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Cournot-Bayesian General Equilibrium: A Radon Measure Approach

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Abstract: In this paper, we consider a Cournot duopoly, in which any firm does not know the marginal costs of production of the other player, as a Bayesian game. In our game, the marginal costs depend on two infinite continuous sets of states of the world. We shall study, before the general case, an intermediate case in which only one player, the second one, shows infinitely many types. Then, we shall generalize to the case in which both players show infinitely many types depending on the marginal costs, where the marginal costs are given by the nature and each actual marginal cost is known only by the respective player. We find, in both cases, the general Nash equilibrium.

Keywords: Cournot duopoly; game theory; Nash–Cournot equilibrium; marginal costs; Bayesian games; infinite dimensional strategy space; probability measure; Radon measures

1. Introduction

1.1. Bayesian-Cournot Games

In this paper, we consider a Cournot duopoly, in which any firm does not know the marginal costs of production of the other player, as a Bayesian game. In our game, the marginal costs depend on two infinite continuous sets of states of the world. We shall study, before the general case, an intermediate case in which only one player, the second one, shows infinitely many types. Then, we shall generalize to the case in which both players show infinitely many types depending on the marginal costs, where the marginal costs are given by the nature and each actual marginal cost is known only by the respective player. We find, in both cases, the general Nash equilibrium.

1.2. Preliminaries: Bayesian Games

We adopt a highly general definition of Bayesian game.

Definition 1. *In a two-player Bayesian game, we need to specify:*

1. *two type spaces S, T;*
2. *two families of individual strategy spaces,*

$$E = (E_s)_{s \in S},$$

for the first player, and:

$$F = (F_t)_{t \in T},$$

for the second player;

3. *two strategy spaces:*

$$\mathbf{E} \subset \times_{s\in S} E_s,$$

for the first player, and:

$$\mathbf{F} \subset \times_{t\in T} F_t;$$

for the second player, subsets of the Cartesian products of the respective families E and F;

4. *two payoff functions:*

$$e, f : (S \times T) \times (\mathbf{E} \times \mathbf{F}) \to \mathbb{R}.$$

5. *two beliefs—one for every player—p, q. These are two probability measures on the type spaces S and T, respectively. The probability measure p is the belief of the second player on the type of the first one; q is the belief of the first player on the type of the second.*

The above game can be formally defined as the system:

$$G = (e, f, p, q),$$

if, for every $(x, y) \in \mathbf{E} \times \mathbf{F}$, for every $s \in S$, and for every $t \in T$, the two functions:

$$e_s(x, y) := e(x, y, s, \cdot)\ ,\ \ f_t(x, y) := f(x, y, \cdot, t)$$

are integrable with respect to q and p, respectively.

Remark 1. *We observe the following points.*

1. *A type space for a player is just the set of all possible "types" of that player.*
2. *The individual strategy space E_s belongs to the type s first player, for every $s \in S$, symmetrically for the second family F.*
3. *According to the definition of the general Cartesian product of a set family, a pure strategy of Player 1 is, by definition, a function:*

$$x : S \to \bigcup E$$

satisfying the property:

$$x(s) \in E_s,$$

for all $s \in S$.

4. *A possible action for the first player is conceivable as a pair:*

$$(s, x_s) \in S \times \bigcup E$$

with $x_s \in E_s$ and such that there exists a pure strategy $x \in \mathbf{E}$ with:

$$x_s = x(s).$$

Therefore, the set of all possible actions of the first player is the union of all graphs $gr(x)$ with $x \in \mathbf{E}$.

5. *The payoff functions e, f are two-place functions of type profiles (s, t) and strategy profiles (x, y). For example, we consider a first player payoff:*

$$e : (S \times T) \times (\mathbf{E} \times \mathbf{F}) \to \mathbb{R} : ((s, t), (x, y)) \mapsto e((x, y), (s, t)),$$

we will see it, under natural integrable conditions, as a family of functions:

$$e_s : \mathbf{E} \times \mathbf{F} \to \mathcal{L}^1_q(T),$$

with $s \in S$.

6. *The beliefs of a player describe the uncertainty of that player about the types of the other players, and often, we can consider p as a probability measure on S and q as a probability measure on T.*

Definition 2. *Consider the above game (e, f, p, q). The expected payoff of Player 1 upon the strategy profile:*

$$(x, y) \in \mathbf{E} \times \mathbf{F},$$

when the first player knows himself/herself to be of type s, is the integral:

$$u_s(x, y) = \mathbb{E}_q(e_s(x, y)) = \int_T e_s(x, y)\ q,$$

whenever the function:

$$e_s(x, y) : T \to \mathbb{R}$$

defined by:

$$e_s(x, y)(t) := e(s, t, x, y),$$

for every $t \in T$, is integrable under the measure q. Analogously, we define the expected payoff function:

$$v_t : \mathbf{E} \times \mathbf{F} \to \mathbb{R} : (x, y) \mapsto \mathbb{E}_p(f_t(x, y)) = \int_S f_t(x, y)\ p,$$

concerning the second player, for every t in T. The families of expected payoff functions, upon the bi-strategy space $\mathbf{E} \times \mathbf{F}$, defined by:

$$u = (u_s)_{s \in S}\ ,\ \ v = (v_t)_{t \in T},$$

determine a game bi-labeled family (u, v). The correspondence:

$$N : S \times T \to \mathbf{E} \times \mathbf{F},$$

sending each type pair (s, t) to the corresponding Nash equilibrium set of the game (u_s, v_t)—between the s-type first player and the t-type second player—can be see as a dynamic multi-path (multi-surface) designing the (non-cooperative) Nash solution of the bi-labeled family game (u, v). A Nash-Bayesian equilibrium for the Bayesian game (e, f, p, q) is defined as the Nash equilibrium multi-surface of the associated game (u, v).

Remark 2. *We note that the certainty of a player of its own type is represented by a Dirac delta measure centered at that type. For instance, we note that:*

$$u_s(x, y) = \mathbb{E}_{\delta_s \otimes q}(e(x, y)) = \int_{S \times T} e(x, y)\ (\delta_s \otimes q),$$

for every s in S.

1.3. Brief Description of the Standard Cournot's Game

The classic model is a non-linear two-player (gain) game $G = (f, >)$. The two players produce and offer the same commodity. In more specific terms: the payoff function f of the game is defined on a subset of the positive cone of the Cartesian plane, interpreted as a space of bi-quantities. We assume that the set of all strategies (of each player) is the interval $E = [0, +\infty]$. The bi-gain function is defined by:

$$f(x, y) = (ax(Q - x - y),\ by(Q - x - y)) - (sx + c, ty + d),$$

for every bi-strategy (x, y) of the game in the positive cone of the Euclidean plane, for convenient positive constants a, b (measured in euro/quantity^2), maximum reasonable quantity Q, marginal costs s, t (measured in euro/quantity), and fixed costs c, d (measured in euros).

1.4. Bayesian Games in Industrial Organization

The novelty and interest of the present paper lie in the use of the Bayesian game analysis for the Cournot duopoly in a scenario of infinitely many possible pairs of marginal costs. a direct, natural generalization of the classic result of microeconomic theory. It seems, indeed, hard to find in the literature such a comprehensive and clear general statement providing a straightforward extension of the Cournot–Nash equilibrium state to the case of complete uncertainty on marginal costs of the other players, which is, by far, the most common and serious issue in the economic duopoly analysis.

Mostly, in the literature, we see developments of the Nash–Cournot equilibrium towards the resolution of other research questions, especially connected with the asymptotic stability of a variously associated dynamical system (see [1]).

Several books and papers introduced, with the classic models of oligopoly (Cournot model), the use of Bayesian games, for the case of finitely many possible marginal costs. We start from those studies and from the non-cooperative game and incomplete information game analysis in industrial organization (see [2–4]).

We have considered also studies about static and dynamic information games and Bayesian Nash equilibrium, the normal form of Bayesian games, the extensive-form of Bayesian games with observable actions, and games with incomplete information played by Bayesian players [5–7]. Some discrete cases of our model could be found in [8]. As concerns the complete analysis of classic duopoly model, see [9,10]. Other applications of game theory to economic duopolies can be found in [11–18]. For a Radon and Schwartz distribution approach to probability, see [19–21].

2. Methods: Derivatives in Infinite Dimensions

In this paper, in order to compute the Nash-Bayesian equilibria, we shall use a convenient type of partial derivation in infinite dimensions for functions of the type:

$$\phi : \mathbf{F} \to \mathbb{R}$$

where:

- $\mathbf{F}$ is the convex strategy sub-space:

  $$\mathbf{F} := \{y \in \mathcal{L}^1_q([0, c_2]) : y(t) \in [0, Q]\},$$

  of all real (bounded) q-integrable functions on a (cost) compact interval $T = [0, c_2]$ with values belonging to $[0, Q]$.

  Our partial derivative:

  $$\partial_t \phi : \mathbf{F} \to \mathbb{R} : y \mapsto \partial_t \phi(y),$$

with $t \in T$, is not standard in infinite dimensional vector spaces and is defined as follows.

Definition 3. *For any t in the compact interval $T = [0, c_2]$ and for any $y \in \mathbf{F}$:*

- *we consider the real function:*

  $$r : U \subset \mathbb{R} \to \mathbb{R} : h \mapsto (\phi(y + h\chi_t) - \phi(y))h^{-1},$$

  where U is a convenient pierced (right) neighborhood of zero;
- *the partial derivative $\partial_t \phi(y)$ is the following limit (if it exists):*

  $$\partial_t \phi(y) = \lim_{h \to 0} (\phi(y + h\chi_t) - \phi(y))h^{-1},$$

where χ_t *is the characteristic function of the singleton* $\{t\}$ *in the interval* $[0, c_2]$*, that is the function:*

$$\chi_t : T \to \mathbb{R} : t' \mapsto \delta(t', t) = \begin{cases} 0 & if \quad t' \neq t \\ 1 & if \quad t' = t \end{cases} .$$

Definition 4. *We can define that derivative also in the following convenient way. Let:*

$$g : [0, Q] \to \mathbb{R}$$

be the function defined by:

$$g(z) = \phi(y + (z - y(t))\chi_t),$$

for every $z \in [0, Q]$*, where* x, y, t *are fixed in their respective spaces. If* g *is differentiable at the point* $y(t)$*, we put:*

$$\partial_t \phi(y) := g'(y(t)).$$

Example 1. *In our game, the payoff function of the second player* t^{th} *type will be defined by:*

$$f_t : [0, Q] \times \mathcal{L}^1_q([0, c_2]) \to \mathbb{R} : f_t(x, y) = y(t)(Q - x - y(t)) - ty(t).$$

Taking into account that:

$$(y + (z - y(t))\chi_t)(t) = z,$$

the above auxiliary function g*—for* $\phi = f_t(x, .)$*—is defined by:*

$$g(z) = f_t(x, y + (z - y(t))\chi_t) = z(Q - x - z) - tz,$$

for every $z \in [0, Q]$*, so that:*

$$g'(z) = Q - x - 2z - t$$

and:

$$\partial_t [f_t(x, .)](y) = g'(y(t)) = Q - x - 2y(t) - t.$$

3. The Model: Uncertainty on the Costs of One Player

Let $Q \in \mathbb{R}_>$ be the maximum reasonable quantity in our symmetric Cournot duopoly.

Strategy set of the first player: The (essential) strategy set $\mathbf{E}$ of our first player is the production compact interval:

$$\mathbf{E} = [0, Q].$$

Type set of the second player: The type-set of the second player is the marginal cost interval:

$$T := [0, c_2].$$

State of the world space of the second player: The state of the world space of the second player is the marginal cost interval:

$$T := [0, c_2].$$

3.1. Strategy Set of the Second Player

The second player t^{th} type adopts the strategy set:

$$F_t := [0, Q];$$

so that the second player's total strategy profile space is the Cartesian product:

$$\times_{t\in T} F_t = [0, Q]^{[0, c_2]},$$

the set of all functions from $[0, c_2]$ into $[0, Q]$. In other terms, any (pure) strategy of the second player is an infinite profile strategy:

$$y : [0, c_2] \to [0, Q],$$

this function specifies, for any type $t \in T$, a certain production $y(t)$ in $[0, Q]$.

3.2. Radon Measure Probability of the Second Player

Assume that:

$$q : C^0([0, c_2]) \to \mathbb{R}$$

is a probability measure (private information of the second player, determined by the nature) on the type space $T = [0, c_2]$ of the second player.

3.3. Reduced Strategy Set of the Second Player

Assumption 1. As a pretty reasonable strategy set of the second player, we restrict our attention to the convex strategy sub-space:

$$\mathbf{F} := \{y \in \mathcal{L}^1_q([0, c_2]) : y(t) \in [0, Q]\},$$

of all real bounded q-integrable functions on $[0, c_2]$ with values into $[0, Q]$.

Remark 3. *Note that the space $\mathcal{L}^1_q(T)$ is a space of functions and not a space of equivalence classes of functions, so that two functions can be different even if they are equal almost everywhere with respect to the measure q.*

Remark 4. *Moreover, we desire to notice that the Lebesgue measure does not play any special role here.*

3.4. Payoff Functions of the First Player

In order to write down the payoff function of the first player, let us indicate the first player stochastic payoff function by e. The payoff function:

$$e : [0, Q] \times \mathbf{F} \to \mathcal{L}^1_q([0, c_2]),$$

is defined, for every bi-strategy $(x, y) \in \mathbf{E} \times \mathbf{F}$, by:

$$e(x, y) = x(Q - x - y) - sx,$$

where s represents the (unique) marginal cost of the first player, known also with certainty by the second player (it is—in this case—public information); we could consider s as belonging to a certain possible cost compact interval $S = [0, c_1]$, but, in this case, s is a fixed constant, the only interesting point of S.

Note, now, that the payoff:

$$e(x, y) : T \to \mathbb{R}$$

is the function defined by:

$$e(x, y)(t) = x(Q - x - y(t)) - sx,$$

for every type $t \in [0, c_2]$ of the second player.

3.5. Expected Payoffs of the First Player

For every bi-strategy $(x, y) \in \mathbf{E} \times \mathbf{F}$, the function:

$$e(x, y) : T \to \mathbb{R} : t \mapsto x(Q - x - y(t)) - sx$$

is therefore a random variable on the probability space (T, q). The expected value of this random variable, with respect to the probability measure q, is:

$$\begin{aligned} u(x, y) &:= \mathbb{E}_q(e(x, y)) = \\ &= \int_T e(x, y)\, q = \\ &= q(e(x, y)) = \\ &= q(x(Q - x - y) - sx) = \\ &= x(Q - x - q(y)) - sx, \end{aligned}$$

for every (x, y) in $\mathbf{E} \times \mathbf{F}$, where by $q(\phi)$, we denoted the integral of any integrable (test) function $\phi \in \mathcal{L}^1_q([0, c_2])$:

$$q(\phi) := \int_T \phi\, q,$$

with respect to the probability measure q.

3.6. Payoff Function(s) of the Second Player

The payoff function f of the second player can be identified with a family:

$$(f_t)_{t \in T}.$$

The payoff function of the second player t^{th} type is defined by:

$$f_t : [0, Q] \times \mathcal{L}^1_q([0, c_2]) \to \mathbb{R} : f_t(x, y) = y(t)(Q - x - y(t)) - ty(t).$$

4. Results: Nash-Bayesian Analysis

In this section, we show the results of our first analysis.

4.1. Best Reply Correspondence of the First Player

Consider the first player. Fixing y in $\mathbf{F}$, let us study the partial derivative of the expected utility function u with respect to its first argument:

$$u_{,1}(x, y) = Q - 2x - q(y) - s.$$

It is positive when:

$$x < 2^{-1}(Q - q(y) - s).$$

Therefore, the best reply of the first player is the function:

$$B_1(y) = \max\{2^{-1}(Q - q(y) - s), 0\},$$

note that the best reply of the first player to the action y of the second player depends on y, and it belongs indeed to the compact interval:

$$[0, (Q - s)/2].$$

4.2. Best Reply Correspondence of the Second Player

We study the best reply correspondence of the second player's t^{th} type. Therefore, fixing $x \in \mathbf{E}$, let us study the partial derivative of the section $f_t(x, \cdot)$ with respect to its t^{th} argument:

$$\partial_t f_t(x, \cdot)(y) = f_t(x, \cdot)_{,t}(y) = Q - x - 2y(t) - t.$$

It is positive when:

$$y(t) < 2^{-1}(Q - x - t).$$

Therefore, the best reply of the second player's t^{th} type is the function:

$$B_{(2,t)}(x) = \max\{2^{-1}(Q - x - t), 0\} = \begin{cases} 2^{-1}(Q - x - t) & if \quad x < Q - t \\ 0 & if \quad x \geq Q - t \end{cases}.$$

Remark 5. *Note that the best reply of the t^{th} type second player, to the action x of the first player, depends on x, and it belongs to the strategy set $F_t = [0, Q]$; more specifically, it belongs to the set:*

$$[0, Q/2 - t/2].$$

4.3. Nash–Cournot-Bayesian Equilibrium

Let $(x^*, y^*) \in \mathbf{E} \times \mathbf{F}$ be a Nash equilibrium, if any of our game $G = (e, f)$, where f is the family $(f_t)_{t \in t}$. By definition, this means that:

$$x^* = B_1(y^*) \ \ \& \ \ y^*(t) = B_{(2,t)}(x^*),$$

for every t in T.

We shall study the case in which every component of the second player equilibrium strategy y^* is strictly positive, as well as the equilibrium strategy x^*. That is, we want, for every $t \in [0, c_2]$,

$$y^*(t) = 2^{-1}(Q - x^* - t) > 0 \ \ \& \ \ x^* = 2^{-1}(Q - q(y^*) - s) > 0.$$

This happens, for sure, when $Q > \max\{2c_1, 2c_2\}$.

Proof. Indeed, we immediately obtain:

$$2x^* = (Q - q(y^*)) - s \geq 2^{-1}Q - t > 2^{-1}Q - c_1 > 0,$$

for every s in $[0, c_1]$, taking into account that:

$$q(y^*) \in [0, 2^{-1}(Q - t)],$$

as concerns the equilibrium strategy x^*. Equivalently, we obtain:

$$2y^*(t) = (Q - x^*) - t \geq 2^{-1}Q - t > 2^{-1}Q - c_2 > 0,$$

for every t in $[0, c_2]$, taking into account that:

$$x^* \in [0, 2^{-1}(Q - s)],$$

if any such equilibrium (x^*, y^*) exists there. □

Theorem 1. *Let $G = (e, f)$ be the Cournot-Bayesian game defined above. Assume:*

$$Q \geq \max\{2c_1, 2c_2\},$$

in other words, we assume that the maximal reasonable individual (and collective) production is greater than the maximal possible double marginal costs. Then, the Nash equilibrium of the game G is the pair:

$$N = (Q/3, Q/3) + \mathbb{E}(q)(1/3, -1/6) + s(-2/3, 1/3) + (0, -\tau/2),$$

for any probability measure q on T, where τ is the identity function on T and:

$$\mathbb{E}(q) := \int_T \tau\, q = q(\tau)$$

is the mean-value of the probability measure q.

Proof of Theorem 1. Therefore, assume $Q \geq \max\{2c_1, 2c_2\}$; we know that:

$$\begin{aligned} x^* &= 2^{-1}(Q - q(y^*) - s) = \\ &= 2^{-1}(Q - q(2^{-1}(Q - x^* - \tau)) - s) = \\ &= 2^{-1}(Q - 2^{-1}(Q - x^* - q(\tau)) - s). \end{aligned}$$

Then, we deduce:

$$3x^* = Q + q(\tau) - 2s.$$

As concerns the equilibrium function y^*, we have only to plug in the equilibrium strategy:

$$x^* = (Q + q(\tau))/3$$

into the expression:

$$y^*(t) = 2^{-1}(Q - x^* - t)$$

obtaining:

$$\begin{aligned} y^*(t) &= 2^{-1}(Q - x^* - t) = \\ &= 2^{-1}(Q - 3^{-1}(Q + q(\tau) - 2s) - t) = \\ &= 6^{-1}(2Q - q(\tau) + 2s - 3t). \end{aligned}$$

Finally, the Nash equilibrium is the pair:

$$N = (3^{-1}(Q + q(\tau) - 2s), 6^{-1}(2Q - q(\tau) + 2s - 3\tau)).$$

In other terms:

$$N = (1/6)\left(2Q + 2\int_T \tau q - 4s,\ 2Q - \int_T \tau q + 2s - 3\tau\right),$$

as we desired. □

Remark 6. *By the way, we notice that:*

$$\begin{aligned} y^*(t) &= 2^{-1}(Q - x^* - t) = \\ &= 2^{-1}(Q - B_{-1}(y^*) - t) = \\ &= 2^{-1}(Q - 2^{-1}(Q - q(y^*)) - t) = \\ &= 4^{-1}(Q + q(y^*) - 2t). \end{aligned}$$

4.4. Examples

Now, we consider here some explicative examples.

Example 2. *Let q be the uniform distribution on T, that is the measure:*

$$q = (1/c_2)\lambda,$$

where λ is the Lebesgue measure on T.

We have:

$$\begin{aligned} N &= (Q/3, Q/3) + \mathbb{E}(q)(1/3, -1/6) + (0, -\tau/2) = \\ &= (Q/3, Q/3) + (c_2/2)(1/3, -1/6) + (0, -\tau/2). \end{aligned}$$

We can see the above equilibrium as a function:

$$N : T \to \mathbb{R}^2 : t \mapsto (Q/3, Q/3) + (c_2/2)(1/3, -1/6) + (0, -t/2).$$

Example 3. *In the conditions of the above Example 1, for $Q = 10$ and $c_2 = 4$, we obtain:*

$$N = (10/3, 10/3) + 2(1/3, -1/6) + (0, -\tau/2) = (4, 3 - \tau/2),$$

in particular $N(0) = (4, 3)$ and $N(4) = (4, 1)$.

Example 4. *Note that the same result is true for every probability distribution q such that $\mathbb{E}(q) = c_2/2$, for instance the discretely-supported Radon measure:*

$$q = 0.5\delta_0 + 0.5\delta_{c_2}$$

(where δ_t is the Dirac delta measure centered at t, for every t in T).

Example 5. *Let us consider the probability measure:*

$$q := a\delta_{t'} + b\delta_t,$$

with $a + b = 1$ and $t, t' \in T$. The mean value of q is:

$$\mathbb{E}(q) = at' + bt,$$

so that we obtain the following Nash equilibrium function profile:

$$N : T \to \mathbb{R}^2 : t'' \mapsto (Q/3, Q/3) + (at' + bt)(1/3, -1/6) + (0, -t''/2).$$

5. Analysis: Uncertainty on the Costs of Both Players

Type set of the first player: The type set of the first player is the marginal cost interval:

$$S := [0, c_1].$$

Type set of the second player: The type set of the second player is the marginal cost interval:

$$T := [0, c_2].$$

5.1. Strategy Set of the First Player

The first player's s^{th} type adopts the strategy set:

$$E_s := [0, Q];$$

so that, the first player's total strategy profile space is the Cartesian product:

$$\times_{s \in S} E_s = [0, Q]^S,$$

the set of all functions from S into $[0, Q]$. In other terms, any strategy of the second player is an infinite profile strategy:

$$x : S \to [0, Q],$$

specifying, for any type $s \in S$, a certain production $x(s)$ in $[0, Q]$.

5.2. Radon Measure Probability of the First Player

Assume that:

$$p : C^0(S) \to \mathbb{R}$$

is a probability measure (private information of the first player, determined by the nature) on the compact type space S of the first player.

5.3. Reduced Strategy Set of the First Player

Assumption 2. As a pretty reasonable strategy set of the first player, we restrict our attention to the convex strategy sub-space:

$$\mathbf{E} := \{x \in \mathcal{L}^1_p(S) : x(s) \in [0, Q]\},$$

of all real bounded p-integrable functions on S with values into $[0, Q]$.

5.4. Strategy Set of the Second Player

The second player's t^{th} type has got the strategy set:

$$F_t := [0, Q];$$

so that the second player's total strategy profile space is the Cartesian product:

$$\times_{t \in T} F_t = [0, Q]^T,$$

the set of all functions from T into $[0, Q]$. In other terms, any strategy of the second player is an infinite profile strategy:

$$y : T \to [0, Q],$$

specifying, for any type $t \in T$, a certain production $y(t)$ in $[0, Q]$.

5.5. Radon Measure Probability of the Second Player

Assume that:

$$q : C^0(T) \to \mathbb{R}$$

is a probability measure (private information of the second player, determined by the nature) on the compact type space T of the second player.

5.6. Reduced Strategy Set of the Second Player

Assumption 3. As a pretty reasonable strategy set of the second player, we restrict our attention to the convex strategy sub-space:

$$\mathbf{F} := \{y \in \mathcal{L}^1_q(T) : y(t) \in [0, Q]\},$$

of all real bounded q-integrable functions on T with values into $[0, Q]$.

Note that the space $\mathcal{L}^1_q(T)$ is a space of functions and not a space of equivalence classes of functions, so that two functions can be different even if they are equal almost everywhere with respect to the measure q.

Now, we can determine the payoff functions of the players.

5.7. Payoff Functions of the First Player

In order to write down the payoff function of the first player, let us indicate the first player's type s by the index s and its stochastic payoff function by e_s. The payoff function:

$$e_s : \mathbf{E} \times \mathbf{F} \to \mathcal{L}^1_q(T),$$

defined, for every $(x, y) \in \mathbf{E} \times \mathbf{F}$ by:

$$e_s(x, y) = x(s)(Q - x(s) - y) - sx(s).$$

Note, now, that the payoff:

$$e_s(x, y) = x(s)(Q - x(s) - y) - sx(s)$$

is the function (random variable):

$$e_s(x, y) : T \to \mathbb{R}$$

defined by:

$$e_s(x, y)(t) = x(s)(Q - x(s) - y(t)) - sx(s),$$

for every type $t \in T$ of the second player.

5.8. Expected Payoffs of the First Player

For every bi-strategy $(x, y) \in \mathbf{E} \times \mathbf{F}$, the function $e_s(x, y)$ is therefore a random variable on the probability space (T, q). The expected value of this random variable, with respect to the probability measure q, is:

$$\begin{aligned}
u_s(x, y) &= \mathbb{E}_q(e_s(x, y)) = \\
&= \int_T q\, e_s(x, y) = \\
&= q(e_s(x, y)) = \\
&= q(x(s)(Q - x(s) - y) - sx(s)) = \\
&= x(s)(Q - x(s) - q(y)) - sx(s),
\end{aligned}$$

for every (x, y) in $\mathbf{E} \times \mathbf{F}$, where by $q(\phi)$, we denoted the integral of any function $\phi \in \mathcal{L}^1_q(T)$:

$$q(\phi) := \int_T \phi \, q.$$

5.9. Payoff Functions of the Second Player

The payoff function of the second player's t^{th} type is defined by:

$$f_t : \mathbf{E} \times \mathbf{F} \to \mathcal{L}^1_p(S) : f_t(x, y) = y(t)(Q - x - y(t)) - ty(t),$$

for every $(x, y) \in \mathbf{E} \times \mathbf{F}$.

5.10. Expected Payoffs of the Second Player

For every bi-strategy $(x, y) \in \mathbf{E} \times \mathbf{F}$, the function:

$$f_t(x, y) : S \to \mathbb{R}$$

is therefore a random variable on the probability space (S, p). The expected value of this random variable, with respect to the probability measure p, is:

$$\begin{aligned} v_t(x, y) &= \int_S p \, f_t(x, y) = \\ &= \mathbb{E}_p(f_t(x, y)) = \\ &= p(f_t(x, y)) = \\ &= p(y(t)(Q - x - y(t)) - ty(t)) = \\ &= y(t)(Q - p(x) - y(t)) - ty(t), \end{aligned}$$

for every (x, y) in $\mathbf{E} \times \mathbf{F}$, where by $p(\phi)$, we denoted the integral of any function $\phi \in \mathcal{L}^1_p(S)$:

$$p(\phi) := \int_S \phi \, p.$$

6. Results: Nash–Cournot Bayesian Analysis

In this section, we state and prove the main general result of the paper.

6.1. Best Reply Correspondence of the First Player

Consider the first player's s-type. Fixing y in the function strategy space $\mathbf{F}$, let us study the partial derivative of the section function:

$$u_s(\cdot, y) : \mathbf{E} \to \mathbb{R} : x \mapsto x(s)(Q - x(s) - q(y)) - sx(s)$$

with respect to its s^{th} argument:

$$u_s(\cdot, y)_{,s}(x) = \partial_s[u_s(\cdot, y)](x) = Q - 2x(s) - q(y) - s;$$

it is positive when:

$$x(s) < 2^{-1}(Q - q(y) - s).$$

Therefore, consider the part $\mathbf{F}_{(s,<)}$ of $\mathbf{F}$ below:

$$\mathbf{F}_{(s,<)} := \{y \in \mathbf{F} : Q - q(y) - s \geq 0\}.$$

The best reply correspondence of the first player's s-type is the function:

$$B_{(1,s)} : \mathbf{F} \to E_s,$$

defined by:

$$B_{(1,s)}(y) = 2^{-1}(Q - q(y) - s),$$

for every integrable profile strategy $y \in \mathbf{F}_{(s,<)}$ and zero otherwise.

Remark 7. *Note that the best reply of the first player to the action y of the second player depends only on the average:*

$$q(y) = \int_T y\, q,$$

of the random variable y with respect to the probability measure q, and it belongs indeed to the compact interval:

$$2^{-1}[0, Q - s].$$

6.2. Best Reply Correspondence of the Second Player

Consider the second player $(T, q, \mathbf{F}, f)$, and fix a type t of him/her. Fixing an integrable strategy profile x, in the first player's function convex space $\mathbf{E}$, let us study the partial derivative of the section:

$$v_t : (x, y) \mapsto p(f_t(x, y)) = y(t)(Q - p(x) - y(t)) - ty(t),$$

with respect to its t^{th} argument.

We immediately obtain:

$$v_{t,t}\,(x, \cdot)(y) = \partial_t[v_t(x, \cdot)](y) = Q - p(x) - 2y(t) - t,$$

for every y in $\mathbf{F}$.

This derivative is positive when:

$$y(t) < 2^{-1}(Q - p(x) - t).$$

Therefore, consider the part $\mathbf{E}_{(t,<)}$ of $\mathbf{E}$ below:

$$\mathbf{E}_{(t,<)} := \{y \in \mathbf{F} : Q - p(x) - t \geq 0\}.$$

The best reply correspondence of the second player's t-type is the function:

$$B_{(2,t)} : \mathbf{E} \to F_t,$$

defined by:

$$B_{(2,t)}(x) = 2^{-1}(Q - p(x) - t).$$

for every integrable profile strategy $x \in \mathbf{E}_{(t,<)}$ and zero otherwise.

Remark 8. *Note that the best reply of the first player to the action x of the second player depends on the average $p(x)$, and it belongs, indeed, to the compact sub-interval:*

$$2^{-1}[0, Q - t].$$

6.3. Nash–Cournot-Bayesian Equilibrium

Let (x^*, y^*) be a Nash equilibrium, if any of our game $G = (u, v)$. This is equivalent to asserting that $(x^*, y^*) \in \mathbf{E} \times \mathbf{F}$ is a Bayesian-Nash equilibrium, if any of our game $G = (e, f)$, where e, f are the random variable families:

$$e = (e_s)_{s \in S}$$

and

$$f = (f_t)_{t \in t}.$$

By definition, that means that:

$$x^*(s) = B_{(1,s)}(y^*) \ \ \& \ \ y^*(t) = B_{(2,t)}(x^*),$$

for every types s, t in S, T, respectively.

We consider here the case in which, for every $(s, t) \in S \times T$, we observe the two conditions:

$$y^*(t) = 2^{-1}(Q - p(x^*) - t) \geq 0,$$

and

$$x^*(s) = 2^{-1}(Q - q(y^*) - s) \geq 0.$$

The above situation happens, for sure, when:

$$Q > \max\{2c_1, 2c_2\}.$$

Indeed,

$$2y^*(t) = (Q - p(x^*)) - t \geq 2^{-1}Q - t \geq 2^{-1}Q - c_2 > 0,$$

for every t in $[0, c_2]$, taking into account that:

$$p(x^*) \in [0, 2^{-1}Q],$$

and:

$$2x^*(s) = (Q - q(y^*)) - s \geq 2^{-1}Q - s \geq 2^{-1}Q - c_1,$$

for every s in $[0, c_1]$, taking into account that:

$$q(y^*) \in [0, 2^{-1}Q],$$

if any equilibrium strategies exist there.

Now, we can state and prove our main general result.

Theorem 2. *Let G be the Cournot-Bayesian game $G = (u, v)$ defined above. Assume:*

$$Q > \max\{2c_1, 2c_2\}.$$

In other words, we assume that the limit production is greater than both the double maximum marginal cost. Then, the Nash equilibrium of G is the function pair (belonging to the bi-strategy space $\mathbf{E} \times \mathbf{F}$):

$$N = (Q/3, Q/3) + (1/6)(2\mathbb{E}(q) - \mathbb{E}(p), 2\mathbb{E}(p) - \mathbb{E}(q)) - (1/2)(\sigma, \tau),$$

for any probability measure p on S and q on T, where:

$$\sigma : S \to S : s \mapsto s$$

is the identity function on the type set S,

$$\tau : T \to T : t \mapsto t$$

is the identity function on the type set T, and:

$$\mathbb{E}(p) := p(\sigma),\ \mathbb{E}(q) := q(\tau)$$

are the mean-values of the probability Radon measures p, q.

Proof of Theorem 2. Therefore, assume:

$$Q > \max\{2c_1, 2c_2\}.$$

We know that:

$$x^*(s) = 2^{-1}(Q - q(y^*) - s),$$

from which:

$$x^* = 2^{-1}(Q - q(y^*) - \sigma). \tag{1}$$

Symmetrically, for the second equilibrium strategy function y^*, we have:

$$y^* = 2^{-1}(Q - p(x^*) - \tau). \tag{2}$$

We immediately deduce, by applying p to (1) and q to (2):

$$p(x^*) = 2^{-1}(Q - q(y^*) - p(\sigma))$$

and:

$$q(y^*) = 2^{-1}(Q - p(x^*) - q(\tau)).$$

From the latter and from:

$$x^* = 2^{-1}(Q - q(y^*) - \sigma)$$

we deduce:

$$4x^* = 2Q - (Q - p(x^*) - q(\tau)) - 2\sigma = Q + p(x^*) + q(\tau) - 2\sigma.$$

By applying p to the preceding, we have:

$$4p(x^*) = Q + p(x^*) + q(\tau) - 2p(\sigma),$$

that is:

$$3p(x^*) = Q + q(\tau) - 2p(\sigma).$$

By substituting this last expression of $p(x^*)$ in (2), we obtain:

$$\begin{aligned} y^* &= (1/2)(Q - (1/3)(Q + q(\tau) - 2p(\sigma)) - \tau) = \\ &= (1/6)(2Q - q(\tau) + 2p(\sigma) - 3\tau). \end{aligned}$$

Symmetrically, we have:

$$x^* = (1/6)(2Q - p(\sigma) + 2q(\tau) - 3\sigma).$$

Finally, the Nash equilibrium is the pair:

$$N = (1/6)(2Q - p(\sigma) + 2q(\tau) - 3\sigma, 2Q - q(\tau) + 2p(\sigma) - 3\tau).$$

In other terms:

$$N = (1/6)\left(2Q + 2\int_T \tau q - \int_S \sigma p - 3\sigma, 2Q + 2\int_S \sigma p - \int_T \tau q - 3\tau\right),$$

as we desired. □

6.4. Examples

Now, we consider here some explicative examples.

Example 6 (Uniform probability distributions). *Let p and q be the uniform distributions on S and T, respectively, that is the measure:*

$$p = (1/c_1)\lambda_S$$

and:

$$q = (1/c_2)\lambda_T,$$

where λ_S and λ_T are the Lebesgue measure on S and T, respectively.

We obtain:

$$\begin{aligned} N &= (Q/3, Q/3) + \mathbb{E}(q)(1/3, -1/6) + \mathbb{E}(p)(-1/6, 1/3) + (-\sigma/2, -\tau/2) = \\ &= (Q/3, Q/3) + (c_1/2)(1/3, -1/6) + (c_2/2)(-1/6, 1/3) + (-\sigma/2, -\tau/2). \end{aligned}$$

We can see the above equilibrium (pair of two real functions) as a compact parametric surface in the Cartesian plane:

$$\mathcal{N} : S \times T \to \mathbb{R}^2$$

defined by:

$$(s, t) \mapsto (Q/3, Q/3) + (c_1/2)(1/3, -1/6) + (c_2/2)(-1/6, 1/3) + (-s/2, -t/2).$$

Example 7. *In the conditions of the above examples. For $Q = 10$, $c_1 = 6$ and $c_2 = 4$, we have:*

$$N = (10/3, 10/3) + 3(1/3, -1/6) + 2(-1/6, 1/3) + (-\sigma/2, -\tau/2) = (4 - \sigma/2, 7/2 - \tau/2),$$

in particular $\mathcal{N}(0, 0) = (4, 7/2)$ and $\mathcal{N}(4, 4) = (2, 3/2)$.

Example 8. *Note that the same result is true for every probability distribution q such that:*

$$\mathbb{E}(q) = \int_T \tau q = c_2/2,$$

and:

$$\mathbb{E}(p) = \int_S \sigma p = c_1/2,$$

for instance, the discretely-supported Radon measure:

$$q = 0.5\delta_0 + 0.5\delta_{c_2}$$

and:

$$p = (2/5)\delta_{c_1/4} + (3/5)\delta_{2c_1/3}$$

(where δ_t is the Dirac delta measure centered at a point t in T).

Example 9. *Let us consider the probability measure:*

$$q := a\delta_{t'} + b\delta_t,$$

and:

$$p := (2/5)\delta_{c_1/4} + (3/5)\delta_{2c_1/3},$$

with a, b positive real numbers such that $a + b = 1$. The mean value of q is:

$$\mathbb{E}(q) = at' + bt,$$

and:

$$\mathbb{E}(p) = c_1/2$$

so that, from the formula:

$$N = (Q/3, Q/3) + (1/6)(-\mathbb{E}(p), 2\mathbb{E}(p)) + 1/6(2\mathbb{E}(q), -\mathbb{E}(q)) - (1/2)(\sigma, \tau),$$

we obtain the associated compact parametric surface:

$$\mathcal{N} : S \times T \to \mathbb{R}^2$$

defined by:

$$(s, t'') \mapsto (Q/3, Q/3) + (c_1/2)(-1/6, 1/3) + (at' + bt)(1/3, -1/6) + (-s/2, -t''/2).$$

7. Discussion and Conclusions

We have considered a Cournot duopoly, in which any firm does not know the marginal costs of production of the other player, as a Bayesian game. In our game, the marginal costs depend on two infinite continuous sets of states of the world. We studied, before the general case, an intermediate case in which only one player, the second one, shows infinitely many types. Then, we have generalized to the case in which both players show infinitely many types depending on the marginal costs, where the marginal costs are given by the nature and each actual marginal cost is known only by the respective player. We found, in both cases, the general Nash equilibrium.

From the perspective of previous studies, we observe that our model constitutes a real step forward in the economic applicability of the duopoly model (see for the classical models [9,10]). It is indeed clear that the marginal costs and other characteristic coefficients and constants of a production process are completely known only by the owner of the process itself and not by the competitors. Future research directions may also be devised for the use of others stochastic variables instead of the other characteristic constants of the classic duopoly games.

Possible Application to the Auto Insurance Market

One possible field of application is the auto insurance market. There, uncertain costs arise from a variety of reasons: uncertain claims; fluctuating input prices; legal risk; exchange rate uncertainty; etc. Now, the marginal costs are particularly important for insurance companies, because the firm's solvency directly depends on its ability to accurately predict losses. On the other hand, insurance companies exist because the pooling of many risks reduces aggregate uncertainty, thereby aggregate losses could become predictable and loss sharing feasible. In practice, however, predictability appears incomplete and unsatisfactory, even when the number of insured individuals is large, leading to marginal cost uncertainty, for all the insurance players in the market. Provided that the marginal costs (of a certain insurance), at the end of the day, come (although approximately) from internal deep statistical, managerial, and business analysis, any insurance needs to consider the other insurance's costs as random variables with a forecasted probability distribution: here, they could use our model to calculate the expected competitive equilibrium strategies and relative payoffs.

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