Higher-order relaxation magnetic phenomena and a hierarchy of first-order relaxation variables

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Aim and motivation

A theory for magnetic relaxation phenomena was developed by G. A. Kluitenberg and the author in the framework of thermodynamics of irreversible processes with internal variables. It was shown that if n different types of microscopic irreversible phenomena give rise to magnetic relaxation, it is possible to describe these microscopic phenomena introducing n macroscopic axial vectorial internal variables in the expression of the entropy. The total specific magnetization m is split in n+1 parts $\mathbf{m}^{(k)}$ $(k=1,...,n)$ **, i.e.**

$$\mathbf{m}=\mathbf{m}^{(0)}+\sum_{k=1}^{n}\mathbf{m}^{(k)}. \tag{1}$$

The following set C of independent variables was assumed

$$C=C\left(u,\epsilon_{\alpha\beta},\mathbf{m},\mathbf{m}^{(1)},...,\mathbf{m}^{(n)}\right), \tag{2}$$

where u is the specific internal energy and $\epsilon_{\alpha\beta}$ **is the strain tensor. Using the same procedure applied in [?], by eliminating the internal variables the following relaxation equation generalizing Snoek equation was obtained**

$$\chi^{(0)}_{BM}\mathbf{B}+\chi^{(1)}_{BM}\frac{d\mathbf{B}}{dt}+...+\chi^{(n-1)}_{BM}\frac{d^{n-1}\mathbf{B}}{dt^{n-1}}+\frac{d^{n}\mathbf{B}}{dt^{n}}=$$

$$\chi^{(0)}_{MB}\mathbf{M}+\chi^{(1)}_{MB}\frac{d\mathbf{M}}{dt}+...+\chi^{(n)}_{MB}\frac{d^{n}\mathbf{M}}{dt^{n}}+\chi^{(n+1)}_{MB}\frac{d^{n+1}\mathbf{M}}{dt^{n+1}}, \tag{3}$$

where $\chi^{(m)}_{BM}$ $(m=0,1,...,n-1)$ and $\chi^{(m)}_{MB}$ $(m=0,1,...,n+1)$ **are characteristic constants of the particular material.**

This magnetic relaxation relation has the same mathematical structure of the following stress-strain relation for mechanical distortional phenomena in isotropic media, derived in 1968 by G. A. K., assuming that n microscopic phenomena give rise to inelastic strains (slip, dislocations, etc.) and the total strain tensor $\epsilon_{\alpha\beta}$ **is split in n+1 parts,** $\epsilon^{(0)}_{\alpha\beta}$ and $\epsilon^{(k)}_{\alpha\beta}$ $(k=1,...,n)$,

$$R^{(\tau)}_{(d)0}\tilde{\tau}_{\alpha\beta}+R^{(\tau)}_{(d)1}\frac{d\tilde{\tau}_{\alpha\beta}}{dt}+...+R^{(\tau)}_{(d)n-1}\frac{d^{n-1}\tilde{\tau}_{\alpha\beta}}{dt^{n-1}}+\frac{d^{n}\tilde{\tau}_{\alpha\beta}}{dt^{n}}=$$

$$R^{(\epsilon)}_{(d)0}\tilde{\epsilon}_{\alpha\beta}+R^{(\epsilon)}_{(d)1}\frac{d\tilde{\epsilon}_{\alpha\beta}}{dt}+...+R^{(\epsilon)}_{(d)n}\frac{d^{n}\tilde{\epsilon}_{\alpha\beta}}{dt^{n}}+R^{(\epsilon)}_{(d)n+1}\frac{d^{n+1}\tilde{\epsilon}_{\alpha\beta}}{dt^{n+1}}. \tag{4}$$

$R^{(\tau)}_{(d)m}$ $(m=0,1,...,n-1)$ $R^{(\epsilon)}_{(d)m}$ $(m=0,1,...,n+1)$ **are material constants and** $\tilde{\tau}_{\alpha\beta}$ and $\tilde{\epsilon}_{\alpha\beta}$ **are the deviators of the stress tensor and of the strain tensor, respectively.**

If n arbitrary microscopic phenomena give rise to the total polarization vector, by introducing n partial polarization vectors as n macroscopic vectorial internal variables in the expression of the entropy, the following dielectric relaxation equation was obtained by the author and G. A. K. in the isotropic case

$$\chi^{(0)}_{EP}\mathbf{E}+\chi^{(1)}_{EP}\frac{d\mathbf{E}}{dt}+...+\chi^{(n-1)}_{EP}\frac{d^{n-1}\mathbf{E}}{dt^{n-1}}+\frac{d^{n}\mathbf{E}}{dt^{n}}=$$

$$\chi^{(0)}_{PE}\mathbf{P}+\chi^{(1)}_{PE}\frac{d\mathbf{P}}{dt}+...+\chi^{(n)}_{PE}\frac{d^{n}\mathbf{P}}{dt^{n}}+\chi^{(n+1)}_{PE}\frac{d^{n+1}\mathbf{P}}{dt^{n+1}}, \tag{5}$$

where E and P are the electric strength field and the polarization vector, respectively, and $\chi^{(k)}_{EP}$ $(k=0,1,...,n-1)$ **and** $\chi^{(k)}_{PE}(k=0,1,...,n+1)$ **are constant quantities.**

The aim of the present work is to relate the n-th order relaxation equation (1) (involving time derivatives of the magnetic field B up to the n-th order, and time derivatives of the magnetization M up to (n + 1)-th order) to a hierarchy of first-order relaxation equations. In this way we relate the general equation to the microscopic structure of the system. Finally, we obtain the form of the entropy and its consequences on the hierarchy of relaxation equations. We try the paper to be as simple and pedagogical as possible, with a minimum of physical complexity related to the mathematical structure of the equations.

Description of the model: a dynamical hierarchy

In the model to be considered we focus our attention on a magnetic system composed of n sets of spins $(i=1,2,...,n)$**, such that there are** $N^{(i)}$ **magnetic particles of the kind** i**, each of them having mass** m_i**, radius** r_i**, spin** S_i**, and so on. Thus, their inertia, relaxation time, magnetization and susceptibility will be different for each set of particles. These differences will show up especially in dynamical phenomena. Here, we will discuss a simple situation in which the time scales of the several variables** $\mathrm{M}^{(i)}$**, the magnetizations of the i-th set, are sufficiently separated to be considered as a hierarchy of equations with minimal couplings amongst them. We propose the following hierarchy of dynamical equations**

$$\frac{d\mathbf{M}^{(1)}}{dt}+\frac{1}{\tau_1}\mathbf{M}^{(1)}=\frac{\chi_1}{\tau_1}\mathbf{B}+\beta_1\mathbf{M}^{(2)},$$

$$\frac{d\mathbf{M}^{(2)}}{dt}+\frac{1}{\tau_2}\mathbf{M}^{(2)}=\frac{\chi_2}{\tau_2}\mathbf{B}+\gamma_1\mathbf{M}^{(1)}+\beta_2\mathbf{M}^{(3)},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{d\mathbf{M}^{(i)}}{dt}+\frac{1}{\tau_i}\mathbf{M}^{(i)}=\frac{\chi_i}{\tau_i}\mathbf{B}+\gamma_{(i-1)}\mathbf{M}^{(i-1)}+\beta_i\mathbf{M}^{(i+1)},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{d\mathbf{M}^{(n)}}{dt}+\frac{1}{\tau_n}\mathbf{M}^{(n)}=\frac{\chi_n}{\tau_n}\mathbf{B}+\gamma_{(n-1)}\mathbf{M}^{(n-1)}. \tag{6}$$

We have assumed that $\tau_1>\tau_2>\tau_3>\ >\tau_n$**. We will denote** $\frac{\chi_i}{2\,\chi_{(i-1)}\,\tau_i}\equiv\gamma_i$**. In these equations,** χ_i **is the magnetic susceptibility of particles** i**,** τ_i **the magnetic relaxation time,** β_i **a coefficient that couples variables i and i + 1, and** γ_i **a coefficient coupling variables** i **to** $i-1$**. This coupling may be physically realized, for instance, through the magnetization of the slower of the couple of the variables** i **and** $i-1$**, namely** $i-1$**, which adds to the external applied magnetic field B acting on** $\mathrm{M}^{(i)}$**. Since** $\mathrm{M}^{(i-1)}$ **is much slower than** $\mathrm{M}^{(i)}$**, the value of** $\mathrm{M}^{(i-1)}$ **will not appreciably change during the relaxation of** $\mathrm{M}^{(i)}$**. On the other side, since** $\mathrm{M}^{(i+1)}$ **is much faster than** $\mathrm{M}^{(i)}$**,** $\mathrm{M}^{(i+1)}$ **will relax in a very short time and will also keep practically constant in its final relaxed value during the relaxation on** $\mathrm{M}^{(i)}$**.**

Differentiating equation $(6)_1$**, using** $(6)_2$ **for the time derivative of** $\mathrm{M}^{(2)}$**, and using** $(6)_1$ **to express** $\mathrm{M}^{(2)}$ **in terms of** $\mathrm{M}^{(1)}$, $\frac{d\mathbf{M}^{(1)}}{dt}$ **and B, one gets**

$$\frac{d^2\mathbf{M}^{(1)}}{dt^2}+\xi^{(11)}_{M}\frac{d\mathbf{M}^{(1)}}{dt}+\xi^{(01)}_{M}\mathbf{M}^{(1)}=\xi^{(11)}_{B}\frac{d\mathbf{B}}{dt}+\xi^{(01)}_{B}\mathbf{B}+\beta_1\,\beta_2\,\mathbf{M}^{(3)},$$

with the coefficients

$$\xi^{(11)}_{M}\equiv\frac{1}{\tau_1}+\frac{1}{\tau_2},$$

$$\xi^{(01)}_{M}\equiv\frac{1}{\tau_1\tau_2}-\beta_1\gamma_1,$$

$$\xi^{(11)}_{M}\equiv\frac{\chi_1}{\tau_1},$$

$$\xi^{(01)}_{B}\equiv\frac{\chi_1}{\tau_1\tau_2}+\beta_1\frac{\chi_2}{\tau_2}. \tag{7}$$

We now differentiate equation $(7)_1$**, use the corresponding evolution equation of hierarchy (6) for** $\frac{d\mathbf{M}^{(3)}}{dt}$**, and use** $(6)_1$ **and** $(6)_2$ **to express** $\mathrm{M}^{(2)}$ **and** $\mathrm{M}^{(3)}$ **in terms of** $\mathrm{M}^{(1)}$**,** $\frac{d\mathbf{M}^{(1)}}{dt}$ **and B, and we get**

$$\frac{d^3\mathbf{M}^{(1)}}{dt^3}+\xi^{(22)}_{M}\frac{d^2\mathbf{M}^{(1)}}{dt^2}+\xi^{(12)}_{M}\frac{d\mathbf{M}^{(1)}}{dt}+\xi^{(02)}_{M}\mathbf{M}^{(1)}=$$

$$\xi^{(22)}_{B}\frac{d^2\mathbf{B}}{dt^2}+\xi^{(12)}_{B}\frac{d\mathbf{B}}{dt}+\xi^{(02)}_{B}\mathbf{B}+\beta_1\beta_2\beta_3\mathbf{M}^{(4)}. \tag{8}$$

Note that $\xi^{(ab)}_{M}$ **is the coefficient multiplying the** a**-the derivative of** $\mathrm{M}^{(1)}$ **in the equation corresponding to the** b**-th order of approximation, and analogously for** $\xi^{(ab)}_{B}$**, but for the** a**-th derivative of the magnetic field B. The corresponding coefficients are given by**

$$\xi^{(22)}_{M}\equiv\xi^{(11)}_{M}+\frac{1}{\tau_3},$$

$$\xi^{(12)}_{M}\equiv\xi^{(01)}_{M}+\gamma_2\beta_2+\frac{1}{\tau_1\tau_3}+\frac{1}{\tau_2\tau_3},$$

$$\xi^{(02)}_{M}\equiv\frac{\gamma_2\beta_2}{\tau_1}+\frac{1}{\tau_1\tau_2\tau_3},$$

$$\xi^{(22)}_{B}\equiv\xi^{(11)}_{B},$$

$$\xi^{(12)}_{B}\equiv\xi^{(01)}_{B}+\frac{\chi_1}{\tau_1\tau_3}$$

$$\xi^{(02)}_{B}\equiv\frac{\gamma_2\beta_2\chi_1}{\tau_1}+\frac{\beta_1\beta_2\chi_3}{\tau_3}+\frac{\chi_1}{\tau_1\tau_2\tau_3}. \tag{9}$$

In principle, one may obtain a series of recurrence relations for the coefficients corresponding to higher-order equations. Here, we stop our calculations, which become increasingly lengthy and cumbersome, as they already illustrate the basic concept that hierarchy (6) may be written in the form of higher-order relation equation of the form we are considering.

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