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PROPAGATION OF COUPLED POROSITY AND FLUID-CONCENTRATION WAVES IN ISOTROPIC POROUS MEDIA

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Abstract. This article presents an application of a theory, previously formulated in the framework of rational extended irreversible thermodynamics, to describe the thermal, mechanical and transport properties of a porous medium filled by a fluid. Starting from the anisotropic rate equations for the porosity field, its flux, and for the heat and fluid-concentration fluxes, the isotropic case is studied when the body has symmetry properties invariant for all rotations and inversions of the frame axes. Furthermore, the phenomenological tensors have special symmetry properties coming from the used theoretic model. Then, the propagation in one direction of coupled porosity and fluid-concentration waves is investigated. The dispersion relation is carried out and the wave propagation velocities as functions of the wavenumber are calculated and represented in a diagram for a given numerical set of the several coefficients characterizing the considered porous media. The results obtained in this article can be applied in several sciences such as seismology, medical sciences, geology and nanotechnology, where there is propagation of high-frequency waves.

1. Introduction

In this article we apply a thermodynamic theory (see [5, 26, 27, 29, 30]), formulated in the framework of rational extended thermodynamics [1, 3, 11, 12, 13, 15, 16, 18, 20, 22, 25, 32], with internal variables for the description of the behaviour of porous media, to the study of a problem of propagation of coupled porosity and fluid-concentration waves in isotropic media. The characterization of the media taken into account is based on an approach à la Kubik [17], that considers an elementary sphere volume Ω of a structure with porous channels filled by a fluid, large enough to use volume and area averaging procedures, being $\Omega = \Omega^s + \Omega^p$, with Ω^s and Ω^p the solid space and the pore space of this volume. Kubik introduces also the central sphere section Γ of Ω, given by $\Gamma = \Gamma^s + \Gamma^p$, with Γ^s and Γ^p the solid area and the pore area, and defines a so called *structural permeability tensor*, r_{ij}, by the equation

$$\bar{\alpha}_i(\mathbf{x}) = r_{ij}(\mathbf{x})\hat{\alpha}_j(\mathbf{x}), \tag{1.1}$$

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that gives a linear mapping between the average of a property of some physical field $\bar{\boldsymbol{\alpha}}(\mathbf{x})$ calculated in the bulk-volume Ω and the average of the same quantity $\hat{\boldsymbol{\alpha}}(\mathbf{x})$ calculated on the pore area Γ^s. It is assumed that such physical quantity is zero in the solid space Ω^s and on Γ^s, and that the volume porosity f_v, defined as $f_v = \frac{\Omega^p}{\Omega}$, is constant. The tensor r_{ij} is symmetric, gives a macroscopic characterization of the geometric structure of the porous medium and has unit m^{-2}.

The introduction of r_{ij} in the thermodynamic state vector besides its gradient, its flux and the other thermal and mechanical variables allows us to investigate the behaviour of the considered media. Here, we focus our interest on the study of their transport properties in the case of perfect isotropy. Isotropic media were investigated also in [5]. The studies of phenomena regarding porous structures saturated by a fluid have great importance (see also [4, 6, 31]) and the obtained results can be used in several technological fields such as seismic waves, medical sciences, biology, geology and nanotechnology (where the Knudsen number $Kn = l/L$, with L the volume element size along a direction of a considered nanostructure and l the free mean path of the heat carriers, is such that $l/L \gg 1$, namely $L \ll l$, i.e. L is so small that it becomes comparable or smaller than l). Furthermore, in nanosystems (such as porous semiconductors [8]) there are high-frequency waves propagation and the transport properties of these systems have a rate variation faster than the time scale of the relaxation times of the fluxes to their equilibrium values.

The organization of this article is the following. In Section 2 we introduce the model with the fundamental laws, derived in [26] in the framework of extended thermodynamics with internal variables and describing the mechanical, thermal and transport properties of a solid structure with porous channels saturated by a fluid. Sections 3 and 4 are addressed to an application of the presented theory to a problem of wave propagation in a porous medium, supposed at rest, when only the porosity field, its flux, the fluid-concentration field and its flux are taken into account. In particular, starting from the anisotropic case (see [29, 30]) we derive in a special case a system of equations describing the propagation of coupled porosity and fluid-concentration waves in a porous isotropic medium, having symmetry properties invariant with respect to all rotations and inversions of frame axes (see [10, 14]). The dispersion relation is obtained and the values of the wave propagation velocities are worked out as functions of the wavenumber k. The dispersion curves are represented in a diagram for a given numerical set of the several coefficients characterizing the considered porous media. The Appendices deal with the achievement of particular forms for fourth and sixth order isotropic tensors having special symmetry properties and the detailed derivation of the transport equations for the porosity field and fluid-concentration flux. A similar propagation problem of coupled waves was studied by one the authors (L. R.) in isotropic n-type semiconductors (see [28]). The difference between both situations is that in [28] the considered media were semiconductors with dislocation lines, described as thin channels by an internal variable, the dislocation core tensor [21] (defined on the basis of the structural permeability tensor à la Kubik), and the fluid-concentration flux field was the flux of the concentration of electronic charge carriers.

2. Fundamental laws

Let us consider, in the framework of extended thermodynamics of irreversible processes, a model for media with porous channels filled by a fluid, deduced in [26], where it was assumed that the following fields interact with each other: the elastic field described by the symmetric stress tensor τ_{ij} and the small strain tensor ε_{ij}, defined by $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, with u_i the displacement field; the thermal field described by the temperature T, its gradient and the heat flux q_i; the field of the fluid-concentration c, its gradient and its flux j_i^c; the porosity field described by the structural permeability tensor r_{ij}, its gradient and its flux $\mathcal{V}_{ijk}$.

We suppose that the fluid filling the porous channels of the medium and the same medium form a two-components mixture of total mass density ρ and we indicate by ρ_1 the fluid mass density and by ρ_2 the mass density of the elastic porous structure, such that we have

$$\rho = \rho_1 + \rho_2. \tag{2.1}$$

The concentration of the fluid is defined by $c = \rho_1/\rho$.

We assume that the following continuity equations are valid for the mixture of the fluid and the porous skeleton as a whole and for each constituent

$$\dot{\rho} + \rho v_{i,i} = 0, \tag{2.2}$$

$$\frac{\partial \rho_1}{\partial t} + (\rho_1 v_{1i})_{,i} = 0, \tag{2.3}$$

$$\frac{\partial \rho_2}{\partial t} + (\rho_2 v_{2i})_{,i} = 0, \tag{2.4}$$

where we have disregarded the source terms in each continuity equation, a superimposed dot indicates the material derivative, v_{1i} and v_{2i} are the velocities of the fluid particles and the velocities of the elastic porous skeleton particles, respectively, the velocity v_i represents the barycentric velocity of the mixture, defined by

$$\rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i} \tag{2.5}$$

and j_i^c by

$$j_i^c = \rho_1(v_{1i} - v_i). \tag{2.6}$$

In the following we will use the standard Cartesian tensor notation in rectangular coordinate systems and we consider a current configuration K_t at the time t.

We assume that the physical processes occurring in the above-defined situation are governed by the following fundamental laws:

the continuity equation, obtained from (2.1)-(2.6) (see [3] and also [29]),

$$\rho \dot{c} + j^c_{i,i} = 0; \tag{2.7}$$

the momentum balance

$$\rho \dot{v}_i - \tau_{ji,j} = 0; \tag{2.8}$$

the internal energy balance

$$\rho \dot{e} - \tau_{ji} v_{i,j} + q_{i,i} = 0, \tag{2.9}$$

where the mass density ρ is supposed constant, e is the specific internal energy (i.e. internal energy per unit mass) and the body force and the heat source have been neglected in (2.8) and (2.9), respectively;

the rate equations for the structural permeability field r_{ij}, its flux $\mathcal{V}_{ijk}$, the heat flux q_i and the fluid-concentration flux j_i^c, constructed in such a way that they are obeying the objectivity and frame-indifference principles (see [9, 23, 24]) and supposed having the form

$$\overset{*}{r}_{ij} + \mathcal{V}_{ijk,k} - \mathcal{R}_{ij}(C) = 0, \tag{2.10}$$

$$\overset{*}{\mathcal{V}}_{ijk} - V_{ijk}(C) = 0, \tag{2.11}$$

$$\overset{*}{q}_i - Q_i(C) = 0, \tag{2.12}$$

$$\overset{*}{j}{}_i^c - J_i^c(C) = 0, \tag{2.13}$$

where the fluxes of $\mathcal{V}_{ijk}$, q_i and j_i^c are not taken into consideration, to close the system of equations describing the media under consideration, and $\mathcal{R}_{ij}(C)$, $V_{ijk}(C)$, $Q_i(C)$ and $J_i^c(C)$ are the source terms regarding the porosity field, its flux, the heat and the fluid-concentration fluxes, respectively. These sources are constitutive functions of the independent variables of the thermodynamic state vector chosen as follows

$$C = \{\varepsilon_{ij}, c, T, r_{ij}, j_i^c, q_i, \mathcal{V}_{ijk}, c_{,i}, T_{,i}, r_{ij,k}\}.$$

In (2.10)-(2.13) the superimposed asterisk defines the Zaremba-Jaumann derivative, i.e.

$$\overset{*}{q}_i = \dot{q}_i - \Omega_{ik} q_k, \quad \overset{*}{j}{}_i^c = \dot{j}_i^c - \Omega_{ik} j_k^c, \quad \overset{*}{r}_{ij} = \dot{r}_{ij} - \Omega_{ik} r_{kj} - \Omega_{jk} r_{ik},$$
$$\overset{*}{\mathcal{V}}_{ijk} = \dot{\mathcal{V}}_{ijk} - \Omega_{il}\mathcal{V}_{ljk} - \Omega_{jl}\mathcal{V}_{ilk} - \Omega_{kl}\mathcal{V}_{ijl},$$

where $\Omega_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the antisymmetric part of the velocity gradient $v_{i,j}$ and v_i the barycentric velocity field of the whole body.

All the admissible solutions of the proposed equations should be restricted by the entropy inequality

$$\rho \dot{S} + \phi_{k,k} - \frac{\rho h}{T} \geq 0, \tag{2.14}$$

where S denotes the entropy per unit mass, ϕ_k is the entropy flux and $\frac{\rho h}{T}$ is the external entropy production source (in the following neglected). In [26] Liu's theorem [19], establishing that all balance equations and rate equations are mathematical constraints for the validity of (2.14), was applied and the state laws, the generalized affinities, the entropy flux density and the functional form of the free energy F were derived. In particular the entropy flux density ϕ_k was given by the expression

$$\phi_k = \frac{1}{T}(q_k - \Pi^c j_k^c - \Pi_{ij}^r \mathcal{V}_{ijk}), \tag{2.15}$$

where Π^c is the chemical potential of the concentration field and Π_{ij}^r is the potential related to the structural permeability tensor. In [29] (see also [30]) we have worked out the following rate equations in the anisotropic case for the structural permeability tensor, its flux, the heat and fluid-concentration fluxes

$$\begin{aligned} \dot{r}_{ij} + \mathcal{V}_{ijk,k} = \beta^1_{ijkl}\varepsilon_{kl} + \beta^2_{ijkl} r_{kl} + \beta^3_{ijk} j_k^c + \beta^4_{ijk} q_k + \beta^5_{ijklm}\mathcal{V}_{klm} + \beta^6_{ijk} c_{,k} \\ + \beta^7_{ijk} T_{,k} + \beta^8_{ijklm} r_{kl,m}, \end{aligned} \tag{2.16}$$

$$\dot{\mathcal{V}}_{ijk} = \gamma^1_{ijkl} j_l^c + \gamma^2_{ijkl} q_l + \gamma^3_{ijklmn}\mathcal{V}_{lmn} + \gamma^4_{ijkl} c_{,l} + \gamma^5_{ijkl} T_{,l} + \gamma^6_{ijklmn} r_{lm,n}, \tag{2.17}$$

$$\tau^q \dot{q}_i = \chi^1_{ij} j_j^c - q_i + \chi^3_{ijkl}\mathcal{V}_{jkl} + \chi^4_{ij} c_{,j} - \chi^5_{ij} T_{,j} + \chi^6_{ijkl} r_{jk,l}, \tag{2.18}$$

$$\tau^{j^c} \dot{j}_i^c = -j_i^c + \xi^2_{ij} q_j + \xi^3_{ijkl}\mathcal{V}_{jkl} - \xi^4_{ij} c_{,j} + \xi^5_{ij} T_{,j} + \xi^6_{ijkl} r_{jk,l}. \tag{2.19}$$

In equations (2.16)-(2.19) we have considered physical situations where it is possible to replace the Zaremba-Jaumann derivative by the material derivative. Equations (2.16)-(2.19) describe disturbances having finite velocity of propagation and own relaxation times to reach the respective thermodynamic equilibrium values and show interactions among different fields.

In equation (2.18), generalizing Maxwell-Vernotte-Cattaneo relation $\tau^q \dot{q}_i = -q_i - \chi^5_{ij} T_{,j}$ (where only the influences of the heat flux and the temperature gradient field on the evolution in time of q_i are taken into consideration), τ^q is the relaxation time of the field q_i and χ^1_{ij}, χ^4_{ij} and χ^5_{ij} are the thermo-diffusive kinetic tensor, the thermo-diffusive tensor and the heat conductivity tensor, respectively. The phenomenological tensors χ^3_{ijkl} and χ^6_{ijkl} describe the influences of the flux and the gradient of the porosity field on the time derivative of the thermal flux. The field $r_{jk,l}$ describe non-local effects of the porosity field.

When equation (2.18) reduce to $q_i = -\chi^5_{ij} T_{,j}$, we obtain Fourier's law in the anisotropic case, leading to propagation infinite velocities of thermal signals, being the relaxation time τ^q null (see [2, 7]).

In equation (2.19), generalizing Fick-Nonnenmacher's law $\tau^{j^c} \dot{j}^c_i = -j^c_i - \xi^4_{ij} c_{,j}$ (where only the influences of the fluid-concentration flux and the fluid-concentration gradient field on the evolution in time of j^c_i are taken into account), τ^{j^c} is the relaxation time of the field j^c_i, ξ^4_{ik} is the diffusion tensor and ξ^5_{ij} is the thermo-diffusive tensor. Furthermore, the phenomenological tensors ξ^2_{ij}, ξ^3_{ijkl} and ξ^6_{ijkl} describe the influences of the heat flux, the porosity flux and the porosity gradient field on the time derivative of the fluid-concentration flux, respectively.

When equation (2.19) reduces to $j^c_i = -\xi^4_{ij} c_{,j}$, we have Fick's law in the anisotropic case, where the velocity of propagation of fluid-concentration flux is infinite and the relaxation time τ^{j^c} is null.

Equations (2.16) and (2.17) describe the evolution in time of the structural permeability tensor and its flux and in their right hand sides there are present the fields that have influence on them in the considered physical situation.

3. Equations governing the evolution of porosity and fluid-concentration fields and their fluxes in a porous medium in a special case

In this Section we consider the system of equations (2.7), (2.16), (2.17) and (2.19) in a special case. In particular, in (2.16), (2.17) and (2.19) we neglect the influence of the thermal phenomena, i.e. the presence of the fields q_i and $T_{,i}$. Furthermore, in equation (2.16) we disregard the effects of the field ε_{ij} and the porosity field r_{ij}, in the rate equation (2.17) the contribution of the fluid-concentration flux j^c_i and in the rate equation (2.19) the influence of the porosity flux $\mathcal{V}_{ijk}$. Thus, we obtain

$$\rho \frac{\partial c}{\partial t} = -j^c_{k,k}, \tag{3.1}$$

$$\frac{\partial r_{ij}}{\partial t} + \mathcal{V}_{ijk,k} = \beta^3_{ijk} j^c_k + \beta^5_{ijklm} \mathcal{V}_{klm} + \beta^6_{ijk} c_{,k} + \beta^8_{ijklm} r_{kl,m}, \tag{3.2}$$

$$\frac{\partial \mathcal{V}_{ijk}}{\partial t} = \gamma^3_{ijklmn} \mathcal{V}_{lmn} + \gamma^4_{ijkl} c_{,l} + \gamma^6_{ijklmn} r_{lm,n}, \tag{3.3}$$

$$\tau^{j^c} \frac{\partial j^c_i}{\partial t} = -j^c_i - \xi^4_{ij} c_{,j} + \xi^6_{ijkl} r_{jk,l}. \tag{3.4}$$

In the rate equation (3.2), because of the symmetry of r_{ij}, i.e. $r_{ij} = r_{ji}$, the phenomenological coefficients β^3_{ijk}, β^5_{ijklm}, β^6_{ijk}, β^8_{ijklm} have the following symmetries

$$\begin{gathered}\beta^3_{ijk} = \beta^3_{jik}, \quad \beta^5_{ijklm} = \beta^5_{jiklm}, \quad \beta^6_{ijk} = \beta^6_{jik}, \\ \beta^8_{ijklm} = \beta^8_{jiklm} = \beta^8_{ijlkm} = \beta^8_{jilkm}.\end{gathered} \tag{3.5}$$

From the symmetry property of r_{ij} and (3.5), also the divergence of the porosity field $\mathcal{V}_{ijk,k}$ is symmetric in the indexes $\{i,j\}$

$$\mathcal{V}_{ijk,k} = \mathcal{V}_{jik,k}. \tag{3.6}$$

Also, from the symmetry property of r_{ij} in the rate equations (3.3) and (3.4) we have for the phenomenological tensors γ^6_{ijklmn} and ξ^6_{ijkl} the symmetries

$$\gamma^6_{jiklmn} = \gamma^6_{ijkmln}, \quad \xi^6_{ijkl} = \xi^6_{ikjl}. \tag{3.7}$$

The symmetry relations (3.6) and (3.7) reduce the number of the significant components of the considered phenomenological tensors. The number of these significant components has a further reduction if we establish some other assumptions. Being r_{ij} a second order tensor, we can introduce its deviator, $\tilde{r}_{ij}$, and its scalar (or spherical) part, r, in the following way

$$\tilde{r}_{ij} = r_{ij} - \frac{1}{3} r\delta_{ij}, \quad r = \frac{1}{3} r_{kk}, \quad (i,j,k = 1,2,3)\,, \tag{3.8}$$

where Einstein convention for the dummy indices is used, and r_{ij} can be written in the form

$$r_{ij} = \tilde{r}_{ij} + r\delta_{ij}, \quad \text{with } \tilde{r}_{kk} = 0, \tag{3.9}$$

where, being r_{ij} symmetric, also $\tilde{r}_{ij}$ is symmetric.

Furthermore, we consider the case in which $\mathcal{V}_{ijk}$ can be written as the sum of three symmetric contributions

$$\mathcal{V}_{ijk} = \mathcal{V}_k\delta_{ij} + \mathcal{V}_i\delta_{jk} + \mathcal{V}_j\delta_{ik}. \tag{3.10}$$

For the sake of simplicity in the following we will consider only the spherical part $r_{ij} = r\delta_{ij}$ of the porosity field and the contribution $\mathcal{V}_k\delta_{ij}$ of its flux, i. e.

$$r_{ij} = r\delta_{ij}, \quad \mathcal{V}_{ijk} = \mathcal{V}_k\delta_{ij}, \tag{3.11}$$

where $\mathcal{V}_k\delta_{ij}$ is symmetric in the indexes $\{i,j\}$.

Thus, by (3.11), the rate equations (3.2)-(3.4) keep the form

$$\frac{\partial r}{\partial t}\delta_{ij} + \mathcal{V}_{k,k}\delta_{ij} = \beta^3_{ijk} j^c_k + \beta^5_{ijklm}\mathcal{V}_m\delta_{kl} + \beta^6_{ijk} c_{,k} + \beta^8_{ijklm} r_{,m}\delta_{kl}, \tag{3.12}$$

$$\frac{\partial \mathcal{V}_k}{\partial t}\delta_{ij} = \gamma^3_{ijklmn}\mathcal{V}_n\delta_{lm} + \gamma^4_{ijkl} c_{,l} + \gamma^6_{ijklmn} r_{,n}\delta_{lm}, \tag{3.13}$$

$$\tau^{j^c}\frac{\partial j^c_i}{\partial t} = -j^c_i - \xi^4_{ij} c_{,j} + \xi^6_{ijkl} r_{,l}\delta_{jk}. \tag{3.14}$$

In (3.13) the following symmetries are valid

$$\begin{gathered}\gamma^4_{ijkl} = \gamma^4_{jikl}, \quad \gamma^3_{ijklmn} = \gamma^3_{jiklmn} = \gamma^3_{ijkmln} = \gamma^3_{jikmln}, \\ \gamma^6_{ijklmn} = \gamma^6_{jiklmn} = \gamma^6_{ijkmln} = \gamma^6_{jikmln}.\end{gathered} \tag{3.15}$$

Properties $(3.15)_1$ and $(3.15)_2$ come from the symmetry of $\mathcal{V}_k\delta_{ij}$ and from the fact that in γ^3_{ijklmn} and in γ^6_{ijklmn} the indexes $\{l,m\}$ are dummy indexes with the indexes of the tensors $\mathcal{V}_n\delta_{lm}$ and $r_{,n}\delta_{lm}$, symmetric in $\{l,m\}$. In $(3.15)_3$ the last two symmetry properties for the tensor γ^6_{ijklmn} are equal to the symmetry properties (3.7).

Also, in (3.14) we have

$$\xi^6_{ijkl} = \xi^6_{ikjl}, \tag{3.16}$$

because in ξ^6_{ijkl} the indexes $\{j,k\}$ are dummy indexes with the indexes of the tensor $r_{,l}\delta_{jk}$, symmetric in $\{j,k\}$.

4. System of equations describing the propagation of coupled porosity and fluid-concentration waves in an isotropic medium

In this Section we apply the theory presented in the previous Section to a problem of propagation of coupled porosity and fluid-concentration waves in a porous perfect isotropic medium, supposed at rest. The existence of spatial symmetry properties in a material system may simplify the form of the rate equations in such a way that the number of the significant Cartesian components of the phenomenological tensors present in them has a further reduction. Here, we consider perfect isotropic systems for which the symmetry properties are invariant with respect to all rotations and the inversion of the frame of axes (i.e. under orthogonal transformations). Thus, in this case of perfect isotropy we have (see [10, 14]):
The tensors of odd order vanish, i.e.

$$L_{ijk} = 0, \quad L_{ijklm} = 0, \tag{4.1}$$

so that in equation (3.12) the tensors

$$\beta^3_{ijk} = \beta^5_{ijklm} = \beta^6_{ijk} = \beta^8_{ijklm} = 0 \tag{4.2}$$

vanish;
The tensors of order two keep the form

$$L_{ij} = L\delta_{ij}, \tag{4.3}$$

so that the phenomenological tensor ξ^4_{ij} takes the form

$$\xi^4_{ij} = \xi^4\delta_{ij}; \tag{4.4}$$

The tensors of order four must have the form

$$L_{ijkl} = L_1\delta_{ij}\delta_{kl} + L_2\delta_{ik}\delta_{jl} + L_3\delta_{il}\delta_{jk}, \tag{4.5}$$

where L_r $(r = 1,2,3)$ are the 3 significant components of L_{ijkl}, so that γ^4_{ijkl} and ξ^6_{ijkl} have only three significant components;
The tensors of order six (see γ^3_{jikmln} and γ^6_{ijklmn} present in (3.13)) assume the following form [14]:

$$\begin{aligned} L_{ijklmn} = {} & L_1\delta_{ij}\delta_{kl}\delta_{mn} + L_2\delta_{ij}\delta_{km}\delta_{ln} + L_3\delta_{ij}\delta_{kn}\delta_{lm} + L_4\delta_{ik}\delta_{jl}\delta_{mn} \\ & + L_5\delta_{ik}\delta_{jm}\delta_{ln} + L_6\delta_{ik}\delta_{jn}\delta_{lm} + L_7\delta_{il}\delta_{jk}\delta_{mn} + L_8\delta_{il}\delta_{jm}\delta_{kn} \\ & + L_9\delta_{il}\delta_{jn}\delta_{km} + L_{10}\delta_{im}\delta_{jk}\delta_{ln} + L_{11}\delta_{im}\delta_{jl}\delta_{kn} + L_{12}\delta_{im}\delta_{jn}\delta_{kl} \\ & + L_{13}\delta_{in}\delta_{jk}\delta_{lm} + L_{14}\delta_{in}\delta_{jl}\delta_{km} + L_{15}\delta_{in}\delta_{jm}\delta_{kl}, \end{aligned} \tag{4.6}$$

where L_r $(r = 1,2,\dots,15)$ are the 15 significant components of L_{ijklmn}.

Taking into account the isotropic form (4.2), (4.4), (4.5) and (4.6) of the phenomenological tensors and their symmetry properties (3.15) and (3.16) we derive from (2.7), (3.12), (3.13) and (3.14) (see detailed calculations in Section 5) the following simplified system of equations governing the evolution of porosity and fluid-concentration fields and their fluxes

$$\rho\frac{\partial c}{\partial t}+j^c_{k,k}=0, \tag{4.7}$$

$$\frac{\partial r}{\partial t}+\mathcal{V}_{k,k}=0, \tag{4.8}$$

$$\tau^\nu\frac{\partial\mathcal{V}_k}{\partial t}=-\mathcal{V}_k-D_\nu r_{,k}+\alpha_\nu c_{,k}, \tag{4.9}$$

$$\tau^{j^c}\frac{\partial j^c_i}{\partial t}=-j^c_i+\alpha_c r_{,i}-\rho D_c c_{,i}, \tag{4.10}$$

where τ^ν is the relaxation time of the field $\mathcal{V}_{ijk}=\mathcal{V}_k\delta_{ij}$, given by relation $(6.4)_1$ of Section 6, D_ν and D_c are the diffusion coefficients of porosity field and fluid-concentration flux, respectively, given by the relations (6.5) and $(7.3)_2$ of Sections 6 and 7, α_ν and α_c are coupling coefficients given by relations $(6.4)_2$ and $(7.3)_1$ of Sections 6 and 7, respectively, being

$$\tau^\nu\geq 0,\quad \tau^{j^c}\geq 0,\quad D_\nu\geq 0,\quad D_c\geq 0. \tag{4.11}$$

A detailed derivation of equations (4.9) and (4.10) has been obtained in Sections 6 and 7.

From equation (4.7), its derivative with respect to time and (4.10) we obtain

$$\tau^{j^c}\frac{\partial^2 c}{\partial t^2}+\frac{\partial c}{\partial t}+\bar{\alpha}_c r_{,ii}-D_c c_{,ii}=0, \tag{4.12}$$

where $\bar{\alpha}_c=\frac{\alpha_c}{\rho}$. In analogous way, from equation (4.8), its derivative with respect to time and (4.9) we have

$$\tau^\nu\frac{\partial^2 r}{\partial t^2}+\frac{\partial r}{\partial t}-D_\nu r_{,ii}+\alpha_\nu c_{,ii}=0. \tag{4.13}$$

The system (4.12), (4.13) describes the coupled porosity and fluid-concentration waves in a perfect isotropic medium. The aim of this paper is to find, from the dispersion relation, the wave propagation velocities as functions of the wavenumber and to obtain some particular propagation mathematical conditions.

We confine our considerations to one-dimensional plane waves. We suppose that the porous medium occupies the whole space and we consider the propagation of the coupled waves along x direction. Thus, assuming that the solutions of the set of equations (4.12) and (4.13) have the form

$$r(x,t)=\widehat{r}e^{ik(x-vt)}, \tag{4.14}$$

$$c(x,t)=\widehat{c}e^{ik(x-vt)}, \tag{4.15}$$

with $\widehat{r}$ and $\widehat{c}$ the amplitudes of the waves $r(x,t)$ and $c(x,t)$, k the wavenumber, v the wave velocity, defined by $v=\frac{\omega}{k}\,[\mathrm{m\,s^{-1}}]$, with ω the angular frequency, $\omega=2\pi f\,[\mathrm{s^{-1}}]$, being f the wave frequency and $k=\frac{2\pi}{\lambda}\,[\mathrm{m^{-1}}]$, with λ the wavelength.

Thus, using the relations (4.14), (4.15) and their derivatives in (4.12)-(4.13) we obtain the following system of equations

$$\big(D_c k-\tau^{j^c}kv^2-iv\big)\widehat{c}-\bar{\alpha}_c k\widehat{r}=0, \tag{4.16}$$

$$\alpha_\nu k^2\widehat{c}+\big(\tau^\nu k^2v^2-D_\nu k^2+ikv\big)\,\widehat{r}=0, \tag{4.17}$$

that has non-trivial solutions only if its determinant vanishes, i.e.

$$\mathcal{D} = \begin{vmatrix} D_c k - \tau^{j^c} k v^2 - iv & -\bar{\alpha}_c k \\ \alpha_\nu k^2 & \tau^\nu k^2 v^2 - D_\nu k^2 + ikv \end{vmatrix} = 0. \tag{4.18}$$

Developing $\mathcal{D}$ we derive the following *dispersion relation* for the wave propagation velocity v, concerning four possible modes:

$$\begin{aligned} &\tau^{j^c}\tau^\nu k^2 v^4 + ik\big(\tau^{j^c} + \tau^\nu\big)v^3 - \big[\big(D_c\tau^\nu + D_\nu\tau^{j^c}\big)k^2 + 1\big]v^2 \\ &- ik\left(D_c + D_\nu\right)v + k^2\left(D_c D_\nu - \bar{\alpha}_c\alpha_\nu\right) = 0. \end{aligned} \tag{4.19}$$

From the real part of the dispersion relation (4.19), we obtain

$$\tau^{j^c}\tau^\nu k^2 v^4 - \big[\big(D_c\tau^\nu + D_\nu\tau^{j^c}\big)k^2 + 1\big]v^2 + k^2\left(D_c D_\nu - \bar{\alpha}_c\alpha_\nu\right) = 0, \tag{4.20}$$

from which we have two possible modes

$$v_{(1)} = \sqrt{\mathcal{G}_1 + \sqrt{\mathcal{G}_1^2 - \mathcal{G}_2}}, \quad v_{(2)} = \sqrt{\mathcal{G}_1 - \sqrt{\mathcal{G}_1^2 - \mathcal{G}_2}}, \tag{4.21}$$

where

$$\mathcal{G}_1 = \frac{D_c\tau^\nu + D_\nu\tau^{j^c}}{2\tau^{j^c}\tau^\nu} + \frac{1}{2\tau^{j^c}\tau^\nu k^2}, \quad \text{being } \mathcal{G}_1 > 0, \tag{4.22}$$

$$\mathcal{G}_2 = \frac{D_c D_\nu - \bar{\alpha}_c\alpha_\nu}{\tau^{j^c}\tau^\nu}. \tag{4.23}$$

From the imaginary part of the dispersion relation (4.19), we derive

$$k\Big(\tau^{j^c} + \tau^\nu\Big)v^3 - k\left(D_c + D_\nu\right)v = 0, \tag{4.24}$$

from which we obtain the other two values for v

$$v_{(3)} = 0, \quad v_{(4)} = \sqrt{\frac{D_c + D_\nu}{\tau^{j^c} + \tau^\nu}}, \quad \text{being} \quad \frac{D_c + D_\nu}{\tau^{j^c} + \tau^\nu} > 0. \tag{4.25}$$

From $(4.25)_3$ and (4.11) *the velocity* $v_{(4)}$ *is always real*, whereas the velocity $v_{(1)}$ is real when

$$\mathcal{G}_1^2 - \mathcal{G}_2 \geq 0, \tag{4.26}$$

namely when

$$\Big[\Big(D_c\tau^\nu - D_\nu\tau^{j^c}\Big)^2 + 4\tau^{j^c}\tau^\nu\bar{\alpha}_c\alpha_\nu\Big]k^4 + 2\Big(D_c\tau^\nu + D_\nu\tau^{j^c}\Big)k^2 + 1 \geq 0, \tag{4.27}$$

that is always true because sum of positive quantities, and then also the *velocity* $v_{(1)}$ *is always real.* From $(4.25)_3$ the velocity $v_{(2)}$ is real when

$$\mathcal{G}_1 - \sqrt{\mathcal{G}_1^2 - \mathcal{G}_2} \geq 0, \tag{4.28}$$

from which we obtain

$$\mathcal{G}_2 \geq 0, \tag{4.29}$$

and thus

$$D_c D_\nu \geq \bar{\alpha}_c\alpha_\nu, \tag{4.30}$$

Thus, in the assumption that (4.29) (or (4.30)) holds $v_{(2)}$ is real.

In Figure 1 the wave propagation speeds as functions of k are represented for a given numerical set of the several coefficients present in the examined problem: $D_c = 10^{-1}\,\mathrm{m^2\,s^{-1}}$, $D_\nu = 10^{-1}\,\mathrm{m^2\,s^{-1}}$, $\tau^{j^c} = 10^{-2}\,\mathrm{s}$, $\tau^\nu = 10^{-3}\,\mathrm{s}$, $\alpha_\nu = 10^{-2}\,\mathrm{s^{-1}}$ and $\bar{\alpha}_c = 10^{-1}\,\mathrm{m^4\,s^{-1}}$, being $\alpha_c = \bar{\alpha}_c\rho$, with $\rho = 10^3\,\mathrm{kg\,m^{-3}}$ and $\alpha_c = 10^{-2}\,\mathrm{kg\,m\,s^{-1}}$. In this assumption the condition (4.30) is satisfied and thus the velocity $v_{(4)}$ is real.

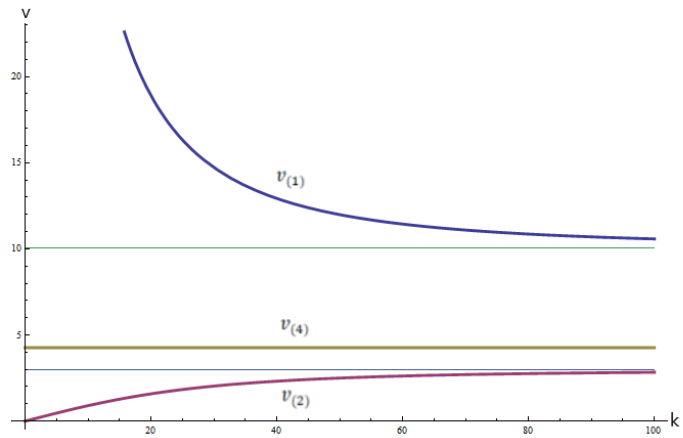

FIGURE 1. Representation of the three wave propagation speeds: $v_{(1)}$, $v_{(2)}$ and $v_{(4)}$ as functions of k, for a given numerical set of several coefficients present in the studied problem. The two horizontal lines are the horizontal asymptotes of the wave propagation velocities $v_{(1)}$ and $v_{(2)}$, respectively

The results presented in Fig. 1 show that for bigger values of k (for shorter wave lengths λ) the propagation velocity $v_{(1)}$ decreases, while the propagation velocity $v_{(2)}$ increases and the velocity $v_{(4)}$ remains constant.

Conclusions

In this article a theoretical approach was used, developed in previous papers in the framework of rational extended irreversible thermodynamics. It was supposed that the media with porous channels filled by a fluid can be studied as a mixture of two components. An internal variable, the structural permeability tensor r_{ij}, its gradient $r_{ij,k}$ and its flux $\mathcal{V}_{ijk}$ were introduced in the thermodynamic state vector besides the other classical variables to describe the mechanical, porous and transport properties.

Here, the rate equations for the porosity field, its flux, the heat and fluid-concentration fluxes, previously obtained in the anisotropic case, were considered in a special case for perfect isotropic media having symmetry properties invariant under orthogonal transformations. It was assumed that the mass density of the mixture of the porous skeleton and the fluid is constant. The body force, the heat source and the external entropy production source were negligible. The obtained results were applied to the study of the propagation in one direction x of coupled porosity and fluid-concentration waves when the body is supposed occupying the whole space. The dispersion relation was carried out and three possible propagation modes were found, with particular propagation mathematical conditions. The wave propagation velocities as functions of the wavenumber k were represented for a given numerical set of the several coefficients characterizing in an example the porous media under consideration. The study of propagation of these coupled waves has several application fields, such as hydrology, biology, nanotechnology, physiology and seismic waves.

5. Appendix: Perfect isotropic tensors with special symmetry properties

In the following Subsections we will consider perfect isotropic tensors of fourth and sixth order, having special symmetry properties, and thus a reduced number of independent significant components (see [10, 14]).

5.1. Special form for forth order perfect isotropic tensors.

In this Subsection we treat special symmetry properties of the fourth order tensors γ^4_{ijkl} and ξ^6_{ijkl} and demonstrate that these tensors can be expressed only by 2 significant components.

Case 1. A fourth order perfect isotropic tensor L_{ijkl} has the symmetry

$$L_{ijkl} = L_{jikl}, \tag{5.1}$$

(valid for the tensor γ^4_{ijkl} in equation (3.3)), from relation (4.5) we have

$$L_{jikl} = L_1\delta_{ji}\delta_{kl} + L_2\delta_{jk}\delta_{il} + L_3\delta_{jl}\delta_{ik}. \tag{5.2}$$

Adding equations (4.5) and (5.2), with the help of (5.1), and multiplying by 1/2 we have

$$L_{ijkl} = A_1\delta_{ij}\delta_{kl} + A_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \text{with} \quad A_1 = L_1,\ A_2 = (L_2 + L_3)/2. \tag{5.3}$$

Thus, the tensor γ^4_{ijkl} keeps the form

$$\gamma^4_{ijkl} = \gamma^4_1\delta_{ij}\delta_{kl} + \gamma^4_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{5.4}$$

Case 2. A fourth order perfect isotropic tensor L_{ijkl} has the symmetry

$$L_{ijkl} = L_{ikjl}, \tag{5.5}$$

(valid for the tensor ξ^6_{ijkl} in equation (3.4)), from relation (4.5) we have

$$L_{ikjl} = L_1\delta_{ik}\delta_{jl} + L_2\delta_{ij}\delta_{kl} + L_3\delta_{il}\delta_{kj}. \tag{5.6}$$

Using the same procedure seen in the case 1, we obtain

$$L_{ijkl} = A_1\delta_{il}\delta_{jk} + A_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}), \quad \text{with} \quad A_1 = L_3,\ A_2 = (L_1 + L_2)/2. \tag{5.7}$$

Thus, the tensor ξ^6_{ijkl} can be written as

$$\xi^6_{ijkl} = \xi^6_1\delta_{il}\delta_{jk} + \xi^6_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}). \tag{5.8}$$

5.2. Special form for sixth order perfect isotropic tensors.

In the case where a sixth order perfect isotropic tensor L_{ijklmn} has the two symmetries

$$L_{ijklmn} = L_{jiklmn}, \quad L_{ijklmn} = L_{ijkmln}, \tag{5.9}$$

equivalent to

$$L_{ijklmn} = L_{jiklmn} = L_{ijkmln} = L_{ijkmln}, \tag{5.10}$$

(valid for the tensors γ^3_{ijklmn} and γ^6_{ijklmn} in equation (3.3)) we show that the number of significant components of this tensors reduce from 15 to 6. In fact, writing relation (4.6) in the case of L_{jiklmn} (i.e. changing the index i with j), we have

$$\begin{aligned} L_{jiklmn} = {} & L_1\delta_{ji}\delta_{kl}\delta_{mn} + L_2\delta_{ji}\delta_{km}\delta_{ln} + L_3\delta_{ji}\delta_{kn}\delta_{lm} + L_4\delta_{jk}\delta_{il}\delta_{mn} \\ & + L_5\delta_{jk}\delta_{im}\delta_{ln} + L_6\delta_{jk}\delta_{in}\delta_{lm} + L_7\delta_{jl}\delta_{ik}\delta_{mn} + L_8\delta_{jl}\delta_{im}\delta_{kn} \\ & + L_9\delta_{jl}\delta_{in}\delta_{km} + L_{10}\delta_{jm}\delta_{ik}\delta_{ln} + L_{11}\delta_{jm}\delta_{il}\delta_{kn} + L_{12}\delta_{jm}\delta_{in}\delta_{kl} \\ & + L_{13}\delta_{jn}\delta_{ik}\delta_{lm} + L_{14}\delta_{jn}\delta_{il}\delta_{km} + L_{15}\delta_{jn}\delta_{im}\delta_{kl}. \end{aligned} \tag{5.11}$$

Matching expressions (5.11) and (4.6), by (5.9), we obtain

$$\begin{aligned} L_{ijklmn} = {} & B_1\delta_{ij}\delta_{kl}\delta_{mn} + B_2\delta_{ij}\delta_{km}\delta_{ln} + B_3\delta_{ij}\delta_{kn}\delta_{lm} + B_4(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} \\ & + B_5(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln} + B_6(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} \\ & + B_7(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} + B_8(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} \\ & + B_9(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}; \end{aligned} \tag{5.12}$$

with

$$\begin{gathered} B_1 = L_1, \quad B_2 = L_2, \quad B_3 = L_3, \quad B_4 = L_4 = L_7, \quad B_5 = L_5 = L_{10}, \\ B_6 = L_6 = L_{13}, \quad B_7 = L_8 = L_{11}, \quad B_8 = L_9 = L_{14}, \quad B_9 = L_{12} = L_{15}. \end{gathered} \tag{5.13}$$

Writing relation (4.6) in the case of L_{ijkmln} (i.e. changing the index l with m), we have

$$\begin{aligned} L_{ijkmln} = {} & L_1\delta_{ij}\delta_{km}\delta_{ln} + L_2\delta_{ij}\delta_{kl}\delta_{mn} + L_3\delta_{ij}\delta_{kn}\delta_{ml} + L_4\delta_{ik}\delta_{jm}\delta_{ln} \\ & + L_5\delta_{ik}\delta_{jl}\delta_{mn} + L_6\delta_{ik}\delta_{jn}\delta_{ml} + L_7\delta_{im}\delta_{jk}\delta_{ln} + L_8\delta_{im}\delta_{jl}\delta_{kn} \\ & + L_9\delta_{im}\delta_{jn}\delta_{kl} + L_{10}\delta_{il}\delta_{jk}\delta_{mn} + L_{11}\delta_{il}\delta_{jm}\delta_{kn} + L_{12}\delta_{il}\delta_{jn}\delta_{km} \\ & + L_{13}\delta_{in}\delta_{jk}\delta_{ml} + L_{14}\delta_{in}\delta_{jm}\delta_{kl} + L_{15}\delta_{in}\delta_{jl}\delta_{km}. \end{aligned} \tag{5.14}$$

Matching relations (5.14) and (4.6) and using (5.10), we obtain

$$\begin{aligned} & L_{ijklmn} \\ & = C_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + C_2\delta_{ij}\delta_{kn}\delta_{lm} + C_3(\delta_{jl}\delta_{mn} + \delta_{jm}\delta_{ln})\delta_{ik} \\ & \quad + C_4\delta_{ik}\delta_{jn}\delta_{lm} + C_5(\delta_{il}\delta_{mn} + \delta_{im}\delta_{ln})\delta_{jk} + C_6(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} \\ & \quad + C_7(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl})\delta_{jn} + C_8\delta_{in}\delta_{jk}\delta_{lm} + C_9(\delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl})\delta_{in}, \end{aligned} \tag{5.15}$$

with

$$\begin{gathered} C_1 = L_1 = L_2, \quad C_2 = L_3, \quad C_3 = L_4 = L_5, \quad C_4 = L_6; \quad C_5 = L_7 = L_{10}, \\ C_6 = L_8 = L_{11}, \quad C_7 = L_9 = L_{12}, \quad C_8 = L_{13}, \quad C_9 = L_{14} = L_{15}. \end{gathered} \tag{5.16}$$

From the match of relations (5.12) and (5.15), we obtain the special form of a sixth order perfect isotropic tensor having the symmetries (5.10) with 6 significant components

$$\begin{aligned} & L_{ijklmn} \\ & = D_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + D_2\delta_{ij}\delta_{kn}\delta_{lm} + D_3[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} \\ & \quad + (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln}] + D_4(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} \\ & \quad + D_5(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} + D_6[(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} + \\ & \quad + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}], \end{aligned} \tag{5.17}$$

with

$$\begin{gathered}D_1 = B_1 = B_2 = C_1 = L_1 = L_2, \quad D_2 = B_3 = C_2 = L_3, \\ D_3 = B_4 = B_5 = C_3 = C_5 = L_4 = L_5 = L_7 = L_{10}, \\ D_4 = B_6 = C_4 = C_8 = L_6 = L_{13}, \quad D_5 = B_7 = C_6 = L_8 = L_{11}, \\ D_6 = B_8 = B_9 = C_7 = C_9 = L_9 = L_{12} = L_{14} = L_{15},\end{gathered} \tag{5.18}$$

where we have used expressions (5.13) and (5.16).

6. Appendix: Derivation of the rate equation for the porosity field flux

To obtain equation (4.9), we use (3.11) and the special forms (5.4) and (5.17), assumed by the forth order tensor γ^4_{ijkl} and the sixth order tensors γ^r_{ijklmn} ($r = 3, 6$), so that equation (3.3) takes the form

$$\begin{aligned}&\delta_{ij}\frac{\partial \mathcal{V}_k}{\partial t} \\ &= \{\gamma^3_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} + \gamma^3_2\delta_{ij}\delta_{kn}\delta_{lm} + \gamma^3_3[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} \\ &\quad + (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln}] + \gamma^3_4(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} + \gamma^3_5(\delta_{il}\delta_{jm} \\ &\quad + \delta_{im}\delta_{jl})\delta_{kn} + \gamma^3_6[(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}]\}\mathcal{V}_n\delta_{lm} \\ &\quad + [\gamma^4_1\delta_{ij}\delta_{kl} + \gamma^4_2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]c_{,l} + \{\gamma^6_1(\delta_{kl}\delta_{mn} + \delta_{km}\delta_{ln})\delta_{ij} \\ &\quad + \gamma^6_2\delta_{ij}\delta_{kn}\delta_{lm} + \gamma^6_3[(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta_{mn} + (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\delta_{ln}] \\ &\quad + \gamma^6_4(\delta_{ik}\delta_{jn} + \delta_{in}\delta_{jk})\delta_{lm} + \gamma^6_5(\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl})\delta_{kn} \\ &\quad + \gamma^6_6[(\delta_{il}\delta_{jn} + \delta_{in}\delta_{jl})\delta_{km} + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\delta_{kl}]\}r_{,n}\delta_{lm},\end{aligned} \tag{6.1}$$

where γ^3_s and γ^6_s ($s = 1, \dots 6$) are the 6 independent significant components of the sixth order tensors γ^3_{ijklmn} and γ^6_{ijklmn}, respectively, and γ^4_1, γ^4_2 are the 2 independent significant components of the fourth order tensor γ^4_{ijkl}. Then, from (6.1) we obtain

$$\begin{aligned}\delta_{ij}\frac{\partial \mathcal{V}_k}{\partial t} &= \left[\left(2\gamma^3_1 + 3\gamma^3_3 + 2\gamma^3_5\right)\mathcal{V}_k + \left(2\gamma^6_1 + 3\gamma^6_3 + 2\gamma^6_5\right)r_{,k} + \gamma^4_1 c_{,k}\right]\delta_{ij} \\ &\quad + \left[\left(3\gamma^3_2 + 2\gamma^3_4 + 2\gamma^3_6\right)\mathcal{V}_j + \left(3\gamma^6_2 + 2\gamma^6_4 + 2\gamma^6_6\right)r_{,j} + \gamma^4_2 c_{,j}\right]\delta_{ik} \\ &\quad + \left[\left(3\gamma^3_2 + 2\gamma^3_4 + 2\gamma^3_6\right)\mathcal{V}_i + \left(3\gamma^6_2 + 2\gamma^6_4 + 2\gamma^6_6\right)r_{,i} + \gamma^4_2 c_{,i}\right]\delta_{jk}.\end{aligned} \tag{6.2}$$

Thus, when $i = j$ we have

$$\begin{aligned}\frac{\partial \mathcal{V}_k}{\partial t} &= \left(2\gamma^3_1 + 6\gamma^3_2 + 3\gamma^3_3 + 6\gamma^3_4 + 2\gamma^3_5 + 6\gamma^3_6\right)\mathcal{V}_k + \left(\gamma^4_1 + 2\gamma^4_2\right)c_{,k} \\ &\quad + \left(2\gamma^6_1 + 6\gamma^6_2 + 3\gamma^6_3 + 6\gamma^6_4 + 2\gamma^6_5 + 6\gamma^6_6\right)r_{,k},\end{aligned} \tag{6.3}$$

i.e. equation (4.9), $\tau^\nu \frac{\partial \mathcal{V}_k}{\partial t} = -\mathcal{V}_k - D_\nu r_{,k} + \alpha_\nu c_{,k}$, when we introduce the following definitions (with the minus sign coming from physical reasons)

$$2\gamma^3_1 + 6\gamma^3_2 + 3\gamma^3_3 + 6\gamma^3_4 + 2\gamma^3_5 + 6\gamma^3_6 = -\left(\tau^\nu\right)^{-1}, \quad \alpha_\nu = \tau^\nu\left(\gamma^4_1 + 2\gamma^4_2\right), \tag{6.4}$$

$$D_\nu = -\tau^\nu\left(2\gamma^6_1 + 6\gamma^6_2 + 3\gamma^6_3 + 6\gamma^6_4 + 2\gamma^6_5 + 6\gamma^6_6\right). \tag{6.5}$$

7. Derivation of the rate equation for the fluid-concentration flux

To derive (4.10), we use equation (3.4), the assumption $(3.11)_1$ and the special form (4.3) and (5.8) of the tensors ξ^4_{ij} and ξ^6_{ijkl}, so that we obtain

$$\tau^{j^c}\frac{\partial j^c_i}{\partial t} = -j^c_i - \xi^4\delta_{ij}c_{,j} + [\xi^6_1\delta_{il}\delta_{jk} + \xi^6_2(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl})]r_{,l}\delta_{jk}, \tag{7.1}$$

where ξ^6_1, ξ^6_2 are the 2 significant independent components of the fourth tensor ξ^6_{ijkl} and ξ^4 is the only one significant component of the second order tensor ξ^4_{ij}. Then, equation (7.1) keeps the form

$$\tau^{j^c}\frac{\partial j^c_i}{\partial t} = -j^c_i - \xi^4 c_{,i} + (3\xi^6_1 + 2\xi^6_2)r_{,i}, \tag{7.2}$$

i.e. equation (4.10), $\tau^{j^c}\frac{\partial j^c_i}{\partial t} = -j^c_i + \alpha_c r_{,i} - \rho D_c c_{,i}$, when we introduce the following definitions

$$\alpha_c = 3\xi^6_1 + 2\xi^6_2, \quad D_c = \frac{\xi^4}{\rho}. \tag{7.3}$$

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