THERMODYNAMICS OF LOTKA–VOLTERRA DYNAMICS

P. Ván[1,2], A. Ciancio[3], M. Szücs[2]
[1]Department of Theoretical Physics, Wigner Research Centre for Physics,
H-1525 Budapest, Konkoly Thege Miklós u. 29-33., Hungary
[2]Department of Energy Engineering, Faculty of Mechanical Engineering,
Budapest University of Technology and Economics,
1111 Budapest, Műegyetem rkp. 3., Hungary
[3]Department of Mathematical and Computer Sciences,
Physical Sciences and Earth Sciences,
University of Messina, viale F. Stagno d'Alcontres, 31, 98166, Messina, Italy

Abstract: *The thermodynamics of simple mechanical systems and social dynamics is treated in the framework of ordinary thermodynamics. Mechanics provides a non-dissipative framework with Hamiltonian time evolution. Thermodynamics is responsible for stability of equilibrium. The combination of the two theories provides general understanding of stable limit cycles of any dynamical system with Hamiltonian core dynamics. Lotka-Volterra type two species evolution is treated as an example.*

1 INTRODUCTION

Lotka-Volterra model of two species evolution plays a distinguished role in dynamical system theory and in the related thermodynamic considerations. The model has clear interpretation: preys and predators are born and die and predators eat preys. The death rate of the preys is proportional to the number of the predators and the birth rate of the predators is proportional to the number of the preys [1, 2]. The model is originated in chemistry [3], have inspired other models in economy [4, 5, 6], in biology [7, 8], and in many other fields beyond ecology. It has some evident shortcomings when real ecological systems are considered. E.g. a prey does not have a natural extinction rate, can only be eaten by predators, and also the model seems to be qualitatively wrong when contrasted to observations [9].

The lack of firm theoretical background is a serious disadvantage in any generalization. In particular the Lotka-Volterra system of equations in its original form has an integral, a conserved quantity. Therefore the amplitude of the periodic solutions depends on the initial conditions, the relevant equilibrium is a centrum, it is stable and not attractive. In this sense it is similar to ideal mechanical systems without dissipation with a Lagrangian dynamics and Hamiltonian structure [9, 10]. Some extensions of the model are considered dissipative, then the relevant equilibrium becomes attractive, from centrum becomes a focus [11, 12]. These models are motivated by empirical considerations, a kind of extinction rate is introduced. However, beyond the conservative oscillating dynamics and the dissipative extension in Lotka-Volterra systems one expect limit cycles: oscillating behaviour that is not sensitive to small perturbations [4, 13]. This is also thought to be related to a thermodynamic background, to a particular form of dissipation [14]. It is remarkable that actually one may expect real stable limit cycles here.

In non-equilibrium thermodynamics of discrete systems, that is in ordinary thermodynamics, the second law ensures the asymptotic stability of thermodynamic equilibrium [15, 16, 17]. Moreover, recent efforts of a unified modelling of conservative and dissipative continuum systems, like GENERIC [18, 19] or the theory of dual internal variables [20, 21, 22], provide concepts and tools to treat these relations from a mechanical perspective. These concepts may help us to extend this unified treatment into other directions.

Here we are looking for a general and unified dynamics that encompasses ecological and economic expectations with the concepts and methodology of mechanics and thermodynamics. The challenge is that one cannot rely on continuum extension and spacetime structure. Therefore instead of direct derivations we are restricted to firm analogies. In this paper we show the embedding of the Lotka-Volterra dynamics into ordinary thermodynamic framework with the help of interpreting the conservative part as a particular Hamiltonian system. This way we have a clear concept of dissipation as a part of the dynamics that contributes to the entropy production and also the limit cycles are connected to the characteristic evolution of the thermodynamic, relaxational evolution of the generalized internal energy.

First we introduce the classical, simple Lotka-Volterra model, and show a Hamiltonian origin of the dynamics. Then we introduce a dissipative extension with strictly mechanical concepts, using the central idea of dual internal variables. Finally a natural concept of internal energy is introduced and then we demonstrate that this extension may lead to stable oscillations.

2 CLASSICAL LOTKA-VOLTERRA - THE HAMILTONIAN BACKGROUND

The classical Lotka-Volterra system of differential equations is

$$\dot{x} = x(a - by), \qquad \dot{y} = y(-c + dx). \tag{1}$$

Here x is the number of preys, and y is the number of predators in a population.

It is easy to see, that the system is conservative and the conserved quantity is

$$H(x, y) = -dx - by + a\ln(y) + c\ln(x). \tag{2}$$

This quantity is concave, but it is not a Hamiltonian function of the dynamics, because the related Hamilton equations are

$$\dot{x} = \partial_y H = -b + \frac{a}{y}, \qquad \dot{y} = -\partial_x H = d - \frac{c}{x}. \tag{3}$$

Here x plays a role of a coordinate and y the role of momentum. However, one may observe that the system of differential equations (1) is obtained, if the right hand side of the Hamiltonian equations are multiplied by xy:

$$\dot{x} = xy\partial_y H = x(-by + a), \qquad \dot{y} = -xy\partial_x H = y(dx - c). \tag{4}$$

With the terminology of dynamical systems this is a generalized gradient dynamics with an antisymmetric coupling and with nonlinear coefficients [23]. On Figure 1 we demonstrate that in case of some typical paremeters, that is $a = 3,\ b = 2,\ c = 2,\ d = 1$ and initial conditions $x(0) = 1.0;\ y(0) = 0.5$. The equilibria are $(x_{e1} = 0;\ y_{e1} = 0)$ and $(x_{e2} = 2.0;\ y_{e2} = 1.5)$. The first, trivial one is a saddle and the second one is a centrum. The Hamiltonian (2) is a constant, $H(0) = -4.079$, determined already by the initial conditions.

3 DISSIPATIVE LOTKA-VOLTERRA

Dynamical systems are considered dissipative when an attractive set exist in their phase space. Dynamical systems in the theory of ordinary thermodynamics have attractive and stable equilibria: thermodynamic equilibrium is asymptotically stable. They are dissipative also in the sense of dynamical system theory. In case of Hamiltonian dynamics dissipativity is related to

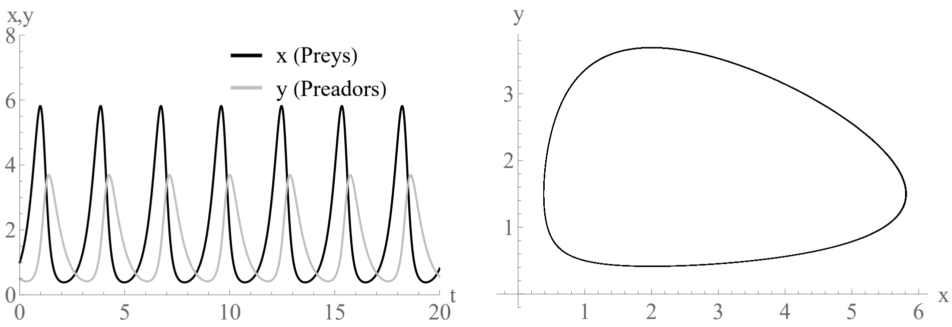

Figure 1: Classical LV, with parameters $a = 3,\ b = 2,\ c = 2,\ d = 1.$ The relevant equilibrium is a centrum. Left: the time evolution of the variables, right: phase space.

the evolution of momentum: damping or friction are special additional terms here. In case of Lotka-Volterra dynamics the dissipative terms are introduced both to prey and predator evolution and they transform the centrum equilibrium to a focus equilibrium [12]. In case of gradient dynamics we can ensure easily that the time evolution of the Hamiltonian become monotonously decreasing, with a symmetric positive definite linear contribution. This idea corresponds to the methodology of classical irreversible thermodynamics with internal variables, where the evolution of internal variables is constructed/derived with the constraint of a nonnegative entropy production.

Let us consider a general autonomous dynamical system with two variables and apply the thermodynamical methodology

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y). \tag{5}$$

The derivative of a scalar function, $L(x, y)$ – a Ljapunov or Hamiltonian candidate – along the differential equation (5) becomes

$$\dot{L} = DL \cdot (f, g) = \partial_x L f + \partial_y L g \tag{6}$$

Our aim is to construct the dynamics that ensures a monotonously decreasing or increasing L along the solution of the differential equation (5). Then an easy way is to introduce a gradient system, that is a linear solution of the resulted inequality assuming that f and g depend on $\partial_x L(x, y)$ and $\partial_y L(x, y)$:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} l_1 & l_{12} \\ l_{21} & l_2 \end{pmatrix} \begin{pmatrix} \partial_x L \\ \partial_y L \end{pmatrix}, \tag{7}$$

where the symmetric part of the matrix of coefficients $l_{ij} = \begin{pmatrix} l_1 & l_{12} // l_{21} & l_2 \end{pmatrix}$ is positive definite. Then, if the gradient generator function L is concave and strictly decreasing in a neighbourhood of an equilibrium point of (7), then the corresponding equilibrium is stable and attractive and our gradient generator is a Ljapunov function of the equilibrium. In case of an antisymmetric coupling our equilibrium is stable but not attractive. If the coupling antisymmetric matrix is the simplest and generates a symplectic dynamics, then our gradient generator is a Hamiltonian.

This is exactly the methodology of non-equilibrium thermodynamics when specified to discrete systems, that is homogeneous thermodynamic bodies. Let us assume that x and y are thermodynamic variables of state. If we do not introduce any further conditions and physical expectations, then this is the case of dual internal variables for discrete systems [20]. Thermodynamics postulate an entropy function, which is concave by thermodynamic stability and increasing along the dynamics. That is the inequality

$$\dot{L}_{th} = \partial_x L_{th} f + \partial_y L_{th} g \geq 0 \tag{8}$$

is a physical requirement for the dynamics and it is solved by the linear specification shown in (7). In non-equilibrium thermodynamics we construct the evolution equations, supplement our balances by constitutive equations according to this inequality requirement. The best method is to introduce thermodynamic fluxes and forces. A flux is always the part of the dynamics: it is to be determined. The force is related to a thermodynamic potential (entropy): it is given. In case of internal variables the fluxes are the evolution equations themselves, in our case they are f and g, and the forces are the derivative of the potential, in our case they are $\partial_x L_{th}$ and $\partial_y L_{th}$ (see table 1).

	Variable 1	Variable 2
Force	$\partial_x L_{th}$	$\partial_y L_{th}$
Flux	f	g

Table 1: Thermodynamic forces and fluxes for dual, discrete internal variables

Then the linear solution of the inequality gives the system of differential equations (7). Therefore, gradient dynamical systems can be considered also thermodynamical systems in this sense. However, from the point of view of thermodynamics three remarks are important:

– It is natural to assume that the linear coupling is not constant, the coupling coefficients can be state dependent, $l_{ij} = l_{ij}(x, y)$, as long as the symmetric part is of the matrix if positive definite.

– Thermodynamics focuses on the dissipative part of the dynamics. Therefore the antysymmetric part of l_{ij} is not very much investigated. Notable exceptions are the theory of dual internal variables [20, 24, 22], GENERIC [25] and the sporadic appearance of the so called gyroscopic forces.

– One may construct the dynamics with different thermodynamic potentials. The convexity and the inequality may be reserved. An energy potential is convex and decreasing, entropy potential is concave and increasing. Partial Legendre transformations result in partial convexity-concavity.

After this preparation it is easy to construct the dissipative part of the dynamics of the Lotka-Volterra system. There one need to introduce the symmetric part of the coupling, and consider the state dependence. In a general form, where the gradient generator of LV is introduced as a concave function $L_{th} = -H_{LV}$. Then it is reasonable to decompose L into symmetric and antisymmetric parts introducing $l^A = (l_{12} - l_{21})/2$ and $l^S = (l_{12} + l_{21})/2$:

$$\begin{aligned}
\dot{x} &= -l_1(x,y)\partial_x H - l_{12}(x,y)\partial_y H = l^A(x,y)\partial_y H \qquad -l_1(x,y)\partial_x H - l^S(x,y)\partial_y H, \\
\dot{y} &= -l_{21}(x,y)\partial_x H - l_2(x,y)\partial_y H = -l^A(x,y)\partial_x H \qquad -l^S(x,y)\partial_x H - l_2(x,y)\partial_y H.
\end{aligned} \tag{9}$$

Substituting (2) with the state variable dependence of the coefficients $l_1(x,y) = \hat{l}_1 x^2$, $l^S(x,y) = \hat{l}^S xy$, $l^A(x,y) = \hat{l}^A xy$ and $l_2(x,y) = \hat{l}_2 y^2$ one obtains that

$$\begin{aligned}
\dot{x} &= x(A - By) - Ex^2, \\
\dot{y} &= y(-C + Dx) - Fy^2,
\end{aligned} \tag{10}$$

where $A = \hat{l}_1 c + \left(\hat{l}^S - \hat{l}^A\right) a$, $B = \left(\hat{l}^S - \hat{l}^A\right) b$, $C = -\hat{l}_2 a - \left(\hat{l}^S - \hat{l}^A\right) c$, $D = -\left(\hat{l}^S - \hat{l}^A\right) d$, $E = \hat{l}_1 d$, $F = \hat{l}_2 b$. One can see that the antisymmetric part cannot be separated, it is a mere multiplier of the classical Lotka-Volterra dynamics, and it may be merged into the potential itself, generating a pure Hamiltonian conserved dynamics. One can see an example time evolution with extending classical dynamics of Figure 1 on Figure 2.

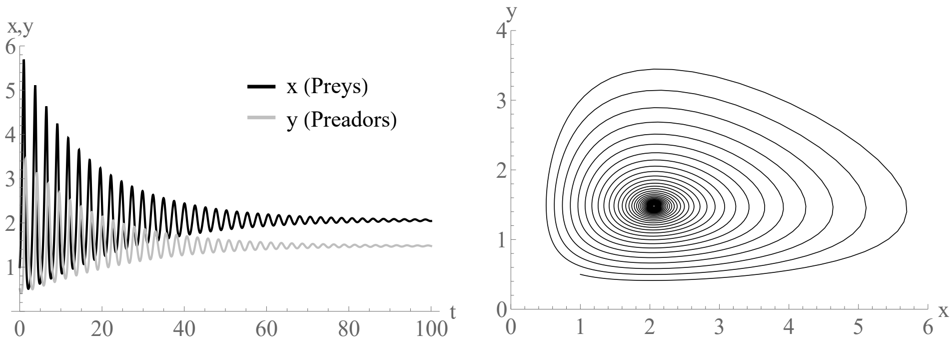

Figure 2: Dissipative LV, with classical parameters $A = 3$, $B = 2$, $C = 2$, $D = 1$ and dissipative extension $E = 0.02$, $F = 0.04$ The relevant equilibrium is a focus. Left: the time evolution of the variables, right: phase space.

3.1 Limit cycles

After the identification the relation of the conservative and dissipative parts of the dynamics one can recognise that the limit cycles can appear if the dissipative part vanishes at a particular value of the gradient potential. That can be easily constructed by redefining the dissipative extension coefficients $l_1(x,y)$ and l_2 as $l_1(x,y) = \hat{l}_1(x,y)[H_0 - H(x,y)]$ and $l_2 = \hat{l}_2(x,y)[H_0 - H(x,y)]$. Therefore a dissipative dynamics with limit cycle becomes

$$\dot{x} = l^A(x,y)\partial_y H - \hat{l}_1(x,y)[H_0 - H(x,y)]\partial_x H - l^S(x,y)\partial_y H,$$
$$\dot{y} = -l^A(x,y)\partial_x H - l^S(x,y)\partial_x H - \hat{l}_2(x,y)[H_0 - H(x,y)]\partial_y H. \quad (11)$$

This modification will stabilize the $\{(x,y)|H_0 = H(x,y)\}$ set. The proof is easy, either by linearisation or by direct nonlinear reasoning.

Substituting here (2) and the dissipative coefficients l_{ij} with the state variable dependence $l_1(x,y) = \hat{l}_1 x^2 y$, $l^S(x,y) = \hat{l}^S xy$, $l^A(x,y) = \hat{l}^A xy$ and $l_2(x,y) = \hat{l}_2 xy^2$, the above equations can be written in the form

$$\dot{x} = x(A - By) + \hat{l}_1 xy[H_0 - H(x,y)](-c + dx),$$
$$\dot{y} = y(-C + Dx) + \hat{l}_2 xy[H_0 - H(x,y)](a - bx), \quad (12)$$

where $A = (\hat{l}^A - \hat{l}^S)a$, $B = (\hat{l}^A - \hat{l}^S)b$, $C = (\hat{l}^A + \hat{l}^S)c$ and $D = (\hat{l}^A + \hat{l}^S)d$. Note, that the equations and their solutions depend on the choice of the state variable dependence of the dissipative coefficients. For example if we choose $l_1(x,y) = \hat{l}_1 x$ and $l_2(x,y) = \hat{l}_2 y$, then we get similar trajectories.

We demonstrate the limit cycle dynamics with the previous parameters of the dissipative LV system except introducing the quasilinear coupling. On Figure 3 the asymptotic value, H_0,

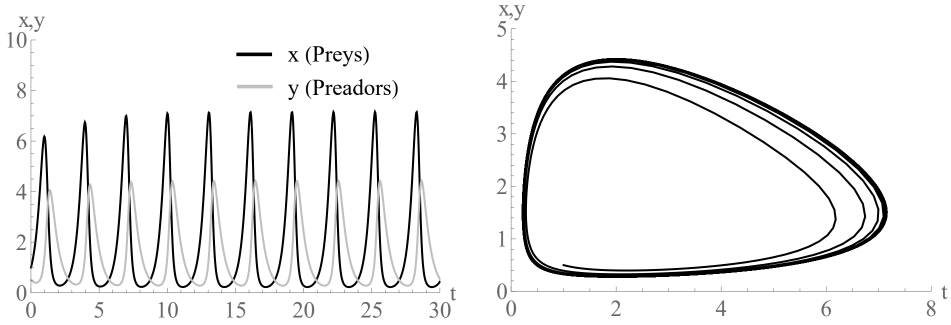

Figure 3: Limit cycle LV, with classical parameters $A = a = 3,\ B = b = 2,\ C = c = 2,\ D = d = 1$ and dissipative extension $l_1 = l_2 = 0.05$; the asymptotic gradient generator is set to $H_0 = -5$ and its initial value is $H_i = -4.079$. The limit cycle is attracting from inside. Left: the time evolution of the variables, right: phase space.

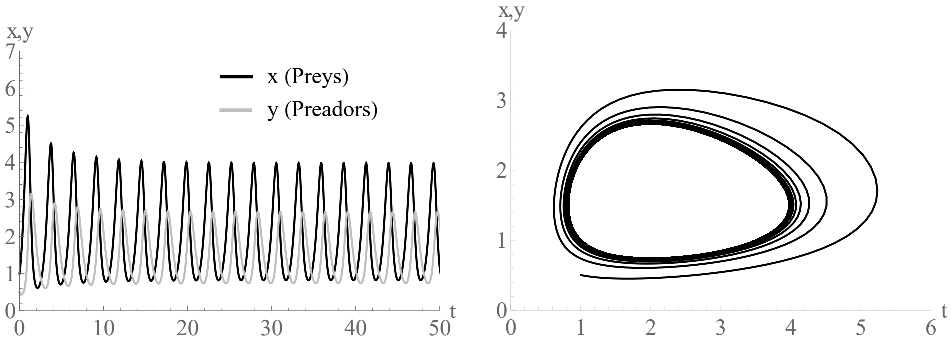

Figure 4: Limit cycle LV, with classical parameters $A = a = 3,\ B = b = 2,\ C = c = 2,\ D = d = 1$ and dissipative extension $l_1 = l_2 = 0.1$; the asymptotic gradient generator is set to $H_0 = -3$ and its initial value is $H_i = -4.079$. The limit cycle is attracting from outside. Left: the time evolution of the variables, right: phase space.

of gradient generator is greater than its initial value, H_i and on Figure 4 it is smaller. In the second case we have increased the dissipation in l_2 also for faster relaxation.

This kind of mathematical stabilisation of the limit cycle is artificial from a physical point of view. It is hard to imagine any realistic evolutionary, biological or mechanical mechanisms that could introduce directly Hamiltonian dependent coefficients in the dynamics. However, up to now we have discussed only the pure mechanical part of the dual internal variable evolution. The complete thermodynamical structure includes also an energetic part, therefore requires a further extension.

4 THERMODYNAMIC LOTKA-VOLTERRA

Up to now the interpretation of our conserved quantity and also the introduced stability concepts are based mostly on mechanical ideas, H being a kind of Hamiltonian. Pushing this interpretation forward one needs to distinguish between the energy of mechanical origin, that is the Hamiltonian, and the total energy, E. The difference of the total and mechanical energies is the internal energy, $U = E - H(x, y)$. An entropy depends directly on the internal energy, $S(U) = S(E - H(x, y))$, and its derivative by the internal energy is the reciprocal temperature: $\frac{\mathrm{d}S}{\mathrm{d}U} = \frac{1}{T}$. Then, the internal variable dependent form of the entropy is the function $\hat{S}(E, x, y) =$

$S(E - H(x, y))$ with the following partial derivatives:

$$\frac{\partial \hat{S}}{\partial E} = \frac{1}{T}, \qquad \frac{\partial \hat{S}}{\partial x} = -\frac{1}{T}\frac{\partial H}{\partial x}, \qquad \frac{\partial \hat{S}}{\partial y} = -\frac{1}{T}\frac{\partial H}{\partial y}. \tag{13}$$

This functional dependence can be expressed also in the form of differentials, as it is customary in thermodynamic literature:

$$\mathrm{d}E = T\, d\hat{S} + \partial_x H \mathrm{d}x + \partial_y H \mathrm{d}y. \tag{14}$$

We can see, that the energetic intensive quantities related to the internal variables x and y are the derivatives of the Hamiltonian, as it is expected.

The internal energy dependence of the entropy is the usual one: $\mathrm{d}U = T\mathrm{d}S$. Let us assume also the simplest possible caloric equation of state, $U = cT$, c being a constant heat capacity. That implies a logarithmic entropy function, $\hat{S}(E - H) = c\ln(E - H) + S_0$, with an auxiliary constant S_0, and the heat capacity $c > 0$ for thermodynamic stability.

In order to consider energy exchange with an environment, one assumes that the sum of the total body energy E and the energy of the environment E_e is conserved:

$$E + E_e = E_0 = constant. \tag{15}$$

The environment has a constant temperature, $T_0 = constant$, therefore its entropy is simply $S_e(E_e) = \frac{E_e}{T_0}$.

Now we can repeat our construction method of evolution equations assuming an extension of the previous dynamics (7) with an energetic part:

$$\dot{U} = (E - \dot{H}) = Q(E, x, y), \qquad \dot{x} = f(E, x, y), \qquad \dot{y} = g(E, x, y). \tag{16}$$

Here, the "heat" Q and also the functions f and g are to be determined according to the requirement of nondecreasing total entropy, according to thermodynamic expectations about the second law. The derivative of the total entropy along the system of differential equations (16) is

$$\dot{\hat{S}}(E, x, y) + \dot{S}_e(E_e) = \left(\frac{1}{T} - \frac{1}{T_0}\right) Q - \frac{\dot{H}}{T_0} \geq 0. \tag{17}$$

We may look for a solution by multiplying this inequality by TT_0 and obtain

$$TT_0\dot{\hat{S}}(E, x, y) + TT_0\dot{S}_e(E_e) = (T_0 - T)\, Q - T\partial_x H f - T\partial_y H g \geq 0. \tag{18}$$

This way a linear solution will lead to a version of the familiar Newton's cooling law for the energy part of the inequality, in the form $Q = -\alpha(T - T_0)$. The thermodynamic forces and fluxes are

	Thermal	Internal 1	Internal 2
Force	$T_0 - T$	$T\partial_x H$	$T\partial_y H$
Flux	Q	f	g

Table 2: Thermodynamic forces and fluxes for the thermal case

An incomplete simple linear solution of (18) is

$$\begin{pmatrix} Q \\ f \\ g \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & l_1 & l_{12} \\ 0 & l_{21} & l_2 \end{pmatrix} \begin{pmatrix} T_0 - T \\ T\partial_x H \\ T\partial_y H \end{pmatrix} \tag{19}$$

where the coefficient matrix is positive definite, that is $\alpha > 0$, $l_1 > 0$, $l_2 > 0$ and $l_1 l_2 - (l_{12} + l_{21})/4 > 0$. This is not the most general solution, but definitely the one where every nonzero term is well known and interpreted in the coupling matrix. Let us separate the symmetric and antisymmetric parts similarly to (9), introducing $l^A = (l_{12} - l_{21})/2$ and $l^S = (l_{12} + l_{21})/2$. Then substituting into the differential equation one obtains

$$\begin{aligned} \dot{E} &= -\alpha(T - T_0) + \partial_x H \dot{x} + \partial_y H \dot{y}, \\ \dot{x} &= l^A T\partial_y H - l_1 T\partial_x H - l^S T\partial_y H, \\ \dot{y} &= l^A T\partial_x H - l^S T\partial_x H - l_2 T\partial_y H. \end{aligned} \tag{20}$$

The remarkable property here is the temperature dependence of the coefficients. Because the temperature is proportional to the internal energy, the difference between the total body energy and the Hamiltonian. Our differential equation is a gradient dynamical system, where the generating potential is the total entropy of the body and the environment $\hat{S}(E, x, y) + S_e(E_e)$. The equilibrium of (20) is asymptotically stable as long as the entropy is concave, and increasing. The first condition is ensured if the heat capacity c is positive and the Hamiltonian $H(x, y)$ is convex. The second condition is also valid, by the construction of the dynamics, fulfilling the entropy inequality. The only interesting point is, when compared to equilibrium or ordinary thermodynamics, that temperature, as the difference of the body energy and Hamiltonian, is not necessarily positive. However, the evolution equations ensure, that if the initial temperature is positive it cannot become negative and contrary, if the initial temperature is negative it cannot become positive. Nevertheless, the typical equilibrium solution of the differential equations is a focus, but the domain of attraction varies. This is exactly that one expects from a szstem model in the framework of ordinary thermodynamics [17].

In the absence of a heat reservoir, $T_0 = 0$, the zero temperature is an equilibrium condition of the evolution equations. At zero temperature the system is ”frozen”. Moreover, cyclic nondissipative behaviour is typical. Let us investigate the corresponding conditions by the example of Lotka-Volterra Hamiltonian. Therefore the qualitative behaviour of the solutions depends also on the initial conditions.

4.1 Stable oscillations in LV systems

Let us assume again that the Hamiltonian is the conserved quantity of the Lotka-Volterra system and let us insert the dynamics into a thermodynamic framework. As we have seen previously we obtain the original LV evolution by introducing a particular, quasilinear, state dependent coefficient matrix, where the coefficients are proportional by xy: $l_1 = \hat{l}_1 xy$, $l_2 = \hat{l}_2 xy$, $a = \hat{a}xy$, and $l = \hat{l}xy$. Moreover, in order to obtain the classical dynamics in the nondissipative limit one need to introduce temperature independent antisymmetric coupling, by $\bar{a} = \hat{a}/T$. This way the original LV dynamics is the nondissipative subsystem of the thermodynamic body, when the symmetric part of the conductivity matrix is zero. Thus the complete system of differential equations becomes

$$\dot{E} = -\alpha(T - T_0) + \dot{H}(x, y),$$
$$\dot{x} = x(a - by) + xyT(l_1 \partial_x H + l \partial_y H),$$
$$\dot{y} = y(-c + dx) + xyT(l \partial_x H + l_2 \partial_y H). \quad (21)$$

In case of zero temperature and nonnegative initial temperature a fast equilibration of the Hamiltonian part leads to limit cycle like behaviour. This is demonstrated on figure 5

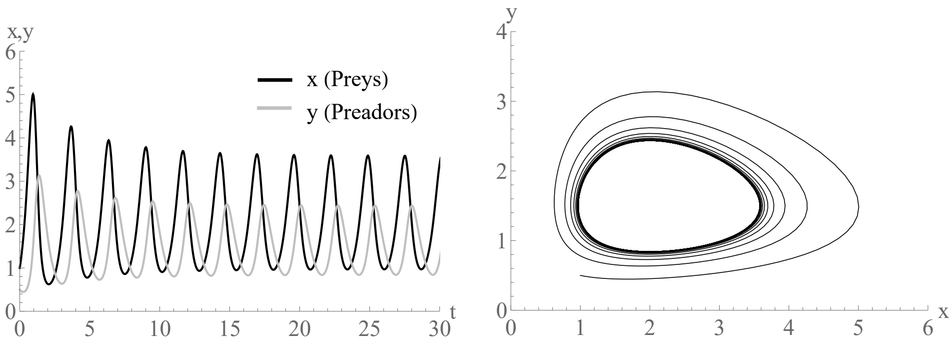

Figure 5: Thermodynamic LV, with limit cycle like dynamics. The classical parameters are $a = 3, b = 2, c = 2, d = 1$ and for the dissipative extension $l_1 = l_2 = 0.02; l = 0$. The thermal relaxation coefficient $\alpha = 0.2$. The environmental temperature $T_0 = 0$ and the initial total body energy is $E_0 = -2$. Left: the time evolution of the variables, right: phase space.

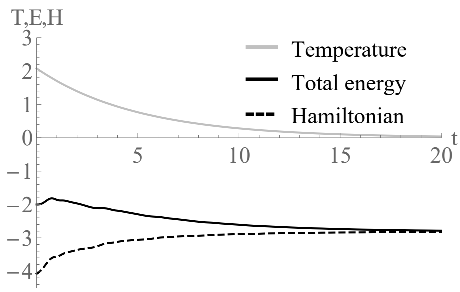

Figure 6: Temperature and energy relaxations of the LV system with parameters given in 5. The grey curve shows that temperature relaxes to zero as the body energy and the Hamiltonian, the black and dashed curves, equilibrate.

If the temperature equilibration is slower than that of the Hamiltonian part the oscillatory behaviour is erased as one can see on figure 7 with different parameters.

Finally, on figures 9 and 10 show an example when the initial temperature is negative, with the same equilibrium solution of the differential equations.

5 DISCUSSION

We have seen that Lotka-Volterra dynamical systems can be treated as dissipative Hamiltonian systems and also as true time dependent homogeneous thermodynamic systems. In both cases ordinary thermodynamics, the non-equilibrium thermodynamics of discrete systems

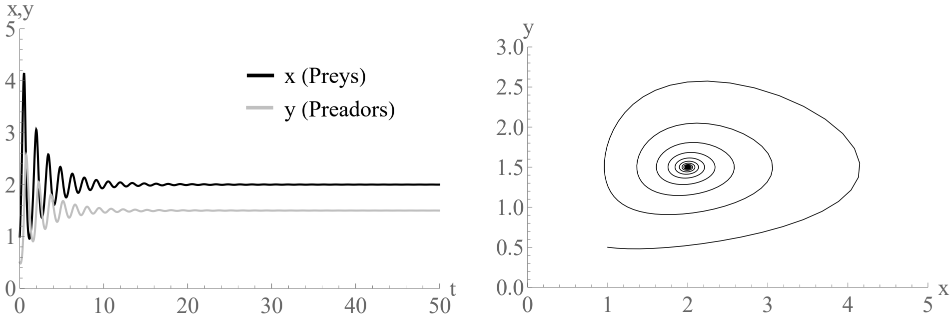

Figure 7: Thermodynamic LV, with limit cycle like dynamics. The classical parameters are $a = 3, b = 2, c = 2, d = 1$ and for the dissipative extension $l_1 = .01; l_2 = 0.1; l = 0$. The thermal relaxation coefficient $\alpha = 0.05$. The environmental temperature $T_0 = 0$ and the initial body energy is $E_0 = -2$. Left: the time evolution of the variables, right: phase space.

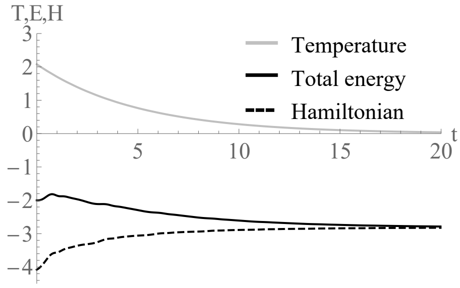

Figure 8: Temperature and energy relaxations of the LV system with parameters given in 7. The grey curve shows that temperature relaxes to zero as the body energy and the Hamiltonian, the black and dashed curves, equilibrate. Here the equilibration of the Hamiltonian (dashed) is faster than the body energy (black)

where the total entropy is a Ljapunov function of the thermodynamic equilibrium is the key for the proper generalization.

With the help of termodynamic concepts one could understood, there is no natural explanation of stable limit cycles in the framework of classical gradient system dynamics. Hovewer, thermodynamically compatible energy balance and the related temperature concept can lead to stable oscillating internal variables. The nondissipative cyclic behaviour depends on the initial conditions as it can be expected in real ecological, biological and economical systems, too.

Thermodynamic conditions of limiting oscillations are natural:

- Zero environmental temperature,
- Classical Newtonian heat exchange,
- Temperature independent ideal dynamics.

In these models the second law does not lead to globally stable equilibrium without any conditions. However, the basin of attraction changes according to the nonlinearities and oscillations may be stabilized by thermodynamics.

It is particularly interesting to understand the implications of the presented ideas in economic models, where thermodynamic concepts are somehow traditionally interesting. The usual interpretations of a possible role of thermodynamics in economy are based on the expected

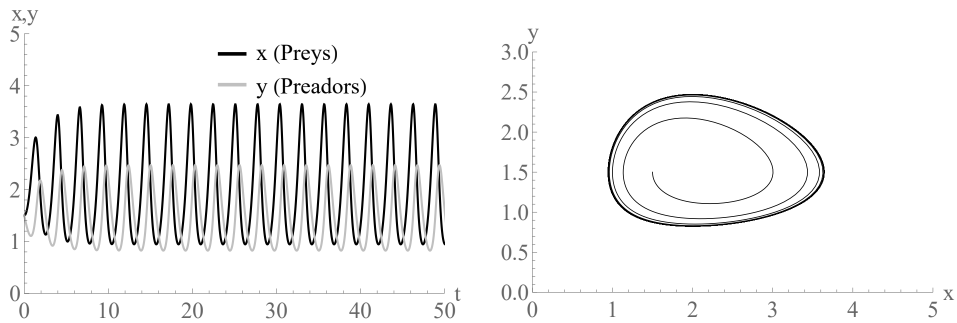

Figure 9: Thermodynamic LV, with limit cycle like dynamics. The classical parameters are $a = 3, b = 2, c = 2, d = 1$ and for the dissipative extension $l_1 = 0.01; l_2 = 0.1; l = 0$. The thermal relaxation coefficient $\alpha = 0.5$. The environmental temperature $T_0 = 0$ and the initial conditions are $x(0) = 1.5$, $y(0) = 1.5$ and $E(0) = -4.5$. Left: the time evolution of the variables, right: phase space.

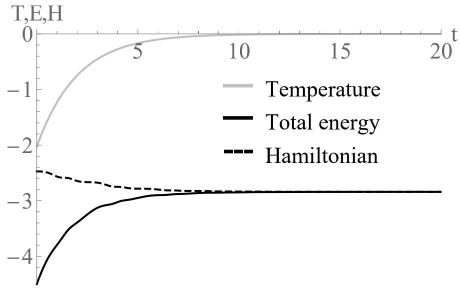

Figure 10: Temperature and energy relaxations of the LV system with parameters given in 10. The grey curve shows that initially negative temperature relaxes to zero as the body energy and the Hamiltonian, the black and dashed curves, equilibrate.

analogy with the potential structure of thermodynamics and looking for the concepts of wealth, value and goods. Also they are accounting the balance structure of irreversible thermodynamics [26, 27, 28]. On the other hand dynamical systems play and important role in economic model generation [5, 4]. There the concepts of stability are not connected to any fundamental aspect of the system. These two approaches to economical dynamics are essentially separated, and early attempts of their connection are essentially failed Here we have seen that ordinary thermodynamics of Matolcsi and the theory of dual internal variables opens a way of unification and new fruitful analogies. Moreover, a clear concept of dissipation is also have an explanatory value.

Thermodynamics is universal as long as the theory is based only on general concepts. This universality may explain the hopes and beliefs in thermodynamic concepts in very different areas. In this work we have shown that recent advances in thermodynamics as a dynamical system theory may justify these expectations extend also the region of its validity. In particular we have seen how conservative dynamical systems may be interpreted in a thermodynamic framework and how thermodynamic concepts like dissipation or temperature emerge in this approach.

6 ACKNOWLEDGEMENT

V. P. thank the support of the NKFIH 116197, NKFIH 124366 and NKFIH 123815 grants.

REFERENCES

[1] A.J. Lotka. *Elements of Physical Biology*. William and Wilkins, Baltimore, 1925.

[2] V. Volterra. Variazioni e fluttuazioni del numero d'individui in specie animali conviventi. *Accad. Naz. dei Lincei*, 2:31–113, 1926.

[3] A.J. Lotka. Contribution to the theory of periodic reactions. *The Journal of Physical Chemistry*, 14(3):271–274, 1910.

[4] Paul A Samuelson. Generalized predator-prey oscillations in ecological and economic equilibrium. *Proceedings of the National Academy of Sciences*, 68(5):980–983, 1971.

[5] R. M. Goodwin. A growth cycle. In C.H. Feinstein and M. Dobb, editors, *Socialism, capitalism & economic growth*, pages 195–200. Cambridge University Press, Cambridge, 1967.

[6] R. M. Goodwin. A growth cycle. In *Essays in Economic Dynamics*, pages 165–170. Springer, 1982.

[7] Carlo Cattani and Armando Ciancio. A class of solutions for the hybrid kinetic model in the tumor-immune system competition. *Mathematical Problems in Engineering*, 2013:430486, 2013.

[8] C Cattani and A Ciancio. Generalized hybrid kinetic mathematical model for population dynamics. *Applied Sciences*, 17:46–56, 2015.

[9] J. D. Murray. *Mathematical Biology*. Springer-Verlag, Berlin-etc.., 1990.

[10] J. Móczár and Márkus F. An economic application of the lie symmetries). *Alkalmazott Matematikai Lapok*, 28:77–93, 2011. in Hungarian.

[11] Xiaohua Zhao and Jigui Luo. Classification and dynamics of stably dissipative lotka–volterra systems. *International Journal of Non-Linear Mechanics*, 45(6):603–607, 2010.

[12] Pedro Duarte, Rui L Fernandes, and Waldyr M Oliva. Dynamics of the attractor in the lotka–volterra equations. *Journal of differential equations*, 149(1):143–189, 1998.

[13] Paul A Samuelson. Gibbs in economics. In D.G. Caldi and G.d. Mostow, editors, *Proceedings of the Gibbs Symposium*, pages 255–267, Yale University, 1989. AMS and AIoP.

[14] G. Lebon, D. Jou, and J. Casas-Vázquez. *Understanding non-equilibrium thermodynamics*. Springer, 2008.

[15] T. Matolcsi. Dynamical laws in thermodynamics. *Physics Essays*, 5(3):320–327, 1992.

[16] P. Ván. Other Dynamic Laws in thermodynamics. *Physics Essays*, 8(4):457–465, 1995.

[17] T. Matolcsi. *Ordinary thermodynamics*. Akadémiai Kiadó (Publishing House of the Hungarian Academy of Sciences), Budapest, 2005.

[18] H. C. Öttinger and M. Grmela. Dynamics and thermodynamics of complex fluids. II. Illustrations of a general formalism. *Physical Review E*, 56(6):6633–6655, 1997.

[19] H. C. Öttinger. *Beyond equilibrium thermodynamics*. Wiley-Interscience, 2005.

[20] P. Ván, A. Berezovski, and J. Engelbrecht. Internal variables and dynamic degrees of freedom. *Journal of Non-Equilibrium Thermodynamics*, 33(3):235–254, 2008. cond-mat/0612491.

[21] A. Berezovski, J. Engelbrecht, and P. Ván. Weakly nonlocal thermoelasticity for microstructured solids: Microdeformation and microtemperature. *Archive of Applied Mechanics*, 84:1249–1261, 2014.

[22] A. Berezovski and Ván P. *Internal Variables in Thermoelasticity*. Springer, 2017.

[23] R. I. McLachlan, GRW Quispel, and N. Robidoux. Unified approach to hamiltonian systems, poisson systems, gradient systems, and systems with lyapunov functions or first integrals. *Physical Review Letters*, 81(12):2399, 1998.

[24] Cs. Asszonyi, T. Fülöp, and P. Ván. Distinguished rheological models for solids in the framework of a thermodynamical internal variable theory. *Continuum Mechanics and Thermodynamics*, 27:971–986, 2015. arXiv:1407.0882.

[25] M. Grmela and H. C. Öttinger. Dynamics and thermodynamics of complex fluids. I. Development of a general formalism. *Physical Review E*, 56(6):6620–6632, 1997.

[26] S. Amir. The role of thermodynamics in the study of economic and ecological systems. *Ecological Economics*, 10(2):125–142, 1994.

[27] R.U. Ayres, L.W. Ayres, and K Martinás. Exergy, waste accounting, and life-cycle analysis. *Energy*, 23(5):355–363, 1998.

[28] R.U. Ayres and Martinás K. *On the reappraisal of microeconomics*. Edward Elgar Publishing, 2005.