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The Doyen-Wilson theorem for 3-sun systems*

Giovanni Lo Faro , Antoinette Tripodi

Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra, Università di Messina, Messina, Italia

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Abstract

A solution to the existence problem of G-designs with given subdesigns is known when G is a triangle with $p = 0, 1$, or 2 disjoint pendent edges: for $p = 0$, it is due to Doyen and Wilson, the first to pose such a problem for Steiner triple systems; for $p = 1$ and $p = 2$, the corresponding designs are kite systems and bull designs, respectively. Here, a complete solution to the problem is given in the remaining case where G is a 3-sun, i.e. a graph on six vertices consisting of a triangle with three pendent edges which form a 1-factor.

Keywords: 3*-sun systems, embedding, difference set.*

Math. Subj. Class.: 05B05, 05B30

1 Introduction

If G is a graph, then let $V(G)$ and $E(G)$ be the vertex-set and edge-set of G, respectively. The graph K_n denotes the complete graph on n vertices. The graph $K_m \setminus K_n$ has vertex-set $V(K_m)$ containing a distinguished subset H of size n; the edge-set of $K_m \setminus K_n$ is $E(K_m)$ but with the $\binom{n}{2}$ edges between the n distinguished vertices of H removed. This graph is sometimes referred to as a complete graph of order m with a *hole* of size n.

Let G and Γ be finite graphs. A *G-design* of Γ is a pair $(X, \mathcal{B})$ where $X = V(\Gamma)$ and $\mathcal{B}$ is a collection of isomorphic copies of G (*blocks*), whose edges partition $E(\Gamma)$. If $\Gamma = K_n$, then we refer to such a design as a *G-design of order n*.

A G-design $(X_1, \mathcal{B}_1)$ of order n is said to be *embedded* in a G-design $(X_2, \mathcal{B}_2)$ of order m provided $X_1 \subseteq X_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$ (we also say that $(X_1, \mathcal{B}_1)$ is a *subdesign* (or *subsystem*) of $(X_2, \mathcal{B}_2)$ or $(X_2, \mathcal{B}_2)$ contains $(X_1, \mathcal{B}_1)$ as subdesign). Let $N(G)$ denote the set of integers n such that there exists a G-design of order n. A natural question to ask is: given $n, m \in N(G)$, with $m > n$, and a G-design $(X, \mathcal{B})$ of order n, does exists a G-design of order m containing $(X, \mathcal{B})$ as subdesign? Doyen and Wilson were the first to

*Supported by I.N.D.A.M. (G.N.S.A.G.A.).
E-mail addresses: lofaro@unime.it (Giovanni Lo Faro), atripodi@unime.it (Antoinette Tripodi)

pose this problem for $G = K_3$ (Steiner triple systems) and in 1973 they showed that given $n, m \in N(K_3) = \{v : v \equiv 1, 3 \pmod 6\}$, then *any Steiner triple system of order n can be embedded in a Steiner triple system of order m if and only if $m \geq 2n+1$ or $m = n$* (see [3]). Over the years, any such problem has come to be called a "Doyen-Wilson problem" and any solution a "Doyen-Wilson type theorem". The work along these lines is extensive ([1, 4, 5, 6, 7, 8, 9, 10, 13]) and the interested reader is referred to [2] for a history of this problem.

In particular, taking into consideration the case where G is a triangle with $p = 0, 1, 2,$ or 3 mutually disjoint pendent edges, a solution to the Doyen-Wilson problem is known when $p = 0$ (Steiner triple systems, [3]), $p = 1$ (kite systems, [9, 10]) and $p = 2$ (bull designs, [4]). Here, we deal with the remaining case ($p = 3$) where G is a 3-*sun*, i.e. a graph on six vertices consisting of a triangle with three pendent edges which form a 1-factor, by giving a complete solution to the Doyen-Wilson problem for G-designs where G is a 3-sun (3-*sun systems*).

2 Notation and basic lemmas

The 3-sun consisting of the triangle (a, b, c) and the three mutually disjoint pendent edges $\{a, d\}, \{b, e\}, \{c, f\}$ is denoted by $(a, b, c; d, e, f)$. A 3-sun system of order n (briefly, 3SS(n)) exists if and only if $n \equiv 0, 1, 4, 9 \pmod{12}$ and if $(X, \mathcal{S})$ is a 3SS(n), then $|\mathcal{S}| = \frac{n(n-1)}{12}$ (see [14]).

Let $n, m \equiv 0, 1, 4, 9 \pmod{12}$, with $m = u + n$, $u \geq 0$. The Doyen-Wilson problem for 3-sun systems is equivalent to the existence problem of decompositions of $K_{u+n} \setminus K_n$ into 3-suns.

Let r and s be integers with $r < s$, define $[r, s] = \{r, r+1, \ldots, s\}$ and $[s, r] = \emptyset$. Let $Z_u = [0, u-1]$ and $H = \{\infty_1, \infty_2, \ldots, \infty_t\}$, $H \cap Z_u = \emptyset$. If $S = (a, b, c; d, e, f)$ is a 3-sun whose vertices belong to $Z_u \cup H$ and $i \in Z_u$, let $S + i = (a+i, b+i, c+i; d+i, e+i, f+i)$, where the sums are modulo u and $\infty + i = \infty$, for every $\infty \in H$. The set $(S) = \{S + i : i \in Z_u\}$ is called the *orbit of S under Z_u* and S is a *base block* of (S).

To solve the Doyen-Wilson problem for 3-sun systems we use the *difference method* (see [11, 12]). For every pair of distinct elements $i, j \in Z_u$, define $|i - j|_u = \min\{|i - j|, u - |i - j|\}$ and set $D_u = \{|i - j|_u : i, j \in Z_u\} = \{1, 2, \ldots, \lfloor \frac{u}{2} \rfloor\}$. The elements of D_u are called *differences* of Z_u. For any $d \in D_u$, $d \neq \frac{u}{2}$, we can form a single 2-factor $\{\{i, d+i\} : i \in Z_u\}$, while if u is even and $d = \frac{u}{2}$, then we can form a 1-factor $\{\{i, i + \frac{u}{2}\} : 0 \leq i \leq \frac{u}{2} - 1\}$. It is also worth remarking that 2-factors obtained from distinct differences are disjoint from each other and from the 1-factor.

If $D \subseteq D_u$, denote by $\langle Z_u \cup H, D \rangle$ the graph with vertex-set $V = Z_u \cup H$ and the edge-set $E = \{\{i, j\} : |i - j|_u = d,\ d \in D\} \cup \{\{\infty, i\} : \infty \in H,\ i \in Z_u\}$. The graph $\langle Z_u \cup H, D_u \rangle$ is the complete graph $K_{u+t} \setminus K_t$ based on $Z_u \cup H$ and having H as a hole. The elements of H are called *infinity points*.

Let X be a set of size $n \equiv 0, 1, 4, 9 \pmod{12}$. The aim of the paper is to decompose the graph $\langle Z_u \cup X, D_u \rangle$ into 3-suns. To obtain our main result the $\langle Z_u \cup X, D_u \rangle$ will be regarded as a union of suitable edge-disjoint subgraphs of type $\langle Z_u \cup H, D \rangle$ (where $H \subseteq X$ may be empty, while $D \subseteq D_u$ is always non empty) and then each subgraph will be decomposed into 3-suns by using the lemmas given in this section. From here on suppose $u \equiv 0, 1, 3, 4, 5, 7, 8, 9, 11 \pmod{12}$.

Lemmas 2.1 – 2.4 give decompositions of subgraphs of type $\langle Z_u \cup H, D \rangle$ where D

contains particular differences, more precisely, $D = \{2\}$, $D = \{2, 4\}$ or $D = \{1, \frac{u}{3}\}$.

Lemma 2.1. *Let* $u \equiv 0 \pmod 4$*,* $u \geq 8$*. Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2\}, \{2\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 2+4i, 4i; 3+4i, 4+4i, \infty_2),$$
$$(\infty_2, 3+4i, 1+4i; 2+4i, 5+4i, \infty_1),$$

for $i = 0, 1, \ldots, \frac{u}{4} - 1$. □

Lemma 2.2. *Let* $u \equiv 0 \pmod{12}$*. Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 12i, 2+12i; 7+12i, , \infty_3, \infty_4),$$
$$(\infty_1, 4+12i, 6+12i; 9+12i, \infty_3, \infty_4),$$
$$(\infty_1, 8+12i, 10+12i; 11+12i, \infty_3, \infty_4),$$
$$(\infty_2, 2+12i, 4+12i; 1+12i, \infty_3, \infty_4),$$
$$(\infty_2, 6+12i, 8+12i; 7+12i, \infty_3, \infty_4),$$
$$(\infty_2, 10+12i, 12+12i; 11+12i, \infty_3, \infty_4),$$
$$(\infty_3, 1+12i, 3+12i; 9+12i, \infty_1, \infty_2),$$
$$(\infty_3, 5+12i, 7+12i; 11+12i, \infty_1, 9+12i),$$
$$(\infty_4, 3+12i, 5+12i; 1+12i, \infty_1, \infty_2),$$
$$(\infty_4, 9+12i, 11+12i; 7+12i, \infty_2, 13+12i),$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

Lemma 2.3. *The graph* $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle$*,* $u \geq 7$*,* $u \neq 8$*, can be decomposed into* 3*-suns.*

Proof. Let $u = 4k + r$, with $r = 0, 1, 3$, and consider the 3-suns

$$(\infty_1, 4+4i, 6+4i; 5+4i, 8+4i, \infty_4),$$
$$(\infty_2, 5+4i, 7+4i; 6+4i, 9+4i, \infty_1),$$
$$(\infty_3, 6+4i, 8+4i; 7+4i, 10+4i, \infty_2),$$
$$(\infty_4, 7+4i, 9+4i; 8+4i, 11+4i, \infty_3),$$

for $i = 0, 1, \ldots, k-3$, $k \geq 3$, plus the following blocks as the case may be.

If $r = 0$,

$$(\infty_1, 0, 2; 1, 4, \infty_4),$$
$$(\infty_2, 1, 3; 2, 5, \infty_1),$$
$$(\infty_3, 2, 4; 3, 6, \infty_2),$$
$$(\infty_4, 3, 5; 4, 7, \infty_3),$$
$$(\infty_1, 4k-4, 4k-2; 4k-3, 0, \infty_4),$$

$$(\infty_2, 4k-3, 4k-1; 4k-2, 1, \infty_1),$$
$$(\infty_3, 4k-2, 0; 4k-1, 2, \infty_2),$$
$$(\infty_4, 4k-1, 1; 0, 3, \infty_3).$$

If $r = 1$,

$$(\infty_1, 0, 2; 1, 4, \infty_2),$$
$$(\infty_2, 1, 3; 0, 5, \infty_1),$$
$$(\infty_3, 2, 4; 3, 6, \infty_2),$$
$$(\infty_4, 3, 5; 4, 7, \infty_3),$$
$$(\infty_1, 4k-4, 4k-2; 4k-3, 4k, \infty_2),$$
$$(\infty_2, 4k-3, 4k-1; 4k, 0, \infty_1),$$
$$(\infty_3, 4k-2, 4k; 4k-1, 1, \infty_1),$$
$$(\infty_4, 4k-1, 0; 4k-2, 2, \infty_3),$$
$$(\infty_4, 4k, 1; 2, 3, \infty_3).$$

If $r = 3$,

$$(\infty_1, 0, 2; 1, 4, \infty_4),$$
$$(\infty_2, 1, 3; 2, 5, \infty_1),$$
$$(\infty_3, 2, 4; 3, 6, \infty_2),$$
$$(\infty_4, 3, 5; 4, 7, \infty_3),$$
$$(\infty_1, 4k-4, 4k-2; 4k-3, 4k, \infty_4),$$
$$(\infty_2, 4k-3, 4k-1; 4k-2, 4k+1, \infty_1),$$
$$(\infty_3, 4k-2, 4k; 4k-1, 4k+2, \infty_2),$$
$$(\infty_4, 4k-1, 4k+1; 4k, 0, \infty_3),$$
$$(\infty_1, 4k, 4k+2; 4k+1, 1, \infty_4),$$
$$(\infty_2, 4k+1, 0; 4k+2, 2, \infty_4),$$
$$(\infty_3, 4k+2, 1; 0, 3, \infty_4).$$

With regard to the difference 4 in Z_7, note that $|4|_7 = 3$ and the seven distinct blocks obtained for $k = 1$ and $r = 3$ gives a decomposition of $\langle Z_7 \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 3\}\rangle$ into 3-suns. □

Lemma 2.4. *Let $u \equiv 0 \pmod 3$, $u \geq 12$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_8\}, \{1, \frac{u}{3}\}\rangle$ can be decomposed into 3-suns.*

Proof. If $u \equiv 0 \pmod 6$ consider the 3-suns:

$$(\infty_1, 2i, \tfrac{u}{3} + 2i; 2\tfrac{u}{3} + 2i, \infty_5, \infty_6),\ i = 0, 1, \dots, \tfrac{u}{6} - 1,$$
$$(\infty_1, 1 + 2i, \tfrac{u}{3} + 1 + 2i; 2\tfrac{u}{3} + 1 + 2i, \infty_6, \infty_5),\ i = 0, 1, \dots, \tfrac{u}{6} - 1,$$
$$(\infty_2, 2\tfrac{u}{3} + 2i, \tfrac{u}{3} + 2i; 2 + 2i, 2i, \infty_5),\ i = 0, 1, \dots, \tfrac{u}{6} - 2,$$
$$(\infty_2, 2\tfrac{u}{3} + 1 + 2i, \tfrac{u}{3} + 1 + 2i; 3 + 2i, 1 + 2i, \infty_6),\ i = 0, 1, \dots, \tfrac{u}{6} - 2,$$
$$(\infty_2, u - 2, 2\tfrac{u}{3} - 2; 0, \tfrac{u}{3} - 2, \infty_5),$$

$$(\infty_2, u-1, 2\tfrac{u}{3}-1; 1, \tfrac{u}{3}-1, \infty_6),$$
$$(\infty_3, 2i, 1+2i; 2\tfrac{u}{3}+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u}{6}-1,$$
$$(\infty_3, \tfrac{u}{3}+2i, \tfrac{u}{3}+1+2i; 2\tfrac{u}{3}+1+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u}{6}-1,$$
$$(\infty_4, 1+2i, 2+2i; 2\tfrac{u}{3}+2+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u}{6}-1,$$
$$(\infty_4, \tfrac{u}{3}+1+2i, \tfrac{u}{3}+2+2i; 2\tfrac{u}{3}+1+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u}{6}-1,$$
$$(\infty_5, 2\tfrac{u}{3}+2i, 2\tfrac{u}{3}+1+2i; 1+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u}{6}-1,$$
$$(\infty_6, 2\tfrac{u}{3}+3+2i, 2\tfrac{u}{3}+4+2i; 2+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u}{6}-2,$$
$$(\infty_6, 2\tfrac{u}{3}+1, 2\tfrac{u}{3}+2; 2\tfrac{u}{3}, \infty_7, \infty_8).$$

If $u \equiv 3 \pmod 6$ consider the 3-suns:

$$(\infty_1, 2i, \tfrac{u}{3}+2i; 2\tfrac{u}{3}+2i, \infty_5, \infty_6),\ i=0,1,\dots,\tfrac{u-3}{6},$$
$$(\infty_1, 1+2i, \tfrac{u}{3}+1+2i; 2\tfrac{u}{3}+1+2i, \infty_6, \infty_5),\ i=0,1,\dots,\tfrac{u-9}{6},$$
$$(\infty_2, 2\tfrac{u}{3}+2i, \tfrac{u}{3}+2i; 2+2i, 2i, \infty_5),\ i=0,1,\dots,\tfrac{u-9}{6},$$
$$(\infty_2, u-1, 2\tfrac{u}{3}-1; 0, \tfrac{u}{3}-1, \infty_5),$$
$$(\infty_2, 2\tfrac{u}{3}+1+2i, \tfrac{u}{3}+1+2i; 3+2i, 1+2i, \infty_6),\ i=0,1,\dots,\tfrac{u-15}{6},$$
$$(\infty_2, u-2, 2\tfrac{u}{3}-2; 1, \tfrac{u}{3}-2, \infty_6),$$
$$(\infty_3, 2i, 1+2i; 2\tfrac{u}{3}+2i, \infty_7, \infty_8),\ i=2,3,\dots,\tfrac{u-3}{6},$$
$$(\infty_3, 0, 1; 2\tfrac{u}{3}, \infty_6, \infty_8),$$
$$(\infty_3, 2, 3; 2\tfrac{u}{3}+2, \infty_6, \infty_8),$$
$$(\infty_3, \tfrac{u}{3}+1+2i, \tfrac{u}{3}+2+2i; 2\tfrac{u}{3}+1+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u-9}{6},$$
$$(\infty_4, 1+2i, 2+2i; 2\tfrac{u}{3}+2+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u-9}{6},$$
$$(\infty_4, \tfrac{u}{3}+2i, \tfrac{u}{3}+1+2i; 2\tfrac{u}{3}+1+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u-3}{6},$$
$$(\infty_5, 2\tfrac{u}{3}+2i, 2\tfrac{u}{3}+1+2i; 1+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u-9}{6},$$
$$(\infty_6, 2\tfrac{u}{3}+1+2i, 2\tfrac{u}{3}+2+2i; 4+2i, \infty_7, \infty_8),\ i=0,1,\dots,\tfrac{u-15}{6},$$
$$(\infty_6, u-2, u-1; 2\tfrac{u}{3}, \infty_7, \infty_8),$$
$$(\infty_7, u-1, 0; 2, \infty_5, \infty_8).$$

□

Lemmas 2.5 – 2.9 allow to decompose $\langle Z_u \cup H, D\rangle$ where u is even and D contains the difference $\frac{u}{2}$.

Lemma 2.5. *Let u be even, $u \geq 8$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 2i, 1+2i; \tfrac{u}{2}+2+2i, \tfrac{u}{2}+2i, \infty_3),\ i=0,1,\dots,\tfrac{u}{4}-2,$$
$$(\infty_1, \tfrac{u}{2}-2, \tfrac{u}{2}-1; \tfrac{u}{2}, u-2, \infty_3),$$
$$(\infty_2, 1+2i, \tfrac{u}{2}+1+2i; 2i, 2+2i, \infty_1),\ i=0,1,\dots,\tfrac{u}{4}-1,$$
$$(\infty_3, \tfrac{u}{2}+1+2i, \tfrac{u}{2}+2i; 2i, \tfrac{u}{2}+2+2i, \infty_2),\ i=0,1,\dots,\tfrac{u}{4}-1.$$

□

Lemma 2.6. *Let $u \equiv 0 \pmod{12}$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$
\begin{aligned}
&(\infty_1, 6i, \tfrac{u}{2}+6i; 4+6i, \infty_3, \infty_2),\\
&(\infty_1, 1+6i, \tfrac{u}{2}+1+6i; 5+6i, \infty_4, \infty_2),\\
&(\infty_1, 2+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+3+6i, \infty_4, \infty_3),\\
&(\infty_2, 1+6i, 6i; \tfrac{u}{2}+3+6i, \infty_3, \infty_4),\\
&(\infty_2, 2+6i, 3+6i; \tfrac{u}{2}+4+6i, 1+6i, \infty_4),\\
&(\infty_2, 5+6i, 4+6i; \tfrac{u}{2}+5+6i, 6+6i, 3+6i),\\
&(\infty_3, 3+6i, \tfrac{u}{2}+3+6i; 2+6i, \infty_1, \tfrac{u}{2}+2+6i),\\
&(\infty_3, 4+6i, \tfrac{u}{2}+4+6i; \tfrac{u}{2}+6i, \infty_4, \infty_1),\\
&(\infty_3, 5+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+1+6i, \infty_4, \infty_1),\\
&(\infty_4, \tfrac{u}{2}+1+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+3+6i, \tfrac{u}{2}+6i, \infty_2),\\
&(\infty_4, \tfrac{u}{2}+4+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+6i, \tfrac{u}{2}+3+6i, \tfrac{u}{2}+6+6i),
\end{aligned}
$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

Lemma 2.7. *Let u be even, $u \geq 8$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$
\begin{aligned}
&(\infty_1, 2i, 1+2i; \tfrac{u}{2}+2+2i, \tfrac{u}{2}+2i, \infty_3),\ i = 0, 1, \ldots, \tfrac{u}{4}-2,\\
&(\infty_1, \tfrac{u}{2}-2, \tfrac{u}{2}-1; \tfrac{u}{2}, u-2, \infty_3),\\
&(\infty_2, 1+2i, \tfrac{u}{2}+1+2i; 2i, \infty_6, \infty_1),\ i = 0, 1, \ldots, \tfrac{u}{4}-1,\\
&(\infty_3, \tfrac{u}{2}+1+2i, \tfrac{u}{2}+2i; 2i, \infty_6, \infty_2),\ i = 0, 1, \ldots, \tfrac{u}{4}-1,\\
&(\infty_4, 1+2i, 2+2i; \tfrac{u}{2}+2+2i, \infty_5, \infty_6),\ i = 0, 1, \ldots, \tfrac{u}{4}-1,\\
&(\infty_5, \tfrac{u}{2}+1+2i, \tfrac{u}{2}+2+2i; 2+2i, \infty_4, \infty_6),\ i = 0, 1, \ldots, \tfrac{u}{4}-1.
\end{aligned}
$$

□

Lemma 2.8. *Let $u \equiv 0 \pmod{12}$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_7\}, \{1, \frac{u}{2}\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the 3-suns

$$
\begin{aligned}
&(\infty_1, 6i, \tfrac{u}{2}+6i; 4+6i, \infty_7, \infty_2),\\
&(\infty_1, 1+6i, \tfrac{u}{2}+1+6i; \tfrac{u}{2}+3+6i, \infty_7, \infty_4),\\
&(\infty_1, 2+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+5+6i, \infty_5, \infty_2),\\
&(\infty_2, 3+6i, \tfrac{u}{2}+3+6i; 6i, \infty_1, \infty_4),\\
&(\infty_2, 4+6i, \tfrac{u}{2}+4+6i; 2+6i, \infty_7, \infty_1),\\
&(\infty_2, 5+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+1+6i, \infty_1, \infty_7),\\
&(\infty_3, 6i, 1+6i; \tfrac{u}{2}+6i, \infty_5, \infty_6),\\
&(\infty_3, 2+6i, 3+6i; \tfrac{u}{2}+2+6i, \infty_7, \infty_6),\\
&(\infty_3, 4+6i, 5+6i; \tfrac{u}{2}+5+6i, \infty_5, \infty_6),\\
&(\infty_4, 1+6i,\ 2+6i; \tfrac{u}{2}+6+6i, \infty_2, \infty_6),\\
&(\infty_4, 3+6i, 4+6i; \tfrac{u}{2}+4+6i, \infty_7, \infty_6),
\end{aligned}
$$

$$(\infty_4, 5+6i, 6+6i; \tfrac{u}{2}+5+6i, \infty_7, \infty_6),$$
$$(\infty_5, \tfrac{u}{2}+6i, \tfrac{u}{2}+1+6i; 1+6i, \infty_7, \infty_3),$$
$$(\infty_5, \tfrac{u}{2}+2+6i, \tfrac{u}{2}+3+6i; 3+6i, \infty_7, \infty_3),$$
$$(\infty_5, \tfrac{u}{2}+4+6i, \tfrac{u}{2}+5+6i; 5+6i, \infty_7, \tfrac{u}{2}+6+6i),$$
$$(\infty_6, \tfrac{u}{2}+1+6i, \tfrac{u}{2}+2+6i; \tfrac{u}{2}+5+6i, \infty_7, \infty_4),$$
$$(\infty_6, \tfrac{u}{2}+3+6i, \tfrac{u}{2}+4+6i; \tfrac{u}{2}+6+6i, \infty_7, \infty_3),$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

Lemma 2.9. *Let* $u \equiv 0 \pmod{12}$. *Then the graph* $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \frac{u}{2}\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. Consider the 3-suns

$$(\infty_1, 6i, 1+6i; \tfrac{u}{2}+1+6i, \tfrac{u}{2}+6i, 3+6i),$$
$$(\infty_1, 2+6i, 3+6i; \tfrac{u}{2}+5+6i, \tfrac{u}{2}+2+6i, 5+6i),$$
$$(\infty_1, 4+6i, 5+6i; \tfrac{u}{2}+2+6i, \tfrac{u}{2}+4+6i, 7+6i),$$
$$(\infty_1, \tfrac{u}{2}+3+6i, \tfrac{u}{2}+4+6i; \tfrac{u}{2}+6i, \tfrac{u}{2}+2+6i, \infty_2),$$
$$(\infty_2, 1+6i, \tfrac{u}{2}+1+6i; \tfrac{u}{2}+3+6i, 2+6i, \tfrac{u}{2}+2+6i),$$
$$(\infty_2, 3+6i, 4+6i; 2+6i, \tfrac{u}{2}+3+6i, 6+6i),$$
$$(\infty_2, 5+6i, \tfrac{u}{2}+5+6i; \tfrac{u}{2}+2+6i, 6+6i, \tfrac{u}{2}+6+6i),$$
$$(\infty_3, 2+6i, 6i; 1+6i, 4+6i, \infty_2),$$
$$(\infty_3, \tfrac{u}{2}+2+6i, \tfrac{u}{2}+6i; 4+6i, \tfrac{u}{2}+4+6i, \infty_2),$$
$$(\infty_3, \tfrac{u}{2}+1+6i, \tfrac{u}{2}+3+6i; 3+6i, \tfrac{u}{2}+6i, \tfrac{u}{2}+5+6i),$$
$$(\infty_3, \tfrac{u}{2}+5+6i, \tfrac{u}{2}+4+6i; 5+6i, \tfrac{u}{2}+7+6i, \tfrac{u}{2}+6+6i),$$

for $i = 0, 1, \ldots, \frac{u}{12} - 1$. □

The following lemma "combines" one infinity point with one difference $d \neq \frac{u}{2}, \frac{u}{3}$ such that $\frac{u}{\gcd(u,d)} \equiv 0 \pmod 3$ (therefore, $u \equiv 0 \pmod 3$).

Lemma 2.10. *Let* $u \equiv 0 \pmod 3$ *and* $d \in D_u \setminus \{\frac{u}{2}, \frac{u}{3}\}$ *such that* $p = \frac{u}{\gcd(u,d)} \equiv 0 \pmod 3$. *Then the graph* $\langle Z_u \cup \{\infty\}, \{d\}\rangle$ *can be decomposed into* 3*-suns.*

Proof. The subgraph $\langle Z_u, \{d\}\rangle$ can be decomposed into $\frac{u}{p}$ cycles of length $p = 3q$, $q \geq 2$.

If $q > 2$, let $(x_1, x_2, \ldots, x_{3q})$ be a such cycle and consider the 3-suns

$$(\infty, x_{2+3i}, x_{3+3i}; x_{7+3i}, x_{1+3i}, x_{4+3i}),$$

for $i = 0, 1, \ldots, q-1$ (where the sum is modulo $3q$).

If $q = 2$, let $(x_1^{(j)}, x_2^{(j)}, x_3^{(j)}, x_4^{(j)}, x_5^{(j)}, x_6^{(j)})$, $j = 0, 1, \ldots, \frac{u}{6} - 1$, be the 6-cycles decomposing $\langle Z_u, \{d\}\rangle$ and consider the 3-suns

$$(\infty, x_2^{(j)}, x_3^{(j)}; x_1^{(j+1)}, x_1^{(j)}, x_4^{(j)}),$$
$$(\infty, x_5^{(j)}, x_6^{(j)}; x_4^{(j+1)}, x_4^{(j)}, x_1^{(j)}),$$

for $j = 0, 1, \ldots, \frac{u}{6} - 1$ (where the sums are modulo $\frac{u}{6}$). □

Subsequent Lemmas 2.11 – 2.14 allow to decompose $\langle Z_u \cup H, D\rangle$, where $|H| = 1, 2, 3, 5$, $|D| = 6 - |H|$ and $\frac{u}{2} \notin D$; here, u and D are any with the unique condition that if D contains at least three differences d_1, d_2, d_3, then $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$.

Lemma 2.11. *Let $d_1, d_2, d_3, d_4, d_5 \in D_u \setminus \{\frac{u}{2}\}$ such that $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. Then the graph $\langle Z_u \cup \{\infty\}, \{d_1, d_2, d_3, d_4, d_5\}\rangle$ can be decomposed into 3-suns.*

Proof. If $d_3 = d_2 - d_1$, consider the orbit of

$$(d_1, d_2, 0; \infty, d_2 + d_5, d_4)$$

(or $(d_1, d_2, 0; \infty, d_2 + d_5, -d_4)$, if $d_2 + d_5 = d_4$) under Z_u. If $d_1 + d_2 + d_3 = u$, consider the orbit of

$$(-d_1, d_2, 0; \infty, d_2 + d_5, d_4)$$

(or $(-d_1, d_2, 0; \infty, d_2 + d_5, -d_4)$, if $d_2 + d_5 = d_4$) under Z_u. □

Lemma 2.12. *Let $d_1, d_2, d_3, d_4 \in D_u \setminus \{\frac{u}{2}\}$ such that $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2\}, \{d_1, d_2, d_3, d_4\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the orbit of $(d_1, d_2, 0; \infty_1, \infty_2, d_4)$ or $(-d_1, d_2, 0; \infty_1, \infty_2, d_4)$ under Z_u when, respectively, $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. □

Lemma 2.13. *Let $d_1, d_2, d_3 \in D_u \setminus \{\frac{u}{2}\}$ such that $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. Then the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{d_1, d_2, d_3\}\rangle$ can be decomposed into 3-suns.*

Proof. Consider the orbit of $(d_1, d_2, 0; \infty_1, \infty_2, \infty_3)$ or $(-d_1, d_2, 0; \infty_1, \infty_2, \infty_3)$ under Z_u when, respectively, $d_3 = d_2 - d_1$ or $d_1 + d_2 + d_3 = u$. □

Lemma 2.14. *Let $d \in D_u \setminus \{\frac{u}{2}\}$, the graph $\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}, \{d\}\rangle$ can be decomposed into 3-suns.*

Proof. The subgraph $\langle Z_u, \{d\}\rangle$ is regular of degree 2 and so can be decomposed into l-cycles, $l \geq 3$. Let $(x_1, x_2, \dots, x_l)$ be a such cycle. Put $l = 3q + r$, with $r = 0, 1, 2$, and consider the 3-suns with the sums modulo l

$$(\infty_1, x_{1+3i}, x_{2+3i}; x_{3+3i}, \infty_4, \infty_5),$$
$$(\infty_2, x_{2+3i}, x_{3+3i}; x_{4+3i}, \infty_4, \infty_5),$$
$$(\infty_3, x_{3+3i}, x_{4+3i}; x_{5+3i}, \infty_4, \infty_5),$$

for $i = 0, 1, \dots, q - 2$, $q \geq 2$, plus the following blocks as the case may be.

If $r = 0$,

$$(\infty_1, x_{3q-2}, x_{3q-1}; x_{3q}, \infty_4, \infty_5),$$
$$(\infty_2, x_{3q-1}, x_{3q}; x_1, \infty_4, \infty_5),$$
$$(\infty_3, x_{3q}, x_1; x_2, \infty_4, \infty_5).$$

If $r = 1$,

$$(\infty_1, x_{3q-2}, x_{3q-1}; x_{3q+1}, \infty_4, \infty_5),$$
$$(\infty_2, x_{3q-1}, x_{3q}; x_1, \infty_4, \infty_1),$$
$$(\infty_3, x_{3q}, x_{3q+1}; x_2, \infty_4, \infty_2),$$
$$(\infty_5, x_{3q+1}, x_1; x_{3q}, \infty_4, \infty_3).$$

If $r = 2$,

$$(\infty_1, x_{3q-2}, x_{3q-1}; x_{3q+2}, \infty_4, \infty_5),$$
$$(\infty_2, x_{3q-1}, x_{3q}; x_1, \infty_4, \infty_5),$$
$$(\infty_3, x_{3q}, x_{3q+1}; x_2, \infty_1, \infty_2),$$
$$(\infty_4, x_{3q+1}, x_{3q+2}; x_{3q}, \infty_1, \infty_3),$$
$$(\infty_5, x_{3q+2}, x_1; x_{3q+1}, \infty_2, \infty_3).$$

□

Finally, after settling the infinity points by using the above lemmas, if u is large we need to decompose the subgraph $\langle Z_u, L\rangle$, where L is the set of the differences unused (*difference leave*). Since by applying Lemmas 2.1 – 2.13 it could be necessary to use the differences 1, 2 or 4, while Lemma 2.14 does not impose any restriction, it is possible to combine infinity points and differences in such a way that the difference leave L is the set of the "small" differences, where 1, 2 or 4 could possibly be avoided.

Lemma 2.15. *Let $\alpha \in \{0, 4, 8\}$ and u, s be positive integers such that $u > 12s + \alpha$. Then there exists a decomposition of $\langle Z_u, L\rangle$ into 3-suns, where:*

i) $\alpha = 0$ *and* $L = [1, 6s]$*;*

ii) $\alpha = 4$ *and* $L = [3, 6s + 2]$*;*

iii) $\alpha = 8$ *and* $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$.

Proof.

i) Consider the orbits (S_j) under Z_u, where $S_j = (5s + 1 + j,\ 5s - j,\ 0; 3s,\ s,\ u - 2 - 2j)$, $j = 0, 1, \ldots, s - 1$.

ii) Consider the orbits in i), where (S_0) is replaced with the orbit of $(6s+1,\ 4s,\ 0;\ s,\ 9s,\ 6s + 2)$.

iii) Consider the orbits in i), where the orbits (S_0) and (S_1) are replaced with the orbits of $(6s + 1,\ 4s,\ 0; s,\ 9s,\ 6s + 4)$ and $(5s + 2,\ 5s - 1,\ 0; 3s,\ s,\ 6s + 2)$. □

3 The main result

Let $(X, \mathcal{S})$ be a $3\mathrm{SS}(n)$ and $m \equiv 0, 1, 4, 9 \pmod{12}$.

Lemma 3.1. *If $(X, \mathcal{S})$ is embedded in a 3-sun system of order $m > n$, then $m \geq \frac{7}{5}n + 1$.*

Proof. Suppose $(X, \mathcal{S})$ is embedded in $(X', \mathcal{S}')$, with $|X'| = m = n + u$ (u positive integer). Let c_i be the number of 3-suns of $\mathcal{S}'$ each of which contains exactly i edges in $X' \setminus X$. Then $\sum_{i=1}^{6} i \times c_i = \binom{u}{2}$ and $\sum_{i=1}^{5}(6 - i)c_i = u \times n$, from which it follows $6c_2 + 12c_3 + 18c_4 + 24c_5 + 30c_6 = \frac{u(5u-2n-5)}{2}$ and so $u \geq \frac{2}{5}n + 1$ and $m \geq \frac{7}{5}n + 1$. □

By previous Lemma:

1. if $n = 60k + 5r$, $r = 0, 5, 8, 9$, then $m \geq 84k + 7r + 1$;
2. if $n = 60k + 5r + 1$, $r = 0, 3, 4, 7$, then $m \geq 84k + 7r + 3$;
3. if $n = 60k + 5r + 2$, $r = 2, 7, 10, 11$, then $m \geq 84k + 7r + 4$;
4. if $n = 60k + 5r + 3$, $r = 2, 5, 6, 9$, then $m \geq 84k + 7r + 6$;

5. if $n = 60k + 5r + 4$, $r = 0, 1, 4, 9$, then $m \geq 84k + 7r + 7$.

In order to prove that the necessary conditions for embedding a 3-sun system $(X, \mathcal{S})$ of order n in a 3-sun system of order $m = n + u$, $u > 0$ are also sufficient, the graph $\langle Z_u \cup X, D_u \rangle$ will be expressed as a union of edge-disjoint subgraphs $\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup X, D \rangle \cup \langle Z_u, L \rangle$, where $L = D_u \setminus D$ is the difference leave, and $\langle Z_u \cup X, D \rangle$ (if necessary, expressed itself as a union of subgraphs) will be decomposed by using Lemmas 2.1 – 2.14, while if $L \neq \emptyset$, $\langle Z_u, L \rangle$ will be decomposed by Lemma 2.15. To obtain our main result we will distinguish the five cases 1. – 5. listed before by giving a general proof for any $k \geq 0$ with the exception of a few cases for $k = 0$, which will be indicated by a star $\star$ and solved in Appendix. Finally, note that:

a) $u \equiv 0, 1, 4, \text{ or } 9 \pmod{12}$, if $n \equiv 0 \pmod{12}$;

b) $u \equiv 0, 3, 8, \text{ or } 11 \pmod{12}$, if $n \equiv 1 \pmod{12}$;

c) $u \equiv 0, 5, 8, \text{ or } 9 \pmod{12}$, $n \equiv 4 \pmod{12}$;

d) $u \equiv 0, 3, 4, \text{ or } 7 \pmod{12}$, if $n \equiv 9 \pmod{12}$.

Proposition 3.2. *For any $n = 60k + 5r$, $r = 0, 5, 8, 9$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k + 2r + 1$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r}\}$, $r = 0, 5, 8, 9$, and $u = 24k + 2r + 1 + h$, with $h \geq 0$. Set $h = 12s + l$, $0 \leq l \leq 11$ (l depends on r), and distinguish the following cases.

Case 1: $r = 0, 5, 8, 9$ and $l = 0$ (odd u).

Write $\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup X, D \rangle \cup \langle Z_u, L \rangle$, where $D = [6s + 1, 12k + r + 6s]$, $|D| = 12k + r$, and $L = [1, 6s]$, and apply Lemmas 2.14 and 2.15.

Case 2: $r = 0, 9$ and $l = 8$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{2, 6s+3, 6s+5\} \rangle \cup \\ \langle Z_u \cup \{\infty_4\}, \{1\} \rangle \cup \langle Z_u \cup \{\infty_5\}, \{6s+4\} \rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D' \rangle \cup \langle Z_u, L \rangle,\end{aligned}$$

where $D' = [6s + 6, 12k + r + 6s + 4]$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 2]$, and apply Lemmas 2.13, 2.10, 2.14 and 2.15.

Case 3: $r = 5, 8$ and $l = 4$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\} \rangle \cup \langle Z_u \cup \{\infty_5\}, \{1\} \rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D' \rangle \cup \langle Z_u, L \rangle,\end{aligned}$$

where $D' = [6s + 3, 12k + r + 6s + 2] \setminus \{6s + 4\}$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.3, 2.10, 2.14 and 2.15.

Case 4: $r = 0, 8$ and $l = 3$ (even u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\} \rangle \cup \langle Z_u \cup \{\infty_4, \infty_5\}, \{2\} \rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D' \rangle \cup \langle Z_u, L \rangle,\end{aligned}$$

where $D' = [6s+3, 12k+r+6s+1]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.1, 2.14 and 2.15.

Case 5: $r = 0$ and $l = 11$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup \{\infty_4, \infty_5\}, \{4, 6s+3, 6s+5, 6s+7\}\rangle \cup \\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+6s+5] \setminus \{6s+7\}$, $|D'| = 12k-1$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.9, 2.12, 2.14 and 2.15.

Case 6: $r = 5$ and $l = 1$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_{10}\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+5]$, $|D'| = 12k+3$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.2, 2.14 and 2.15.

Case 7: $r = 5, 9$ and $l = 9$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup \{\infty_4, \infty_5\}, \{2, 6s+3, 6s+4, 6s+5\}\rangle \cup \\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+r+6s+4]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.12, 2.14 and 2.15.

Case 8: $r = 8$ and $l = 7$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup \{\infty_4\}, \{4\}\rangle \cup \langle Z_u \cup \{\infty_5\}, \{6s+5\}\rangle \cup \\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+11] \setminus \{6s+4, 6s+5\}$, $|D'| = 12k+7$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.9, 2.10, 2.14 and 2.15.

Case 9: $r = 9$ and $l = 5$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{2\}\rangle \cup \\ &\langle Z_u \cup \{\infty_5\}, \{4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+11] \setminus \{6s+4\}$, $|D'| = 12k+8$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.10, 2.14 and 2.15. □

Proposition 3.3. *For any* $n = 60k + 5r + 1$, $r = 0, 3, 4, 7$, *there exists a decomposition of* $K_{n+u} \setminus K_n$ *into 3-suns for every admissible* $u \geq 24k + 2r + 2$.

Proof. Let $X = \{\infty_1, \infty_2, \dots, \infty_{60k+5r+1}\}$, $r = 0, 3, 4, 7$, and $u = 24k + 2r + 2 + h$, with $h \geq 0$. Set $h = 12s + l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 0, 3$ and $l = 1$ (odd u).

Write $\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty\}, \{6s + 2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty\}), D'\rangle \cup \langle Z_u, L\rangle$, where $D' = [6s + 1, 12k + r + 6s + 1] \setminus \{6s + 2\}$, $|D'| = 12k + r$, and $L = [1, 6s]$, and apply Lemmas 2.10, 2.14 and 2.15.

Case 2: $r = 0, 3, 4, 7$ and $l = 9$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s + 3, 6s + 4\}\rangle \cup \\ \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s + 5, 6s + 7\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+r+6s+5] \setminus \{6s+7\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.14 and 2.15.

Case 3: $r = 4^\star, 7$ and $l = 5$ (odd u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \langle Z_u \cup \{\infty_5\}, \{1\}\rangle \cup \\ \langle Z_u \cup \{\infty_6\}, \{6s + 8\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s + 3, 12k + r + 6s + 3] \setminus \{6s + 4, 6s + 8\}$, $|D'| = 12k + r - 1$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.3, 2.10, 2.14 and 2.15.

Case 4: $r = 0, 4$ and $l = 6$ (even u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s + 3, 6s + 5\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+4, 12k+r+6s+3] \setminus \{6s+5\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.13, 2.14 and 2.15.

Case 5: $r = 0$ and $l = 10$ (even u).

Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ \langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \\ \langle Z_u \cup \{\infty_{11}\}, \{4, 6s + 3, 6s + 5, 6s + 6, 6s + 7\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_{11}\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s + 8, 12k + 6s + 5]$, $|D'| = 12k - 2$, and $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.7, 2.2, 2.11, 2.14 and 2.15.

Case 6: $r = 3, 7$ and $l = 0$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s+3, 12k+r+6s]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.14 and 2.15.

Case 7: $r = 3$ and $l = 4$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5\}, \{2\}\rangle \cup \langle Z_u \cup \{\infty_6\}, \{6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+6s+5] \setminus \{6s+5\}$, $|D'| = 12k+2$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.10, 2.14 and 2.15.

Case 8: $r = 4$ and $l = 2$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+6s+5]$, $|D'| = 12k+3$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.1, 2.14 and 2.15.

Case 9: $r = 7$ and $l = 8$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{4, 6s+3, 6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+6s+11] \setminus \{6s+7\}$, $|D'| = 12k+6$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.9, 2.13, 2.14 and 2.15. □

Proposition 3.4. *For any* $n = 60k+5r+2$, $r = 2, 7, 10, 11$, *there exists a decomposition of* $K_{n+u} \setminus K_n$ *into* 3*-suns for every admissible* $u \geq 24k+2r+2$.

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r+2}\}$, $r = 2, 7, 10, 11$, and $u = 24k+2r+2+h$, with $h \geq 0$. Set $h = 12s+l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 2, 11$ and $l = 3$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1\}, \{6s+2\}\rangle \cup \langle Z_u \cup \{\infty_2\}, \{6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+1, 12k+r+6s+2] \setminus \{6s+2, 6s+4\}$, $|D'| = 12k+r$, and $L = [1, 6s]$, and apply Lemmas 2.10, 2.14 and 2.15.

Case 2: $r = 2, 7, 10, 11$ and $l = 7$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2\}, \{1, 2, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+r+6s+4]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.12, 2.14 and 2.15.

Case 3: $r = 7, 10$ and $l = 11$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s+5, 6s+7\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{6s+8\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+r+6s+6] \setminus \{6s+7, 6s+8\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.10, 2.14 and 2.15.

Case 4: $r = 2$ and $l = 6$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6, \infty_7\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+4, 12k+6s+5] \setminus \{6s+5\}$, $|D'| = 12k+1$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.13, 2.14 and 2.15.

Case 5: $r = 2, 10$ and $l = 10$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{2, 6s+3, 6s+4, 6s+5, 6s+6\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+7, 12k+r+6s+5]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.11, 2.14 and 2.15.

Case 6: $r = 7, 11$ and $l = 4$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6, \infty_7\}, \{2, 4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+r+6s+2] \setminus \{6s+4\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.3, 2.14 and 2.15.

Case 7: $r = 7$ and $l = 8$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+4, 12k+6s+11] \setminus \{6s+5, 6s+7\}$, $|D'| = 12k+6$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.13, 2.10, 2.14 and 2.15.

Case 8: $r = 10$ and $l = 2$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+3, 12k+6s+11]$, $|D'| = 12k+9$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.10, 2.14 and 2.15.

Case 9: $r = 11$ and $l = 0$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_7\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_7\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s+3, 12k+6s+11]$, $|D'| = 12k+10$, and $L = [3, 6s+2]$, and apply Lemmas 2.8, 2.14 and 2.15. □

Proposition 3.5. *For any* $n = 60k+5r+3$, $r = 2, 5, 6, 9$, *there exists a decomposition of* $K_{n+u} \setminus K_n$ *into* 3*-suns for every admissible* $u \geq 24k+2r+3$.

Proof. Let $X = \{\infty_1, \infty_2, \dots, \infty_{60k+5r+3}\}$, $r = 2, 5, 6, 9$, and $u = 24k+2r+3+h$, with $h \geq 0$. Set $h = 12s+l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 2, 5, 6, 9$ and $l = 4$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s+5, 12k+r+6s+3]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.14 and 2.15.

Case 2: $r = 2, 5$ and $l = 8$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2\}, \{1, 6s+3, 6s+4, 6s+5\}\rangle \cup \langle Z_u \cup \{\infty_3\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+r+6s+5]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.12, 2.10, 2.14 and 2.15.

Case 3: $r = 6, 9$ and $l = 0$ (odd u).

If $s = 0$, then write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_8\}, \{1, \tfrac{u}{3}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle,$$

where $D' = [2, 12k+r+1] \setminus \{\frac{u}{3}\}$, $|D'| = 12k+r-1$, and apply Lemmas 2.4 and 2.14.

If $s > 0$, then write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 5s, 5s+1\}\rangle \cup \langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s+1, 6s+3\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{6s+2\}\rangle \cup \langle Z_u \cup \{\infty_8\}, \{6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2s+1, 4s\} \cup [6s+5, 12k+r+6s+1]$, $|D'| = 12k+r-1$, and $L = [3, 6s] \setminus \{2s+1, 4s, 5s, 5s+1\}$, and apply Lemmas 2.13, 2.10 and 2.14 to decompose the first five subgraphs, while to decompose the last one apply Lemma 2.15 i) and delete the orbit (S_0).

Case 4: $r = 2, 6$ and $l = 1$ (even u).

Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = \{2\} \cup [6s+3, 12k+r+6s+1]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.14 and 2.15.

Case 5: $r = 2^\star$ and $l = 5$ (even u).

Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \langle Z_u \cup \{\infty_{11}, \infty_{12}, \infty_{13}\}, \{4, 6s+3, 6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_{13}\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+6s+5] \setminus \{6s+7\}$, $|D'| = 12k$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.7, 2.2, 2.13, 2.14 and 2.15.

Case 6: $r = 5, 9$ and $l = 7$ (even u).

Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7, \infty_8\}, \{2, 6s+3, 6s+4, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+r+6s+4]$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.12, 2.14 and 2.15.

Case 7: $r = 5$ and $l = 11$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup \{\infty_5, \infty_6\}, \{2, 6s+3, 6s+5, 6s+6\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{4\}\rangle \cup \\ &\langle Z_u \cup \{\infty_8\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+8, 12k+6s+11]$, $|D'| = 12k+4$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.6, 2.12, 2.10, 2.14 and 2.15.

Case 8: $r = 6$ and $l = 9$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup \{\infty_4, \infty_5, \infty_6\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup \{\infty_7\}, \{4\}\rangle \cup \\ &\langle Z_u \cup \{\infty_8\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_8\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+6, 12k+6s+11] \setminus \{6s+7\}$, $|D'| = 12k+5$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.13, 2.10, 2.14 and 2.15.

Case 9: $r = 9$ and $l = 3$ (even u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 2, \tfrac{u}{2}\}\rangle \cup \\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = [6s+3, 12k+6s+11]$, $|D'| = 12k+9$, and $L = [3, 6s+2]$, and apply Lemmas 2.9, 2.14 and 2.15. □

Proposition 3.6. *For any $n = 60k+5r+4$, $r = 0, 1, 4, 9$, there exists a decomposition of $K_{n+u} \setminus K_n$ into 3-suns for every admissible $u \geq 24k+2r+3$.*

Proof. Let $X = \{\infty_1, \infty_2, \ldots, \infty_{60k+5r+4}\}$, $r = 0, 1, 4, 9$, and $u = 24k+2r+3+h$, with $h \geq 0$. Set $h = 12s+l$, $0 \leq l \leq 11$, and distinguish the following cases.

Case 1: $r = 0, 1^\star, 4, 9$ and $l = 2$ (odd u).
Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = &\langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \\ &\langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

where $D' = \{1, 6s+3\} \cup [6s+5, 12k+r+6s+2]$, $|D'| = 12k+r$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.3, 2.14 and 2.15.

Case 2: $r = 0, 9$ and $l = 6$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, 6s+3, 6s+4\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{2\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+5, 12k+r+6s+4]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.13, 2.10, 2.14 and 2.15.

Case 3: $r = 1, 4$ and $l = 10$ (odd u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2\}, \{1, 6s+3, 6s+5, 6s+6\}\rangle \cup \langle Z_u \cup \{\infty_3\}, \{2\}\rangle \cup \langle Z_u \cup \{\infty_4\}, \{6s+4\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+7, 12k+r+6s+6]$, $|D'| = 12k+r$, and $L = [3, 6s+2]$, and apply Lemmas 2.12, 2.10, 2.14 and 2.15.

Case 4: $r = 0, 4$ and $l = 5$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_6\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_7, \infty_8, \infty_9\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_9\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+4, 12k+r+6s+3] \setminus \{6s+5\}$, $|D'| = 12k+r-1$, and $L = [3, 6s+2]$, and apply Lemmas 2.7, 2.13, 2.14 and 2.15.

Case 5: $r = 0$ and $l = 9$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_5, \infty_6, \infty_7\}, \{2, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup \{\infty_8\}, \{4\}\rangle \cup \langle Z_u \cup \{\infty_9\}, \{6s+7\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_9\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+6s+5] \setminus \{6s+7\}$, $|D'| = 12k-1$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.6 , 2.13, 2.10, 2.14 and 2.15.

Case 6: $r = 1$ and $l = 7$ (even u).
Write

$$\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \dots, \infty_7\}, \{1, \tfrac{u}{2}\}\rangle \cup \langle Z_u \cup \{\infty_8, \infty_9\}, \{2, 4, 6s+3, 6s+5\}\rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \dots, \infty_9\}), D'\rangle \cup \langle Z_u, L\rangle,$$

where $D' = [6s+6, 12k+6s+5]$, $|D'| = 12k$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.8, 2.12, 2.14 and 2.15.

Case 7: $r = 1, 9$ and $l = 11$ (even u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\} \rangle \cup \langle Z_u \cup \{\infty_4\}, \{2, 4, 6s+3, 6s+5, 6s+6\} \rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D' \rangle \cup \langle Z_u, L \rangle,$$

where $D' = [6s+7, 12k+r+6s+6]$, $|D'| = 12k+r$, and $L = [3, 6s+4] \setminus \{4, 6s+3\}$, and apply Lemmas 2.5, 2.11, 2.14 and 2.15.

Case 8: $r = 4$ and $l = 1$ (even u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{1, \tfrac{u}{2}\} \rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D' \rangle \cup \langle Z_u, L \rangle,$$

where $D' = \{2\} \cup [6s+3, 12k+6s+5]$, $|D'| = 12k+4$, and $L = [3, 6s+2]$, and apply Lemmas 2.6, 2.14 and 2.15.

Case 9: $r = 9$ and $l = 3$ (even u).
Write

$$\langle Z_u \cup X, D_u \rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3\}, \{1, \tfrac{u}{2}\} \rangle \cup \langle Z_u \cup \{\infty_4\}, \{2\} \rangle \cup \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \infty_3, \infty_4\}), D' \rangle \cup \langle Z_u, L \rangle,$$

where $D' = [6s+3, 12k+6s+11]$, $|D'| = 12k+9$, and $L = [3, 6s+2]$, and apply Lemmas 2.5, 2.10, 2.14 and 2.15. □

Combining Lemma 3.1 and Propositions 3.2 – 3.6 gives our main theorem.

Theorem 3.7. *Any* 3SS(n) *can be embedded in a* 3SS(m) *if and only if*

$$m \geq \tfrac{7}{5}n + 1 \quad \text{or} \quad m = n.$$

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Appendix

- $n = 21$, $u = 12s + 15$

  Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \\ \langle Z_u \cup \{\infty_5\}, \{1\}\rangle \cup \langle Z_u \cup \{\infty_6\}, \{6s + 7\}\rangle \cup \\ \langle Z_u \cup (X \setminus \{\infty_1, \infty_2, \ldots, \infty_6\}), \{6s + 3, 6s + 5, 6s + 6\}\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

  where $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.3, 2.10, 2.14 and 2.15.

- $n = 13$, $u = 12s + 12$

  Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \ldots, \infty_6\}, \{1, 6s + 6\}\rangle \cup \\ \langle Z_u \cup \{\infty_7, \infty_8, \infty_9, \infty_{10}\}, \{2\}\rangle \cup \\ \langle Z_u \cup \{\infty_{11}, \infty_{12}, \infty_{13}\}, \{4, 6s + 3, 6s + 5\}\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

  where $L = [3, 6s + 4] \setminus \{4, 6s + 3\}$, and apply Lemmas 2.7, 2.2, 2.13 and 2.15.

- $n = 9$, $u = 12s + 7$

  Write

$$\begin{aligned}\langle Z_u \cup X, D_u\rangle = \langle Z_u \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}, \{2, 4\}\rangle \cup \\ \langle Z_u \cup \{\infty_5, \infty_6, \infty_7, \infty_8, \infty_9\}, \{1\}\rangle \cup \langle Z_u, L\rangle,\end{aligned}$$

  where $L = [3, 6s + 3] \setminus \{4\}$, and apply Lemmas 2.3, 2.14 and decompose $\langle Z_u, L\rangle$ as in Lemma 2.15 iii), taking in account that $|6s + 4|_{12s+7} = 6s + 3$.