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Existence of solutions of infinite system of nonlinear sequential fractional differential equations

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Abstract

In a recent paper (Filomat 32:4577–4586, 2018) the authors have investigated the existence and uniqueness of a solution for a nonlinear sequential fractional differential equation. To present an analytical improvement for Fazli–Nieto's results with some conditions removed based on a new technique is the main objective of this paper. In addition, we introduce an infinite system of nonlinear sequential fractional differential equations and discuss the existence of a solution for them in the classical Banach sequence spaces c_0 and ℓ_p by applying the Darbo fixed point theorem. Moreover, the proposed method is applied to several examples to show the clarity and effectiveness.

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1 Introduction and preliminaries

As is well known, the fractional differential equations (FDEs) is a fundamental topic that considered as a powerful tool in many fields, for example, dynamic systems, rheology, blood flow phenomena, biophysics, electrical networks, modeled by different fractional order derivatives equations; see for details [2–5] and the references therein. Also, in the last two decades, FDEs have been used to model various stable physical phenomena [6–8]. For example, when the random oscillation force is assumed to be white noise, Brown's motion is well described by some fractional differential equations. On the other hand, during the last years, many studies have been done on the existence and uniqueness of solution of nonlinear initial fractional differential equations by the use of some fixed point theorems; see [9–20].

Recently, Fazli and Nieto [1] investigated the existence and uniqueness of the following interesting problem, which is a model of physical phenomena:

$$\begin{cases} \mathcal{D}^{2\alpha} u(x) = f(x, u(x), \mathcal{D}^{\alpha} u(x)), & x \in (0, T], \\ \lim_{x\to 0} x^{1-\alpha} u(x) = u_0, & \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha} u(x) = u_1, \end{cases} \tag{1}$$

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where $0 < \alpha \leq 1$, $0 < T < \infty$. The term $\mathcal{D}^{2\alpha}$ is for the sequence fractional derivative presented by Miller and Ross [21],

$$\begin{cases} \mathcal{D}^{\alpha} u = D^{\alpha} u, \\ \mathcal{D}^{k\alpha} u = \mathcal{D}^{\alpha} \mathcal{D}^{(k-1)\alpha} u \quad (k = 2, 3, \ldots), \end{cases} \tag{2}$$

where $\mathcal{D}^{\alpha}$ is the classical Riemann–Liouville fractional derivative of order α.

Before giving the weighted Cauchy type problem obtained in [1], let us recall some notions introduced in that work. Let

$$C_{1-\alpha}[0, T] = \left\{ u \in C[0, T] : x^{1-\alpha} u \in C[0, T] \right\} \tag{3}$$

be the weighted spaces of continuous functions with the following norm:

$$\|u\|_{C_{1-\alpha}[0,T]} = \max_{0 \leq x \leq T} \left| x^{1-\alpha} u(x) \right|.$$

We define the following spaces of functions:

$$C_{1-\alpha}^{\alpha}[0, T] = \left\{ u \in C[0, T] : x^{1-\alpha} u \in C_{1-\alpha}[0, T], \mathcal{D}^{\alpha} u \in C_{1-\alpha}[0, T] \right\},$$

with the norm

$$\|u\|_{C_{1-\alpha}^{\alpha}[0,T]} = \|u\|_{C_{1-\alpha}[0,T]} + \left\| \mathcal{D}^{\alpha} u \right\|_{C_{1-\alpha}[0,T]},$$

which are Banach spaces.

A function $\underline{u} \in C_{1-\alpha}^{\alpha}[0, T]$ is called a lower solution of the initial value problem (1), if $\mathcal{D}^{2\alpha} \underline{u}(x) \leq f(x, \underline{u}(x), \mathcal{D}^{\alpha} \underline{u}(x))$ for every $x \in (0, T]$ and

$$\lim_{x \to 0} x^{1-\alpha} \underline{u}(x) \leq u_0, \qquad \lim_{x \to 0} x^{1-\alpha} \mathcal{D}^{\alpha} \underline{u}(x) \leq u_1.$$

Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be a function satisfying the following axioms:

(H_1) for every $u \in C_{1-\alpha}^{\alpha}[0, T]$, $f(x, u(x), \mathcal{D}^{\alpha} u(x)) \in C_{\gamma}[0, T]$ for some $0 \leq \gamma < 1$.

(H_2) f is non-decreasing in all its arguments except for the first argument and

$$f(x, u, v) - f(x, \tilde{u}, \tilde{v}) \leq L_1(u - \tilde{u}) + L_2(v - \tilde{v})$$

for some $L_1, L_2 > 0$ whenever $x \in (0, T]$ and $u \geq \tilde{u}$, $v \geq \tilde{v}$.

The weighted Cauchy type problem presented in [1] is given by the following result.

Theorem 1.1 *Assume that* (H_1)–(H_2) *hold. Then there exists* $0 < \delta \leq T$ *such that the existence of a lower solution for* (1)–(2) *in* $C_{1-\alpha}^{\alpha}[0, \delta]$ *provides the existence of a unique solution* $u \in C_{1-\alpha}^{\alpha}[0, \delta]$ *for* (1).

Moreover, the authors in [1] defined the generalization of (1) and obtained some results for it as follows:

$$\begin{cases} \mathcal{D}^{n\alpha} u(x) = f(x, u(x), \mathcal{D}^{\alpha} u(x), \mathcal{D}^{2\alpha} u(x), \ldots, \mathcal{D}^{(n-1)\alpha} u(x)), \quad x \in (0, T], \\ \lim_{x \to 0} x^{1-\alpha} \mathcal{D}^{k\alpha} u(x) = u_k \quad (k = 0, 1, \ldots, n-1), \end{cases} \tag{4}$$

where $0 < \alpha \leq 1$.

Definition 1.1 For $0 < \alpha \le 1$, we define the space

$$C^{n\alpha}_{1-\alpha}[0,T] = \{u \in C_{1-\alpha}[0,T] : \mathcal{D}^{k\alpha}u \in C_{1-\alpha}[0,T], k = 1,2,\dots,n-1\},$$

equipped with the norm

$$\|u\|_{C^{n\alpha}_{1-\alpha}[0,T]} = \sum_{k=0}^{n-1} \|\mathcal{D}^{k\alpha}u\|_{C_{1-\alpha}[0,T]}.$$

To prove the main results, we need the following assumptions:

(H_3) $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that, for every $u \in C^{n\alpha}_{1-\alpha}[0,T]$,

$$f(\cdot, u(\cdot), \mathcal{D}^{\alpha}u(x), \mathcal{D}^{2\alpha}u(x), \dots, \mathcal{D}^{(n-1)\alpha}u(x)) \in C_{\gamma}[0,T]$$

for some $0 \le \gamma < 1$.

(H_4) f is non-decreasing in all its arguments except for the first argument and there exists $L > 0$ such that

$$f(x,u_1,\dots,u_n) - f(x,\tilde{u}_1,\dots,\tilde{u}_n) \le L \sum_{i=1}^{n} (u_i - \tilde{u}_i), u_i \ge \tilde{u}_i, \quad i = 1,2,\dots,n.$$

Theorem 1.2 *Assume that* (H_3)–(H_4) *hold. Then there exists* $0 < \delta \le T$ *such that the existence of a lower solution for* (4) *in* $C^{\alpha}_{1-\alpha}[0,\delta]$ *provides the existence of a unique solution* $u \in C^{\alpha}_{1-\alpha}[0,\delta]$ *for* (4).

In present paper, we address the following questions.

(Q_1) Is it possible to remove the non-decreasing conditions of the mappings f in Theorem 1.1 and Theorem 1.2?

(Q_2) Is it possible to remove assumption of the existence of a lower solution of the problems (1) and (4)?

(Q_3) Is it possible to define the problem (1) as an infinite system and discuss the existence results of the solution to it in spaces c_0 and ℓ_p?

In the sequel, we prove that the non-decreasing condition of function f in Theorem 1.1 and Theorem 1.2 is not necessary. Also, in Theorem 1.1 and Theorem 1.2, we need to find a lower solution of (1) and (4), respectively, while we show that do not need to this assumptions. In fact, by removing some of the assumptions and even with the weakening of other conditions of the main results of [1], using the new technique, we get the same results. Moreover, we present some remarks and examples to support the results herein and we compare the main results of Fazli and Nieto [1] and our results. In addition, since the theory of infinite systems of differential equations is an attractive research topic of the theory of differential equations in Banach spaces (for details, see [22–24]), we consider the problem (1) as an infinite system as follows:

$$\begin{cases} \mathcal{D}^{2\alpha}u_n(x) = f_n(x,u_n(x),\mathcal{D}^{\alpha}u_n(x)), & x \in (0,T], \\ \lim_{x\to 0} x^{1-\alpha}u_n(x) = u^0_n, \quad \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}u_n(x) = u^1_n, & n = 1,2,\dots, \end{cases} \tag{5}$$

where $0 < T < \infty$, α and $\mathcal{D}^{2\alpha}$ are defined in (2), and also $f_n(x,u(x),\mathcal{D}^{\alpha}u(x))$, $i = 1,2,\dots,$ are real valued functions. Actually, we study the existence of the solution for the infinite system (5) in the spaces c_0 and ℓ_p, $1 \leq p < \infty$, which c_0 is the space of sequences tends to zero. For this purpose, we use the Darbo fixed point theorem. Finally, illustrative examples are presented to evaluate the realization and effectiveness of our results.

At first, we recall some important definitions, lemmas and theorems that we use in our proofs of the main results. For details see [25, 26].

Definition 1.2 The Riemann–Liouville fractional integral of order γ of a function $u \in C[0,T]$ is defined as

$$I^{\gamma}u(x) = \frac{1}{\Gamma(\gamma)}\int_0^x \frac{u(s)}{(x-s)^{1-\gamma}}\,ds, \quad 0 \leq x \leq T.$$

Definition 1.3 The Riemann–Liouville fractional derivative D^{γ} of order $0 < \gamma \leq 1$ of a function $u : [0,T) \longrightarrow \mathbb{R}$ is defined by

$$\mathcal{D}^{\gamma}u(x) = \frac{1}{\Gamma(1-\gamma)}\frac{d}{dx}\int_0^x (x-s)^{-\gamma}u(s)\,ds,$$

provided the right-hand side is defined for almost every $x \in (0,T)$. Herein, $\Gamma(\cdot)$ represents the classical Gamma function.

Lemma 1.3 *Let $\alpha, \beta \geq 0$. If $u \in L^1(0,T)$, then $I^{\alpha}I^{\beta}u = I^{\alpha+\beta}u$ almost everywhere on $(0,T)$.*

Lemma 1.4 *Let $\alpha \geq 0$. If $u \in L^1(0,T)$, then $D^{\alpha}I^{\alpha}u = u$ almost everywhere on $(0,T)$.*

Lemma 1.5 *Assume that $u \in C(0,T] \cap L^1(0,T)$ with a fractional derivative of order $0 < \alpha \leq 1$ that belongs to $C(0,T] \cap L^1(0,T)$. Then*

$$I^{\alpha}D^{\alpha}u(x) = u(x) + cx^{\alpha-1}$$

for some $c \in \mathbb{R}$.

Throughout this paper $(X, \|\cdot\|)$ indicates a Banach space, for every $E \subset X$, $\bar{E}$ indicates the closure of E, and $\operatorname{conv}(E)$ indicates the closed convex hull of X. Also, note that $\mathcal{M}_X$ is the family of non-empty bounded subsets of X and $\mathcal{N}_X$ is the family of non-empty and relatively compact subsets of X. The use of the measure of noncompactness(MNC) concepts was first proposed by Kuratowski [27]. Here, we will give a brief overview of this notion, which is used in Sect. 3.

Definition 1.4 ([28]) A mapping $\mu : \mathcal{M}_X \longrightarrow \mathbb{R}^+$ is said to be a measure of the noncompactness in E if it satisfies the following conditions:

(A_1) The family $\operatorname{Ker}\mu = \{X \in \mathcal{M}_X : \mu(E) = 0\}$ is non-empty and $\operatorname{Ker}\mu \subset \mathcal{N}_X$;
(A_2) $X_1 \subset E_2 \Rightarrow \mu(E_1) \leq \mu(E_2)$;
(A_3) $\mu(\bar{E}) = \mu(E)$;
(A_4) $\mu(\operatorname{conv} E) = \mu(E)$;
(A_5) $\mu(\gamma E_1 + (1-\gamma)E_2) \leq \gamma\mu(E_1) + (1-\gamma)\mu(E_2)$ for $0 \leq \gamma \leq 1$;
(A_6) if (E_n) is a sequence of closed sets from $\mathcal{M}_E$ such that $E_{n+1} \subset E_n$ and $\lim_{n\to\infty}\mu(E_n) = 0$, then the intersection set $E_\infty = \bigcap_{n=1}^{\infty} E_n$ is non-empty.

In addition, the definition of the Hausdorff measure of noncompactnesss χ which can be found in [27] is expressed as follows:

$$\chi(s) = \inf\{\epsilon > 0 : S \text{ has finite } \epsilon\text{-net in } X\}.$$

Lemma 1.6 ([29]) *Let Λ be a non-empty, closed, bounded and convex subset of a Banach space X and let $H : \Lambda \longrightarrow \Lambda$ be a continuous mapping such that there exists a constant $L \in [0,1)$ with the property $\mu(H(\Lambda)) \le L\mu(\Lambda)$. Then H has a fixed point in Λ.*

Proposition 1.7 ([30]) *If $W \subset C(I,X)$ for all continuous functions on I to E is bounded and equicontinuous, then the set $\mu(W(x))$ is continuous on I and*

$$\mu(W) = \sup_{x\in I} \mu\bigl(W(x)\bigr), \qquad \mu\left(\int_0^x W(\eta)\,d\eta\right) \le \int_0^x \mu\bigl(W(\eta)\bigr)\,d\eta.$$

Theorem 1.8 ([30]) *Let Q be a bounded subset of the Banach space $X = c_0$. As $(e^{(1)}, e^{(2)}, \ldots)$ is a Schauder basis for c_0, the Hausdorff MNC χ for Q is given by*

$$\chi_{c_0} = \lim_{n\to\infty} \left\{ \sup_{x\in Q} \left(\max_{k\ge n} |x_k| \right) \right\}.$$

Theorem 1.9 ([30]) *Let Q be a bounded subset of the Banach space $X = \ell_p$. As $(e^{(1)}, e^{(2)}, \ldots)$ is a Schauder basis for ℓ_p, the Hausdorff MNC χ for Q is given by*

$$\chi_{\ell_p} = \lim_{n\to\infty} \left\{ \sup_{x\in Q} \left(\sum_{k\ge n} |x_k|^p \right)^{\frac{1}{p}} \right\}.$$

2 An improvement of the existence and uniqueness of solutions to the initial value problem

In the following theorems, we remove some of the hypotheses of Theorems 1.1 and 1.2. Moreover, we show that under our assumptions (1) and (4) have a unique solution. This gives a partial answer to (Q_1) and (Q_2).

Theorem 2.1 *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a function such that, for every $u \in C^{\alpha}_{1-\alpha}[0,T]$, $f(x,u(x),\mathcal{D}^{\alpha}u(x)) \in C_{\gamma}[0,T]$ for some $0 \le \gamma < 1$, and also*

$$\bigl|f(x,u,v) - f(x,\tilde{u},\tilde{v})\bigr| \le L_1|u - \tilde{u}| + L_2|v - \tilde{v}|, \quad \forall x \in (0,t], \forall (u,v), (\tilde{u},\tilde{v}) \in \mathbb{R}^2$$

for some $L_1, L_2 > 0$. Then there exists $0 < \delta \le T$ such that the problem (1) possesses a unique solution in $C^{\alpha}_{1-\alpha}[0,\delta]$.

Proof Fix $\delta > 0$ such that

$$l = \max\{L_1, L_2\} \left(\delta^{2\alpha} \frac{\Gamma(\alpha)}{\Gamma(3\alpha)} + \delta^{\alpha} \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \right) < 1.$$

Consider the operator A defined on $C^{\alpha}_{1-\alpha}[0,\delta]$ by

$$Au(x) = u_0 x^{\alpha-1} + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + I^{2\alpha} f\bigl(x, u(x), \mathcal{D}^{\alpha}u(x)\bigr)$$

for each $u \in C^{\alpha}_{1-\alpha}[0,\delta]$ and $x \in [0,\delta]$. It is obvious that u is a solution of the problem (1) if and only if u is fixed point of the operator A. By the same arguments as given in the proof of Theorem 4.2 of [1], we draw the conclusion that the operator A is well defined. Now, we only need to show that A is a contraction mapping. For each $u, \tilde{u} \in C^{\alpha}_{1-\alpha}[0,\delta]$, we have

$$\begin{aligned}
&\bigl\|Au(x) - A\tilde{u}(x)\bigr\|_{C^{\alpha}_{1-\alpha}[0,\delta]} \\
&\quad = \bigl\|I^{2\alpha}\bigl[f\bigl(x, u(x), \mathcal{D}^{\alpha}u(x)\bigr) - f\bigl(x, \tilde{u}(x), \mathcal{D}^{\alpha}\tilde{u}(x)\bigr)\bigr]\bigr\|_{C^{\alpha}_{1-\alpha}[0,\delta]} \\
&\quad \le L_1\bigl\|I^{2\alpha}\bigl|\bigl(u(x) - \tilde{u}(x)\bigr)\bigr|\bigr\|_{C^{\alpha}_{1-\alpha}[0,\delta]} + L_2\bigl\|I^{2\alpha}\bigl|D^{\alpha}\bigl(u(x) - \tilde{u}(x)\bigr)\bigr|\bigr\|_{C^{\alpha}_{1-\alpha}[0,\delta]} \\
&\quad = L_1\bigl\|I^{2\alpha}\bigl|\bigl(u(x) - \tilde{u}(x)\bigr)\bigr|\bigr\|_{C_{1-\alpha}[0,\delta]} + L_1\bigl\|I^{\alpha}\bigl|\bigl(u(x) - \tilde{u}(x)\bigr)\bigr|\bigr\|_{C_{1-\alpha}[0,\delta]} \\
&\qquad + L_2\bigl\|I^{2\alpha}\bigl|D^{\alpha}\bigl(u(x) - \tilde{u}(x)\bigr)\bigr|\bigr\|_{C_{1-\alpha}[0,\delta]} + L_2\bigl\|I^{\alpha}\bigl|D^{\alpha}\bigl(u(x) - \tilde{u}(x)\bigr)\bigr|\bigr\|_{C_{1-\alpha}[0,\delta]} \\
&\quad \le \max\{L_1, L_2\}\left(\delta^{2\alpha}\frac{\Gamma(\alpha)}{\Gamma(3\alpha)} + \delta^{\alpha}\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\right)\|u - \tilde{u}\|_{C^{\alpha}_{1-\alpha}[0,\delta]} \\
&\quad \le L\|u - \tilde{u}\|_{C^{\alpha}_{1-\alpha}[0,\delta]}.
\end{aligned}$$

Since the space $C^{\alpha}_{1-\alpha}[0,\delta]$ is a complete metric space, applying the Banach contraction, the operator A has a unique fixed point and this fixed point is the unique solution of the problem (1). □

Theorem 2.2 *Let $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that, for every $u \in C^{n\alpha}_{1-\alpha}[0,T]$,*

$$f\bigl(x, u(x), \mathcal{D}^{\alpha}u(x), \mathcal{D}^{2\alpha}u(x), \dots, \mathcal{D}^{(n-1)\alpha}u(x)\bigr) \in C_{\gamma}[0,T]$$

for some $0 \le \gamma < 1$, and also

$$\bigl|f(x, u_1, \dots, u_n) - f(x, \tilde{u}_1, \dots, \tilde{u}_n)\bigr| \le L\sum_{i=1}^{n}|u_i - \tilde{u}_i| u_i \ge \tilde{u}_i, \quad i = 1, 2, \dots, n$$

for some $L > 0$. Then there exists $0 < \delta \le T$ such that the problem (4) possesses a unique solution in $C^{n\alpha}_{1-\alpha}[0,\delta]$.

Proof The proof is the same as Theorem 2.1. □

Remark 2.1 Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be a non-decreasing function in all its arguments except for the first argument such that

$$f(x, u, v) - f(x, \tilde{u}, \tilde{v}) \le L_1(u - \tilde{u}) + L_2(v - \tilde{v})$$

for some $L_1, L_2 > 0$ for all $u \ge \tilde{u}$, $v \ge \tilde{v}$ and $x \in (0,T]$. Then

$$\bigl|f(x, u, v) - f(x, \tilde{u}, \tilde{v})\bigr| \le L_1|u - \tilde{u}| + L_2|v - \tilde{v}|$$

for all $(u, v), (\tilde{u}, \tilde{v}) \in \mathbb{R}^2$ and $x \in (0,T]$.

Remark 2.2 Let $f:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ be a non-decreasing function in all its arguments except for the first argument such that

$$f(x,u_1,\dots,u_n)-f(x,\tilde{u}_1,\dots,\tilde{u}_n)\le L\sum_{i=1}^{n}(u_i-\tilde{u}_i)u_i\ge\tilde{u}_i,\quad i=1,2,\dots,n$$

for $L>0$ and $x\in(0,T]$. Then

$$\bigl|f(x,u_1,\dots,u_n)-f(x,\tilde{u}_1,\dots,\tilde{u}_n)\bigr|\le L\sum_{i=1}^{n}|u_i-\tilde{u}_i|$$

for all $(u_1,\dots,u_n),(\tilde{u}_1,\dots,\tilde{u}_n)\in\mathbb{R}^n$ and $x\in(0,T]$.

Remark 2.3 By using Remark 2.1 and Remark 2.2, we can conclude that our theorems are really generalizations of Theorem 1.1 and Theorem 1.2.

Remark 2.4 In Theorem 1.1 and Theorem 1.2, we need to find a lower solution of (1) and (4), respectively, while we do not need this assumption in Theorem 2.1 and Theorem 2.2. In general, finding a lower solution of (1) and (4) is difficult.

Remark 2.5 The mapping f in Theorem 1.1 and Theorem 1.2 is non-decreasing in all its arguments except for the first argument, while this assumption is not required in our theorems.

Now, with the following examples, we show that our main theorems are generalizations of the main theorems of [1] that are Theorem1.1 and Theorem1.2.

Example 2.1 The linear initial value problem is given as follows:

$$\begin{cases}\mathcal{D}u(x)=x^2-\frac{\nu_1\sin(2k)\mathcal{D}^\alpha u(x)}{8}-\frac{\nu_2\tanh(x)u(x)}{4}, & x\in(0,T],\\ \lim_{x\to0}x^{\frac12}u(x)=a, & \lim_{x\to0}x^{\frac12}\mathcal{D}^{\frac12}u(x)=b.\end{cases}\tag{6}$$

This problem is a special case of (1) with $\alpha=\frac12$, $T,a,b,\nu_1,\nu_2>0$, $\max\{\nu_1,\nu_2\}<\frac{4}{(2+\sqrt{\pi})}$ and $f(x,u(x),\mathcal{D}^\alpha u(x))=x^2-\frac{\nu_1\sin(2k)\mathcal{D}^\alpha u(x)}{8}-\frac{\nu_2\tanh(x)u(x)}{4}$. It is easy to see that

$$\bigl|f\bigl(x,u(x),\mathcal{D}^\alpha u(x)\bigr)-f\bigl(x,v(x),\mathcal{D}^\alpha v(x)\bigr)\bigr|\le\frac{\nu_2}{4}\bigl|v(x)-u(x)\bigr|+\frac{\nu_1}{8}\bigl|\mathcal{D}^\alpha v(x)-\mathcal{D}^\alpha u(x)\bigr|.$$

Applying Theorem 2.1 the linear initial value problem (6) possesses a unique solution in $C_{1-\alpha}^{\alpha}[0,\gamma]$. It is simple to verify that Theorem 1.1 cannot be applied to our example. Because f is not increasing in all its arguments except for the first argument, that is, the condition (H_2) of Theorem 1.1 is not satisfied.

Example 2.2 Let $\alpha=\frac23$, $n=4$, $a_i\ge0$, $i=1,2,3,4$ and

$$\begin{aligned}f\bigl(x,u(x),\mathcal{D}^\alpha u(x),\mathcal{D}^{2\alpha}u(x),\mathcal{D}^{3\alpha}u(x)\bigr)&=e^x-\frac{\zeta_1}{e^{(x+3\alpha)}}u(x)-\frac{\zeta_2}{e^{(x+3\alpha)}}\mathcal{D}^\alpha u(x)\\&\quad+\frac{\zeta_3}{e^{(x+3\alpha)}}\mathcal{D}^{2\alpha}u(x)-\frac{\zeta_4}{e^{(x+3\alpha)}}\mathcal{D}^{3\alpha}u(x).\end{aligned}$$

If $\max_{1\leq i\leq 4}\{\zeta_i\} < \frac{e^4}{5.86}$, then, by applying Theorem (4), the problem

$$\begin{cases} \mathcal{D}^{\frac{8}{3}}u(x) = e^x - \frac{\zeta_1}{e^{(x+2)}}u(x) - \frac{\zeta_2}{e^{(x+2)}}\mathcal{D}^{\alpha}u(x) + \frac{\zeta_3}{e^{(x+2)}}\mathcal{D}^{2\alpha}u(x) - \frac{\zeta_4}{e^{(x+2)}}\mathcal{D}^{3\alpha}u(x), \\ \lim_{x\to 0} x^{\frac{1}{3}}u(x) = a_1, \qquad \lim_{x\to 0} x^{\frac{1}{3}}\mathcal{D}^{\frac{2}{3}}u(x) = a_2, \\ \lim_{x\to 0} x^{\frac{1}{3}}\mathcal{D}^{\frac{4}{3}}u(x) = a_3, \qquad \lim_{x\to 0} x^{\frac{1}{3}}\mathcal{D}^{2}u(x) = a_4, \end{cases} \tag{7}$$

where $x \in (0, T]$, possesses a unique solution. On the other hand, since f is not increasing, Theorem 2.2 is not applicable here.

3 Solution of infinite system (5)

In this section, we give a partial answer to (Q_3). For this purpose, firstly, we present some weighted continuous spaces. Then we discuss the existence of solution of infinite system (5) in the Banach space c_0 and ℓ_p in Sects. 3.1 and 3.2, respectively.

Definition 3.1 Let X be a norm space and $C(I,X)$ be the family of all continuous functions on I to X. We define a weighted spaces of continuous functions as follows:

$$C_{1-\alpha}(I,X) = \bigl\{u \in C((0,T],X) : x^{1-\alpha}u \in C(I,X)\bigr\},$$

with the norm $\|u\|_{C_{1-\alpha}(I,X)} = \max_{0\leq x\leq T}\|x^{1-\alpha}u(x)\|_X$.

Definition 3.2 We denote the spaces of continuous functions

$$C^{\alpha}_{1-\alpha}(I,X) = \bigl\{u \in C_{1-\alpha}(I,X) : \mathcal{D}^{\alpha}u \in C_{1-\alpha}(I,X)\bigr\},$$

with the norm $\|u\|_{C^{\alpha}_{1-\alpha}(I,X)} = \|u\|_{C_{1-\alpha}(I,X)} + \|\mathcal{D}^{\alpha}u\|_{C_{1-\alpha}(I,X)}$.

Throughout this section, we define $I = (0,T]$, $u_0 = \{u^0_n\}_{n=1}^{\infty}$, $u_1 = \{u^1_n\}_{n=1}^{\infty}$, $u(x) = \{u_n(x)\}_{n=1}^{\infty}$ and $f(x,u(x),\mathcal{D}^{\alpha}u(x)) = \{f_n(x,u(x),\mathcal{D}^{\alpha}u(x))\}_{n=1}^{\infty}$, which belongs to some Banach space $(X,\|\cdot\|)$. Therefore, one has system (5) as follows:

$$\begin{cases} \mathcal{D}^{2\alpha}u(x) = f(x,u(x),\mathcal{D}^{\alpha}u(x)), \quad x \in (0,T], \\ \lim_{x\to 0} x^{1-\alpha}u(x) = u^0, \qquad \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}u(x) = u^1, \quad n = 1,2,\ldots, \end{cases} \tag{8}$$

where $u_n(x)$, $n = 1,2,3,\ldots$, are continuous on I, f is defined on $I \times X \times X \longrightarrow X$ and f_i is a real valued function.

3.1 Solution in space c_0

In this subsection, let $X = c_0$. We intend to show the existence of a solution of the infinite system (5) in the Banach space c_0 with the norm $\|u\| = \sup\{|u_i| : i = 1,2,3,\ldots\}$.

Suppose that the following conditions are satisfied:

(C_1) $\{u^n_0\}_{n=1}^{\infty}$ and $\{u^n_1\}_{n=1}^{\infty}$ belong to c_0;

(C_2) for any fixed u, $f(x,u(x),\mathcal{D}^{\alpha}u(x))$ is measurable;

(C_3) for each $x \in I$, $u(x) \in c_0$ and $i = 1,2,\ldots$, we have

$$\bigl|f_i\bigl(x,u(x),\mathcal{D}^{\alpha}u(x)\bigr)\bigr| \leq j_i(x) + k_i(x)\sup\bigl\{|u_n| : n \geq i\bigr\},$$

where $j_i(x)$ and $k_i(x)$ are continuous real valued functions on I such that the sequence $\{k_i(x)\}_{i=1}^{\infty}$ is equibounded on I and the sequence $(j_i(x))$ converges uniformly on I to the zero function identically;

(C_4) the family of $\{f_x(u)\}_{x\in I}$ where $f_x(u) = f(x, u(x), \mathcal{D}^{\alpha}u(x))$ is equicontinuous at any point of the space c_0.

Theorem 3.1 *Under the conditions* (C_1)–(C_4), *with* $(\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T^{2\alpha}}{\Gamma(2\alpha+1)})K < 1$, *where* $\max_i \sup_{x\in I} |k_i(x)| \leq K$, *the infinite system* (5) *possesses at least one solution* $\{u_n(x)\}_{n=1}^{\infty} = u(x) \in c_0$ *for any* $x \in I$.

Proof Suppose that $u(x) = \{u_n(x)\}_{n=1}^{\infty}$ satisfies the boundary conditions of the infinite system (8). We define the operator $\mathcal{A} : C_{1-\alpha}^{\alpha}(I, c_0) \longrightarrow C_{1-\alpha}^{\alpha}(I, c_0)$ by

$$\mathcal{A}u(x) = u_0 x^{\alpha-1} + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} + I^{2\alpha} f\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big).$$

Applying (C_2), $\mathcal{A}$ is well defined. We show that $\mathcal{A}$ is bounded on $C_{1-\alpha}^{\alpha}(I, c_0)$,

$$\begin{aligned}
&\big\|x^{1-\alpha}\mathcal{A}u(x)\big\|_{c_0} + \big\|x^{1-\alpha}\mathcal{D}^{\alpha}\mathcal{A}u(x)\big\|_{c_0} \\
&\quad = \left\| u_0 + u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{\alpha} + x^{1-\alpha} I^{2\alpha} f\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big)\right\|_{c_0} \\
&\qquad + \big\| u_1 + x^{1-\alpha} I^{\alpha} f\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big)\big\|_{c_0} \\
&\quad \leq \sup_{i\geq 1}\left| u_0^i + u_1^i \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{\alpha} + x^{1-\alpha} I^{2\alpha} f_i\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big)\right| \\
&\qquad + \sup_{i\geq 1}\big| u_1^i + x^{1-\alpha} I^{\alpha} f_i\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big)\big| \\
&\quad \leq \sup_{i\geq 1}\big|u_0^i\big| + \sup_{i\geq 1}\big|u_1^i\big| \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{\alpha} + 1\right) \\
&\qquad + \frac{x^{1-\alpha}}{\Gamma(2\alpha)} \sup_{i\geq 1}\left| \int_0^x (x-\eta)^{2\alpha-1} f_i\big(\eta, u(\eta), \mathcal{D}^{\alpha}u(\eta)\big)\, d\eta\right| \\
&\qquad + \frac{x^{1-\alpha}}{\Gamma(\alpha)} \sup_{i\geq 1}\left| \int_0^x (x-\eta)^{\alpha-1} f_i\big(\eta, u(\eta), \mathcal{D}^{\alpha}u(\eta)\big)\, d\eta\right| \\
&\quad \leq \|u_0\|_{c_0} + \|u_1\|_{c_0} \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha} + 1\right) \\
&\qquad + \frac{T^{1-\alpha}}{\Gamma(2\alpha)} \sup_{i\geq 1}\left| \int_0^x (x-\eta)^{2\alpha-1} \big(j_i(x) + k_i(x) \sup\big\{|u_n| : n \geq i\big\}\big)\, d\eta\right| \\
&\qquad + \frac{T^{1-\alpha}}{\Gamma(\alpha)} \sup_{i\geq 1}\left| \int_0^x (x-\eta)^{\alpha-1} \big(j_i(x) + k_i(x) \sup\big\{|u_n| : n \geq i\big\}\big)\, d\eta\right|.
\end{aligned}$$

Using (C_3), there exists $J = \max_i \sup_{x\in I} |j_i(x)|$, therefore

$$\begin{aligned}
&\max_{x\in I}\big\|x^{1-\alpha}\mathcal{A}u(x)\big\|_{c_0} + \max_{x\in I}\big\|x^{1-\alpha}\mathcal{D}^{\alpha}\mathcal{A}u(x)\big\|_{c_0} \\
&\quad \leq \max_{x\in I}\left(\|u_0\|_{c_0} + \|u_1\|_{c_0} \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{\alpha} + 1\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\frac{x^{1-\alpha}}{\Gamma(2\alpha)}\sup_{i\geq 1}\int_0^x (x-\eta)^{2\alpha-1}\left(\left|j_i(x)\right|+\left|k_i(x)\right|\sup\left\{|u_n|:n\geq i\right\}\right)d\eta\\
&+\frac{x^{1-\alpha}}{\Gamma(\alpha)}\sup_{i\geq 1}\int_0^x (x-\eta)^{\alpha-1}\left(j_i(x)+k_i(x)\sup\left\{|u_n|:n\geq i\right\}\right)d\eta.\Big)\\
\leq{}& \|u_0\|_{c_0}+\|u_1\|_{c_0}\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^{\alpha}+1\right)\\
&+\frac{1}{\Gamma(2\alpha)}\max_{x\in I}\sup_{i\geq 1}x^{1-\alpha}\int_0^x (x-\eta)^{2\alpha-1}\left(\left|j_i(x)\right|+\left|k_i(x)\right|\sup\left\{|u_n|:n\geq i\right\}\right)d\eta\\
&+\frac{1}{\Gamma(\alpha)}\max_{x\in I}\sup_{i\geq 1}x^{1-\alpha}\int_0^x (x-\eta)^{\alpha-1}\left(\left|j_i(x)\right|+\left|k_i(x)\right|\sup\left\{|u_n|:n\geq i\right\}\right)d\eta\\
\leq{}& \|u_0\|_{c_0}+\|u_1\|_{c_0}\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}+1\right)+\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(\alpha+1)}\right)J\\
&+\left(\frac{T^{\alpha}}{\Gamma(2\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(2\alpha+1)}\right)K\|u\|_{c_0},
\end{aligned}
$$

and so

$$
\begin{aligned}
\|\mathcal{A}u\| \leq{}& \|u_0\|_{c_0}+\|u_1\|_{c_0}\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}+1\right)+\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(2\alpha+1)}\right)J\\
&+\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(2\alpha+1)}\right)K\|u\|.
\end{aligned}
$$

Then we conclude that

$$
r=\frac{\|u_0\|_{c_0}+\|u_1\|_{c_0}(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}+1)+(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(2\alpha+1)})J}{1-(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(2\alpha+1)})K}
$$

is the optimal solution of the inequality

$$
\begin{aligned}
&\|u_0\|_{c_0}+\|u_1\|_{c_0}\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}+1\right)+\left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(\alpha+1)}\right)J\\
&\quad+\left(\frac{T^{\alpha}}{\Gamma(2\alpha+1)}+\frac{T^{2\alpha}}{\Gamma(2\alpha+1)}\right)Kr^{'}\leq r^{'}.
\end{aligned}
$$

Define the closed, bounded and convex set

$$
B_r=\left\{u\in C_{1-\alpha}^{\alpha}(I,c_0):\|u\|\leq r,\lim_{x\to 0}x^{1-\alpha}u_n(x)=u_0^n,\lim_{x\to 0}x^{1-\alpha}\mathcal{D}^{\alpha}u_n(x)=u_1^n,\right\},
$$

where $n=1,2,\ldots$. Clearly, $\mathcal{A}$ is bounded on B_r. In the following, we show that $\mathcal{A}$ is continuous on B_r. We can write

$$
\begin{aligned}
&\left\|\mathcal{A}u(x)-\mathcal{A}v(x)\right\|_{c_0}\\
&\quad\leq \sup_{i\geq 1}\frac{1}{\Gamma(2\alpha)}\int_0^x (x-\eta)^{2\alpha-1}\left|f_i\left(\eta,u(\eta),\mathcal{D}^{\alpha}u(\eta)\right)-f_i\left(\eta,v(\eta),\mathcal{D}^{\alpha}v(\eta)\right)\right|d\eta\\
&\quad\leq \frac{1}{\Gamma(2\alpha)}\int_0^x (x-\eta)^{2\alpha-1}\left\|f\left(\eta,u(\eta),\mathcal{D}^{\alpha}u(\eta)\right)-f\left(\eta,v(\eta),\mathcal{D}^{\alpha}v(\eta)\right)\right\|_{c_0}d\eta.
\end{aligned}
$$

The family of $\{f_x(u)\}_{x\in I}$ where $f_x(u) = f(x, u(x), \mathcal{D}^\alpha u(x))$ is equicontinuous on c_0. Bearing (C_4) in mind, we have

$$\forall v, u \in B_r \text{ and } \forall \epsilon > 0, \quad \exists \delta > 0: \quad \|u - v\| \le \delta \quad \Longrightarrow \quad \|f_x(u) - f_x(v)\|_{c_0} \le \frac{\epsilon}{z}, \quad \forall x \in I,$$

where $z = \frac{T^2}{\Gamma(2\alpha+1)}$. Therefore, we conclude that

$$\|\mathcal{A}u(x) - \mathcal{A}v(x)\|_{c_0} \le \frac{1}{\Gamma(2\alpha)} \int_0^x (x-\eta)^{2\alpha-1} \|f_\eta(u) - f_\eta(v)\|_{c_0}\, d\eta < \epsilon,$$

which means that $\mathcal{A}$ is continuous. Without loss of generality, we can suppose $x_1 > x_2$. There exist m_1, m_2 and m_3 in $\mathbb{R}^+$ such that

$$\begin{aligned}
&\left|x_1^{\alpha-1} - x_2^{\alpha-1}\right| \le m_1|x_1 - x_2|, \quad \text{for all } x_1 \le \eta \le x_2, \\
&\left|x_1^{\alpha-1} - x_2^{\alpha-1}\right| \le m_2|x_1 - x_2|, \quad \text{for all } x_1 \le \eta \le x_2, \\
&\left|(x_2-\eta)^{2\alpha-1} - (x_1-\eta)^{2\alpha-1}\right| < m_3|x_1?x_2|, \quad \text{for all } x_1 \le \eta \le x_2.
\end{aligned} \tag{9}$$

Applying (9), for any $u \in B_r$, we have

$$\begin{aligned}
&\|\mathcal{A}u(x_1) - \mathcal{A}u(x_2)\|_{c_0} \\
&\quad = \sup_{i\ge1}\left| u_0^i\left(x_1^{\alpha-1} - x_2^{\alpha-1}\right) + u_1^i \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left(x_1^{2\alpha-1} - x_2^{2\alpha-1}\right)\right. \\
&\qquad + \frac{1}{\Gamma(2\alpha)}\left(\int_0^{x_1} (x_1-\eta)^{2\alpha-1} f_i\left(\eta, u(\eta), \mathcal{D}^\alpha u(\eta)\right) d\eta\right. \\
&\qquad \left.\left. - \int_0^{x_2} (x_2-\eta)^{2\alpha-1} f_i\left(\eta, u(\eta), \mathcal{D}^\alpha u(\eta)\right) d\eta\right)\right| \\
&\quad \le \sup_{i\ge1}\left|u_0^i\left(x_1^{\alpha-1} - x_2^{\alpha-1}\right)\right| + \sup_{i\ge1}\left|u_1^i \frac{\Gamma(\alpha)}{\Gamma(2\alpha)}\left(x_1^{2\alpha-1} - x_2^{2\alpha-1}\right)\right| \\
&\qquad + \frac{1}{\Gamma(2\alpha)} \sup_{i\ge1}\left|\int_0^{x_1} \left((x_2-\eta)^{2\alpha-1} - (x_1-\eta)^{2\alpha-1}\right) f_i\left(\eta, u(\eta), \mathcal{D}^\alpha u(\eta)\right) d\eta\right. \\
&\qquad \left. + \int_{x_1}^{x_2} (x_2-\eta)^{2\alpha-1} f_i\left(\eta, u(\eta), \mathcal{D}^\alpha u(\eta)\right) d\eta\right| \\
&\quad \le \|u_0\||x_2 - x_1|m_1 + \|u_1\| \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} m_2|x_2 - x_1| \\
&\qquad + \frac{1}{\Gamma(2\alpha)} \Big(\sup_{i\ge1} \int_0^{x_1} |x_2 - x_1| m_3\big(j_i(\eta) + k_i(\eta)\sup\left\{\left|u_n(\eta)\right| : n \ge i\right\} d\eta \\
&\qquad + \int_{x_1}^{x_2} (x_2-\eta)^{2\alpha-1}\left(j_i(\eta) + k_i(\eta)\sup\left\{\left|u_n(\eta)\right| : n \ge i\right\} d\eta\right) \\
&\quad \le \|u_0\||x_2 - x_1|m_1 + \|u_1\| \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} m_2|x_2 - x_1| \\
&\qquad + \frac{1}{\Gamma(2\alpha)}\left(m_3 J + K\|u\|\right)|x_1 - x_2| + \frac{1}{\Gamma(2\alpha+1)}\left(J + K\|u\|\right)(x_1 - x_2)^{2\alpha},
\end{aligned}$$

which tends to zero when $x_1 \longrightarrow x_2$. Thus, we deduce that $\mathcal{A}$ is equicontinuous on B_r.

Setting $\bar{B} = \operatorname{conv}(\mathcal{A}(B_r))$, clearly $\bar{B} \subset B_r$. Let $Y \subset \bar{B}$, then $\mathcal{A}$ is continuous on Y and the functions from the set of Y are equicontinuous on I. In view of the definition of the Hausdorff MNC χ on the space $C^{\alpha}_{1-\alpha}(I, c_0)$, Proposition 1.7 and Theorem 1.8, we have

$$\chi_{C^{\alpha}_{1-\alpha}(I,c_0)}(Y) = \sup_{x\in I} \chi_{c_0}\big(Y(x)\big).$$

Recalling Theorem 1.8, for any $u \in Y$, we observe

$$\begin{aligned}
\chi_{c_0}\big(\mathcal{A}u(x)\big) &= \lim_{i\to\infty}\Big\{\sup_{u\in B}\Big(\max_{n\geq i}\mathcal{A}u_n(x)\Big)\Big\} \\
&\leq \lim_{i\to\infty}\Big\{\sup_{u\in B}\Big(\max_{n\geq i}\Big|u_0^n x^{\alpha-1} u_1^n \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} x^{2\alpha-1} \\
&\quad + \frac{1}{\Gamma(2\alpha)}\int_0^x (x-\eta)^{2\alpha-1} f_n\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big)\,d\eta\Big|\Big)\Big\} \\
&\leq \lim_{i\to\infty}\Big\{\sup_{u\in B}\Big(\max_{n\geq i}\frac{1}{\Gamma(2\alpha)}\Big|\int_0^x (x-\eta)^{2\alpha-1}\big(j_n(\eta) \\
&\quad + k_n(\eta)\sup\big\{\big|u_k(\eta)\big| : k\geq n\big\}\big)\,d\eta\Big|\Big)\Big\} \\
&\leq K\lim_{i\to\infty}\Big\{\sup_{u\in B}\Big(\max_{n\geq i}\frac{1}{\Gamma(2\alpha)}\Big|\int_0^x (x-\eta)^{2\alpha-1}\big(\sup\big\{\big|u_k(\eta)\big| : k\geq n\big\}\big)\,d\eta\Big|\Big)\Big\} \\
&\leq \frac{KT^{2\alpha}}{\Gamma(2\alpha+1)}\lim_{i\to\infty}\Big\{\sup_{u\in B}\Big(\max_{n\geq i}\big|u_n(x)\big|\Big)\Big\}.
\end{aligned}$$

Therefore

$$\sup_{x\in I}\chi_{c_0}\big(\mathcal{A}u(x)\big) \leq \frac{KT^{2\alpha}}{\Gamma(2\alpha+1)}\sup_{x\in I}\lim_{i\to\infty}\Big\{\sup_{u\in B}\Big(\max_{n\geq i}\big|u_n(x)\big|\Big)\Big\}$$

and

$$\chi_{C^{\alpha}_{1-\alpha}(I,c_0)}\big(\mathcal{A}u(x)\big) \leq \frac{KT^{2\alpha}}{\Gamma(2\alpha+1)}\chi_{C^{\alpha}_{1-\alpha}(I,c_0)}(Y).$$

As $\frac{KT^{2\alpha}}{\Gamma(2\alpha+1)} < 1$, applying Lemma 1.6, $\mathcal{A}$ possesses at least one fixed point in $\mathcal{A}$, which is a solution for (5) in the space $C^{\alpha}_{1-\alpha}(I, c_0)$. □

Now, with the following example, we clarify the main result of this subsection.

Example 3.1 The system of fractional differential equation is given as follows:

$$\begin{cases} \mathcal{D}^2 u_n(x) = \frac{x\sin(\frac{1}{n}+1)+\tan(\arctan(x))u_n(x)}{n+1}, & x\in(0,1], \\ \lim_{x\to 0} x^{1-\alpha}u_n(x) = \frac{a}{n}, & \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha}u_n(x) = \frac{b}{n}, n = 1, 2, \ldots, \end{cases} \tag{10}$$

where $u_n(x)$, $n = 1, 2, 3, \ldots$ are continuous on I. This system is a special case of (5) with $\alpha = T = 1$,

$$f_n\big(x, u(x), \mathcal{D}^{\alpha}u(x)\big) = \frac{x\sin(\frac{1}{n}+1)+\tan(\arctan(x))u_n(x)}{n+1},$$

and $a, b \geq 0$. Obviously, the conditions (C_1) and (C_2) are satisfied. Hereafter, we show that $f(x, u(x), \mathcal{D}^\alpha u(x)) \in c_0$. For any $x \in (0,1]$ and $u \in c_0$, we conclude that

$$\begin{aligned} \lim_{n\to\infty} f_n\bigl(x, u(x), \mathcal{D}^\alpha u(x)\bigr) &= \lim_{n\to\infty} \frac{x\sin(\frac{1}{n}+1) + \tan(\arctan(x))u_n(x)}{n+1} \\ &\leq \lim_{n\to\infty} \frac{\sin(\frac{1}{n}+1) + \sup_{n\leq 1}|u_n(x)|}{n+1} = 0. \end{aligned}$$

Also, clearly $|f_n(x, u(x), \mathcal{D}^\alpha u(x))| \leq j_n(x) + k_n(x)\sup\{|u_k| : k \geq n\}$, where

$$j_n(x) = \frac{x\sin(\frac{1}{n}+1)}{n+1}, \qquad k_n(x) = \frac{\tan(\arctan(x))}{n+1}.$$

Moreover, $j_n(x)$ converges uniformly to zero and $k_n(x)$ is equibounded by $K = \frac{1}{2}$. Now, we are going to check the conditions (C_4). For any $x \in (0,1]$ and $u, v \in c_0$ with $\|u(x) - v(x)\| < \delta$, we have

$$\begin{aligned} \bigl\|f_x(u) - f_x(v)\bigr\|_{c_0} &= \bigl\|f\bigl(x, u(x), \mathcal{D}^\alpha u(x)\bigr) - f\bigl(x, v(x), \mathcal{D}^\alpha v(x)\bigr)\bigr\| \\ &= \sup_{n\geq 1}\bigl|f_n\bigl(x, u(x), \mathcal{D}^\alpha u(x)\bigr) - f_n\bigl(x, v(x), \mathcal{D}^\alpha v(x)\bigr)\bigr| \\ &= \sup_{n\geq 1}\biggl|\frac{x\sin(\frac{1}{n}+1) + \tan(\arctan(x))u_n(x)}{n+1} \\ &\quad - \frac{x\sin(\frac{1}{n}+1) + \tan(\arctan(x))v_n(x)}{n+1}\biggr| \\ &\leq \frac{1}{2}\sup_{n\geq 1}\bigl|u_n(x) - v_n(x)\bigr| \\ &\leq \frac{1}{2}\bigl\|u(x) - v(x)\bigr\|_{c_0} < \epsilon. \end{aligned}$$

Applying Theorem 3.1, hence the system of fractional differential equation (5) possesses at least one solution in $C^\alpha_{1-\alpha}(I, c_0)$.

3.2 Solution in space l_p

In this subsection, let $X = \ell_p$. For a real number $p \geq 1$, the space denoted by ℓ_p is the Banach sequence space, when equipped with the following norm:

$$\|u\|_P = \Biggl(\sum_{i=1}^{\infty}|u_i|^P\Biggr)^{\frac{1}{p}}.$$

In the following, we show that the infinite system (5) has at least on solution in the space ℓ_p, when the following conditions are satisfied:

(C_1') u_0 and u_1 belong to ℓ_p;

(C_2') $f : I \times \ell_p \longrightarrow \ell_p$ is continuous;

(C_3') for each $x \in [0,t]$, $u(x) \in \ell_p$ and $i = 1, 2, \ldots,$ we have

$$\bigl|f_i\bigl(x, u(x), \mathcal{D}^\alpha u(x)\bigr)\bigr|^p \leq j_i(x) + k_i(x)|u_n|^p,$$

where $j_i(x)$ and $k_i(x)$ are continuous nonnegative functions on I such that the sequence $\{k_i(x)\}_{i=1}^{\infty}$ is equibounded on I, $\lim_{i\longrightarrow\infty}\sup k_i(x)$ is integrable over I and the series $\sum_{i=1}^{\infty} j_i(x)$ converges uniformly on I;

(C_4') the family of $\{f_x(u)\}_{x\in I}$, where $f_x(u) = f(x,u(x),\mathcal{D}^{\alpha}u(x))$ is equicontinuous at any point of the space ℓ_p.

Theorem 3.2 *Under the conditions* (C_1')–(C_4'), *if*

$$\left(\frac{T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} + \frac{T^{\alpha}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}}\right)K^{\frac{1}{p}} < 1,$$

the infinite system (5) *possesses at least one solution* $\{u_n(x)\}_{n=1}^{\infty} = u(x) \in \ell_P$ *for any* $x \in I$, *where* $J = \sup_{x\in I}|j(x)|$, $j(x) = \sum_{i=1}^{\infty} j_i(x)$ *and* $k_i(x)$ *is equibounded by* K.

Proof Suppose that $u(x) = \{u_n(x)\}_{n=1}^{\infty}$ satisfies the boundary conditions of the infinite system (5). We define the operator $\mathcal{A}: C_{1-\alpha}^{\alpha}(I,\ell_p) \longrightarrow C_{1-\alpha}^{\alpha}(I,\ell_p)$ by

$$\mathcal{A}u(x) = u_0 x^{\alpha-1} + u_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}x^{2\alpha-1} + I^{2\alpha}f\bigl(x,u(x),\mathcal{D}^{\alpha}u(x)\bigr).$$

Applying (C_2'), $\mathcal{A}$ is well defined. We show that $\mathcal{A}$ is bounded on $C_{1-\alpha}^{\alpha}(I,\ell_p)$.

$$\begin{aligned}
\bigl\|x^{1-\alpha}\mathcal{A}u(x)\bigr\|_p &= \left\|u_0 + u_1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}x^{\alpha} + x^{1-\alpha}I^{2\alpha}f\bigl(x,u(x),\mathcal{D}^{\alpha}u(x)\bigr)\right\|_p \\
&\leq \|u_0\|_p + \|u_1\|_p\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}+1\right) \\
&\quad + \bigl\|x^{1-\alpha}I^{2\alpha}f\bigl(x,u(x),\mathcal{D}^{\alpha}u(x)\bigr)\bigr\|_p + \bigl\|x^{1-\alpha}I^{\alpha}f\bigl(x,u(x),\mathcal{D}^{\alpha}u(x)\bigr)\bigr\|_p \\
&\leq \|u_0\|_p + \|u_1\|_p\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}\right) \\
&\quad + T^{\frac{p-1}{p}}\frac{x^{1-\alpha}}{\Gamma(2\alpha)}\left(\sum_{i\geq 1}\int_0^x \bigl|(x-\eta)^{2\alpha-1}\bigr|^p \bigl|f_i\bigl(\eta,u(\eta),\mathcal{D}^{\alpha}u(\eta)\bigr)\bigr|^P\,d\eta\right)^{\frac{1}{p}} \\
&\leq \|u_0\|_p + \|u_1\|_p\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}\right) \\
&\quad + T^{\frac{p-1}{p}}\frac{x^{1-\alpha}}{\Gamma(2\alpha)}\left(\sum_{i\geq 1}\int_0^x \bigl|(x-\eta)^{2\alpha-1}\bigr|^p \bigl(j_i(\eta)+k_i(\eta)\bigl|u_i(\eta)\bigr|^P\bigr)\,d\eta\right)^{\frac{1}{p}} \\
&\leq \|u_0\|_p + \|u_1\|_p\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}\right) \\
&\quad + T^{\frac{p-1}{p}}\frac{x^{1-\alpha}}{\Gamma(2\alpha)}\left(J\frac{x^{(2\alpha-1)p+1}}{(2\alpha-1)p+1} + K\bigl\|u(x)\bigr\|_p^p\frac{x^{(2\alpha-1)p+1}}{(2\alpha-1)p+1}\right)^{\frac{1}{p}} \\
&\leq \|u_0\|_p + \|u_1\|_p\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^{\alpha}\right) \\
&\quad + \frac{T^{\frac{p-1}{p}}x^{1-\alpha}}{\Gamma(2\alpha)}\left(J^{\frac{1}{p}}\frac{x^{(2\alpha-1)+\frac{1}{p}}}{((2\alpha-1)p+1)^{\frac{1}{p}}} + K^{\frac{1}{p}}\bigl\|u(x)\bigr\|_p\frac{x^{(2\alpha-1)+\frac{1}{p}}}{((2\alpha-1)p+1)^{\frac{1}{p}}}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \|u_0\|_p + \|u_1\|_p \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha} \right) \\
&\quad + \frac{T^{(1+\alpha)} J^{\frac{1}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} + \frac{T^{(2\alpha)} K^{\frac{1}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} \left\| x^{1-\alpha} u(x) \right\|_p
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| x^{1-\alpha} \mathcal{D}^{\alpha} \mathcal{A}u(x) \right\|_p \\
&\quad \leq \|u_1\|_p + \left\| x^{1-\alpha} I^{\alpha} f\big(x, u(x), \mathcal{D}^{\alpha} u(x)\big) \right\|_p \\
&\quad \leq \|u_1\|_p + \frac{x^{1-\alpha}}{\Gamma(\alpha)} \left(\sum_{i\geq 1} \left| \int_0^x (x-\eta)^{\alpha-1} f_i\big(\eta, u(\eta), \mathcal{D}^{\alpha} u(\eta)\big)\, d\eta \right|^p \right)^{\frac{1}{p}} \\
&\quad \leq \|u_1\|_p + T^{\frac{p-1}{p}} \frac{x^{1-\alpha}}{\Gamma(\alpha)} \left(\sum_{i\geq 1} \int_0^x \left| (x-\eta)^{\alpha-1} \right|^p \left| f_i\big(\eta, u(\eta), \mathcal{D}^{\alpha} u(\eta)\big) \right|^p d\eta \right)^{\frac{1}{p}} \\
&\quad \leq \|u_1\|_p + T^{\frac{p-1}{p}} \frac{x^{1-\alpha}}{\Gamma(\alpha)} \left(\sum_{i\geq 1} \int_0^x \left| (x-\eta)^{\alpha-1} \right|^p \big(j_i(\eta) + k_i(\eta) \left| u_i(\eta) \right|^p \big)\, d\eta \right)^{\frac{1}{p}} \\
&\quad \leq \|u_1\|_p + T^{\frac{p-1}{p}} \frac{x^{1-\alpha}}{\Gamma(\alpha)} \left(J \frac{x^{(\alpha-1)p+1}}{(\alpha-1)p+1} + K \left\| u(x) \right\|_p^p \frac{x^{(\alpha-1)p+1}}{(\alpha-1)p+1} \right)^{\frac{1}{p}} \\
&\quad \leq \|u_1\|_p + T^{\frac{p-1}{p}} \frac{x^{1-\alpha}}{\Gamma(\alpha)} \left(J^{\frac{1}{p}} \frac{x^{(\alpha-1)+\frac{1}{p}}}{((\alpha-1)p+1)^{\frac{1}{p}}} + K^{\frac{1}{p}} \left\| u(x) \right\|_p \frac{x^{(\alpha-1)+\frac{1}{p}}}{((\alpha-1)p+1)^{\frac{1}{p}}} \right) \\
&\quad \leq \|u_1\|_p + \frac{T J^{\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}} + \frac{T^{\alpha} K^{\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}} \left\| x^{1-\alpha} u(x) \right\|_p.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\max_{x\in I} \left\| x^{1-\alpha} \mathcal{A}u(x) \right\|_p + \max_{x\in I} \left\| x^{1-\alpha} \mathcal{D}^{\alpha} \mathcal{A}u(x) \right\|_p \\
&\quad \leq \max_{x\in I} \|u_0\|_p + \max_{x\in I} \|u_1\|_p \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha} + 1 \right) + \frac{T^{(1+\alpha)} J^{\frac{1}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} \\
&\qquad + \frac{T^{(2\alpha)} K^{\frac{1}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} \max_{x\in I} \left\| x^{1-\alpha} u(x) \right\|_p \\
&\qquad + \frac{T J^{\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}} + \frac{T^{\alpha} K^{\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}} \max_{x\in I} \left\| x^{1-\alpha} u(x) \right\|_p,
\end{aligned}
$$

and so

$$
\begin{aligned}
\|\mathcal{A}u\| &\leq \|u_0\|_p + \|u_1\|_p \left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^{\alpha} + 1 \right) \\
&\quad + \left(\frac{T^{1+\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} + \frac{T}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}} \right) J^{\frac{1}{p}} \\
&\quad + \left(\frac{T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} + \frac{T^{\alpha}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}} \right) K^{\frac{1}{p}} \left\| u(x) \right\|.
\end{aligned}
$$

Then we see that

$$r=\frac{\|u_0\|_p+\|u_1\|_p(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^\alpha+1)+(\frac{T^{1+\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}+\frac{T}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}})J^{\frac{1}{p}}}{1-(\frac{T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}+\frac{T^{\alpha}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}})K^{\frac{1}{p}}}$$

is the optimal solution of the inequality

$$\begin{aligned}&\|u_0\|_p+\|u_1\|_p\left(\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}T^\alpha+1\right)\\&\quad+\left(\frac{T^{1+\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}+\frac{T}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}}\right)J^{\frac{1}{p}}\\&\quad+\left(\frac{T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}+\frac{T^{\alpha}}{\Gamma(\alpha)((\alpha-1)p+1)^{\frac{1}{p}}}\right)K^{\frac{1}{p}}r'\le r'.\end{aligned}$$

Define the closed, bounded and convex set

$$C_r=\left\{u\in C^\alpha_{1-\alpha}(I,\ell_p):\|u\|\le r,\lim_{x\to0}x^{1-\alpha}u_n(x)=u_0^n,\lim_{x\to0}x^{1-\alpha}\mathcal{D}^\alpha u_n(x)=u_1^n,\right\},$$

where $n=1,2,\ldots$. Clearly, $\mathcal{A}$ is bounded on C_r. In the following, we show that $\mathcal{A}$ is continuous on C_r. For any $u,v\in C_r$, applying Hölder's inequality, we can write

$$\begin{aligned}&\left\|\mathcal{A}u(x)-\mathcal{A}v(x)\right\|_p^p\\&\quad\le\sum_{i\ge1}\left|\int_0^x\frac{(x-\eta)^{2\alpha-1}}{\Gamma(2\alpha)}\left(f_i(\eta,u(\eta),\mathcal{D}^\alpha u(\eta))-f_i(\eta,v(\eta),\mathcal{D}^\alpha v(\eta))\right)d\eta\right|^p\\&\quad\le T^{p-1}\sum_{i\ge1}\int_0^x\left|\frac{(x-\eta)^{2\alpha-1}}{\Gamma(2\alpha)}\right|^p\left|f_i(\eta,u(\eta),\mathcal{D}^\alpha u(\eta))-f_i(\eta,v(\eta),\mathcal{D}^\alpha v(\eta))\right|^p d\eta\\&\quad\le T^{p-1}\int_0^x\frac{(x-\eta)^{(2\alpha-1)p}}{\Gamma(2\alpha)^p}d\eta\left\|f(\eta,u(\eta),\mathcal{D}^\alpha u(\eta))-f(\eta,v(\eta),\mathcal{D}^\alpha v(\eta))\right\|_p^p\\&\quad\le\frac{T^{2\alpha}}{\Gamma(2\alpha)^p((2\alpha-1)p+1)}\left\|f(\eta,u(\eta),\mathcal{D}^\alpha u(\eta))-f(\eta,v(\eta),\mathcal{D}^\alpha v(\eta))\right\|_p^p.\end{aligned}$$

The family of $\{f_x(u)\}_{x\in I}$ where $f_x(u)=f(x,u(x),\mathcal{D}^\alpha u(x))$ is equicontinuous on ℓ_p. Bearing (C_4') in mind, for all $x\in I$, we have

$$\forall v,u\in B_r\text{ and }\forall\epsilon>0,\quad\exists\delta>0\text{ with }\|u-v\|\le\delta\quad\text{such that}\quad\left\|f_x(u)-f_x(v)\right\|_p\le\frac{\epsilon}{z},$$

where $z=\frac{T^{\frac{2\alpha}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}$. Therefore, we see that

$$\left\|\mathcal{A}u(x)-\mathcal{A}v(x)\right\|_p\le\frac{t^{\frac{2\alpha}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}\left\|f_\eta(u)-f_\eta(v)\right\|_p<\epsilon,$$

which means that $\mathcal{A}$ is continuous.

Without loss of generality, we can suppose $x_1 > x_2$. Applying (9), for any $u \in C_r$, we have

$$
\begin{aligned}
&\left\| \mathcal{A}u(x_1) - \mathcal{A}u(x_2) \right\|_p \\
&\quad \le \left\| u_0 \left(x_1^{\alpha-1} - x_2^{\alpha-1} \right) \right\|_p + \left\| u_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} \left(x_1^{2\alpha-1} - x_2^{2\alpha-1} \right) \right\|_p \\
&\qquad + \left\| \frac{1}{\Gamma(2\alpha)} \int_0^{x_1} \left((x_2-\eta)^{2\alpha-1} - (x_1-\eta)^{2\alpha-1} \right) f\left(\eta, u(\eta), \mathcal{D}^\alpha u(\eta)\right) d\eta \right\|_p \\
&\qquad + \left\| \int_{x_1}^{x_2} (x_2-\eta)^{2\alpha-1} f\left(\eta, u(\eta), \mathcal{D}^\alpha u(\eta)\right) d\eta \right\|_p \\
&\quad \le \|u_0\|_p |x_2 - x_1| m_1 + \|u_1\|_p \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} m_2 |x_2 - x_1| \\
&\qquad + \frac{T^{\frac{p-1}{p}}}{\Gamma(2\alpha)} \left(\sum_{i \ge 1} \int_0^{x_1} |x_2 - x_1|^P m_3^P \left(j_i(\eta) + k_i(\eta) \left| u_i(\eta) \right|^P \right) d\eta \right)^{\frac{1}{p}} \\
&\qquad + \frac{T^{\frac{p-1}{p}}}{\Gamma(2\alpha)} \left(\sum_{i \ge 1} \int_{x_1}^{x_2} (x_2-\eta)^{(2\alpha-1)P} \left(j_i(\eta) + k_i(\eta) \left| u_i(\eta) \right|^P \right) d\eta \right)^{\frac{1}{p}} \\
&\quad \le \|u_0\|_p |x_2 - x_1| m_1 + \|u_1\|_p \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} m_2 |x_2 - x_1| \\
&\qquad + \frac{T^{\frac{p-1}{p}}}{\Gamma(2\alpha)} \left(m_3^P \int_0^{x_1} |x_2 - x_1|^P \sum_{i \ge 1} j_i(\eta) \, d\eta \right. \\
&\qquad \left. + \int_0^{x_1} |x_2 - x_1|^P \lim_{n \to \infty} \sup k_i(\eta) \sum_{i \ge 1} \left| u_i(\eta) \right|^P d\eta \right)^{\frac{1}{p}} \\
&\qquad + \frac{T^{\frac{p-1}{p}}}{\Gamma(2\alpha)} \left(\int_{x_1}^{x_2} (x_2-\eta)^{(2\alpha-1)P} \sum_{i \ge 1} j_i(\eta) \, d\eta \right. \\
&\qquad \left. + \int_{x_1}^{x_2} (x_2-\eta)^{(2\alpha-1)p} \lim_{n \to \infty} \sup k_i(\eta) \sum_{i \ge 1} \left| u_i(\eta) \right|^P d\eta \right)^{\frac{1}{p}} \\
&\quad \le \|u_0\|_p |x_2 - x_1| m_1 + \|u_1\|_p \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} m_2 |x_2 - x_1| \\
&\qquad + \frac{T^{\frac{p-1}{p}}}{\Gamma(2\alpha)} \left(TJm_3^P |x_2 - x_1|^P + TK\|u\|^p |x_2 - x_1|^P \right)^{\frac{1}{p}} \\
&\qquad + \frac{T^{\frac{p-1}{p}}}{\Gamma(2\alpha)} \left(\frac{|x_2 - x_1|^{(2\alpha-1)p+1}}{(2\alpha-1)p+1} J + \frac{|x_2 - x_1|^{(2\alpha-1)p+1}}{(2\alpha-1)p+1} K \|u\|^p \right)^{\frac{1}{p}} \\
&\quad \le \|u_0\|_p |x_2 - x_1| m_1 + \|u_1\|_p \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} m_2 |x_2 - x_1| \\
&\qquad + \frac{TJ^{\frac{1}{p}} m_3}{\Gamma(2\alpha)} |x_2 - x_1| + \frac{TJK^{\frac{1}{p}}}{\Gamma(2\alpha)} |x_2 - x_1| \\
&\qquad + \frac{J^{\frac{1}{p}} T^{1-\frac{1}{p}}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} |x_2 - x_1|^{(2\alpha-1)p+1},
\end{aligned}
$$

which tends to zero when $x_1 \longrightarrow x_2$. Thus, we deduce that $\mathcal{A}$ is equicontinuous on C_r.

Setting $\bar{C} = \operatorname{conv}(\mathcal{A}(C_r))$, obviously $\bar{C} \subset C_r$. Let $Y \subset \bar{C}$, then $\mathcal{A}$ is continuous on Y and the functions from the set of Y are equicontinuous on I. In view of the definition of the Hausdorff MNC χ on the space $C^{\alpha}_{1-\alpha}(I, c_0)$, Proposition 1.7 and Theorem 1.9, we have

$$\chi_{C^{\alpha}_{1-\alpha}(I,\ell_p)}(Y) = \sup_{x\in I} \chi_{\ell_p}\bigl(Y(x)\bigr).$$

For any $u \in Y$, we obtain

$$\begin{aligned}
\chi_{\ell_p}\bigl(\mathcal{A}u(x)\bigr) &= \lim_{i\to\infty}\left\{\sup_{u\in B}\left(\sum_{n\geq i}\bigl|\mathcal{A}u_n(x)\bigr|^p\right)^{\frac{1}{p}}\right\} \\
&\leq \lim_{i\to\infty}\left\{\sup_{u\in B}\left(\sum_{n\geq i}\left|u_n^0 x^{\alpha-1} + u_n^1\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}x^{2\alpha-1}\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{1}{\Gamma(2\alpha)}\int_0^x (x-\eta)^{2\alpha-1} f_n\bigl(x, u(x), \mathcal{D}^{\alpha}u(x)\bigr)\,d\eta\right|^p\right)^{\frac{1}{p}}\right\} \\
&\leq \lim_{i\to\infty}\left\{\sup_{u\in B}\frac{1}{\Gamma(2\alpha)}\left(\sum_{n\geq i}\left|\int_0^x (x-\eta)^{2\alpha-1} f_n\bigl(x, u(x), \mathcal{D}^{\alpha}u(x)\bigr)\,d\eta\right|^p\right)^{\frac{1}{p}}\right\} \\
&\leq \lim_{i\to\infty}\left\{\sup_{u\in B}\frac{T^{\frac{1-p}{p}}}{\Gamma(2\alpha)}\left(\sum_{n\geq i}\int_0^x \bigl|(x-\eta)^{2\alpha-1}\bigr|^p\bigl(j_i(\eta) + k_i(\eta)\bigl|u_i(\eta)\bigr|^p\bigr)\,d\eta\right)^{\frac{1}{p}}\right\} \\
&\leq \lim_{i\to\infty}\left\{\sup_{u\in B}\frac{T^{\frac{1-p}{p}}}{\Gamma(2\alpha)}\left(\int_0^x \bigl|(x-\eta)^{2\alpha-1}\bigr|^p\sum_{n\geq i} j_i(\eta)\,d\eta\right.\right. \\
&\quad \left.\left. + \int_0^x \bigl|(x-\eta)^{2\alpha-1}\bigr|^p k_i(\eta)\sum_{n\geq i}\bigl|u_i(\eta)\bigr|^p\right)d\eta)^{\frac{1}{p}}\right\}.
\end{aligned}$$

Then we get

$$\sup_{x\in I}\chi_{\ell_p}\bigl(\mathcal{A}u(x)\bigr) \leq \sup_{x\in I}\frac{T^{\frac{1-p}{p}}}{\Gamma(2\alpha)}\lim_{i\to\infty}\left\{\sup_{u\in B}\left(\frac{T^{(2\alpha-1)p+1)}}{((2\alpha-1)p+1)}K\sum_{n\geq i}\bigl|u_i(\eta)\bigr|^p\right)^{\frac{1}{p}}\right\}.$$

Therefore

$$\sup_{x\in I}\chi_{\ell_p}\bigl(\mathcal{A}u(x)\bigr) \leq \frac{K^{\frac{1}{p}}T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}\sup_{x\in I}\lim_{i\to\infty}\left\{\sup_{u\in B}\left(\max_{n\geq i}\bigl|u_i(\eta)\bigr|^p\right)^{\frac{1}{p}}\right\}$$

and

$$\chi_{C^{\alpha}_{1-\alpha}(I,\ell_p)}\bigl(\mathcal{A}u(x)\bigr) \leq \frac{K^{\frac{1}{p}}T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}}\chi_{C^{\alpha}_{1-\alpha}(I,\ell_p)}(Y).$$

As $\frac{K^{\frac{1}{p}}T^{2\alpha}}{\Gamma(2\alpha)((2\alpha-1)p+1)^{\frac{1}{p}}} < 1$, hence, applying Lemma 1.6, $\mathcal{A}$ admits at least one fixed point in $\mathcal{A}$ which is a solution for (5) in the space $C^{\alpha}_{1-\alpha}(I, \ell_p)$. □

Example 3.2 The system of fractional differential equation in the space ℓ_2 is given as follows:

$$\begin{cases} \mathcal{D}u_n(x) = x^n \tanh(-x) + \sum_{m=n}^{\infty} \frac{u_m(x)e^x}{mn}, & x \in (0, \frac{1}{2}], \\ \lim_{x\to 0} x^{1-\alpha} u_n(x) = \frac{a}{n}, \qquad \lim_{x\to 0} x^{1-\alpha}\mathcal{D}^{\alpha} u_n(x) = \frac{b}{n}, & n = 1, 2, \dots. \end{cases} \tag{11}$$

This system is a special case of (5) with $a, b \geq 0$, $\alpha = 1$, $T = \frac{1}{2}$ and

$$f_n\big(x, u(x), \mathcal{D}^{\alpha} u(x)\big) = x^n \tanh(-x) + \sum_{m=n}^{\infty} \frac{u_m(x)e^x}{mn}.$$

Obviously, the conditions (C_1') and (C_2') are satisfied. For every $x \in (0, \frac{1}{2}]$ and $u \in \ell_2$, we have

$$\begin{aligned} \big|f_n\big(x, u(x), \mathcal{D}^{\alpha} u(x)\big)^2\big| &= \left| x^n \tanh(-x) + \sum_{m=n}^{\infty} \frac{u_m(x)e^x}{mn} \right| \\ &\leq \big|x^n \tanh(-x)\big|^2 + \sum_{m=n}^{\infty} \left| \frac{u_m(x)e^x}{mn} \right|^2 \\ &\leq x^{2n} + \frac{\pi^2 e^{2x}}{6n^2} \big|u_n(x)\big|^2. \end{aligned}$$

Therefore, f satisfies condition (C_3') with $j_i(x) = x^{2n}$ and $k_i(t) = \frac{\pi^2 e^{2x}}{6n^2}$ in which the functions $j_i(x)$ are continuous, $\sum_{i\geq 1} j_i(x)$ converges uniformly to $\frac{1}{1-x^2}$ and $\lim_{i\to\infty} k_i(x) = 0$, that is, it is integrable over I. Now, we are going to check condition (C_4'). For any $\epsilon > 0$, $x \in (0, 1]$ and $u, v \in \ell_2$, choose $\delta = \frac{\epsilon\sqrt{6}}{e^2\pi}$ with $\|u(x) - v(x)\| < \delta$, we have

$$\begin{aligned} \sum_{n\geq 1} \big|f_n\big(x, u(x), \mathcal{D}^{\alpha} u(x)\big) - f_n\big(x, v(x), \mathcal{D}^{\alpha} v(x)\big)\big|^2 &\leq \sum_{n\geq 1} \left| \sum_{m\geq n} \frac{(u_m(x) - v_m(x))e^{2x}}{mn} \right|^2 \\ &\leq \sum_{n\geq 1} \frac{e^{2x}}{n^2} \sum_{m\geq n} \frac{|u_m(x) - v_m(x)|^2}{m^2} \\ &\leq \sum_{n\geq 1} \frac{e^{2x}}{n^2} \big\|u(x) - v(x)\big\|_{\ell_2}^2 \\ &\leq \big\|u(x) - v(x)\big\|_{\ell_2}^2 \frac{e^2\pi^2}{6} < \epsilon. \end{aligned}$$

Applying Theorem 3.2, hence the system of fractional differential equation (11) possesses at least one solution in $C_{1-\alpha}^{\alpha}(I, \ell_2)$.

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