Dissipative processes in porous solids filled by a fluid flow

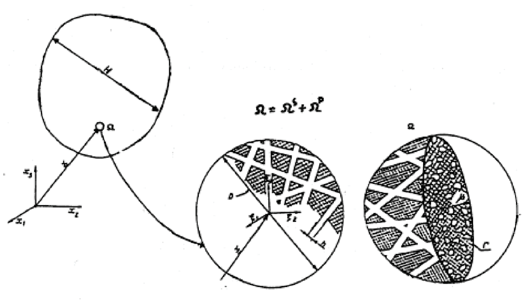
L. Restuccia[a], L. Palese[b], M. T. Caccamo[a] and A. Famà[a]

[a]University of Messina, Department of Mathematical and Computer Sciences, Physical Sciences and Earth Sciences, Italy

[b]University of Bari, Department of Mathematics, Italy

Aim and motivation The models of porous solids may have relevance in important advances in the description of problems and phenomena that accompany flow of mass in porous structures. They find applications in many fundamental sectors: ***geology, biology, medical sciences, technology of materials and other applied sciences.*** The porous channels sometime can self propagate because of changed conditions and surrounding conditions that are favorable and this propagation can provoke a premature fracture of these porous media. Among the various descriptions of porous media, that one based on a macroscopic characterization of a skeleton pore structure à la Kubik, coming from the use of volume and area averaging procedures, was choosen. In this paper, using the methods of the non-equilibrium thermodynamics with internal variables, the deformation of fluid-satured porous solids is considered and its influence on state laws and evolution equations of heat and porous tensor fluxes is investigated. Also, the state laws to construct the constitutive relations are derived, that close the system of equations describing the behaviour of these complex materials. Also the heat equation is derived.

The structural permeability tensor model Let us consider a representative elementary sphere volume Ω of a porous skeleton filled with fluid, the diameter D of which is much greater than the characteristic lenght of the entire porous medium. Ω is large enough to provide a representation of all the statistical properties of the pore space. $\Omega = \Omega^s + \Omega^p$, with Ω^s and Ω^p the solid space and the pore space of the elementary volume. In Fig. 1 we see the averaging scheme regarding a pore structure (see [1]).



Since all pores are considered to be interconnected the effective volume porosity is completely defined as $f_v = \frac{\Omega^p}{\Omega}$, where Ω^p represents the pore space of Ω. The present analysis is restricted to media which homogeneous with respect to volume porosity f_v , i.e. f_v is coonstant. To avoid confusion we describe all microscopic quantities with the xi_i coordinate system, while microscopic quantities are assigned to the x_i axes: there is non difference between these systems apart from possible translations. In such a medium Kubik introduces a so called structural permeabily tensor , responsible for the structure of a system of pores, in the following way

$$\bar{v}(\mathbf{x})_i = r_{ij}(\mathbf{x}, \mu) \overset{*}{v}_j (\mathbf{x}, \mu),$$

where $\bar{\mathbf{v}}(\mathbf{x}) = \frac{1}{\Omega} \int_{\tilde{\Omega}} \mathbf{v}(\xi) d\tilde{\Omega},\ \xi \in \Omega^p$, is the bulk-volume average of fluid velocity and the corresponding pore-area fluid velocity is $\overset{*}{\mathbf{v}}(\mathbf{x}, \mu) = \frac{1}{\Gamma^p} \int_{\tilde{\Gamma}} \mathbf{v}(\xi) d\tilde{\Gamma},\ \xi \in \Gamma^p$. The orientation of the central sphere section Γ in Ω is given by the normal vector $\boldsymbol{\mu}$.

The tensor r_{ij} describes a structure of the porous media. Equation above gives a linear mapping between the bulk volume average fluid velocity and the local velocity of fluid particles passing through the pore area Γ^p of central sphere section with normal vector $\boldsymbol{\mu}$. In [1] Kubik establishes the geometrical interpretation of r_{ij} considering a fluid flow, having the average bulk average velocity $\bar{\mathbf{v}}$, as the superposition of three one-dimensional fluid flows along three mutually perpendicular channels, having average velocities on the areas of these channels $v_i^*(\mathbf{x}, \boldsymbol{\mu})$. Only part of the fluid can flow unimpeded while the rest is trapped in the porous skeleton.

Governing equations In [2] a thermodynamical model for describing reciprocal interactions between a fluid flow and a structural permeability field coming from a network of porous in an elastic body was developed. The assumption that an anisotropic pore structure is continuously distributed within the medium was done. Since during deformation the porous structure evolves in time, the structural permeability field is described both by the state tensor r_{ij} which relates to the density of porous and by the flux of this tensor $\mathcal{V}_{ijk}$. Similarly the fluid flow is described by two variables: the concentration of the fluid c and the flux of this fluid j_i.

The mass of density ρ_1 comes from the mass of the fluid transported through the elastic porous body of density ρ_2. The mass of the fluid and the elastic solid form a two-components mixture of density $\rho = \rho_1 + \rho_2$, where $\rho_1 \ll \rho_2$. We define the concentration c as follows $\boldsymbol{c} = \frac{\rho_1}{\rho}$.

For the mixture of continua as a whole and also for each constituent separately the continuity equations are satisfied

$$\dot{\rho} + \rho v_{i,i} = 0, \quad \frac{\partial \rho_1}{\partial t} + (\rho_1 v_{1i})_{,i} = h_1,$$
$$\frac{\partial \rho_2}{\partial t} + (\rho_2 v_{2i})_{,i} = h_2, \quad \rho v_i = \rho_1 v_{1i} + \rho_2 v_{2i}$$

where a superimposed dot denotes the material derivative, h_1 and h_2 are the sources, $v_{1,i}$ and $v_{2,i}$ are the velocities of the fluid particles and the particles of the elastic body, respectively. Then the fluid flux must be taken in the form $j_i^c = \rho_1(v_{1,i} - v_i)$.

The proposed model is based on the extended irreversible thermodynamics with internal variables. The thermal field is governed by the temperature and the heat flux q_i. The vector space is chosen as follows

$$C = \{\varepsilon_{ij}, c, T, r_{i,j}, j_i, q_i, \mathcal{V}_{ijk}, c_{,i}, T_{,i}, r_{ij,k}\},$$

where, we have taken into consideration the gradients $c_{,i}, T_{,i}$ and $r_{ij,k}$. We ignore the viscoelastic effects, so that τ_{ij} is not in the set C. All the processes occurring in the considered body are governed by two groups of laws. The first group concerns the classical balance equations:

The balance of mass $\quad \rho\dot{c} + j_{i,i}^c = 0$

The momentum balance $\quad \rho\dot{v}_i - \tau_{ij,j} - f_i = 0,$

where f_i denotes a body force.

The internal energy balance $\quad \rho\dot{e} - \tau_{ji} v_{i,j} + q_{i,i} - \rho h = 0,$

where h is the heat source distribution. The second groups of laws deals with the rate properties of the internal variable and the fluxes of the mass, heat and porous field. The evolution equations for the internal variable r_{ij} and for the fluxes q_i and j_i, read

$$\dot{r}_{ij} + \mathcal{V}_{ijk,k} - \mathcal{R}_{ij}(C) = 0, \quad \dot{j}_i - J_i(C) = 0,$$
$$\dot{q}_i - Q_i(C) = 0, \quad \dot{\mathcal{V}}_{ijk} - V_{ijk}(C) = 0,$$

Constitutive relations and rate equations To be sure that our consideration deal with the real physical processes occuring in the considered body where all the assumed fields can interact with each other, all the admissible solutions of the proposed evolution equations should be restricted by the following entropy inequality (see [2])

$$\rho\dot{S} + (\phi_i + k_i)_{,i} - \frac{\rho h}{T} \geq 0,$$

where S is the entropy density, $\rho h T^{-1}$ is the external entropy production source, ϕ_i is the entropy flux and $k_i = k_i(C)$ is an extra entropy flux which is added because the nonequilibrium fluxes are adjoined to the set of independent variables.

The following set of constitutive functions (dependent variables) is formed

$$W = \{\tau_{ij}, \pi^c, \pi^r_{ij}, g_i, e, \mathcal{R}_{ij}, J_i, Q_i, V_{ijk}, S, \phi_i F, k_i\},$$

where F denotes the free energie density given by $F = e - TS$, Π^r_{ij} is a potential related to the porous field, defined in the sequel and Pi^c denotes the chemical potential of mass. Then we must look for general constitutive equations in the form $W = \tilde{W}(C)$, where both C and W are evaluated at the same point and time. In [3] the entropy inequality was analyzed by Liu theorem obtaining the following results: *the state laws* $\tau_{ij} = \rho\frac{\partial F}{\partial \varepsilon_{ij}}, \quad \pi^c = \frac{\partial F}{\partial c}, \quad \frac{\partial F}{\partial c_{,i}} = 0,\ S = -\frac{\partial F}{\partial T}, \quad \pi^r_{ij} = \frac{\partial F}{\partial r_{ij}}, \quad \frac{\partial F}{\partial T_{,i}} = 0, \quad \frac{\partial F}{\partial r_{ij,k}} = 0,$ *the affinities conjugate to the fluxes* $\Pi^j_i \equiv \rho\frac{\partial F}{\partial j_i}, \quad \Pi^q_i \equiv \rho\frac{\partial F}{\partial q_i}, \quad \Pi^{\mathcal{V}}_{ijp} \equiv \rho\frac{\partial F}{\partial \mathcal{V}_{ijp}},$
where Π^c denotes the chemical potential of the diffusion mass field and Π^r_{ij} is the similar constitutive quantity related to the porous field. Finally *the energy flux* $\phi_k = \frac{1}{T}(q_k - \pi^c j_k + \pi^r_{ij}\mathcal{V}_{ijk})$.
τ_{ij} is symmetric and then the couple g_i vanishes and the free energy is the function $F = F(\varepsilon_{ij}, c, T, r_{ij}, j_i, q_i, \mathcal{V}_{ijk})$.

Also in [4], expanding the free energy about the equilibrium state, denoting by $\theta = T - T_0$, $\left|\frac{\theta}{T_0}\right| \ll 1$, $\mathcal{C} = c - (c)_0$, $\left|\frac{\mathcal{C}}{(c)_0}\right| \ll 1$, $R_{ij} = r_{ij} - (r_{ij})_0$, $\left|\frac{R_{ij}}{(r_{ij})_0}\right| \ll 1$, deviations with respect to equilibrium (with the subscript "$_0$" referring to the equilibrium state and assuming that $(\varepsilon_{ij})_0 = 0$, $(\mathcal{V}_{ijk})_0 = 0$, $(q_i)_0 = 0$, $(j_i^c)_0 = 0$. The following relations are derived:

The constitutive relations

$$\tau_{ij} = c_{ijlm}\varepsilon_{lm} - \lambda^\theta_{ij}\theta + \alpha^{r\epsilon}_{ijlm}R_{lm} + \lambda^{c\varepsilon}_{ij}\mathcal{C},$$
$$S = \frac{\lambda^\theta_{ij}}{\rho}\varepsilon_{ij} + \frac{c_v}{\theta_0}\theta - \frac{\alpha^{r\theta}_{ij}}{\rho}R_{ij} + \lambda^{\theta c}\mathcal{C},$$
$$\pi^r_{ij} = \lambda^{r\epsilon}_{ijlm}\varepsilon_{lm} + \lambda^{r\theta}_{ij}\theta + \lambda^{rr}_{ijlm}R_{lm} + \lambda^{rc}_{ij}\mathcal{C}$$

The rate equations for the fluxes and the internal variables

$$\tau\dot{q}_i = -q_i - \chi^1_{ij}\theta_{,j} + \chi^2_{ij}j^c_j + \chi^3_{ijk}r_{jk} + \chi^4_{ijk} + \varepsilon_{jk} + \chi^5_{ijkl}\mathcal{V}_{jkl} + \chi^6_{ijkl}r_{jk,l}$$
$$\dot{j}^c_i = \eta^1_{ij}\theta_{,j} + \eta^2_{ij}q_j + \eta^3_{ij}j^c_j + \eta^4_{ijk}r_{jk} + \eta^5_{ijk}\varepsilon_{jk} + \eta^6_{ijkl}\mathcal{V}_{jkl} + \eta^7_{ijkl}r_{jk,l},$$
$$\dot{r}_{ij} + \mathcal{V}_{ijk,k} = \beta^2_{ijk}\theta_{,k} + \beta^3_{ijk}q_k + \beta^4_{ijk}j^c_k + \beta^6_{ijkl}\varepsilon_{kl} + \beta^5_{ijkl}r_{kl} + \beta^6_{ijklm}\mathcal{V}_{klm} + \beta^7_{ijklm}r_{kl,m},$$
$$\dot{\mathcal{V}}_{ijk} = \gamma^2_{ijkl}\theta_{,l} + \gamma^3_{ijkl}q_l + \gamma^4_{ijkl}j^c_l + \gamma^6_{ijklm}\varepsilon_{lm} + \gamma^7_{ijklm}r_{lm} + \gamma^8_{ijklmn}\mathcal{V}_{lmn} + \gamma^9_{ijklmn}r_{lm,n}.$$

where the first is the rate equation for the heat flux generalizes Vernotte - Cattaneo relation and χ^1_{ij} are heat conductivity coefficients.

Heat equation and dissipative processes To consider and solve analytically and/or numerically particular problems, we linearize the developed theory, obtaining a mathematical model to describe the physical reality in many situations. Introducing the Legendre transformation, considering the material derivative of the free energy F i.e. $\rho T\dot{S} = \rho\dot{U} - \rho S\dot{T} - \rho\dot{F}$, and taking into consideration the balance energy equation $\rho\dot{U} = \tau_{ij}\dot{\varepsilon}_{ij} - q_{i,i}$, calculating the material derivative of the free energy and using the laws of state, the definitions of the affinities and the rate equations for the heat and dislocation fluxes and for the core dislocation tensor, in the linear case we work out the following equation:

$$\tau\rho_0 T_0\left(\frac{\lambda^\theta_{ij}}{\rho_0}\ddot{\varepsilon}_{ij} + \frac{c_v}{T_0}\ddot{\theta} - \frac{\alpha^{a\theta}_{ij}}{\rho_0}\ddot{r}_{ij} + \lambda^{\theta c}\ddot{\mathcal{C}}\right) = -\rho_0 T_0\left(\frac{\lambda^\theta_{ij}}{\rho_0}\dot{\varepsilon}_{ij} + \frac{c_v}{T_0}\dot{\theta} - \frac{\alpha^{a\theta}_{ij}}{\rho_0}\dot{r}_{ij} + \lambda^{\theta c}\dot{\mathcal{C}}\right) + \chi^1_{ij}\theta_{,ji} - \chi^3_{ijk}r_{jk,i} - \chi^4_{ijk}\varepsilon_{jk,i} - \chi^5_{ijkl}\mathcal{V}_{jkl,i} - \chi^6_{ijkl}r_{jk,li} - \chi^7_{ij}j^c_{j,i}.$$

The above equations allow to describe the thermal behaviour of anisotropic porous media in the linear approximation. In the case when we consider perfect ***isotropic porous media***, for which the symmetry properties are invariant under orthogonal transformations with respect to all rotations and to inversion of the frame of axes, it can be see that the heat equation becomes

$$\tau\ddot{\theta} + \dot{\theta} + \frac{\gamma}{c}(\tau\ddot{\varepsilon}_{ii} + \dot{\varepsilon}_{ii}) - \frac{\eta}{c}(\tau\ddot{r}_{ii} + \dot{r}_{ii})\frac{\varphi}{c}\left(\tau\ddot{\mathcal{C}} + \dot{\mathcal{C}}\right) = \nu^1 j^c_{,ii} + -\nu^2\mathcal{C}_{,ii} - \frac{1}{2}(k_1 r_{jj,ii} + k_2 r_{ij,ji}),$$

with all the phenomenological constants, leading to finite speeds of propagation of disturbances. It is a generalized telegraphic equation for isotropic defective solids. In the case that the the deformation and dislocation effects can be neglected equation above reduces to the telegrafic equation $\tau\ddot{\theta} + \dot{\theta} = \frac{k}{c}\theta_{,ii}$.

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