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MONOTONE NORMALITY AND RELATED PROPERTIES

Fortunata Aurora Basile [a], Maddalena Bonanzinga [a*]
and Maria Vittoria Cuzzupè [a]

ABSTRACT. In this paper monotone versions of some results on normality and on property (a) are investigated.

1. Introduction

By a space we mean a topological space. Assume all spaces to be T_1. If $A \subseteq X$ and $\mathscr{U}$ is a collection of subsets of X, then $St(A,\mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}$. A cover $\mathscr{V}$ of a space X *refines* another cover $\mathscr{U}$ of the same space if for every $V \in \mathscr{V}$ there exists a $U \in \mathscr{U}$ such that $V \subset U$. We say that a cover $\mathscr{V}$ of a space X is a *star refinement* of another cover $\mathscr{U}$ of the same space if for every $V \in \mathscr{V}$ there exists a $U \in \mathscr{U}$ such that $St(V,\mathscr{V}) \subset U$.

Recall that a space X is *monotonically normal* if for each pair (H,K) of disjoint closed subsets of X, one can assign an open set $r(H,K)$ such that

(1) $H \subset r(H,K) \subset \overline{r(H,K)} \subset X \setminus K$
(2) if $H_1 \subset H_2$ and $K_1 \supset K_2$ then $r(H_1,K_1) \subset r(H_2,K_2)$.

The function r is called a *monotone normality operator for* X. All linearly ordered and generalized ordered spaces are monotonically normal and monotonically normal spaces are countably paracompact and collectionwise normal (Gruenhage 1984).

Monotone normality was first examined by Borges (1973) and later widely studied in several papers (see, for example, Heath *et al.* 1973; Moody and Roscoe 1992; Gartside 1993; Rudin 1993). We are interested in monotone versions of two characterizations of normality: the Urysohn's Lemma (Engelking 1989) that gives a characterization of normality in terms of functions and the following characterization of normality in terms of stars:

Theorem 1. (Engelking 1989) The following conditions are equivalent for a topological space X

(1) X is normal;
(2) Every two-element open cover of X has an open star-refinement;
(3) Every finite open cover of X has a finite open star-refinement.

This paper is dedicated to Mikhail "Misha" Matveev.

In Section 2 we prove a monotone version of Urysohn's Lemma (this result was first mentioned without proof by Borges (1973)), and give a partial solution about finding a monotone version of Theorem **3**.

The style of the definition of monotone normality was adapted and applied to other kinds of properties. Hart (1993) described a process for obtaining a monotone version of any well-known covering property: "by requiring that there is an operator, r, assigning to every open cover a refinement in such a way that $r(\mathscr{V})$ refines $r(\mathscr{U})$ whenever $\mathscr{V}$ refines $\mathscr{U}$". Monotone versions of covering properties have been studied in literature (see Gartside and Moody 1993; Bennett *et al.* 2005; Levy and Matveev 2008; Popvassilev 2009; Bennett *et al.* 2010; Bonanzinga *et al.* 2011).

Matveev (1997) introduced the following covering property: A space X has X has *property (a)* if for every open cover $\mathscr{U}$ of X and every dense $D \subset X$ there is a closed in X and discrete $F \subset D$ such that $St(F,\mathscr{U}) = X$. Rudin *et al.* (1997) proved that "every monotonically normal space has property (a)". It is natural to ask if it is possible to define a monotone version of property (a) in order to prove a monotone version of the previous result. In Section 3 we consider all possible monotone versions of property (a) and give a negative answer to that problem. We also consider monotone versions of covering properties strictly related to property (a).

2. Around monotone normality

Recall the following definition due to P. Zenor:

Definition 2. (Zenor 1970) A space X is monotonically normal if for every closed set $F \subset X$ and every neighborhood $U \supset F$, one can find a neighborhood $r(U,F)$ such that

(1) $F \subset r(U,F) \subset \overline{r(U,F)} \subset U$
(2) if $F_1 \subset F_2$ and $U_1 \subset U_2$ then $r(F_1,U_1) \subset r(F_2,U_2)$.

Equivalently (see Heath *et al.* 1973), a space X is monotonically normal iff one can assign to every $x \in X$ and every neighborhood $U \ni x$ a neighborhood $H(x,U)$ so that:

(1) if $H(x,U) \cap H(y,V) \neq \emptyset$ then either $x \in V$ or $y \in U$;
(2) if $x \in U \subset W$ then $H(x,U) \subset H(x,W)$.

We will give a characterization of monotone normality in terms of functions. Recall that, by Urysohn's Lemma (see Engelking 1989), X is normal iff for every pair of disjoint non empty closed sets F and H there is an $f \in C(X,I)$ such that $f(x) = 0$ for every $x \in F$ and $f(x) = 1$ for every $x \in H$.

Theorem 3. A space X is monotonically normal iff one can assign to every pair of disjoint non empty closed sets F and H a function $f_{F,H} \in C(X,I)$ so that

(1) $f_{F,H}(x) = 0$ for every $x \in F$ and $f_{F,H}(x) = 1$ for every $x \in H$;
(2) $f_{F_2,H_2} \leq f_{F_1,H_1}$ whenever $F_1 \subset F_2$ and $H_1 \supset H_2$.

Proof. Suppose that the function $f_{F,H}$ such as in the theorem is given. Define

$$r(F,H) = f_{F,H}^{-1}([0,1/2)).$$

Assume that we have two pairs of disjoint closed sets F_1, H_1 and F_2, H_2 such that $F_1 \subset F_2$ and $H_2 \subset H_1$. Then if $p \in r(F_1, H_1)$ we have $f_{F_1,H_1}(p) < 1/2$. It means that $f_{F_2,H_2}(p) \leq f_{F_1,H_1}(p) < 1/2$, i.e., $p \in r(F_2, H_2)$.

For the other one, fix an enumeration $\{q_n : n \in \mathbb{N}\}$ of $Q = \mathbb{Q} \cap [0,1]$; we assume that all q_n are different and moreover that $q_1 = 0$ and $q_2 = 1$. Let F and H be arbitrary disjoint closed subsets of X. As in the standard proof of Urysohn's Lemma, we will construct for every $n \in \mathbb{N}$ an open subset V_{q_n} of X such that

(1) $\overline{V}_{q_n} \subset V_{q_m}$ whenever $q_n < q_m$,
(2) $F \subset V_{q_1}$ and $H \subset X \setminus V_{q_1}$.

We put $V_{q_1} = r(F,H)$ and $V_{q_2} = X \setminus H$. Then (1) and (2) are clearly satisfied. Let $n > 2$ and assume that V_{q_i} are defined for all $i < n$. Put

$$r = \max\{q_i : i < n, q_i < q_n\} \text{ and } s = \min\{q_i : i < n, q_n < q_i\}.$$

Then by our inductive hypothesis we have $\overline{V}_r \subseteq V_s$. Put $V_{q_n} = r(\overline{V}_r, X \setminus V_s)$. Then our inductive hypotheses are clearly satisfied. We now define, as in the proof of Urysohn's Lemma, the function $f_{F,H}$ by the formula:

$$f_{F,H}(x) = \begin{cases} \inf\{q_n \in Q : x \in V_{q_n}\} & \text{if } x \in V_{q_2}, \\ 1 & \text{if } x \notin V_{q_2}. \end{cases}$$

Then $f_{F,H}$ is continuous.

To prove that this assignment of functions is 'monotone', consider two pairs of disjoint closed sets F_1, H_1 and F_2, H_2 such that $F_1 \subset F_2$ and $H_2 \subset H_1$. For F_1, H_1 we use the above notation V_{q_n}, and for F_2, H_2 we use the notation W_{q_n}. Observe that $V_{q_1} = r(F_1, H_1)$ and $V_{q_2} = X \setminus H_1$. Moreover, $W_{q_1} = r(F_2, H_2)$ and $W_{q_2} = X \setminus H_2$. Observe that

$$V_{q_1} \subset W_{q_1} \text{ and } V_{q_2} \subset W_{q_2}.$$

Claim. For every $n \in \mathbb{N}$, $V_{q_n} \subset W_{q_n}$.

For $n = 1, 2$ there is nothing to prove. Let $n > 2$ and assume that we have what we want for all $i < n$. In the above inductive construction, we put

$$r = \max\{q_i : i < n, q_i < q_n\} \text{ and } s = \min\{q_i : i < n, q_n < q_i\},$$

and $V_{q_n} = r(\overline{V}_r, X \setminus V_s)$ and $W_{q_n} = r(\overline{W}_r, X \setminus W_s)$. By our inductive hypothesis we have

$$V_r \subset W_r \text{ and } V_s \subset W_s,$$

hence

$$\overline{V}_r \subset \overline{W}_r \text{ and } X \setminus W_s \subset X \setminus V_r,$$

and so

$$V_{q_n} = r(\overline{V}_r, X \setminus V_s) \subset r(\overline{W}_r, X \setminus W_s) = W_{q_n}.$$

This completes the proof of the claim.

Now assume that $x \in X$. We want to prove that $f_{F_2,H_2}(x) \leq f_{F_1,H_1}(x)$. Assume first that $f_{F_1,H_1}(x) < 1$. If $x \notin V_{q_2}$, then $f_{F_1,H_1}(x) = 1$, which is impossible. Hence $x \in V_{q_2}$ and so $x \in W_{q_2}$. For every q_n such that $x \in V_{q_n}$ we also have that $x \in W_{q_n}$. From this we see that

$$\{q_n \in Q : x \in V_{q_n}\} \subset \{q_n \in Q : x \in W_{q_n}\},$$

and so

$$f_{F_2,H_2}(x) = \inf\{q_n \in Q : x \in W_{q_n}\} \leq \inf\{q_n \in Q : x \in V_{q_n}\} = f_{F_1,H_1}(x).$$

If $f_{F_1,H_1}(x) = 1$, then there is nothing to prove and so we are done. □

There are a lot of properties that can be defined or characterized in terms of stars. Theorem **3** gives a star-characterization of normality. It is natural to ask if it is possible to give a monotone version of Theorem **3**. In this section we give a partial solution to this problem.

The following definitions give monotone versions of (2) and (3) in Theorem 1, respectively.

Definition 4. A space X is *2-monotonically star-normal* if there exists an operator that assigns to every two-element open cover $\mathscr{U}$ an open star refinement $r(\mathscr{U})$ so that $r(\mathscr{U})$ refines $r(\mathscr{V})$ whenever $\mathscr{U}$ refines $\mathscr{V}$. The function r is called *2-monotone star-normality operator* for X.

Definition 5. A space X is *finitely-monotonically star-normal* if there exists an operator that assigns to every finite open cover $\mathscr{U}$ a finite open star refinement $r(\mathscr{U})$ so that $r(\mathscr{U})$ refines $r(\mathscr{V})$ whenever $\mathscr{U}$ refines $\mathscr{V}$. The function r is called *finite-monotone star-normality operator* for X.

Note that both the previous definitions are weak forms of monotone paracompactness. Recall that a space X is *monotonically paracompact* if there exists a function r which assigns to every open cover $\mathscr{U}$ an open star-refinement $r(\mathscr{U})$ such that if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U})$ refines $r(\mathscr{V})$ (Popvassilev and Porter 2014).

Introduce the following useful definition. For a cover $\mathscr{A}$ of a set X, let $\mathscr{A}^b = \{St(x,\mathscr{A}) : x \in X\}$.

Definition 6. A space X satisfies *property* $(*)$ if for each binary open cover $\mathscr{U}$ of X, there is an open cover $r(\mathscr{U})$ of X such that $r(\mathscr{U})^b$ refines $\mathscr{U}$ (i.e., $r(\mathscr{U})$ is a "barycentric" open refinement of $\mathscr{U}$), and $r(\mathscr{V})$ refines $r(\mathscr{U})$ whenever $\mathscr{V}$ refines $\mathscr{U}$. The function r is called $(*)$*-operator* for X.

The following fact is obvious.

Proposition 7. A 2-monotonically star-normal space has property $(*)$.

Proposition 8. A space with property $(*)$ is monotonically normal.

Proof. Let X be a space having property $(*)$ and let r be a $(*)$-operator for X. For a point $x \in X$ and an open neighborhood U of x, consider the binary open cover $\mathscr{U}(x,U) = \{U, X \setminus \{x\}\}$. Let $H(x,U) = St(x, r(\mathscr{U}(x,U)))$. Obviously $x \in H(x,U) \subset U$. Let U be an open neighborhood of $x \in X$ and let V be an open neighborhood of $y \in X$. Assume $y \notin U$ and $x \notin V$. We show $H(x,U) \cap H(y,V) = \emptyset$. Consider the binary open cover $\mathscr{W} = \{X \setminus \{x\}, X \setminus \{y\}\}$. Since both $\mathscr{U}(x,U)$ and $\mathscr{U}(y,V)$ are refinements of $\mathscr{W}$, both $r(\mathscr{U}(x,U))$ and $r(\mathscr{U}(y,V))$ are refinements of $r(\mathscr{W})$. Hence, we have $H(x,U) \cap H(y,V) \subset St(x, r(\mathscr{W})) \cap St(y, r(\mathscr{W}))$. Assume that there is a point $z \in St(x, r(\mathscr{W})) \cap St(y, r(\mathscr{W}))$. Then there are some $W_0, W_1 \in$

$r(\mathscr{W})$ such that $\{x,z\} \subset W_0$ and $\{y,z\} \subset W_1$. Since $r(\mathscr{W})$ is a barycentric refinement of $\mathscr{W}$, $St(z, r(\mathscr{W}))$ is contained in $X \setminus \{x\}$, or $X \setminus \{y\}$. This is a contradiction, because $\{x,y\} \subset St(z, r(\mathscr{W}))$. □

Corollary 9. A 2-monotonically star-normal space is monotonically normal.

Question 10. Does monotone normality imply property $(*)$?

By propositions 7 and 8, a negative answer to Question 10 permits to prove that monotone normality and 2-monotone star-normality are not equivalent conditions.

Now introduce the following useful definition:

Definition 11. A space X has *property* $(**)$ if for each finite open cover $\mathscr{U}$ of X, there is an open cover $r(\mathscr{U})$ of X such that $r(\mathscr{U})^b$ refines $\mathscr{U}$, and $r(\mathscr{V})$ refines $r(\mathscr{U})$ whenever $\mathscr{V}$ refines $\mathscr{U}$.

The following fact is obvious.

Proposition 12. A finite-monotonically star-normal space has property $(**)$.

Also recall the following property which is stronger than monotone normality.

Definition 13. (Moody and Roscoe 1992) A space X is *acyclically monotonically normal* if it has a monotonically normal operator r such that for distinct points $x_0, \ldots, x_{n-1}$ in X and $x_n = x_0$, $\bigcap_{t=0}^{n-1} r(x_t, X \setminus \{x_{t+1}\}) = \emptyset$.

Proposition 14. A space with property $(**)$ is acyclically monotonically normal.

Proof. Similar to the proof of Proposition 14. □

Since Rudin (1993) constructed a monotonically normal space which is not acyclically monotonically normal, monotone normality does not imply property $(**)$ (hence finite monotone star normality).

Then, monotone normality and finite-monotone star-normality are not equivalent conditions.

Question 15. Are 2-monotone star-normality and finite-monotone star-normality equivalent conditions?

Question 16. Which properties imply the finite-monotonically star-normal property? In particular, is every LOTS a finite-monotonically star-normal space?

3. Monotone versions of property (a) and related spaces

Matveev (1997) gave the following definition:

Definition 17. (Matveev 1997) A space X is an *(a)-space*, or has *property (a)* if for every open cover $\mathscr{U}$ and every dense $D \subset X$ there is a closed in X and discrete $F \subset D$ such that $St(F, \mathscr{U}) = X$.

Matveev asked if monotonically normal spaces have property (a). Rudin *et al.* (1997) answered in the affirmative to this question.

Theorem 18. (Rudin *et al.* 1997) Monotonically normal spaces satisfy property (a).

Motived by the previous result it is natural to pose the following question

Question 19. Is it possible to define a monotone version of property (a) in order to prove that monotone normality implies such a property?

Logically there are four ways to give the definition of the monotone version of the property (a). The following two results shows that two of them are trivial.

Proposition 20. Let X be a space. If there exists a function r that assigns to every open cover $\mathscr{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathscr{U},D) \subset D$ such that $St(r(\mathscr{U},D),\mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U},D) \subseteq r(\mathscr{V},D)$, then X is discrete.

Proof. Let X be a space, D a dense subset of X and r the same as in the hypothesis. Let $\mathscr{U} = \{X\}$ be the trivial cover of X and $F = r(\mathscr{U},D)$ be the closed in X and discrete such that $St(F,\mathscr{U}) = X$.

Claim: $X = F$.

Assume the contrary and fix $a \in X \setminus F$. Put $V = X \setminus \{a\}$ and $U = X \setminus F$. Clearly, the sets U and V are open in X and $\mathscr{C} = \{U,V\}$ covers X. Since $\mathscr{C}$ refines $\mathscr{U}$, we have $E \subset F$, where $E = r(\mathscr{C},D)$. Therefore, $St(E,\mathscr{C}) \neq X$, since $St(E,\mathscr{C})$ does not contain the point a; a contradiction. □

Corollary 21. Let X be a space. If there exists a function r that assigns to every open cover $\mathscr{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathscr{U},D) \subset D$ such that $St(r(\mathscr{U},D),\mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$ and $D \subseteq E$ then $r(\mathscr{U},D) \subseteq r(\mathscr{V},E)$, then X is discrete.

The following two monotone versions of property (a) are less exceptional.

Definition 22. A space X is:

- **•:** $\mathbf{sm}(\mathbf{a})$ or has *strongly monotone property (a)* if there exists a function r that assigns to every open cover $\mathscr{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathscr{U},D) \subset D$ such that $St(r(\mathscr{U},D),\mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$ and $D \subseteq E$ then $r(\mathscr{U},D) \supseteq r(\mathscr{V},E)$. The function r is called $\mathbf{sm}(\mathbf{a})$ *operator* for X.
- **•:** $\mathbf{m}(\mathbf{a})$ or has *monotone property (a)* if there exists a function r that assigns to every open cover $\mathscr{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathscr{U},D) \subset D$ such that $St(r(\mathscr{U},D),\mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U},D) \supseteq r(\mathscr{V},D)$. The function r is called $\mathbf{m}(\mathbf{a})$ *operator* for X. A space having property $\mathbf{m}(\mathbf{a})$ is called $\mathbf{m}(\mathbf{a})-$*space*.

Clearly, $\mathbf{sm}(\mathbf{a}) \Rightarrow \mathbf{m}(\mathbf{a})$. Also every $\mathbf{m}(\mathbf{a})$ space has property (a) but an (a)-space need not be a $\mathbf{m}(\mathbf{a})$ space: consider ω_1 (see Example 31).

Recall that a space X is non-Archimedean if it has an open base which is a tree under reverse inclusion. Bennett *et al.* (2002) called a space monotonically ultraparacompact if there is a monotonically ultraparacompact operator m assigning to every open cover $\mathscr{U}$ of the space an open disjoint refinement $m(\mathscr{U})$ that is also an open cover of the space and such that if $\mathscr{U} \prec \mathscr{V}$ then $m(\mathscr{U}) \prec m(\mathscr{V})$.

Theorem 23. (see Popvassilev and Porter 2014) Monotonically ultraparacompact spaces concide with non-Archimedean spaces.

Theorem 24. Every monotonically ultraparacompact space is **(ma)**.

Proof. Well-order $X = \{x_\alpha : \alpha < k\}$. For every open cover $\mathscr{U}$ of X, every dense subset D of X and every open subset U of X let $x(U)$ be the minimal element of $U \cap D$ with respect to the above well-order. Let $s(\mathscr{U}, D) = \{x(U) : U \in m(\mathscr{U})\}$. It is easy to prove that $s(\mathscr{U}, D)$ is a closed and discrete subset of D, $St(s(\mathscr{U}, D), \mathscr{U}) = X$ and s is a **m(a)**-operator. □

Example 25. A **m(a)** space which is not **sm(a)**.

Let $\mathbb{Q} \subset \mathbb{R}$ be the set of rational numbers with the topology induced by the usual topology on $\mathbb{R}$. By Example 26 (see below) $\mathbb{Q}$ is not **sm(a)**. Since $\mathbb{Q}$ is a metrizable second countable zero-dimensional space it is non-Archimedean (see Nyikos 1999) hence it is monotonically ultraparacompact and then, by Theorem 24, **m**(**a**).

By examples 26 and 31 (see below), we answer in the negative to Question 19.

Example 26. A monotonically normal space which is not **sm(a)**.

Let $E = \mathbb{Q} \subset \mathbb{R}$ be the set of rational numbers with the topology induced by the usual topology on $\mathbb{R}$. Suppose E be a **sm**(**a**) space, and r be a **sm(a)** operator for E. Let $\mathscr{V} = \{\mathbb{Q}\}$, then there exists a closed in $\mathbb{Q}$ and discrete $r(\mathscr{V}, E) \subset E$. Let $\mathscr{U} = \{\mathbb{Q}\}$ and let $D = \mathbb{Q} \setminus r(\mathscr{V}, E)$. Clearly D is dense in $\mathbb{Q}$ and then there exists a closed in $\mathbb{Q}$ and discrete $r(\mathscr{U}, D) \subset D$. Therefore, since $D \subseteq E$, by hypothesis we have $r(\mathscr{U}, D) \supset r(\mathscr{V}, E)$; a contradiction since $r(\mathscr{U}, D) \subset D = \mathbb{Q} \setminus r(\mathscr{V}, E)$.□

Recall the following definition:

Definition 27. (Popvassilev and Porter 2014) A space X is *monotonically star closed-and-discrete* if there exists an operator r which assigns to each open cover $\mathscr{U}$ a subspace $r(\mathscr{U}) \subseteq X$ such that $r(\mathscr{U})$ is closed and discrete in X, $St(r(\mathscr{U}), \mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$, then $r(\mathscr{U}) \supseteq r(\mathscr{V})$.

Matveev (1997) noted that every space is star closed-and-discrete. Recall that a space X is star closed-and-discrete if for every open cover $\mathscr{U}$ of X there is a closed and discrete subset $F \subseteq X$ such that $St(F, \mathscr{U}) = X$. However, monotone version of star closed-and-discrete property turns out to be interesting. Recall that a space X is *protometrizable* if it is a paracompact space with an othobase; Gartside and Moody (1993) showed that a space is protometrizable if and only if it ia monotone paracompact. Also every protometrizable space is monotonically star closed-and-discrete (Popvassilev and Porter 2014).

It is easy to show that

Proposition 28. Every **m(a)**-space is monotonically star closed-and-discrete.

Corollary 29. (Popvassilev and Porter 2014) Every monotonically ultraparacompact space is monotonically star closed-and-discrete.

The following is open

Question 30. Does exist a monotonically star closed-and-discrete space not **m(a)**?

We have the following example

Example 31. A monotonically normal space which is not a $\mathbf{m}(\mathbf{a})$.

Proof. The ordinal space ω_1 is a GO-space, then a monotonically normal space. Since every monotonically star closed-and-discrete GO-space is paracompact (Popvassilev and Porter 2014) and ω_1 is not paracompact, by Proposition 28 it is not **m(a)**. □

Recall the following monotone version of countable compactness.

Definition 32. (Popvassilev 2009) A space X is *monotonically countably compact* (briefly mcc) if there is a function r, called *mcc operator*, that assigns to every countable open cover $\mathscr{U}$ of X a finite open cover $r(\mathscr{U})$ which refines $\mathscr{U}$ in such a way that $r(\mathscr{U})$ refines $r(\mathscr{V})$ whenever $\mathscr{U}$ refines $\mathscr{V}$.

Recall that (see Matveev 1997) in the class of Hausdorff spaces countable compactness is equivalent to star-compactness (a space X is *star-compact* if for every cover $\mathscr{U}$ of X there exists a finite subset $F \subset X$ such that $St(F,\mathscr{U}) = X$).

The following definition of monotonically star-compact space was introduced by Popvassilev and Porter (2014) and called monotonically star-finite space.

Definition 33. (Popvassilev and Porter 2014) A space X is *monotonically star-compact* (briefly msc) if there exists a function r that assigns to every open cover $\mathscr{U}$ of X a finite subset $r(\mathscr{U})$ of X such that $St(r(\mathscr{U}),\mathscr{U}) = X$ and such that if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U}) \supseteq r(\mathscr{V})$.

It is natural to pose the following question

Question 34. Is monotone countable compactness equivalent to monotone star-compactness in the class of Hausdorff spaces?

Note that ω_1 is not monotonically star compact (see Example 26) neither monotonically countably compact (Popvassilev 2009).

A countable compact space X which is not monotonically star-compact is given by Popvassilev and Porter (2014, Example 21). If such a space X is mcc then this example permits to give a negative answer to Question 34.

For sake of completness, note that another possible monotone version of star-compactness could be given requireing that:

($\square$) there exists a function r, called $\square$-*operator*, that assigns to every open cover $\mathscr{U}$ of X a finite subset of X $r(\mathscr{U})$ such that $St(r(\mathscr{U}),\mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U}) \subseteq r(\mathscr{V})$.

However the following result proves the absurdness of the previous definition:

Theorem 35. Every space X having property ($\square$) is finite.

Proof. Let X be a space having property ($\square$) and r be the $\square$-operator. Let $\mathscr{U} = \{X\}$ be the trivial cover of X. Put $F = r(\mathscr{U})$.

Claim: $X = F$.

Assume the contrary, and fix $a \in X \setminus F$. Put $V = X \setminus \{a\}$ and $U = X \setminus F$. Clearly, the sets U and V are open in X and $\mathscr{C} = \{U,V\}$ covers X. Since X has property $\square$ and $\mathscr{C}$ refines $\mathscr{U}$, we have $E \subset F$, where $E = r(\mathscr{C})$. Therefore, $St(E,\mathscr{C}) \neq X$, since $St(E,\mathscr{C})$ does not contain the point a; a contradiction. □

Property (a) is strictly related to the following property:

Definition 36. (see Matveev 1997) A space X is *absolutely countably compact* (briefly acc), or has acc property, if for every open cover $\mathscr{U}$ of X and every dense $D \subset X$ there exists a finite subset F of X such that $St(F, \mathscr{U}) = X$.

Recall that:

Theorem 37. In the class of Hausdorff space, property acc is equivalent to property (a) plus countable compactness.

In order to give a monotone version of the previous theorem, we introduce the following monotone version of acc property:

Definition 38. A space X has *property monotone acc* if there exists a function r that assigns to every open cover $\mathscr{U}$ of X and every dense $D \subset X$ a finite subset of X $r(\mathscr{U}, D) \subset D$ such that $St(r(\mathscr{U}, D), \mathscr{U}) = X$ and such that if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U}, D) \supseteq r(\mathscr{V}, D)$. A space having property macc is called *macc space*.

The following proposition gives a monotone version of Theorem 37

Proposition 39. In the class of Hausdorff space, property macc is equivalent to property **m(a)** + monotone countable compactness.

Proof. $\Rightarrow$) By hypothesis, there exists an operator r that assigns to every open cover $\mathscr{U}$ of X and every dense subset $D \subset X$, a finite set $r(\mathscr{U}, D) \subset D$ such that $St(r(\mathscr{U}, D), \mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$, then $r(\mathscr{U}, D) \supseteq r(\mathscr{V}, D)$. By Hausdorfness, $r(\mathscr{U}, D)$ is closed and discrete and then X is an $\mathbf{m}(\mathbf{a})-$space.

$\Leftarrow$) Let X be a countable compact $\mathbf{m}(\mathbf{a})-$space. Then there exists an operator r that assigns to every open cover $\mathscr{U}$ of X and every dense $D \subset X$ a closed in X and discrete $r(\mathscr{U}, D) \subset D$ such that $St(r(\mathscr{U}, D), \mathscr{U}) = X$ and if $\mathscr{U}$ refines $\mathscr{V}$ then $r(\mathscr{U}, D) \supseteq r(\mathscr{V}, D)$. Since X is mcc, hence countable compact, every closed and discrete is finite and hence it is macc. □

Obviously every macc space is an acc space. The converse is not true:

Example 40. ω_1 is acc but not a mcc (Popvassilev 2009), hence not macc.

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[a] Università degli Studi di Messina
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra
Contrada Papardo, 98166 Messina, Italy

[*] To whom correspondence should be addressed | email: mbonanzinga@unime.it

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