

for all $r_1, r_2 \in R$, we have

$$[au + ub, u](cu + uq) = 0 \tag{2.42}$$

for all $u \in R$. In (2.42) we substitute u with $x + 1$ and apply again (2.42), then we obtain

$$[ax + xb, x](c + q) + [a + b, x](cx + xq) + [a + b, x](c + q) = 0 \tag{2.43}$$

for all $x \in R$. Again for $x = y + 1$ in (2.43) and using (2.43) we get

$$2[a + b, y](c + q) = 0 \tag{2.44}$$

for all $y \in R$. Since $char(R) \neq 2$ and by Fact 2.9, it follows that either $a + b \in C$ or $c + q = 0$.
Suppose first that $c + q \neq 0$ and $a + b = \lambda \in C$, then by (2.43) it follows that $[ax - xa, x](c + q) = 0$, for all $x \in R$. Applying the result in Fact 2.10, one has $a \in C$, as required.
On the other hand, if $a + b \notin C$ and $c + q = 0$, then (2.43) reduces to $[a + b, x][c, x] = 0$ for all $x \in R$. Using the result in Fact 2.11, we have $c \in C$, and we are done.
Finally consider both $a + b = \lambda \in C$ and $c + q = 0$. In this case we write (2.42) as follows:

$$[a, u]_2[c, u] = 0 \tag{2.45}$$

for all $u \in R$.
By contradiction we assume that $a \notin C$ and $c \notin C$. Under this assumption there exist $r_1, r_2 \in R$ such that $ar_1 \neq r_1 a$ and $cr_2 \neq r_2 c$. By Litoff's Theorem in [19] there exist $e^2 = e \in R$ and a positive integer $k = dim_{\mathcal{F}}(Ve)$ such that

$$ar_1, r_1 a, cr_2, r_2 c, r_1, r_2 \in eRe \cong M_k(\mathcal{F}).$$

In the relation (2.45) replace u with $(1 - e)x(1 - e)$, for any $x \in R$ and multiply both on the right and on the left by e. It follows that R satisfies

$$ea(1 - e)x(1 - e)x(1 - e)x(1 - e)ce = 0.$$

Since R is a prime ring with $char(R) \neq 2$, by [32, Theorem] the last relation implies that either $ea(1-e) = 0$ or $(1-e)ce = 0$, that is either $ea = eae$ or $ce = ece$. In any case, eRe satisfies (2.45). Following the matrix-case argument in Lemmas 2.4 and 2.5, we have that either $eae \in Z(eRe)$ or $ece \in Z(eRe)$. Hence, one of the following cases happens:

- $ar_1 = ear_1 = eaer_1 = r_1eae = r_1ae = r_1a$;
- $cr_2 = ecr_2 = ecer_2 = r_2ece = r_2ce = r_2c$.

In any case we have a contradiction and the proof is completed. □

Corollary 2.13. *Let R be a prime ring of characteristic different from 2, $L = [R, R]$, a, c elements of R. If*

$$[a, u]_2 cu = 0$$

for all $u \in L$, then either $a \in Z(R)$ or $c = 0$.

3. The main Theorem

Firstly we need to recall some well known results:

Remark 3.1. Every derivation d of R can be uniquely extended to a derivation of U ([3], Proposition 2.5.1).

Remark 3.2. Let I be a two-sided ideal of R. Then I, R and U satisfy the same generalized polynomial identity with coefficients in U ([9]).

Remark 3.3. We denote by $Der(U)$ the set of all derivations on U. By a derivation word we mean an additive map Δ of the form $\Delta = d_1d_2\ldots d_m$, with each $d_i \in Der(U)$. Then a differential polynomial is a generalized polynomial, with coefficients in U, of the form $\Phi(^{\Delta_j}x_i)$ involving non-commutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential $\Phi(^{\Delta_j}x_i)$ is said to be a differential identity on a subset T of U if it vanishes for any assignment of values from T to its indeterminates x_i.
Let D_{int} be the $C-$subspace of $Der(U)$ consisting of all inner derivations on U and let d be a non-zero derivation on R. By Theorem 2 in [23] we have the following result (see also Theorem 1 in [27]): If $\Phi(x_1, \ldots, x_n, ^d x_1, \ldots, ^d x_n)$ is a differential identity on R, then one of the following holds:

(a) either $d \in D_{int}$;
(b) or R satisfies the generalized polynomial identity $\Phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$

Remark 3.4. Let I be a two-sided ideal of R. Then I, R and U satisfy the same differential identity.([27])

We refer to [[3], Chapter 7] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

We are ready to prove the main result of the paper:

Theorem 3.5. *Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, L a non-central Lie ideal of R, F and G two non-zero generalized derivations of R. If $[F(u), u]G(u) = 0$ for all $u \in L$, then one of the following holds:*

(a) *there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$;*
(b) *$R \subseteq M_2(\mathcal{F})$, the ring of 2×2 matrices over a field $\mathcal{F}$, and there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$, for all $x \in R$.*

Proof. By Theorem 3 in [28] every generalized derivation g on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R, and thus any generalized derivation of R can be implicitely assumed to be defined on the whole U and assumes the form $g(x) = qx + d(x)$ for some $q \in U$ and d a derivation on U. In light of this we may assume that there exist $a, c \in U$ and d, g derivations on U such that

$$F(x) = ax + d(x) \quad \text{and} \quad G(x) = cx + g(x).$$

Moreover it is known that there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$ or $char(R) = 2$ and $R \subseteq M_2(\mathcal{F})$ for some field $\mathcal{F}$ (see [22, pp 4-5], [17, Lemma 2, Proposition 1], [25, Theorem 4]).
Since $char(R) \neq 2$, then by our assumption we have that I satisfies

$$\Phi(x_1, x_2, d(x_1), d(x_2), g(x_1), g(x_2)) = \Big[a[x_1, x_2] + d([x_1, x_2]), [x_1, x_2]\Big]\Big(c[x_1, x_2] + g([x_1, x_2])\Big) \tag{3.1}$$

that is

$$\begin{aligned}\Phi(x_1, x_2, d(x_1), d(x_2), g(x_1), g(x_2)) = &\Big[a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]\Big] \cdot c[x_1, x_2] \\ &+ \Big[a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]\Big] \cdot [g(x_1), x_2] \\ &+ \Big[a[x_1, x_2] + [d(x_1), x_2] + [x_1, d(x_2)], [x_1, x_2]\Big] \cdot [x_1, g(x_2)].\end{aligned} \tag{3.2}$$

Moreover, since I and U satisfy the same generalized polynomial identities as well as the same differential identities (see Remarks 3.2 and 3.4), then U satisfies (3.1). In light of Proposition 2.12, we assume that d and g are not simultaneously inner derivations.

Case 1: Assume that d and g are linearly C-independent modulo U-inner derivations. By Kharchenko's theorem in [23] and by (3.2), U satisfies

$$\begin{aligned}\Phi(x_1, x_2, d(x_1), d(x_2), g(x_1), g(x_2)) = &\Big[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]\Big] \cdot c[x_1, x_2] \\ &+ \Big[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]\Big] \cdot [z_1, x_2] \\ &+ \Big[a[x_1, x_2] + [y_1, x_2] + [x_1, y_2], [x_1, x_2]\Big] \cdot [x_1, z_2].\end{aligned} \tag{3.3}$$

In particular U satisfies the blended component

$$\Big[[y_1, x_2], [x_1, x_2]\Big] \cdot [z_1, x_2] \tag{3.4}$$

that is U is a prime ring satisfying a polynomial identity and hence there exists a field $\mathcal{F}$ such that $U \subseteq M_t(\mathcal{F})$ with $t > 1$. Moreover U and $M_t(\mathcal{F})$ satisfy the same polynomial identity. Assume $t \geq 2$ and choose in (3.4) $y_1 = e_{12}$, $x_1 = e_{21}$, $x_2 = e_{22}$ and $z_1 = e_{12}$. Thus, by computations, the contradiction $e_{12} = 0$ follows. Hence $t = 1$ and U is commutative, a contradiction to the non-commutativity of R.

Case 2: Assume now that d and g are C-dependent modulo U-inner derivations. Thus there exist $\alpha, \beta \in C$ and $q \in U$ such that $\alpha d(x) + \beta g(x) = [q, x]$. In this case we prove that a number of contradictions occurs. Assume first that $\alpha = 0$, so that $g(x) = [p, x]$, for all $x \in U$, where $p = \beta^{-1}q$ and d is not an inner derivation (if not we are done). If $d = 0$, then (3.1) reduces to

$$\Big[a[x_1, x_2], [x_1, x_2]\Big]\Big((c + p)[x_1, x_2] - [x_1, x_2]p\Big) = 0$$

and by Proposition 2.12 we have that either $p \in C$ and $c = 0$ (that is $G = 0$), or $a \in C$. In any case we are done.
Assume $d \neq 0$, hence by (3.1) we have that U satisfies

$$\Big[a[x_1,x_2] + [d(x_1),x_2] + [x_1,d(x_2)],[x_1,x_2]\Big]c[x_1,x_2] + \Big[a[x_1,x_2] + [d(x_1),x_2] + [x_1,d(x_2)],[x_1,x_2]\Big]\Big[p,[x_1,x_2]\Big]. \tag{3.5}$$

Since d is not inner, then by Kharchenko's theorem and (3.5), we have that U satisfies the generalized identity

$$\Big[a[x_1,x_2] + [y_1,x_2] + [x_1,y_2],[x_1,x_2]\Big]c[x_1,x_2] + \Big[a[x_1,x_2] + [y_1,x_2] + [x_1,y_2],[x_1,x_2]\Big]\Big[p,[x_1,x_2]\Big]. \tag{3.6}$$

In particular U satisfies

$$\Big[[y_1,x_2] + [x_1,y_2],[x_1,x_2]\Big]\Big((c+p)[x_1,x_2] - [x_1,x_2]p\Big). \tag{3.7}$$

Let $u \in U$ be such that $u \notin C$ and replace any y_i with $[u,x_i]$ (for $i = 1,2$) in (3.7). Thus it follows that U also satisfies

$$\Big[u,[x_1,x_2]\Big]_2\Big((c+p)[x_1,x_2] - [x_1,x_2]p\Big). \tag{3.8}$$

Thus, application of Proposition 2.12 to (3.8) implies that $c = 0$ and $p \in C$, that is $G = 0$.
Let now $\beta = 0$, so that $d(x) = [v,x]$, for all $x \in U$, where $v = \alpha^{-1}q$ and g is not an inner derivation (if not we are done). If $g = 0$, then (3.1) reduces to

$$\Big[(a+v)[x_1,x_2] - [x_1,x_2]v,[x_1,x_2]\Big]c[x_1,x_2] = 0$$

and by Proposition 2.12 it follows that either $c = 0$ or $a, v \in C$, that is, respectively, either $G = 0$ or $F(x) = ax$, with $a \in C$. In any case we are finished.
Let $g \neq 0$, then (3.1) implies that U satisfies

$$\Big[(a+v)[x_1,x_2] - [x_1,x_2]v,[x_1,x_2]\Big]\Big(c[x_1,x_2] + [g(x_1),x_2] + [x_1,g(x_2)]\Big).$$

By Kharchenko's theorem U satisfies

$$\Big[(a+v)[x_1,x_2] - [x_1,x_2]v,[x_1,x_2]\Big]\Big(c[x_1,x_2] + [y_1,x_2] + [x_1,y_2]\Big)$$

and in particular U satisfies

$$\Big[(a+v)[x_1,x_2] - [x_1,x_2]v,[x_1,x_2]\Big][y_1,x_2].$$

In this case, by [6, Theorem 4.7] we get $U = M_2(C)$ and $a + v = -v + \lambda$, for some fixed $\lambda \in C$, that is $F(x) = -vx - xv + \lambda x$, as required.
Finally we consider the case both $\alpha \neq 0$ and $\beta \neq 0$ and write $g(x) = [w,x] + \gamma d(x)$ for all $x \in R$, where $w = \beta^{-1}q$ and $\gamma = -\beta^{-1}\alpha \neq 0$. Moreover we remark that d is not inner. Notice that, if $d = 0$ then (3.1) reduces to

$$\Big[a[x_1,x_2],[x_1,x_2]\Big]\Big((c+w)[x_1,x_2] - [x_1,x_2]w\Big) = 0.$$

Therefore application of Proposition 2.12 implies that either $w \in C$ and $c = 0$ (that is $G = 0$), or $a \in C$. In any case we get the expected conclusion.
Assume $d \neq 0$, hence by (3.1), U satisfies

$$\begin{aligned}&\Big[a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)],[x_1,x_2]\Big]\Big((c+w)[x_1,x_2]-[x_1,x_2]w\Big)\\&\qquad+\gamma\Big[a[x_1,x_2]+[d(x_1),x_2]+[x_1,d(x_2)],[x_1,x_2]\Big]\Big([d(x_1),x_2]+[x_1,d(x_2)]\Big).\end{aligned} \tag{3.9}$$

By Kharchenko's result and (3.9), U satisfies

$$\begin{aligned}&\Big[a[x_1,x_2]+[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big((c+w)[x_1,x_2]-[x_1,x_2]w\Big)\\&\qquad+\gamma\Big[a[x_1,x_2]+[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big([y_1,x_2]+[x_1,y_2]\Big).\end{aligned} \tag{3.10}$$

In particular U satisfies

$$\Big[a[x_1,x_2],[x_1,x_2]\Big]\Big((c+w)[x_1,x_2]-[x_1,x_2]w\Big). \tag{3.11}$$

Again by Proposition 2.12, it follows that either $w \in C$ and $c = 0$, or $a \in C$. Let the first case occur. Since $\gamma \neq 0$, by (3.9) we have $\Big[a[x_1,x_2]+[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big([y_1,x_2]+[x_1,y_2]\Big)=0$. In particular U satisfies

$$\Big[a[x_1,x_2],[x_1,x_2]\Big][x_1,x_2].$$

By [6, Theorem 4.7] the relation implies that either $a \in C$ or $[x,y]^2 \in Z(R)$. By the non-commutativity of R the second case can not occur. So if $w \in C$ then $a \in c$, as required. In the sequel we assume that $a \in C$. In this final case, (3.10) reduces to

$$\begin{aligned}&\Big[[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big((c+w)[x_1,x_2]-[x_1,x_2]w\Big)\\&\qquad+\gamma\Big[[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big([y_1,x_2]+[x_1,y_2]\Big).\end{aligned} \tag{3.12}$$

Now replace y_i with $[z,x_i]$ for any $i = 1,2$ and for a fixed $z \in U$ such that $z \notin C$. Then by (3.12) one has that U satisfies

$$\Big[z,[x_1,x_2]\Big]_2\Big((c+w+\gamma z)[x_1,x_2]-[x_1,x_2](w+\gamma z)\Big). \tag{3.13}$$

Since $z \notin C$ and by Proposition 2.12 we get $w+\gamma z \in C$ and $c = 0$. Thus by (3.12) it follows that U satisfies

$$\begin{aligned}&\Big[[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big[w,[x_1,x_2]\Big]\\&\qquad+\gamma\Big[[y_1,x_2]+[x_1,y_2],[x_1,x_2]\Big]\Big([y_1,x_2]+[x_1,y_2]\Big)\end{aligned} \tag{3.14}$$

and in particular U satisfies

$$\Big[[x_1,y_2],[x_1,x_2]\Big]\Big[w,[x_1,x_2]\Big]+\gamma\Big[[x_1,y_2],[x_1,x_2]\Big]\Big[x_1,y_2\Big]. \tag{3.15}$$

Notice that, since $\gamma \neq 0$ and $z \notin C$, then $w \notin C$. Therefore (3.14) is a non-trivial generalized polynomial identity for U. By Martindale's theorem [30], U is a primitive ring having a non-zero socle with C as the associated division ring. In light of Jacobson's theorem ([20], page 75) R is isomorphic to a dense ring of linear transformations on some vector space V over C. Of course we may assume $dim_C V \geq 2$, because U is not commutative. Consider first the case $dim_C V \geq 3$. Since $w \notin C$ then there exists $v \in V$ such that $\{w, wv\}$ is linearly C-independent. Thus there exists $v' \in V$ such that $\{w, wv, v'\}$ is linearly C-independent. Moreover, by Jacobson Density Theorem, there exist $r_1, r_2, s_2 \in U$ such that
$r_1 v = r_2 v = s_2 v = v$ and

$$r_1(wv) = 0, \quad r_2(wv) = v', \quad r_1 v' = v', \quad r_2 v' = 0, \quad s_2 v' = wv$$

which imply

$$[r_1, r_2]v = [r_1, s_2]v = 0, \quad [r_1, r_2](wv) = v', \quad [r_1, r_2]v' = 0, \quad [r_1, s_2]v' = -wv.$$

Therefore, by (3.15), it follows the contradiction

$$0 = \Big(\Big[[r_1, s_2], [r_1, r_2]\Big]\Big[w, [r_1, r_2]\Big] + \gamma\Big[[r_1, s_2], [r_1, r_2]\Big]\Big[r_1, s_2\Big]\Big)v = -v' \neq 0.$$

Finally we study the case $dim_C V = 2$, that is $U = M_2(C)$, the 2×2 matrices over C. In this case we make the following choices in (3.15): $x_1 = e_{11}$, $x_2 = e_{21}$, $y_2 = e_{12}$. Thus, both left and right multiplying (3.15) by e_{11}, it follows that $e_{11} w e_{21} = 0$. In a similar way one obtains $e_{22} w e_{12} = 0$. It is easy to see that the previous relations imply that w is a diagonal matrix, and standard argument forces the contradiction $w \in C$. □

Corollary 3.6. *Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, L a non-central Lie ideal of R, d and g two derivations of R. If $[d(u), u]g(u) = 0$ for all $u \in L$, then either $d = 0$ or $g = 0$.*

We would like to conclude our paper with the following result, which is an application of the previous one:

Theorem 3.7. *Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, I a non-zero two-sided ideal of R, F and G two non-zero generalized derivations of R. If $[F(x), x]G(x) = 0$ for all $x \in I$, then there exists $\lambda \in C$ such that $F(x) = \lambda x$, for all $x \in R$.*

Proof. It is known that there exist $c \in U$ and g derivation on U such that $G(x) = cx + g(x)$. Using the previous Theorem, we may assume that $R \subseteq M_2(C)$, the ring of 2×2 matrices over C, and there exist $a \in U$ and $\lambda \in C$ such that $F(x) = ax + xa + \lambda x$, for all $x \in R$. Thus U satisfies

$$[a, x^2]\big(cx + g(x)\big). \tag{3.16}$$

Since R is a PI-ring, its Utumi quotient ring U is a finite dimensional simple central algebra and we may consider $U = M_2(C)$, the ring of 2×2 matrices over C. Moreover R and U satisfies the same generalized differential identities as well as the same polynomial identities.
Assume first that $g(x) = qx - xq$, thus $G(x) = bx - xq$, with $b = c + q$. Hence U satisfies

$$[a, x^2](bx - xq). \tag{3.17}$$

Denote $a = \sum a_{ij} e_{ij}$, $q = \sum q_{ij} e_{ij}$, $c = \sum c_{ij} e_{ij}$, with $a_{ij}, q_{ij}, c_{ij} \in C$. For $x = e_{11}$ in (3.17) and both left and right multiplying by e_{22} we get

$$a_{21} q_{12} = 0. \tag{3.18}$$

We remark that similarly one obtains

$$a_{12} q_{21} = 0. \tag{3.19}$$

Consider now the inner automorphisms $\chi(x) = (1 + e_{21})x(1 - e_{21})$, $\varphi(x) = (1 - e_{21})x(1 + e_{21})$ in $M_2(C)$, and denote $\chi(a) = \sum a'_{ij}e_{ij}$, $\chi(c) = \sum c'_{ij}e_{ij}$, $\chi(q) = \sum q'_{ij}e_{ij}$, $\varphi(a) = \sum a''_{ij}e_{ij}$, $\varphi(c) = \sum c''_{ij}e_{ij}$ and $\varphi(q) = \sum q''_{ij}e_{ij}$. Since

$$[\chi(a), x^2](\chi(b)x - x\chi(q))$$

and

$$[\varphi(a), x^2](\varphi(b)x - x\varphi(q))$$

are identities for $M_2(C)$, then, by (3.18), $a'_{21}q'_{12} = 0$ and $a''_{21}q''_{12} = 0$. By computations it follows that both $(a_{11} - a_{22} - a_{12})q_{12} = 0$ and $(-a_{11} + a_{22} - a_{12})q_{12} = 0$, which imply that

$$a_{12}q_{12} = 0, \qquad (a_{11} - a_{22})q_{12} = 0. \tag{3.20}$$

Therefore, if $q_{12} \neq 0$, then, by (3.18) and (3.20), we get $a \in C$. Let $q_{21} \neq 0$. Now consider the inner automorphisms $\mu(x) = (1 + e_{12})x(1 - e_{12})$ and $\sigma(x)(1 - e_{12})x(1 + e_{12})$ in $M_2(C)$, and denote $\mu(a) = \sum a'''_{ij}e_{ij}$, $\mu(q) = \sum q'''_{ij}e_{ij}$, $\sigma(a) = \sum a^{iv}_{ij}e_{ij}$, $\sigma(q) = \sum q^{iv}_{ij}e_{ij}$. Since

$$[\sigma(a), x^2](\sigma(b)x - x\sigma(q))$$

and

$$[\mu(a), x^2](\mu(b)x - x\mu(q))$$

are identities for $M_2(C)$, then, by (3.19), $a'''_{12}q'''_{21} = 0$ and and $a^{iv}_{12}q^{iv}_{21} = 0$. By computations it follows that both $(a_{22} - a_{11} - a_{21})q_{21} = 0$ and $(-a_{22} + a_{11} - a_{21})q_{21} = 0$, which imply that $a_{21}q_{21} = 0$ and $(a_{22} - a_{11})q_{21} = 0$. Since $q_{21} \neq 0$ we get $a_{21} = 0$ and $a_{22} = a_{11}$. By (3.19), we also have $a_{12} = 0$ which implies that $a \in C$. Consequently if either $q_{21} \neq 0$ or $q_{12} \neq 0$, then we get $a \in C$. So if $q_{21} = 0$ and $q_{12} = 0$ then q is diagonal.
In other words, either q is diagonal or a is central. Moreover, by using the same argument in Lemma 2.5, it follows that either $q \in C$ or $a \in C$. In this last case we are done, so that in the sequel we assume $q \in C$. Hence $G(x) = cx$ and U satisfies

$$[a, x^2]cx = 0. \tag{3.21}$$

Replacing x with $x + 1$ in above relation we have $[a, x^2]c + 2[a, x]cx + 2[a, x] = 0$. Again replacing x with $x + 1$ in the last relation and using $char(R) \neq 2$ we get $[a, x]c = 0$ for all $x \in R$. Since R is a prime ring we have either $a \in C$ or $c = 0$, as required.
Assume finally that g is not an inner derivation of R. If $g = 0$, then $G(x) = cx$ and we conclude by the same above argument. Let $g \neq 0$ be an outer derivation of R. In light of Kharchenko's result and by (3.16)

$$[a, x^2](cx + y) \tag{3.22}$$

is a generalized polynomial identity for U. In particular U satisfies $[a, x^2]y$ and by the primeness of U, it follows that $[a, r^2] = 0$ for all $r \in U$. It is well known that in this case $a \in C$ follows, and we are done. □

The followings are easy consequences of Theorem 3.7.

Corollary 3.8. *Let R be a non-commutative prime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, I a non-zero two-sided ideal of R, d and g two derivations of R. If $[d(x), x]g(x) = 0$ for all $x \in I$, then either $d = 0$ or $g = 0$.*

Corollary 3.9. *Let R be a semiprime ring of characteristic different from 2 with Utumi quotient ring U and extended centroid C, d and g two nonzero derivations of R. If $[d([x, y]), [x, y]]g([x, y]) = 0$ for all $x, y \in R$, then d or g map R into $Z(R)$.*

Proof. Since R is semiprime, U is also semiprime and C, the extended centroid of R, is a von Neumann regular ring, so C includes idempotents elements. Let ε be the set of all idempotents of C. The elements of ε are called central idempotents. Let $\mathcal{B}$ be the complete Boolean algebra of ε. We choose a maximal ideal M of $\mathcal{B}$. By [1], MU is a prime ideal of U. Moreover MU is minimal in U, which is invariant under any derivation of U. We know that $\bigcap_M MU = (0)$ (see [[1], Lemma 1 and Theorem 1]). It is also well known that the pair of derivations d, g on R can be uniquely extended to a pair of derivations on U (see [27]). Let $\overline{d}, \overline{g}$ be a pair of derivations on $\overline{U} = U/MU$ induced by d, g, respectively. Therefore $\overline{d}$ and $\overline{g}$ satisfy the same property of d and g in $\overline{U} = U/MU$. By the hypothesis

$$[d([x,y]),[x,y]]g([x,y]) = 0$$

for all $x, y \in R$. By [[27], Theorem 2] R and U satisfy the same differential identities. Thus U satisfies

$$[d([x,y]),[x,y]]g([x,y]).$$

Furthermore, $\overline{U}$ satisfies

$$[\overline{d}([\overline{x},\overline{y}]),[\overline{x},\overline{y}]]\overline{g}([\overline{x},\overline{y}]).$$

Since $\overline{U}$ is prime ring, by Corollary 3.6 we have either $\overline{d}(\overline{x}) = \overline{0}$ or $\overline{g}(\overline{x}) = \overline{0}$ or $[\overline{U},\overline{U}] = (\overline{0})$. In any case, we get either $d(U)[U,U] \in MU$ or $g(U)[U,U] \in MU$ for all M, that is either $d(U)[U,U] \in \bigcap_M MU = (0)$ or $g(U)[U,U] \in \bigcap_M MU = (0)$. In paticular either $d(R)[R,R] = (0)$ or $g(R)[R,R] = (0)$. By easy calculations we arrive at either $[d(R),R] = (0)$ or $[g(R),R] = (0)$. These imply that either $d(R) \subset Z(R)$ or $g(R) \subset Z(R)$. □

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