

# Periodic, complex and kink-type solitons for the nonlinear model in microtubules

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**Abstract.** In this paper, we use the exponential function method to find some complex travelling wave solutions in the nonlinear dynamics model which describes the dimer's dynamics within microtubules. We obtain some entirely complex and kink-type soliton solutions to this nonlinear model. By choosing some suitable values of parameters, we plot the various dimensional simulations of all the obtained solutions in this study. We observe that our result may be useful in detecting some complex waves behaviors of kink solitons moving along the microtubule

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**Key words:** The EFM; NLEEs; microtubules; periodic; complex; kink-type soliton solutions; contour graphs.

## 1 Introduction

In recent several decades, many nonlinear evolution equations (NLEEs) for explaining more properties of Micro-Tubules (MTs) in solitons have been largely studied. Soliton theory plays an important role in the analysis of many nonlinear models that describe various phenomena in the field of nonlinear media. Dynamics of solitons has been largely studied in literature (see e.g. [1-3]). Moreover, powerful tools such as the exponential function method, the modified simple equation method, the Kudryashov method, Sumudu transform method,  $(G'/G)$ -expansion method and many more techniques have been used (see e.g. [4-12]). In this paper, we propose a soliton nonlinear model for the analysis of proteins. Proteins are indispensable part of living creatures. Major cytoskeletal proteins create Microtubules (MTs) [13]. These MTs are sketched as hollow cylinders usually formed by 13 parallel protofilaments (PFs) covering the cylindrical walls of MTs [13]. Each PFs symbolize a series of proteins called tubulin dimers [13-16]. S.Pospich et al have investigated an optimal tool to study cytoskeletal proteins [17]. P. Drabik et al and E. Nogales et al have studied on Microtube stability and high-resolution model of the Microtubule, and observed that the bonds between dimers within the same PFs are remarkable stronger than the soft bonds between neighbouring PFs [18,19]. This fact has been represented by the nonlinear dynamical equation (NDE) defined as [13,20].

$$(1.1) \quad m\Upsilon_{tt}(x,t) - kl^2\Upsilon_{xx}(x,t) - \alpha\Upsilon(x,t) + \beta\Upsilon^3(x,t) + \gamma\Upsilon_t(x,t) - qe = 0,$$

where  $\Upsilon(x,t)$  symbolizes the real displacement of the dimer along  $x$  axis. This model is used to explain that the longitudinal displacements of pertaining dimers in a single PF should cause the longitudinal wave propagating along PF [13].

In this study, the  $\exp(-\varphi(\zeta))$ -exponential function method (EFM) will be used to find soliton solutions such as complex, dark and kink-type from the Eq.(1.1). Dark soliton describes the solitary waves with lower intensity than the background [21]. The kink-type soliton describes the physical properties of quasi-one-dimensional ferromagnets [22] and the singular soliton solutions is a solitary wave with discontinuous derivatives; examples of such solitary waves include compactons [23,24]. In this sense, many powerful models along with engineering applications have been presented in literature (see e.g. [29-33]).

## 2 The expansion function method (EFM)

Here, we shortly give the main steps of the EFM. Let us consider the nonlinear partial differential equation (NPDE):

$$(2.1) \quad P(\Psi, \Psi_x, \Psi_x\Psi^2, \Psi_{xx}, \Psi_{xxt}, \dots) = 0,$$

where  $\Psi = \Psi(x,t)$  is the unknown function, and  $P$  is a polynomial in  $\Psi(x,t)$ .

**Step 1:** By using the wave transformation:

$$(2.2) \quad \Psi(x,t) = U(\zeta), \quad \zeta = \kappa x - \omega t,$$

from Eq.(2.1), we obtain the nonlinear ordinary differential equation (NODE):

$$(2.3) \quad NODE(U, U'U^2, U', U'', \dots) = 0,$$

where  $NODE$  is a polynomial of  $U$  and its derivatives.

**Step 2:** Let us now assume the solutions of Eq. (2.3) to have the form:

$$(2.4) \quad U(\zeta) = \sum_{i=0}^n A_i \left[ e^{-\varphi(\zeta)} \right]^i = A_0 + A_1 e^{-\varphi} + \dots + A_n e^{-n\varphi},$$

where  $A_i, (0 \leq i \leq n)$  are constants to be obtained later, such that  $A_n \neq 0$ , and  $\varphi = \varphi(\zeta)$  solves the following ODE:

$$(2.5) \quad \varphi'(\zeta) = e^{-\varphi(\zeta)} + \mu e^{\varphi(\zeta)} + \lambda.$$

Eq.(2.5) admits the following set of solutions [25, 26, 28, 29]:

**Set 1:** When  $\mu \neq 0$ ,  $\lambda^2 - 4\mu > 0$ ,

$$(2.6) \quad \varphi(\zeta) = \ln \left( \frac{-\sqrt{\lambda^2 - 4\mu}}{2\mu} \times \tanh \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} (\zeta + E) \right) - \frac{\lambda}{2\mu} \right).$$

**Set 2:** When  $\mu \neq 0$ ,  $\lambda^2 - 4\mu < 0$ ,

$$(2.7) \quad \varphi(\zeta) = \ln \left( \frac{\sqrt{-\lambda^2 + 4\mu}}{2\mu} \times \tan \left( \frac{\sqrt{-\lambda^2 + 4\mu}}{2} (\zeta + E) \right) - \frac{\lambda}{2\mu} \right).$$

**Set 3:** When  $\mu = 0$ ,  $\lambda \neq 0$  and  $\lambda^2 - 4\mu > 0$ ,

$$(2.8) \quad \varphi(\zeta) = -\ln \left( \frac{\lambda}{e^{\lambda(\zeta+E)} - 1} \right).$$

**Set 4:** When  $\mu \neq 0$ ,  $\lambda \neq 0$  and  $\lambda^2 - 4\mu = 0$ ,

$$(2.9) \quad \varphi(\zeta) = \ln \left( -\frac{2\lambda(\zeta + E) + 4}{\lambda^2(\zeta + E)} \right).$$

**Set 5:** When  $\mu = 0$ ,  $\lambda = 0$  and  $\lambda^2 - 4\mu = 0$ ,

$$(2.10) \quad \varphi(\zeta) = \ln(\zeta + E).$$

$A_i$ , ( $0 \leq i \leq n$ ),  $E$ ,  $\lambda$ ,  $\mu$  are coefficients to be obtained, and  $n$ ,  $m$  are positive integers that one can find by the balancing principle.

**Step 3:** Inserting Eq.(2.4) together with its derivatives along with the Eq.(2.5) and simplifying, we find a polynomial equation of  $e^{-\varphi(\zeta)}$ . We extract a set of algebraic equations from this polynomial equation by summing the terms of the same power and equating each summation to zero. We solve this set of equations to find the values of the coefficients involved. By inserting the obtained values of the coefficients along with one of Eqs.(2.6-2.10) into Eq.(2.4), we can obtain the new solitons to the NPDE equation (2.1).

### 3 Application of EFM

In this section, we use the EFM to obtain some new solutions of the nonlinear dynamical equation (1.1).

Consider the following travelling wave transformation:

$$(3.1) \quad \Upsilon(x, t) = \Psi(\zeta), \quad \zeta = \kappa x - \omega t.$$

Substituting Eq.(3.1) into Eq.(1.1), yields the following NODE;

$$(3.2) \quad (m\omega^2 - kl^2\kappa^2)\Psi'' - \gamma\omega\Psi' - \alpha\Psi + \beta\Psi^3 - qe = 0.$$

Balancing the highest power nonlinear term and the highest derivative in Eq.(3.2). Doing so, the value of  $n$  is obtained as  $n = 1$ . Using  $n = 1$  along with Eq.(2.4), yields

$$(3.3) \quad \Psi(\zeta) = A_0 + A_1e^{-\varphi},$$

Putting Eq. (3.3) along with its second derivative into Eq.(3.2), gives a polynomial equation in  $e^{-\varphi}$ . We gather a group of algebraic equations from this polynomials by equating the sum of the coefficients of  $e^{-\varphi}$  with the same power to zero. We solve this group of equations and obtain the values of the coefficients involved. To obtain the solutions of Eq.(1.1), we put the values of the coefficients into Eq. (3.3) along with *Family-1* condition.

**Case-1:**

$$A_0 = \frac{3kl^2\kappa^2\lambda - \omega(\gamma + 3m\lambda\omega)}{3\sqrt{2}\sqrt{\beta}\sqrt{kl^2\kappa^2 - m\omega^2}}, A_1 = \frac{\sqrt{2kl^2\kappa^2 - 2m\omega^2}}{\sqrt{\beta}},$$

$$\alpha = \frac{\gamma^2\omega^2 + 3(\lambda^2 - 4\mu)(kl^2\kappa^2 - m\omega^2)^2}{6kl^2\kappa^2 - 6m\omega^2}, \quad q = \frac{\gamma\omega(\gamma^2\omega^2 - 9(\lambda^2 - 4\mu)(kl^2\kappa^2 - m\omega^2)^2)}{27\sqrt{2}e\sqrt{\beta}(kl^2\kappa^2 - m\omega^2)^{\frac{3}{2}}},$$

with these coefficients, the following set of solutions are obtained:

**Set-1:** When  $\mu \neq 0$ ,  $\lambda^2 - 4\mu > 0$ , Eq.(1.1) gives the following **kink-type soliton** solution:

$$(3.4) \quad \Upsilon_1(x, t) = \frac{2\mu\sqrt{2\varpi} - \lambda\rho\sqrt{\beta} - \rho\sqrt{\beta}\vartheta\tanh(\frac{1}{2}\vartheta(e + \kappa x - \omega t))}{-\lambda\sqrt{\beta} - \vartheta\sqrt{\beta}\tanh(\frac{1}{2}\vartheta(e + \kappa x - \omega t))},$$

$$\text{where, } \varpi = kl^2\kappa^2 - m\omega^2, \quad \vartheta = \sqrt{\lambda^2 - 4\mu}, \quad \rho = \frac{3kl^2\kappa^2\lambda - \omega(\gamma + 3m\lambda\omega)}{3\sqrt{2}\sqrt{\beta}\sqrt{\varpi}}.$$

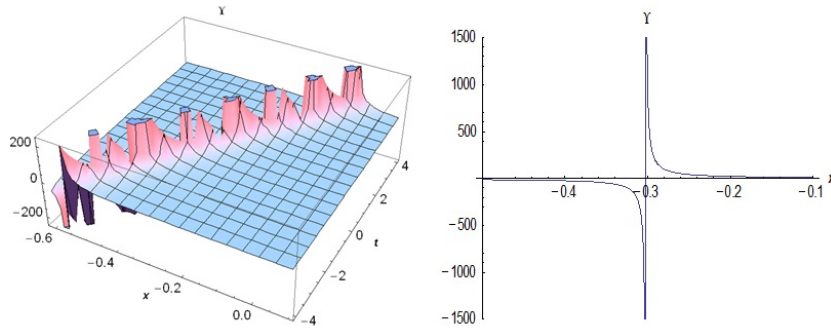


Figure 1: The 2D and 1D ( $t=5$  for 1D) surfaces of Eq.(3.4).

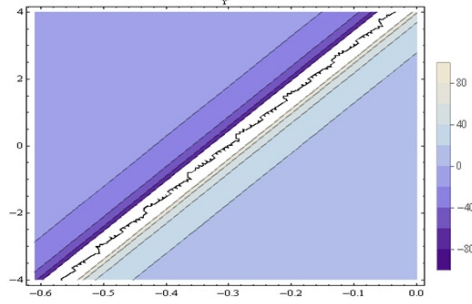


Figure 2: Contour plot of Eq.(3.4).

**Set-2:** When  $\mu = 0$ ,  $\lambda \neq 0$  and  $\lambda^2 - 4\mu < 0$ , Eq.(1.1) is of the following **periodic soliton** solution:

$$(3.5) \quad \Upsilon_2(x, t) = \frac{2\mu\sqrt{2\varpi} - \lambda\varrho\sqrt{\beta} + \varrho\sqrt{\beta}\vartheta\tan(\frac{1}{2}\vartheta(e + \kappa x - \omega t))}{-\lambda\sqrt{\beta} - \vartheta\sqrt{\beta}\tan(\frac{1}{2}\vartheta(e + \kappa x - \omega t))},$$

$$\text{where, } \varpi = kl^2\kappa^2 - m\omega^2, \vartheta = \sqrt{-\lambda^2 + 4\mu}, \varrho = \frac{3kl^2\kappa^2\lambda - \omega(\gamma + 3m\lambda\omega)}{3\sqrt{2}\sqrt{\beta}\sqrt{\varpi}}.$$

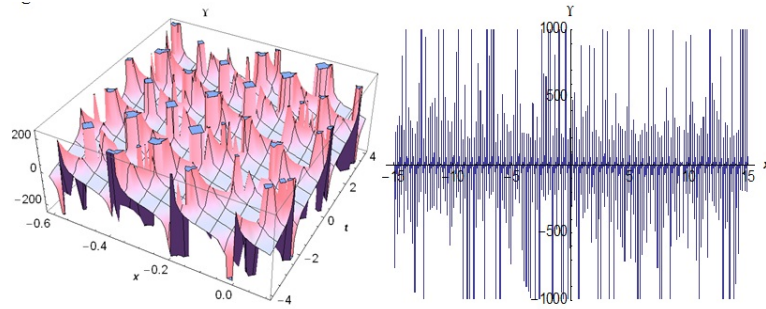


Figure 3: The 2D and 1D (t=0.85 for 1D) surfaces of Eq.(3.5).

**Case-2:**

$$A_0 = \frac{3qe(\alpha + i\sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2})}{4\alpha^2}, \quad A_1 = \frac{-3eq\gamma\omega}{\alpha^2}$$

,

$$\lambda = -\frac{i\sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2}}{2\gamma\omega}, \quad m = \frac{-2\gamma^2}{3\alpha} + \frac{kl^2\kappa^2}{\omega^2}, \quad \beta = \frac{4\alpha^3}{27e^2q^2},$$

with these coefficients, and, for *Family - 1* condition being  $\mu \neq 0$ ,  $\lambda^2 -$

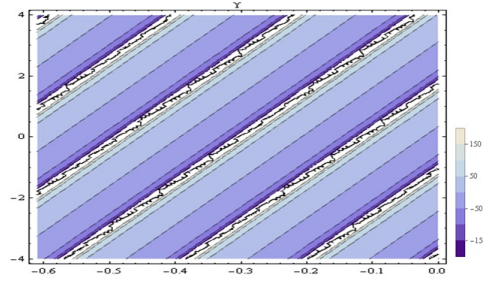


Figure 4: Contour plot of Eq.(3.5).

$4\mu > 0$ , Eq.(1.1) is also solved by this new **complex kink-type soliton**

$$(3.6) \quad \Upsilon_3(x, t) = \frac{-6eq\gamma\mu\omega + \frac{3qe\alpha + 3iqe\tau}{4}(i\varpi - \theta \tanh[\frac{\theta}{2}(e + \kappa x - \omega t)])}{i\alpha^2\varpi - \alpha^2\theta \tanh[\frac{\theta}{2}(e + \kappa x - \omega t)]},$$

where

$$\begin{aligned} \varpi &= \frac{\sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2}}{2\gamma\omega}, \\ \theta &= \sqrt{-4\mu - \frac{-9\alpha^2 - 16\gamma^2\mu\omega^2}{4\gamma^2\omega^2}}, \\ \tau &= \sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2} \end{aligned}$$

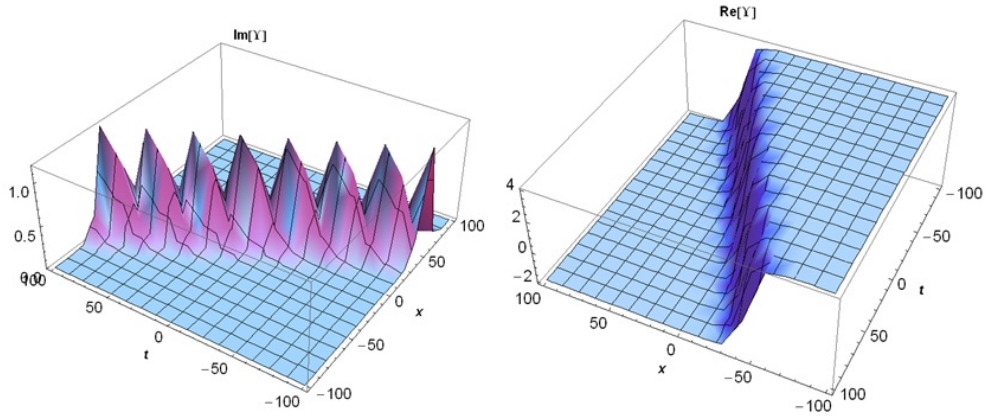


Figure 5: The 2D surfaces imaginary and real part of Eq.(3.6).

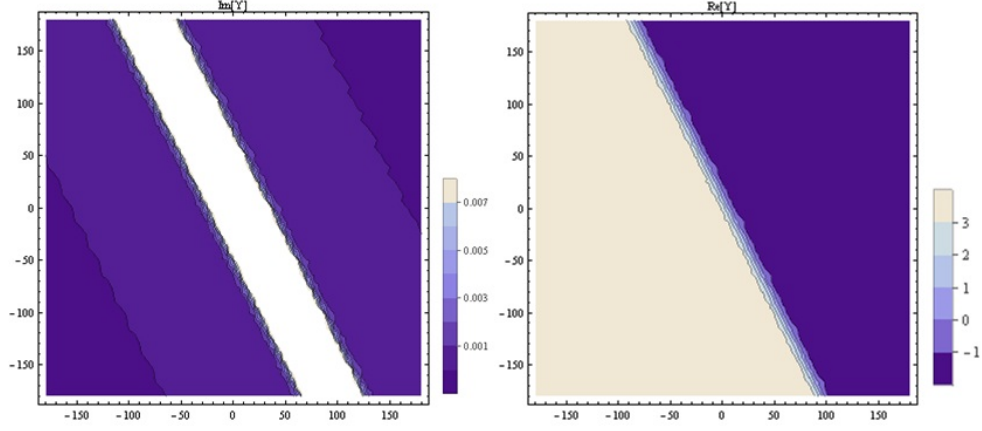


Figure 6: Contour plot of imaginary and real part of Eq.(3.6).

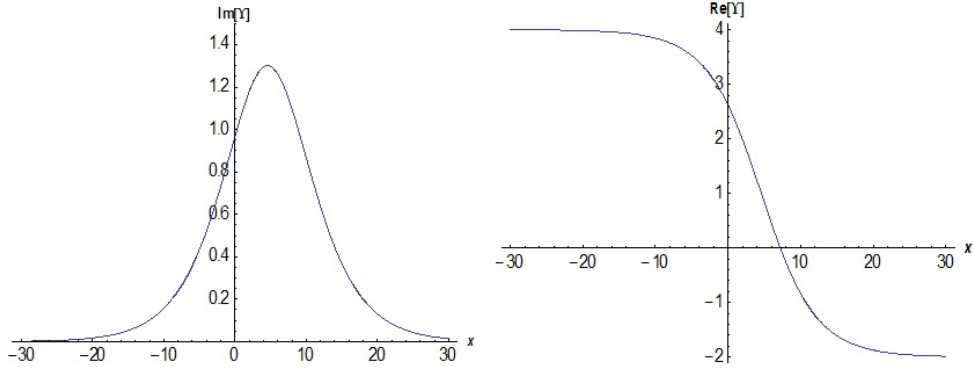


Figure 7: The 1D surfaces of imaginary and real part of Eq.(3.6) (t=0.85).

**Case-3:**  $A_0 = \frac{3qe(\alpha - i\sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2})}{4\alpha^2}$ ,  $A_1 = \frac{-3eq\gamma\omega}{\alpha^2}$ ,  $\lambda = \frac{i\sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2}}{2\gamma\omega}$ ,  $m = \frac{-2\gamma^2}{3\alpha} + \frac{kl^2\kappa^2}{\omega^2}$ ,  $\beta = \frac{4\alpha^3}{27e^2q^2}$ , with these coefficients, and, for *Family - 1* with  $\mu \neq 0$ ,  $\lambda^2 - 4\mu > 0$ , Eq.(1.1) is solved by the new **complex kink-type soliton**

$$(3.7) \quad \Upsilon_4(x, t) = \frac{-6eq\gamma\mu\omega + \frac{3}{4}(qe\alpha - iqe\tau)(-i\varpi - \theta \tanh[\frac{\theta}{2}(e + \kappa x - \omega t)])}{\alpha^2(-i\varpi - \theta \tanh[\frac{\theta}{2}(e + \kappa x - \omega t)])},$$

$$\text{where } \varpi = \frac{\sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2}}{2\gamma\omega}, \theta = \sqrt{-4\mu - \frac{-9\alpha^2 - 16\gamma^2\mu\omega^2}{4\gamma^2\omega^2}}, \tau = \sqrt{-9\alpha^2 - 16\gamma^2\mu\omega^2}.$$

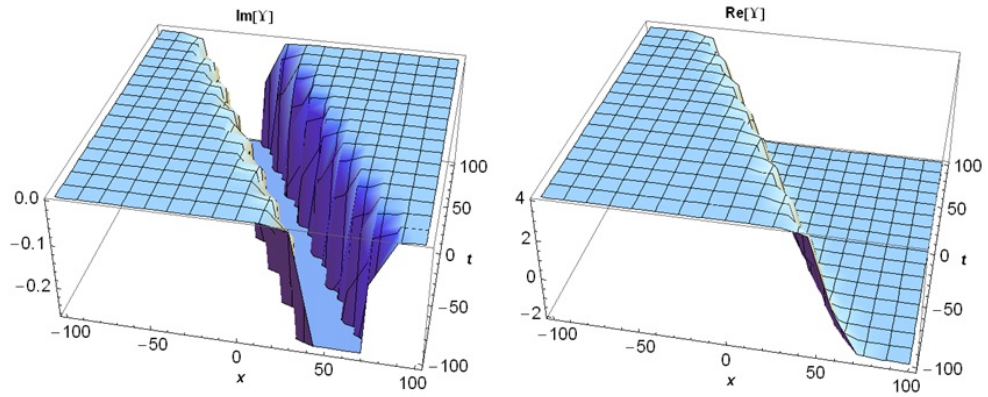


Figure 8: The 2D surfaces imaginary and real part of Eq.(3.7).

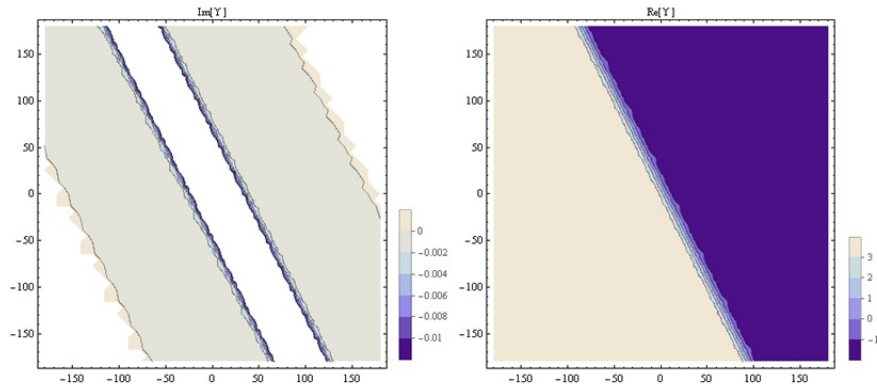


Figure 9: Contour plot of imaginary and real part of Eq.(3.7).

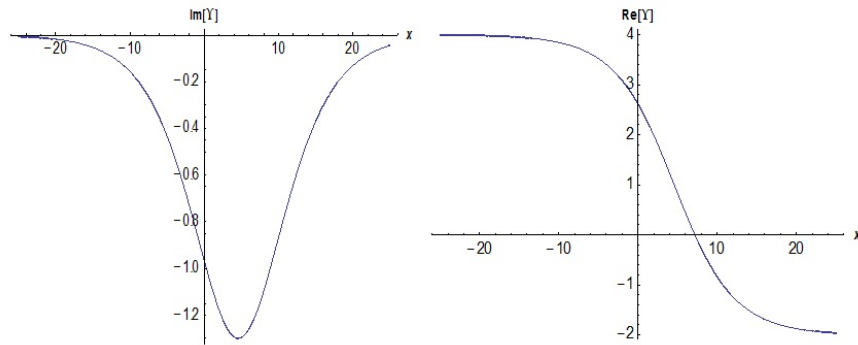


Figure 10: The 1D surfaces of imaginary and real part of Eq.(3.7) ( $t=0.85$ ).



## 4 Remark and comparisons

All analytical solutions obtained in this paper via EFM is completely new with the results of [13], and has been introduced firstly to the literature along with the figures plotted under the suitable values of parameters in solutions.

## 5 Conclusion

In this paper, the  $\exp(-\varphi(\zeta))$ -exponential function method has been successfully used in extracting complex, kink-type and periodic singular soliton solutions to the nonlinear dynamics model (1.1). The constraint conditions for the existence of valid soliton solutions where necessary are also given. The physical meaning of the obtained solutions in relations to the nonlinear dynamics model (1.1) are given as well. Solutions (3.4), (3.6) and (3.7) belong to complex and kink-type soliton solutions. Solution (3.5) belongs to periodic soliton solutions. The solution of Set-2 in Case-2 being  $\lambda^2 - 4\mu < 0$  is not valid because it is always positive as  $\lambda^2 - 4\mu = \frac{9\alpha^2}{4\gamma^2\omega^2} > 0$ .

The soliton solutions obtained in this paper might be physically useful in explaining how the bonds between dimers within the same PFs are remarkable stronger than the soft bonds between neighbouring PFs. As a physical aspects of results, it is observed that the hyperbolic tangent arises in the calculation of magnetic moment and rapidity of special relativity [27]. The results found in here are entirely new when comparing the results presented in [13]. To the best of our knowledge, the application of EFM to the Eq.(1.1) has not been submitted to the literature in advance. Finally, one can be inferred from results that the  $\exp(-\varphi(\zeta))$ -exponential function method is a powerful and efficient mathematical tool that can be used to find many soliton solutions such as complex, kink-type and periodic to various nonlinear partial differential equations with high nonlinearity.

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