

MINIMAL RESOLUTIONS OF GRADED MODULES OVER AN EXTERIOR ALGEBRA

LUCA AMATA ^a AND MARILENA CRUPI ^{a*}

ABSTRACT. Let K be a field, E the exterior algebra of a n -dimensional K -vector space V . We study projective and injective resolutions over E . More precisely, given a category \mathcal{M} of finitely generated \mathbb{Z} -graded left and right E -modules, we give upper bounds for the graded Betti numbers and the graded Bass numbers of classes of modules in \mathcal{M} .

1. Introduction

Let K be a field, $E = K\langle e_1, \dots, e_n \rangle$ the exterior algebra of a K -vector space V with basis e_1, \dots, e_n . We work on the category \mathcal{M} of finitely generated \mathbb{Z} -graded left and right E -modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. It is well-known that, even if E is not commutative, it behaves like a commutative local ring or $*$ local ring (Bruns and Herzog 1998) in many cases. If $M \in \mathcal{M}$, we denote by $\beta_{i,j}(M) = \dim_K \operatorname{Tor}_i^E(M, K)_j$ the *graded Betti numbers* of M and by $\mu_{i,j}(M) = \dim_K \operatorname{Ext}_E^i(K, M)_j$ the *graded Bass numbers* of M . Our aim is to give upper bounds for such invariants.

Many authors were interested in the problem of giving upper bounds for the graded Betti numbers and the graded Bass numbers of graded submodules of a finitely generated graded free module with homogeneous basis, both in the polynomial and in the exterior algebra context (see, for instance, Bigatti 1993; Pardue 1994; Hulett 1995; Pardue 1996; Aramova *et al.* 1997; Crupi and Ferrò 2013, 2015, and references contained therein). A fundamental tool in both contexts is the class of lexicographic submodules (Definition 2.6). Such a class of monomial submodules has been deeply studied by Amata and Crupi (2018a,b).

The paper is organized as follows. Section 2 introduces definitions, notations and gives a short survey on those facts which are relevant in the next sections. Section 3 analyzes the generic initial module of a graded module $M \in \mathcal{M}$. Generic initial modules preserve much information of the original module and, furthermore, they are strongly stable (Definition 2.5). Therefore in many situations it is a successful strategy to pass on to the generic initial module and then exploit the nice properties of strongly stable submodules. In Section 4, if $F = \bigoplus_{i=1}^r E g_i$ is the free E -module with homogeneous basis g_1, \dots, g_r , such that $\deg g_1 \leq \deg g_2 \leq \dots \leq \deg g_r$, we show that the lexicographic submodules give upper bounds for the graded Betti numbers of the class of graded submodules of F with the same

Hilbert function (Theorem 4.8). Our techniques generalize the ones exploited by Aramova *et al.* (1997, 1998). In Section 5, upper bounds for the graded Bass numbers of the class of graded submodules of $F \simeq E^r$ with a given Hilbert function, are given. Moreover, some remarks on the annihilator of classes of monomial submodules in F are given (Theorem 5.4). Finally, Section 6 contains our conclusions and perspectives.

2. Preliminaries and notations

Let K be a field. We denote by $E = K\langle e_1, \dots, e_n \rangle$ the exterior algebra of an n dimensional K -vector space V with basis e_1, \dots, e_n . For any subset $\sigma = \{i_1, \dots, i_d\}$ of $\{1, \dots, n\}$ with $i_1 < i_2 < \dots < i_d$ we write $e_\sigma = e_{i_1} \wedge \dots \wedge e_{i_d}$, and call e_σ a monomial of degree d . We set $e_\sigma = 1$, if $\sigma = \emptyset$. The set of monomials in E forms a K -basis of E of cardinality 2^n .

An element $f \in E$ is called *homogeneous* of degree j if $f \in E_j$, where $E_j = \bigwedge^j V$. An ideal I is called *graded* if I is generated by homogeneous elements. If I is graded, then $I = \bigoplus_{j \geq 0} I_j$, where I_j is the K -vector space of all homogeneous elements $f \in I$ of degree j . We denote by $\text{indeg}(I)$ the *initial degree* of I , i.e., the minimum s such that $I_s \neq 0$.

For any not empty subset S of E , we denote by $\text{Mon}(S)$ the set of all monomials in S , and we denote its cardinality by $|S|$.

From now on, in order to simplify the notation, we put $fg = f \wedge g$ for any two elements f and g in E . Let $e_\sigma = e_{i_1} \cdots e_{i_d} \neq 1$ be a monomial in E . We define

$$\text{supp}(e_\sigma) = \sigma = \{i : e_i \text{ divides } e_\sigma\},$$

and we write

$$m(e_\sigma) = \max\{i : i \in \text{supp}(e_\sigma)\}.$$

We set $m(e_\sigma) = 0$, if $e_\sigma = 1$.

Let \mathcal{M} be the category of finitely generated \mathbb{Z} -graded left and right E -modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for all homogeneous elements $a \in E$, $m \in M$. If $M \in \mathcal{M}$, the function $H_M : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $H_M(d) = \dim_K M_d$ is called the Hilbert function of M .

If $M \in \mathcal{M}$, then M has a unique minimal graded free resolution over E (Herzog and Hibi 2011):

$$F_\bullet : \dots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0,$$

where $F_i = \bigoplus_j E(-j)^{\beta_{i,j}(M)}$. The integers $\beta_{i,j}(M) = \dim_K \text{Tor}_i^E(M, K)_j$ are called the *graded Betti numbers* of M .

Furthermore, M has a unique minimal graded injective resolution:

$$I_\bullet : 0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow I^2 \rightarrow \dots,$$

where $I^i = \bigoplus_j E(n-j)^{\mu_{i,j}(M)}$. The integers $\mu_{i,j}(M) = \dim_K \text{Ext}_E^i(K, M)_j$ are called the *graded Bass numbers* of M (Bruns and Herzog 1998; Kämpf 2010).

Let M^* be the right (left) E -module $\text{Hom}_E(M, E)$. The duality between projective and injective resolutions implies the following relation (Aramova *et al.* 1997, Proposition 5.2) between the graded Bass numbers of a module and the graded Betti numbers of its dual.

Proposition 2.1. *Let $M \in \mathcal{M}$. Then*

$$\beta_{i,j}(M) = \mu_{i,n-j}(M^*), \quad \text{for all } i, j.$$

Let $F \in \mathcal{M}$ be a free module with homogeneous basis g_1, \dots, g_r , where $\deg(g_i) = f_i$ for each $i = 1, \dots, r$, with $f_1 \leq f_2 \leq \dots \leq f_r$. The elements of the form $e_\sigma g_i$, where $e_\sigma \in \text{Mon}(E)$, are called *monomials* of F , and $\deg(e_\sigma g_i) = \deg(e_\sigma) + \deg(g_i)$. In particular, if $F \simeq E^r$ and $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 appears in the i -th place, we assume, as usual, $\deg(e_\sigma g_i) = \deg(e_\sigma)$, i.e., $\deg(g_i) = f_i = 0$.

Throughout this paper, we denote by $F = \bigoplus_{i=1}^r E g_i$ the free E -module with homogeneous basis g_1, \dots, g_r , where $\deg(g_i) = f_i$ ($i = 1, \dots, r$) with $f_1 \leq f_2 \leq \dots \leq f_r$. Furthermore, when we write $F \simeq E^r$, we mean that $F = \bigoplus_{i=1}^r E g_i$ is the free E -module with homogeneous basis g_1, \dots, g_r , where g_i ($i = 1, \dots, r$) is the r -tuple whose only non zero-entry is 1 in the i -th position and such that $\deg(g_i) = 0$, for all i .

Definition 2.2. A graded submodule M of F is a monomial submodule if M is a submodule generated by monomials of F , i.e., M can be written as

$$M = \bigoplus_{i=1}^r I_i g_i,$$

with I_i the monomial ideal of E generated by those monomials e_σ such that $e_\sigma g_i \in M$.

Moreover, if $r = 1$ and $f_1 = 0$, a monomial submodule is a monomial ideal of E .

Definition 2.3. Let I be a monomial ideal of E . I is called *stable* if for each monomial $e_\sigma \in I$ and each $j < m(e_\sigma)$ one has $e_j e_{\sigma \setminus \{m(e_\sigma)\}} \in I$. I is called *strongly stable* if for each monomial $e_\sigma \in I$ and each $j \in \sigma$ one has $e_i e_{\sigma \setminus \{j\}} \in I$, for all $i < j$.

Definition 2.4. A monomial submodule $M = \bigoplus_{i=1}^r I_i g_i$ of F is an *almost (strongly) stable submodule* if I_i is a (strongly) stable ideal of E , for each i .

Definition 2.5. A monomial submodule $M = \bigoplus_{i=1}^r I_i g_i$ of F is a (strongly) *stable submodule* if I_i is a (strongly) stable ideal of E , for each i , and $(e_1, \dots, e_n)^{f_{i+1}-f_i} I_{i+1} \subseteq I_i$, for $i = 1, \dots, r-1$.

If I is a monomial ideal in E , we denote by $G(I)$ the unique minimal set of monomial generators of I , and by $G(I)_d$ the set of all monomials $u \in G(I)$ such that $\deg(u) = d$, $d > 0$. Instead, for every monomial submodule $M = \bigoplus_{i=1}^r I_i g_i$ of F , we set

$$\begin{aligned} G(M) &= \{u g_i : u \in G(I_i), i = 1, \dots, r\}, \\ G(M)_d &= \{u g_i : u \in G(I_i)_{d-f_i}, i = 1, \dots, r\}. \end{aligned}$$

Now, order the monomials of F in the *degree reverse lexicographic order*, $>_{\text{degrevlex}_F}$, as follows: let $e_\sigma g_i$ and $e_\tau g_j$ be monomials of F , then $e_\sigma g_i >_{\text{degrevlex}_F} e_\tau g_j$ if

- $\deg(e_\sigma g_i) > \deg(e_\tau g_j)$, or
- $\deg(e_\sigma g_i) = \deg(e_\tau g_j)$, and either $e_\sigma >_{\text{revlex}} e_\tau$, or $e_\sigma = e_\tau$ and $i < j$;

$>_{\text{revlex}}$ is the usual reverse lexicographic order on E with $e_1 >_{\text{revlex}} \dots >_{\text{revlex}} e_n$ (see, for instance, Aramova and Herzog 2000).

Any element f of F is a unique linear combination of monomials with coefficients in K . The largest monomial in this presentation is called the *initial monomial* of f and it is denoted by $\text{in}(f)$. If M is a graded submodule of F then the submodule of initial terms of M , denoted by $\text{in}(M)$, is the submodule of F generated by the initial terms of elements of M . Using the same arguments as in the polynomial case (Eisenbud 1995, Ch. 15; Miller

and Sturmfels 2005, Ch. 8.3; Herzog 2002; Crupi and Restuccia 2009; see also Aramova *et al.* 1997, for the rank one case), one has

$$H_{F/M} = H_{F/\text{in}(M)} \quad (1)$$

and

$$\beta_{i,j}(F/M) \leq \beta_{i,j}(F/\text{in}(M)), \text{ for all } i, j. \quad (2)$$

Since $\text{in}(M)$ is a monomial submodule of F with the same Hilbert function as M , we may assume M itself is a monomial submodule without changing the Hilbert function.

Now, for every $d \geq 1$, let F_d be the part of degree d of $F = \bigoplus_{i=1}^r E g_i$, i.e., the K -vector space of homogeneous elements of F of degree d . Denote by $\text{Mon}_d(F)$ the set of all monomials of degree d of F . We order such a set by the ordering $>_{\text{lex}_F}$ defined as follows: if $u g_i$ and $v g_j$ are monomials of F such that $\deg(u g_i) = \deg(v g_j)$, then $u g_i >_{\text{lex}_F} v g_j$ if $i < j$ or $i = j$ and $u >_{\text{lex}} v$.

Definition 2.6. Let \mathcal{L} be a monomial submodule of F . \mathcal{L} is a *lexicographic submodule* (lex submodule, for short) if for all $u, v \in \text{Mon}_d(F)$ with $u \in \mathcal{L}$ and $v >_{\text{lex}_F} u$, one has $v \in \mathcal{L}$, for every $d \geq 1$.

The next characterization of lex submodules in F (Crupi and Ferrò 2016; Amata and Crupi 2018b) will be useful in the sequel.

Proposition 2.7. Let \mathcal{L} be a graded submodule of F . Then \mathcal{L} is a lex submodule of F if and only if

- (i) $\mathcal{L} = \bigoplus_{i=1}^r I_i g_i$, with $I_i \subsetneq E$ lex ideals, for $i = 1, \dots, r$, and
- (ii) $(e_1, \dots, e_n)^{\rho_i + f_i - f_{i-1}} \subseteq I_{i-1}$, for $i = 2, \dots, r$, with $\rho_i = \text{indeg } I_i$.

Lexicographic submodules play a fundamental role in the classification of the Hilbert functions of quotient of finitely generated graded free E -modules.

Let a and i be two positive integers. Then a has the unique i -th Macaulay expansion

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \dots + \binom{a_j}{j}$$

with $a_i > a_{i-1} > \dots > a_j \geq j \geq 1$. We define

$$a^{(i)} = \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \dots + \binom{a_j}{j+1}.$$

We also set $0^{(i)} = 0$ for all $i \geq 1$.

Now, for the reader's convenience, we include some comments of Amata and Crupi (2018a) on Hilbert functions of a graded E -algebra F/M , with M submodule of F .

Let us consider the graded E -module $F = \bigoplus_{i=1}^r E g_i$. One can verify that

$$H_F(d) = \dim_K F_d = 0, \quad \text{for } d < f_1 \quad \text{and} \quad d > f_r + n. \quad (3)$$

Hence, if M is a monomial submodule of F , from (3), it follows that

$$H_{F/M}(t) = \sum_{i=f_1}^{f_r+n} H_{F/M}(i) t^i,$$

and we can associate to F/M the following sequence

$$(H_{F/M}(f_1), H_{F/M}(f_1 + 1), \dots, H_{F/M}(f_r + n)) \in \mathbb{N}_0^{f_r + n - f_1 + 1}. \quad (4)$$

For $p, q \in \mathbb{Z}$ with $p < q$, let us define the following set:

$$[p, q] = \{j \in \mathbb{Z} : p \leq j \leq q\}.$$

The next result, which is a generalization of the well-known Kruskal–Katona theorem (Aramova *et al.* 1997), can be found in the paper by Amata and Crupi (2018a). It describes the possible Hilbert functions of quotients of free E -modules.

Theorem 2.8. *Let $(f_1, f_2, \dots, f_r) \in \mathbb{Z}^r$ be an r -tuple such that $f_1 \leq f_2 \leq \dots \leq f_r$ and let $(h_{f_1}, h_{f_1+1}, \dots, h_{f_r+n})$ be a sequence of nonnegative integers. Set*

$$s = \min\{k \in [f_1, f_r + n] : h_k \neq 0\},$$

and

$$\tilde{r}_j = |\{p \in [r] : f_p = s + j\}|, \quad \text{for } j = 0, 1.$$

Then the following conditions are equivalent:

- (a) $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of a graded E -module F/M , with $F = \bigoplus_{i=1}^r E g_i$ finitely generated graded free E -module with the basis elements g_i of degrees f_i ;
- (b) $h_s \leq \tilde{r}_0$, $h_{s+1} \leq n\tilde{r}_0 + \tilde{r}_1$, $h_i = \sum_{j=N+1}^r \binom{n}{i-f_j} + a$, where a is a positive integer less than $\binom{n}{i-f_N}$, $0 < N \leq r$, and $h_{i+1} \leq \sum_{j=N+1}^r \binom{n}{i-f_j+1} + a^{(i-f_N)}$, $i = s+1, \dots, f_r+n$;
- (c) there exists a unique lexicographic submodule L of a finitely generated graded free E -module $F = \bigoplus_{i=1}^r E g_i$ with the basis elements g_i of degrees f_i and such that $\sum_{i=s}^{f_r+n} h_i t^i$ is the Hilbert series of F/L .

Theorem 2.8 points out that if M is a graded submodule of F , then there exists a unique lex submodule of F with the same Hilbert function as M . We will denote it by M^{lex} .

3. The generic initial module

In this Section, we study the generic initial module of a graded module $M \in \mathcal{M}$. Such a module can be defined as in the polynomial case (Pardue 1994, 1996; Aramova and Herzog 2000; Miller and Sturmfels 2005).

Let $\text{GL}(n)$ be the group of $n \times n$ invertible matrices with entries in the field K , or equivalently, the group of K -linear graded automorphisms of E .

If $\varphi = (a_{i,j}) \in \text{GL}(n)$, one can define the action of φ on E_1 as follows:

$$\varphi(e_j) = \sum_{i=1}^n a_{i,j} e_i, \quad a_{i,j} \in K$$

and

$$\varphi\left(\sum_{i=1}^n a_i e_i\right) = \sum_{i=1}^n a_i \varphi(e_i), \quad a_i \in K.$$

Furthermore, such an action can be extended to E_d as follow:

$$\varphi(e_\sigma) = \varphi(e_{i_1}) \cdots \varphi(e_{i_d}), \quad \text{for } e_\sigma = e_{i_1} \cdots e_{i_d} \in \text{Mon}_d(E).$$

The automorphism φ induces a natural compatible action on $F = \bigoplus_{i=1}^r E g_i$ by

$$\varphi\left(\sum_{i=1}^r f_i g_i\right) = \sum_{i=1}^r \varphi(f_i) g_i, \quad f_i \in E.$$

Now, let $\text{GL}(F)$ be the group of E -linear graded automorphism of F . An element of $\text{GL}(F)$ sends g_i to $\sum_{j=1}^r f_{ij} g_j$, where $f_{ij} \in E_{d_i - d_j}$. If $\varphi_1 \in \text{GL}(n)$ and $\varphi_2 \in \text{GL}(F)$, then $\varphi_1 \varphi_2 \varphi_1^{-1}$ is an E -linear graded automorphism of F and so we have an action of $\text{GL}(n)$ on $\text{GL}(F)$. Therefore, we can consider the semidirect product $G = \text{GL}(n) \rtimes \text{GL}(F)$. G acts on F through graded K -vector space automorphisms; this action takes submodules to submodules.

Let B be the subgroup of G consisting of all automorphisms taking g_i to a E -linear combination of g_1, \dots, g_i and e_i to a K -linear combination of e_1, \dots, e_i . B is the Borel group of G and it is naturally realized by upper triangular matrices.

In what follows, if $F = \bigoplus_{i=1}^r E g_i$ is a free graded E -module of rank r we will always use the *degree reverse lexicographic* order on the monomials of F defined in Section 2.

Definition 3.1. A submodule M of F is *Borel-fixed* if $\varphi(M) = M$, for every $\varphi \in B$.

The following result is the analogue of a general result of Galligo's theorem (Eisenbud 1995) on generic initial ideals proved by Pardue (1994). Since its proof is quite similar to the one on submodules of a finitely generated graded free module on a polynomial ring, we omit its proof (see also Aramova *et al.* 1997, Theorem 1.6, for the rank one case).

Proposition 3.2. Assume the base field K is infinite and let G and B as above. Then for each graded submodule M of F there exists a nonempty open subset $U \subseteq G$ such that

- (1) there is a monomial submodule N of F such that $N = \text{in}(\varphi(M))$ for all $\varphi \in U$;
- (2) N is a Borel-fixed submodule of F , that is $\varphi(N) = N$ for all $\varphi \in B$.

The monomial submodule $N = \text{in}(\varphi(M))$ of F is denoted by $\text{Gin}(M)$ and called the *generic initial module* of M .

Proposition 3.3. Let K be infinite and let M be a graded submodule of F . Then $\text{Gin}(M)$ is a strongly stable submodule of F with the same Hilbert function as M .

Proof. Since E is noetherian (see, for instance, Kämpf 2010), using the same arguments as in the paper by Hulett (1993, Lemmas 14, 15), we may assume that $M = I_1 g_1 \oplus \dots \oplus I_r g_r$, is a monomial submodule of F such that $(e_1, \dots, e_n)^{f_{i+1} - f_i} I_{i+1} \subseteq I_i$ ($i = 1, \dots, r-1$), where $f_i = \deg(g_i)$, for all i , without changing the Hilbert function. Moreover, since $\text{in}(P)\text{in}(Q) \subseteq \text{in}(PQ)$, with P, Q graded ideals of E , one has that $(e_1, \dots, e_n)^{f_{i+1} - f_i} \text{in}(\varphi(I_{i+1})) \subseteq \text{in}(\varphi(I_i))$, for all $\varphi \in B$.

Hence, $\text{Gin}(M) = \bigoplus_{i=1}^r J_i g_i$, with J_i monomial ideal of E , for all i , and such that $(e_1, \dots, e_n)^{f_{i+1} - f_i} J_{i+1} \subseteq J_i$, for $i = 1, \dots, r-1$.

Now, we prove that every J_i ($i = 1, \dots, r$) is a strongly stable ideal of E .

Assume there exists an integer $i \in \{1, \dots, r\}$ such that J_i is not a strongly stable ideal of E . Hence, there exist a monomial $e_\sigma \in J_i$ and a pair (h, j) of positive integers with $h < j$, $j \in \text{supp}(e_\sigma)$, such that $e_h e_{\sigma \setminus \{j\}} \notin J_i$. Let $\varphi \in \text{GL}(n)$ with $\varphi(e_j) = e_j + e_h$ and $\varphi(e_k) = e_k$

for $k \neq j$. Then, $\varphi(e_\sigma) = e_\sigma + e_h e_{\sigma \setminus \{j\}}$ and consequently $\varphi(J_i) \not\subseteq J_i$. Therefore, $\varphi(e_\sigma)g_i$ does not belong to $\text{Gin}(M)$. A contradiction.

Finally, from Eq. (1) $\text{Gin}(M)$ is a strongly stable submodule of F with the same Hilbert function as M . \square

From now on, we will assume that the base field K is infinite.

4. Maximal Betti numbers

In this Section we generalize the “higher” Kruskal–Katona Theorem (Aramova *et al.* 1997, Theorem 4.4). We show that if \mathcal{H} is a class of graded submodules of the free E -module $F = \bigoplus_{i=1}^r E g_i$ with a given Hilbert function H , then the unique lex submodule belonging to \mathcal{H} (Theorem 2.8) gives upper bounds for the graded Betti numbers of any graded submodule in \mathcal{H} .

For a monomial $e_\sigma g_i$ of $F = \bigoplus_{i=1}^r E g_i$, setting

$$m_F(e_\sigma g_i) = m(e_\sigma), \quad 1 \leq i \leq r,$$

define

$$G(M : j) = \{e_\sigma g_i \in G(M) : m_F(e_\sigma g_i) = j\},$$

and

$$m_j^F(M) = |G(M : j)|, \quad 1 \leq j \leq n, \quad m_{\leq t}^F(M) = \sum_{j=1}^t m_j^F(M), \quad 1 \leq t \leq n.$$

One can observe that $m_{\leq n}^F(M) = |G(M)|$.

If $M = \bigoplus_{i=1}^r I_i g_i$ is an (almost) stable submodule of F , then we can use the *Aramova–Herzog–Hibi formula* (Aramova *et al.* 1997, Corollary 3.3) for computing the graded Betti numbers of M :

$$\beta_{k,k+\ell}(M) = \sum_{i=1}^r \beta_{k,k+\ell}(I_i g_i) = \sum_{u \in G(M)_\ell} \binom{m_F(u) + k - 1}{m_F(u) - 1}, \quad \text{for all } k. \quad (5)$$

Indeed, one can easily observe that

$$\sum_{u \in G(M)_\ell} \binom{m_F(u) + k - 1}{m_F(u) - 1} = \sum_{i=1}^r \left[\sum_{u \in G(I_i)_{\ell-f_\ell}} \binom{m(u) + k - 1}{m(u) - 1} \right]. \quad (6)$$

As in the case when ideals of a polynomial ring are considered (Aramova *et al.* 1998, Lemma 3.6), next characterization of an almost strongly stable submodule of F easily follows.

Lemma 4.1. *Let M be a monomial submodule of F . Assume $M = M' + M''$, with $M' = \bigoplus_{i=1}^r I'_i g_i$, $M'' = \bigoplus_{i=1}^r I''_i e_n g_i$ and I'_i, I''_i ideals of the exterior algebra $\tilde{E} = K\langle e_1, \dots, e_{n-1} \rangle$ $i = 1, \dots, r$. Set $\tilde{M}'' = \bigoplus_{i=1}^r I''_i g_i$. Then the following conditions are equivalent:*

- (i) M is an almost strongly stable submodule;
- (ii) M', \tilde{M}'' are almost strongly stable submodules, and $I''_i(e_1, \dots, e_{n-1}) \subset I'_i$, for all i .

Remark 4.2. One can quickly verify that if M is a strongly stable submodule of F , then M admits a decomposition of the type defined in Lemma 4.1 with M' strongly stable submodule of M , too; whereas \tilde{M}'' could not be a strongly stable submodule.

Example 4.3. Let

$$M = (e_1e_2, e_1e_3, e_1e_4, e_2e_3)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2$$

be a strongly stable submodule of E^2 , with $E = K\langle e_1, \dots, e_4 \rangle$. We can write M as follows

$$M = M' + M'',$$

where $M' = (e_1e_2, e_1e_3, e_2e_3)g_1 \oplus (e_1e_2e_3)g_2$ and $M'' = (e_1)e_4g_1 \oplus (e_1e_2, e_1e_3, e_2e_3)e_4g_2$.

One can observe that M' is a strongly stable submodule, whereas $\tilde{M}'' = (e_1)g_1 \oplus (e_1e_2, e_1e_3, e_2e_3)g_2$ is an almost strongly stable submodule, which is not strongly stable.

Remark 4.4. One can observe that if I is a strongly stable ideal of the exterior algebra $\tilde{E} = K\langle e_1, \dots, e_{n-1} \rangle$, then I is a strongly stable ideal of the exterior algebra $E = K\langle e_1, \dots, e_n \rangle$; whereas, one can easily find a monomial ideal I which is lex in the exterior algebra $\tilde{E} = K\langle e_1, \dots, e_{n-1} \rangle$, but not in E .

Following Aramova *et al.* (1998), the following map can be defined

$$\alpha : \text{Mon}_d(E) \rightarrow \text{Mon}_d(E),$$

with

- $\alpha(e_\sigma) = e_\sigma$, if $n \notin \text{supp}(e_\sigma)$;
- $\alpha(e_\sigma) = (-1)^{\alpha(\sigma, j)} e_j e_{\sigma \setminus \{n\}}$, if $n \in \text{supp}(e_\sigma)$ and j is the largest integer $< n$ which does not belong to $\text{supp}(e_\sigma)$, $\alpha(\sigma, j) = |\{t \in \sigma : t < j\}|$.

Such a map is order preserving (Aramova *et al.* 1998), i.e., if $e_\sigma, e_\tau \in \text{Mon}_d(E)$ and $e_\sigma \geq_{\text{lex}} e_\tau$, then $\alpha(e_\sigma) \geq_{\text{lex}} \alpha(e_\tau)$. The map α can be extended to $\text{Mon}_d(F)$ as follows:

$$\alpha_F : \text{Mon}_d(F) \rightarrow \text{Mon}_d(F), \quad \alpha_F(e_\sigma g_i) = \alpha(e_\sigma) g_i, \quad 1 \leq i \leq r.$$

The map α_F is order preserving too.

Let $e_\sigma g_i, e_\tau g_j \in \text{Mon}_d(F)$ with $e_\sigma g_i \geq_{\text{lex}_F} e_\tau g_j$. We distinguish two cases: $i = j$, $i \neq j$.

Let $i = j$. If $e_\sigma g_i \geq_{\text{lex}_F} e_\tau g_i$, then $e_\sigma \geq_{\text{lex}} e_\tau$. Since $\alpha_F(e_\sigma g_i) = \alpha(e_\sigma) g_i$, $\alpha_F(e_\tau g_i) = \alpha(e_\tau) g_i$ and α is order preserving, then $\alpha_F(e_\sigma g_i) \geq_{\text{lex}_F} \alpha_F(e_\tau g_i)$.

Let $i \neq j$. If $e_\sigma g_i \geq_{\text{lex}_F} e_\tau g_j$, then $i < j$. Hence, $\alpha_F(e_\sigma g_i) = \alpha(e_\sigma) g_i \geq_{\text{lex}_F} \alpha(e_\tau) g_j = \alpha_F(e_\tau g_j)$.

For a non empty subset M of $\text{Mon}(F)$, let us denote by $\min(M)$ the smallest monomial of M with respect to \leq_{lex_F} .

Lemma 4.5. Let $M = \bigoplus_{i=1}^r I_i g_i = M' + M''$ be an almost strongly stable submodule of F , with $M' = \bigoplus_{i=1}^r I'_i g_i$, $M'' = \bigoplus_{i=1}^r I''_i e_n g_i$ and I'_i, I''_i ($i = 1, \dots, r$) ideals of $\tilde{E} = K\langle e_1, \dots, e_{n-1} \rangle$. Then $\alpha_F(\min(G(M))) = \alpha_F(\min(G(M')))$.

Proof. Since $\min(G(M')) \geq_{\text{lex}} \min(G(M))$, then $\min(G(M')) = \alpha_F(\min(G(M'))) \geq_{\text{lex}_F} \alpha_F(\min(G(M)))$. On the other hand, since M is almost strongly stable, then $\alpha_F(\min(G(M))) \in G(M')$ and $\min(G(M')) \leq_{\text{lex}_F} \alpha_F(\min(G(M)))$. \square

Theorem 4.6. *Let M and L be monomial submodules of F generated in degree s . Assume*

- (1) *M is an almost strongly stable submodule,*
- (2) *L is a lex submodule, and*
- (3) *$\dim_K L_s \leq \dim_K M_s$.*

Then

$$m_{\leq i}^F(L) \leq m_{\leq i}^F(M) \quad (7)$$

for all i .

Proof. Set $\tilde{E} = K\langle e_1, \dots, e_{n-1} \rangle$. We proceed by induction on $n = \dim_K E_1$. By hypotheses, $m_{\leq n}^F(L) = \dim_K L_s \leq \dim_K M_s = m_{\leq n}^F(M)$. In order to prove the inequality in (7) for $i < n$, we write M and L as follows:

$$M = \oplus_{i=1}^r I_i g_i = M' + M'',$$

with $M' = \oplus_{i=1}^r I'_i g_i$, $M'' = \oplus_{i=1}^r I''_i e_n g_i$, and I'_i, I''_i ($i = 1, \dots, r$) ideals of E generated by monomials in e_1, \dots, e_{n-1} , i.e., monomial ideals of \tilde{E} , and

$$L = \oplus_{i=1}^r J_i g_i = L' + L'',$$

with $L' = \oplus_{i=1}^r J'_i g_i$, $L'' = \oplus_{i=1}^r J''_i e_n g_i$ and J'_i, J''_i monomial ideals of \tilde{E} .

It is clear that M' is an almost strongly stable submodule and that L' is a lex submodule.

Hence, if we prove that $\dim_K L'_s \leq \dim_K M'_s$, from the inductive hypothesis the assertion will follow.

Set $\tilde{M}'' = \oplus_{i=1}^r I''_i g_i$. We can assume that M' and \tilde{M}'' are lex submodules.

Indeed, let $\tilde{M} = \oplus_{i=1}^r \tilde{I}_i g_i$ ($\tilde{L} = \oplus_{i=1}^r \tilde{J}_i g_i$, respectively) be the lex submodules of F generated by those monomials $u g_i$ with u monomial of \tilde{E} and such that $\dim_K \tilde{M}_s = \dim_K M'_s$ ($\dim_K \tilde{L}_{s-1} = \dim_K \tilde{M}''_{s-1}$, respectively).

Let $N = \tilde{M} + \tilde{L} = \oplus_{i=1}^r \tilde{I}_i g_i + \oplus_{i=1}^r \tilde{J}_i g_i$. We prove that N is an almost strongly stable submodule.

First of all note that \tilde{I}_i, \tilde{J}_i are lex ideals and so strongly stable ideals, for all i . On the other hand, following Aramova *et al.* (1998, Lemma 3.7, Theorem 3.9), one can verify that $\tilde{J}_i(e_1, \dots, e_{n-1}) \subset \tilde{I}_i$, for all i . Hence, N is an almost strongly stable submodule.

Now, we are in the following situation:

$$M = \oplus_{i=1}^r I'_i g_i + \oplus_{i=1}^r I''_i e_n g_i, \quad L = \oplus_{i=1}^r J'_i g_i + \oplus_{i=1}^r J''_i e_n g_i$$

where M is an almost strongly stable submodule and L is a lex submodule, and in addition $M' = \oplus_{i=1}^r I'_i g_i$, $\tilde{M}'' = \oplus_{i=1}^r I''_i g_i$ are lex submodules. Assuming that $\dim_K L_s \leq \dim_K M_s$ we want to prove that

$$\dim_K L'_s \leq \dim_K M'_s. \quad (8)$$

Thanks to Lemma 4.5 we have

$$\min(G(M')) = \alpha_F(\min(G(M))) \leq_{\text{lex}_F} \min(G(L')) = \alpha_F(\min(G(L))).$$

Since the submodules L' and M' are lex, the inequality (4) holds. Hence, by the inductive hypothesis, the required inequality (7) follows. \square

By using combinatorial arguments one can quickly verify the following lemma.

Lemma 4.7. *Let M be an almost strongly stable submodule of F generated in degree d . If $M_{\langle d+1 \rangle}$ is the submodule of F generated by the elements of M_{d+1} , then*

$$m_i(M_{\langle d+1 \rangle}) = m_{\leq i-1}(M)$$

for all i .

If M is a set of monomials of degree $d < n$ of F , we denote by $M\{e_1, \dots, e_n\}$ the following set of monomials of degree $d+1$ of F (Crupi and Ferrò 2015; Amata and Crupi 2018b):

$$M\{e_1, \dots, e_n\} = \{(-1)^{\alpha(\sigma, j)} e_j e_{\sigma} g_i : e_{\sigma} g_i \in M, j \notin \text{supp}(e_{\sigma}), j = 1, \dots, n, i = 1, \dots, r\},$$

$\alpha(\sigma, j) = |\{r \in \sigma : r < j\}|$. Such a set is usually called the *shadow* of M .

Theorem 4.6 and Lemma 4.7 yield the following result.

Theorem 4.8. *Let M be a graded submodule of F . Then*

$$\beta_{i,j}(M) \leq \beta_{i,j}(M^{\text{lex}}),$$

for all i, j .

Proof. The proof is quite similar to that given by Amata and Crupi (2018b, Theorem 4). Due to (2), from Proposition 3.3, we may assume that M is a strongly stable submodule.

From (5) we have:

$$\beta_{i,i+j}(M) = \sum_{u \in G(M)_j} \binom{m_F(u) + i - 1}{m_F(u) - 1}, \quad (9)$$

for $i \geq 1$.

Since $G(M)_j = G(M_{\langle j \rangle}) - G(M_{\langle j-1 \rangle})\{e_1, \dots, e_n\}$, the above sum can be written as a difference $\beta_{i,i+j}(M) = C - D$, with

$$\begin{aligned} C &= \sum_{u \in G(M_{\langle j \rangle})} \binom{m_F(u) + i - 1}{m_F(u) - 1} \\ &= \sum_{t=1}^n \sum_{u \in G(M_{\langle j \rangle}):t} \binom{t + i - 1}{t - 1} = \sum_{t=1}^n m_t(M_{\langle j \rangle}) \binom{t + i - 1}{t - 1} \\ &= \sum_{t=1}^n (m_{\leq t}(M_{\langle j \rangle}) - m_{\leq t-1}(M_{\langle j \rangle})) \binom{t + i - 1}{t - 1} \\ &= m_{\leq n}(M_{\langle j \rangle}) \binom{n + i - 1}{n - 1} \\ &\quad + \sum_{t=1}^{n-1} m_{\leq t}(M_{\langle j \rangle}) \left[\binom{t + i - 1}{t - 1} - \binom{t + 1 + i - 1}{t} \right] \\ &= m_{\leq n}(M_{\langle j \rangle}) \binom{n + i - 1}{n - 1} - \sum_{t=j}^{n-1} m_{\leq t}(M_{\langle j \rangle}) \binom{t + i - 1}{t} \end{aligned}$$

and

$$\begin{aligned} D &= \sum_{u \in G(M_{\langle j-1 \rangle}) \setminus \{e_1, \dots, e_n\}} \binom{m_F(u) + i - 1}{m_F(u) - 1} \\ &= \sum_{t=2}^n m_{\leq t-1}(M_{\langle j-1 \rangle}) \binom{t + i - 1}{t - 1}, \end{aligned}$$

from Lemma 4.7. On the other hand, since the number of generators of $M_{\langle d \rangle}$ and $M_{\langle d \rangle}^{\text{lex}}$ are equal for all d , we have $m_{\leq n}(M_{\langle d \rangle}) = m_{\leq n}(M_{\langle d \rangle}^{\text{lex}})$. Hence, from Theorem 4.6, $m_{\leq i}(M_{\langle d \rangle}^{\text{lex}}) \leq m_{\leq i}(M_{\langle d \rangle})$ for $1 \leq i \leq n$, and consequently:

$$\begin{aligned} \beta_{i,i+j}(M) &= m_{\leq n}(M_{\langle j \rangle}) \binom{n+i-1}{n-1} - \sum_{t=j}^{n-1} m_{\leq t}(M_{\langle j \rangle}) \binom{t+i-1}{t} \\ &\quad - \sum_{t=2}^n m_{\leq t-1}(M_{\langle j-1 \rangle}) \binom{t+i-1}{t-1} \leq \\ &\leq m_{\leq n}(M_{\langle j \rangle}^{\text{lex}}) \binom{n+i-1}{n-1} - \sum_{t=j}^{n-1} m_{\leq t}(M_{\langle j \rangle}^{\text{lex}}) \binom{t+i-1}{t} \\ &\quad - \sum_{t=2}^n m_{\leq t-1}(M_{\langle j-1 \rangle}^{\text{lex}}) \binom{t+i-1}{t-1} = \beta_{i,i+j}(M^{\text{lex}}). \end{aligned}$$

□

5. Graded Bass numbers

In this section we analyze the graded Bass numbers of graded submodules of F . We are interested in determining upper bounds for such invariants. For the reader's convenience, we recall some notions and results provided by Aramova *et al.* (1997) and Kämpf (2010). Let $M \in \mathcal{M}$ and let M^* be the right (left) E -module $\text{Hom}_E(M, E)$. We quote next a result from the article by Aramova *et al.* (1997, Proposition 5.1).

Lemma 5.1. *Let $M \in \mathcal{M}$. Then*

$$\dim_K M_i^* = \dim_K M_{n-i}, \quad \text{for all } i.$$

Let us consider the dual module $\text{Hom}_E(F/M, E)$, where M is a graded submodule of F . If $\text{rank } F = 1$ with $f_1 = 0$, i.e., $F = E$ and $M = I$ is a graded ideal of E , then

$$\text{Hom}_E(E/I, E) \simeq 0 : I, \quad (10)$$

where $0 : I$ is the annihilator of I , i.e., the set of all elements $b \in E$ such that $ba = 0$, for all $a \in I$. Moreover, from Lemma 5.1 (see also Aramova *et al.* 1997, Corollary 5.3):

$$\dim_K (E/I)_i = \dim_K (0 : I)_{n-i} \quad \text{for all } i. \quad (11)$$

Remark 5.2. *The ideal $0 : I$ is spanned as K -vector space by all monomials $e_{\bar{\sigma}}$ such that $e_{\sigma} \notin I$, where $\bar{\sigma}$ is the complement of σ in the set $\{1, \dots, n\}$ (see Aramova *et al.* 1997, Proposition 5.7, proof). Furthermore, if I is a lex ideal in E , then $0 : I$ is a lex ideal in E , too.*

Note that $0 : I$ is the exterior version of the Alexander dual of a squarefree monomial ideal in a polynomial ring.

The next example will be useful for describing our strategy in Theorem 5.4.

Example 5.3. Let $E = K\langle e_1, e_2, e_3, e_4 \rangle$ and $F = E^3$. Let us consider the lex submodule of F (Amata and Crupi 2018c; Grayson and Stillman 2018):

$$L = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)g_1 \oplus (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)g_2 \oplus (e_1e_2e_3e_4)g_3.$$

Setting $I_1 = (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4)$, $I_2 = (e_1e_2e_3, e_1e_2e_4, e_1e_3e_4, e_2e_3e_4)$ and $I_3 = (e_1e_2e_3e_4)$, one has

$$\begin{aligned} 0 : I_1 &= (e_1e_2, e_1e_3, e_1e_4, e_2e_3e_4), \\ 0 : I_2 &= (e_1e_2, e_1e_3, e_1e_4, e_2e_3, e_2e_4, e_3e_4), \\ 0 : I_3 &= (e_1, e_2, e_3, e_4). \end{aligned}$$

Even though the annihilators above are lex ideals, the submodule $N = \bigoplus_{t=1}^r (0 : I_t)g_t$ is not a lex submodule of F (see for instance Proposition 2.7). Indeed, the monomial $e_2e_4 \notin 0 : I_1$. Equivalently, $e_2e_4g_1 >_{\text{lex}_F} e_2e_4g_2$, but $e_2e_4g_2 \in N$, whereas $e_2e_4g_1 \in F \setminus N$. Conversely,

$$\tilde{N} = (0 : I_3)g_1 \oplus (0 : I_2)g_2 \oplus (0 : I_1)g_3$$

is a lex submodule in F . Note that $(F/L)^* \simeq N \simeq \tilde{N}$ as E -graded modules (see (10)) and $H_{F/N} = (3, 8, 3, 0, 0) = H_{F/\tilde{N}}$.

Theorem 5.4. Let M be a graded submodule of E^r , $r \geq 1$. Then

$$\mu_{i,j}(E^r/M) \leq \mu_{i,j}(E^r/M^{\text{lex}}), \quad \text{for all } i, j.$$

Proof. Set $F = E^r$. The case $r = 1$ has been proved by Aramova *et al.* (1997, Corollary 5.8). Assume $r > 1$.

From Proposition 2.1 and Theorem 4.8, one has

$$\mu_{i,j}(F/M) = \beta_{i,n-j}(\text{Hom}_E(F/M, E)) \leq \beta_{i,n-j}((\text{Hom}_E(F/M, E))^{\text{lex}}). \quad (12)$$

Let us consider the lex submodule M^{lex} . It is $M^{\text{lex}} = \bigoplus_{t=1}^r J_t g_t$, with each J_t lex ideal in E and $(e_1, \dots, e_n)^{\text{ind}_E J_t} \subseteq J_{t-1}$, for $t = 2, \dots, r$. Moreover, from (10),

$$\mu_{i,j}(F/M^{\text{lex}}) = \beta_{i,n-j}(\bigoplus_{t=1}^r (0 : J_t)g_t).$$

Now, consider the submodule $\bigoplus_{t=1}^r (0 : J_t)g_t$ of F . It is not a lex submodule in general (see for instance Example 5.3), nevertheless the behavior of the ideals J_t , together with the fact that $\deg g_t = 0$ for all t , assures that $\bigoplus_{t=1}^r (0 : J_{r-t-1})g_t$ is a lex submodule of F (see also Remark 5.2).

Moreover, it is clear that $\bigoplus_{t=1}^r (0 : J_t)g_t \simeq \bigoplus_{t=1}^r (0 : J_{r-t-1})g_t$. Hence

$$\mu_{i,j}(F/M^{\text{lex}}) = \beta_{i,n-j}(\bigoplus_{t=1}^r (0 : J_{r-t-1})g_t). \quad (13)$$

Claim. The graded E -modules $(\text{Hom}_E(F/M, E))^{\text{lex}}$ and $\bigoplus_{t=1}^r (0 : J_{r-t-1})g_t$ have the same Hilbert function.

Set $P = (\text{Hom}_E(F/M, E))^{\text{lex}}$ and $Q = \bigoplus_{t=1}^r (0 : J_{r-t-1})g_t$. From Lemma 5.1 and (11), we have

$$\begin{aligned} \dim_K P_i &= \dim_K ((\text{Hom}_E(F/M, E))^{\text{lex}})_i = \dim_K (\text{Hom}_E(F/M, E))_i \\ &= \dim_K (F/M)_{n-i} = \dim_K (F/M^{\text{lex}})_{n-i} \\ &= \sum_{t=1}^r \dim_K (0 : J_t)_i = \sum_{t=1}^r \dim_K (0 : J_{r-t-1})_i \\ &= \dim_K Q_i. \end{aligned}$$

The claim follows.

Therefore, since P and Q are lex submodules of F with the same Hilbert function, then they coincide. Finally, from (12) and (13),

$$\mu_{i,j}(F/M) \leq \beta_{i,n-j}(P) = \beta_{i,n-j}(Q) = \mu_{i,j}(F/M^{\text{lex}}),$$

for all i, j . □

We close this Section discussing the annihilator of a submodule of F . The next proposition generalizes some results obtained by Aramova *et al.* (1997, Remark 5.2).

Proposition 5.5. *Let M be a graded submodule of F .*

- (1) *If M is a (strongly) stable submodule, then $0 : M$ is a strongly stable ideal in E .*
- (2) *If M is a lex submodule, then $0 : M$ is a lex ideal in E .*

Proof. (1). Since $M = \bigoplus_{i=1}^r I_i g_i$ is a monomial submodule of F , then

$$0 : M = \bigcap_{i=1}^r (0 : I_i g_i) = \bigcap_{i=1}^r (0 : I_i),$$

and each ideal $0 : I_i$ is strongly stable (Crupi and Ferrò 2013, Lemma 4.1). The definition of a strongly stable submodule assures us that the ideal $0 : M$ is not null and strongly stable.

Similarly, one can verify that (2) holds. □

If I is a graded ideal of E , then $0 : I^{\text{lex}} = (0 : I)^{\text{lex}}$ (Aramova *et al.* 1997, Proposition 5.7). The next example shows that such a property does not hold if I is a graded submodule of F .

Example 5.6. Let $E = K\langle e_1, e_2, e_3, e_4, e_5 \rangle$ and $F = E^2$. Consider the following submodules of F (Amata and Crupi 2018c; Grayson and Stillman 2018):

$$M = (e_1 e_2, e_1 e_3, e_1 e_4 e_5, e_2 e_3 e_4, e_2 e_4 e_5, e_3 e_4 e_5)g_1 \oplus (e_1 e_2, e_2 e_3 e_4)g_2,$$

$$M^{\text{lex}} = (e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3 e_4, e_2 e_3 e_5, e_2 e_4 e_5, e_3 e_4 e_5)g_1 \oplus (e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_2 e_5, e_1 e_3 e_4 e_5)g_2.$$

One has

$$\begin{aligned} 0 : M &= (e_1 e_4, e_1 e_2 e_3, e_1 e_2 e_5, e_1 e_3 e_5, e_2 e_3 e_4, e_2 e_3 e_5), \\ (0 : M)^{\text{lex}} &= (e_1 e_2, e_1 e_3 e_4, e_1 e_3 e_5, e_1 e_4 e_5, e_2 e_3 e_4, e_2 e_3 e_5), \\ 0 : M^{\text{lex}} &= (e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_2 e_5, e_1 e_3 e_4, e_1 e_3 e_5, e_1 e_4 e_5, e_2 e_3 e_4). \end{aligned}$$

Hence, $0 : M^{\text{lex}} \neq (0 : M)^{\text{lex}}$.

6. Conclusions and perspectives

In this paper, we have given upper bounds for the graded Bass numbers of E -modules of the type E^r/M , $r \geq 1$. It would be nice to verify the inequality in Theorem 5.4 for quotients of the type F/M , with $F = \bigoplus_{i=1}^r E g_i$, when the basis elements g_1, \dots, g_r have different degrees. We believe that such a statement should be proved by using a different approach, as next example illustrates.

Example 6.1. Let $E = K\langle e_1, e_2, e_3, e_4 \rangle$ and $F = \bigoplus_{i=1}^r E g_i$ with $\deg g_1 = \deg g_2 = -2$, $\deg g_3 = -1$. Let us consider the lex submodule of F

$$L = (e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3 e_4) g_1 \oplus (e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_3 e_4) g_2 \oplus (e_1 e_2 e_3) g_3,$$

Setting $I_1 = (e_1 e_2, e_1 e_3, e_1 e_4, e_2 e_3 e_4)$, $I_2 = (e_1 e_2 e_3, e_1 e_2 e_4, e_1 e_3 e_4)$, $I_3 = (e_1 e_2 e_3)$, one has

$$0 : I_1 = (e_1 e_4, e_1 e_3, e_1 e_2, e_2 e_3 e_4),$$

$$0 : I_2 = (e_1, e_2 e_3, e_2 e_4, e_3 e_4),$$

$$0 : I_3 = (e_1, e_2, e_3),$$

and $N = \bigoplus_{i=1}^r (0 : I_i) g_i$ is not a lex submodule of F . Proceeding as in Example 5.3, let us consider the module

$$\tilde{N} = (0 : I_3) g_1 \oplus (0 : I_2) g_2 \oplus (0 : I_1) g_3.$$

It is not a lex submodule of F ($e_4 \notin 0 : I_3$).

Consider $F^* = \text{Hom}_E(F, E)$. By using *Macaulay2* (Grayson and Stillman 2018), it is $F^* = \bigoplus_{i=1}^r E \tilde{g}_i$, with $\deg \tilde{g}_1 = 1$, $\deg \tilde{g}_2 = \deg \tilde{g}_3 = 2$ and one can quickly verify that $\bar{N} = (0 : I_3) \tilde{g}_1 \oplus (0 : I_2) \tilde{g}_2 \oplus (0 : I_1) \tilde{g}_3$ is a lex submodule of F^* . Note that, $F \simeq F^*$ as E -modules, but not as graded E -modules. Indeed, $H_F \neq H_{F^*}$.

Hence, the arguments given in Theorem 5.4 do not work in the case of quotients of a free module with basis elements with different degrees.

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^a Università degli Studi di Messina,
Dipartimento di Scienze Matematiche e Informatiche, Scienze Fisiche e Scienze della Terra,
Viale Ferdinando Stagno d’Alcontres 31, 98166 Messina, Italy.

* To whom correspondence should be addressed | email: mcrupi@unime.it

