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Uniformly Resolvable Decompositions of K_v-I into n-Cycles and n-Stars, for Even n

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Abstract: If X is a connected graph, then an X-factor of a larger graph is a spanning subgraph in which all of its components are isomorphic to X. Given a set Γ of pairwise non-isomorphic graphs, a *uniformly resolvable* Γ*-decomposition*of a graph G is an edge decomposition of G into X-factors for some graph $X \in \Gamma$. In this article we completely solve the existence problem for decompositions of K_v-I into C_n-factors and $K_{1,n}$-factors in the case when n is even.

Keywords: graph decomposition; factor; uniform factorization

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1. Introduction and Definitions

For any graph G, let $V(G)$ and $E(G)$ be the vertex-set and the edge-set of G, respectively. Throughout the paper K_v will denote the complete graph on v vertices, while $K_v \setminus K_h$ will denote the graph with $V(K_v)$ as vertex-set and $E(K_v) \setminus E(K_h)$ as edge-set (this graph is sometimes referred to as a complete graph of order v with a *hole* of size h).

Given a set Γ of pairwise non-isomorphic graphs, a Γ-*decomposition* (or Γ-*design*)of a graph G is a decomposition of the edge-set of G into subgraphs (called *blocks*) that are isomorphic to some element of Γ. A Γ-*factor* of G is a spanning subgraph of G whose components are isomorphic to a member of Γ. If $X \in \Gamma$, then an X-factor is a spanning subgraph whose components are isomorphic to X. A Γ-decomposition of G is resolvable if its blocks can be partitioned into Γ-factors and is called a Γ-*factorization* of G. A Γ-factorization of G is called uniform if each factor is an X-factor for some graph $X \in \Gamma$. A K_2-factorization of G is known as a 1-*factorization* and its factors are called 1-*factors*; it is well known that a 1-factorization of K_v exists if and only if v is even ([1]). A C_k-factorization of K_v exists if and only if $3 \le k \le v$, v and k are odd, and $v \equiv 0 \pmod{k}$ ([2]).

A Γ-isofactorization of G is a Γ-factorization with isomorphic factors. If Γ is the set of all possible cycles of K_v, then determining the existence of possible Γ-isofactorizations of K_v with an odd v is known as the *Oberwolfach Problem*. It was first posed in 1967 by Gerhard Ringel and asks whether it is possible to seat an odd number v of mathematicians at n round tables in $(v-1)/2$ meals so that each mathematician sits next to everyone else exactly once. If the n round tables are of sizes $p_1, p_2, \ldots, p_n$ (with $p_1 + p_2 + \cdots + p_n = v$), the Oberwolfach Problem asks for an isofactorization of K_v with factors whose components are isomorphic to cycles of length $p_1, p_2, \ldots, p_n$. It is easy to see that such a factorization can exist only if v is odd. For even v, it is common to instead decompose K_v-I, with the complete graph with the edges of a 1-factor removed. The uniform Oberwolfach problem (all cycles of a factor have the same size) has been completely solved by Alspach and Häggkvist [3] and Alspach, Schellenberg, Stinson and Wagner [2] .

Additional existence problems for Γ-factorizations of K_v or K_v-I have been studied and many results have been obtained, especially on uniformly resolvable Γ-decompositions: when Γ is a set of two complete graphs of an order of at most five in [4–7]; when Γ is a set of two or three paths on two, three or four vertices in [8–10]; for $\Gamma = \{P_3, K_3 + e\}$ in [11]; for $\Gamma = \{K_3, K_{1,3}\}$ in [12]; for $\Gamma = \{C_4, P_3\}$ in [13]; for $\Gamma = \{K_3, P_3\}$ in [14]; for $\Gamma = \{K_2, K_{1,3}\}$ in [15,16]; for $\Gamma = \{K_2, K_{1,4}\}$ in [17]. Most famous is the variation of the Oberwolfach problem known as the Hamilton-Waterloo problem. In this problem the meals for the dining mathematicians take place at two different venues. Hence a decomposition of K_v or K_v-I is sought where the factors can be of either one of two types. In particular, the uniform case asks for a decomposition of K_v or K_v-I into C_p-factors and C_q-factors. Thus the round tables in one venue sit p mathematicians, whereas the tables in the other venue each sit q. Of course, in this case p and q must divide v and $\Gamma = \{C_p, C_q\}$.

A uniformly resolvable $\{X, Y\}$-decomposition of G into exactly r X-factors and s Y-factors is abbreviated as (X, Y)-URD$(G; r, s)$. If $G = K_v$ we simply write (X, Y)-URD$(v; r, s)$. In this paper, we study uniformly resolvable Γ-decompositions in the case when $\Gamma = \{C_n, K_{1,n}\}$. The existence problem of a $(C_n, K_{1,n})$-URD$(v; r, s)$ was solved for $n = 2$ ([9], note that $C_2 = K_2$) and $n = 3$ ([12]). Here we deal with the case when n is even and greater or equal to 4. For an even n, it is known that a $(C_n, K_{1,n})$-URD$(v; 0, s)$ exists if and only if $v \equiv 1 \pmod{2n}$ and $v \equiv 0 \pmod{n+1}$ ([18]), while, when v is even, no $(C_n, K_{1,n})$-URD$(v; r, s)$ exists with $r > 0$ because otherwise, $2(n+1)r + 2ns = (n+1)(v-1)$, which is clearly impossible. Hence we study the existence problem for $(C_n, K_{1,n})$-URD$(K_v - I; r, s)$, which is denoted by $(C_n, K_{1,n})$-URD$^*(v; r, s)$ and, since n and $n+1$ must divide v, we assume that $v \equiv 0 \pmod{n(n+1)}$. Furthermore, since $\frac{(v-2)(n+1)}{2n} \notin \mathbb{N}$, necessarily $r > 0$.

For $v \equiv 0 \pmod{n(n+1)}$, defined the set $J(v)$ according to the following Table 1.

Table 1. The set $J(v)$.

v	$J(v)$
$0 \pmod{2n(n+1)}$	$(\frac{v-2}{2} - nx, (n+1)x), \ \ x = 0, 1, \ldots, \frac{v-2n}{2n}$
$n(n+1) \pmod{2n(n+1)}$	$(\frac{v-2}{2} - nx, (n+1)x), \ \ x = 0, 1, \ldots, \frac{v-n}{2n}$

We completely solve the existence problem of a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ by proving the following result.

Theorem 1. *Let $v \equiv 0 \pmod{n(n+1)}$. There exists a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ if and only if $(r, s) \in J(v)$.*

2. General Constructions and Related Structures

A Γ-decomposition of $K_{u(g)}$, the complete multipartite graph with u parts of size g, is known as a *group divisible decomposition* (Γ-GDD for short) of type g^u; the parts of size g are called the *groups*. (If Γ consists of complete subgraphs, then a GDD is called a *group divisible design*). When $\Gamma = \{G\}$, we simply write G-GDD, and when $G = K_n$, we refer to such a group divisible design as an n-GDD. We denote a (uniformly) resolvable Γ-GDD by Γ-(U)RGDD. Specifically, an (X, Y)-URGDD with r X-factors and s Y-factors is denoted by (X, Y)-URGDD(r, s). It is easy to deduce that the number of G-factors of a G-RGDD is $\alpha = \frac{g(u-1)|V(G)|}{2|E(G)|}$.

If the blocks of a Γ-GDD of type g^u can be partitioned into *partial* factors, each of which contains all vertices except those of one group, we refer to such a decomposition as a Γ-*frame* (an n-*frame* if $\Gamma = \{K_n\}$). For a fixed positive integer d, if Γ is a set of d-regular graphs, then it is easy to deduce that the number of partial factors missing a specified group is $\alpha = \frac{g}{d}$.

A Γ-decomposition of $K_{v+h} \setminus K_h$ is known as an *incomplete Γ-design of order $v + h$ with a hole of size h*. We are interested in incomplete resolvable Γ-designs, which will be used in the "filling" and "frame"-constructions of this section. These designs have two types of factors: *partial* factors, which cover every vertex except the ones in the hole; and *full* factors, which cover every vertex of K_{v+h}.

Specifically, a (X,Y)-IURD$(v+h,h;[r',s'],[r,s])$ is a uniformly resolvable (X,Y)-decomposition of $K_{v+h}\setminus K_h$ with r' partial X-factors and s' partial Y-factors that cover every vertex not in the hole, and r X-factors and s Y-factors that cover every vertex of K_{v+h}.

Given a graph G and a positive integer t, $G_{(t)}$ will denote the graph on $V(G)\times\mathbb{Z}_t$ with edge-set $\{\{x_i,y_j\}:\{x,y\}\in E(G), i,j\in\mathbb{Z}_t\}$, where the subscript notation a_i is used to denote the pair (a,i). The graph $G_{(t)}$ is said to be obtained from G by expanding each vertex t times. When $G=K_n$, the graph $G_{(t)}$ is the complete equipartite graph $K_{\underbrace{t,t,\ldots,t}_{n\text{ times}}}$ with n parts of size t and will be denoted by $K_{n(t)}$; while $C_{n(t)}$ will denote the graph $G_{(t)}$ where G is an n-cycle.

Remark 1. *Note that the graph $G_{(t)}$ admits t 1-factors corresponding to each 1-factor of G; for instance, starting from the two 1-factors of a $2m$-cycle, $2t$ 1-factors of $C_{2m(t)}$ can be obtained (t 1-factors for each 1-factor of the $2m$-cycle).*

For any two pairs of non-negative integers (r,s) and (r',s'), define $(r,s)+(r',s')=(r+r',s+s')$. If X and X' are two sets of pairs of non-negative integers and a is a positive integer, then $X+X'$ will denote the set $\{(r,s)+(r',s'):(r,s)\in X,(r',s')\in X'\}$ and $a*X$ will denote the set of all pairs of non-negative integers that can be obtained by adding any a pairs of X together (repetitions of elements of X are allowed).

Construction 1. *(GDD-Construction) Let $\mathcal{G}$ be a Γ-RGDD of type g^u, where Γ is a set of graphs of order $n\geq 2$, and let t be a positive integer. If for any fixed factor F_i, $i=1,2,\ldots,\alpha$, there exists an (X,Y)-URD$(\bar{r},\bar{s})$ of $B_{(t)}$ for each $B\in F_i$ and for each $(\bar{r},\bar{s})\in J_i$, then so does an (X,Y)-URGDD(r,s) of type $(gt)^u$ for each $(r,s)\in J_1+J_2+\cdots+J_\alpha$.*

Proof. Expand each vertex t times. For $i=1,2,\ldots,\alpha$, for each block B of F_i on $V(B)\times\mathbb{Z}_t$ place a copy of an (X,Y)-URD(r_i,s_i) of $B_{(t)}$ with $(r_i,s_i)\in J_i$. Thus we obtain an (X,Y)-URGDD(r,s) of type $(gt)^u$ with $r=\sum_{i=1}^{\alpha} r_i$ and $s=\sum_{i=1}^{\alpha} s_i$, and so $(r,s)\in J_1+J_2+\cdots+J_\alpha$. □

Construction 2. *(Filling Construction) Suppose there exists a (X,Y)-URGDD(r,s) of type g^u for each $(r,s)\in J$. If there exists an (X,Y)-URD$(g;r',s')$, for each $(r',s')\in J'$, then so does:*

(i) an (X,Y)-IURD$(ug,g;[r',s'],[r,s])$ for each $(r',s')\in J'$ and $(r,s)\in J$;
(ii) an (X,Y)-URD$(ug;\bar{r},\bar{s})$, for each $(\bar{r},\bar{s})\in J'+J$.

Proof. Fix any pairs $(r,s)\in J$ and $(r',s')\in J'$, and start with an (X,Y)-URGDD(r,s) with u groups of size g, G_i, $i=1,2,\ldots,u$. For every $i=2,3,\ldots,u$, place a copy of an (X,Y)-URD$(g;r',s')$ on G_i to obtain an (X,Y)-IURD$(gu,g;[r',s'],[r,s])$ with G_1 as the hole. Finally, on G_1 place a copy of an (X,Y)-URD$(g;r',s')$ to obtain an (X,Y)-URD$(gu;r'+r,s'+s)$. □

Remark 2. *Note that the "filling" technique allows us to construct an (X,Y)-URD$(v+h;r'+r,s'+s)$ whenever an (X,Y)-IURD$(v+h,h;[r',s'],[r,s])$ and an (X,Y)-URD$(h;r',s')$ are given.*

Construction 3. *(Frame-Construction) Let $\mathcal{F}$ be a Γ-frame of type g^u, where Γ is a set of graphs of order $n\geq 2$ and the number of partial factors missing any fixed group is α, and let t, h and v be positive integers such that $v=gtu+h$. If there exists:*

(i) An (X,Y)-URD$(\bar{r},\bar{s})$ of $G_{(t)}$ for each $G\in\Gamma$ and for each $(\bar{r},\bar{s})\in J$;
(ii) An (X,Y)-IURD$(gt+h,h;[r',s'],[\bar{\bar{r}},\bar{\bar{s}}])$ for each $(r',s')\in J'$ and $(\bar{\bar{r}},\bar{\bar{s}})\in\alpha*J$;
(iii) An (X,Y)-URD$(h;r',s')$ for each $(r',s')\in J'$;

then so does an (X,Y)-URD$(v;r,s)$ for each $(r,s)\in J'+u\alpha*J$ exist.

Proof. Let A_i, $i = 1, 2, \ldots, u$, be the groups of $\mathcal{F}$ and for $j = 1, 2, \ldots, \alpha$, let F_{ij} be the j-th partial factor that misses the group A_i. Expand each vertex t times and add a set H of t extra vertices. For $j = 1, 2, \ldots, \alpha$, let F_{ij} be the j-th partial factor that misses the group G_i. For each block $B \in F_{ij}$, on $v(B) \times \mathbb{Z}_t$ place a copy, $\mathcal{D}_{ij}(B)$, of an (X, Y)-URD(r_{ij}, s_{ij}) of $B_{(t)}$ with $(r_{ij}, s_{ij}) \in J$. For $i = 1, 2, \ldots, u$, on $H \cup (A_i \times \mathbb{Z}_t)$ place a copy $\mathcal{D}_i$ of an (X, Y)-IURD$(gt + h, h; [r', s'], [r_i, s_i])$ with $(r', s') \in J'$ and $(r_i, s_i) = \sum_{j=1}^{\alpha} (r_{ij}, s_{ij}) \in \alpha * J$. For every $i = 1, 2, \ldots, u$, combine all of the factors of $\mathcal{D}_{ij}(B)$, $B \in F_{ij}$, along with the full factors of $\mathcal{D}_i$ to obtain $\bar{r}$ X-factors and $\bar{s}$ Y-factors, where $(\bar{r}, \bar{s}) = \sum_{i=1}^{u} (r_i, s_i) \in u\alpha * J$. Now, fill the hole H with a copy $\mathcal{D}$ of an (X, Y)-URD$(h; r', s')$ with $(r', s') \in J'$. Combine the factors of $\mathcal{D}$ with the partial factors of $\mathcal{D}_i$ to obtain further r' X-factors and s' Y-factors with $(r', s') \in J'$. The result is an (X, Y)-URD$(v; r, s)$ where $(r, s) = (r' + \bar{r}, s' + \bar{s}) \in J' + u\alpha * J$. □

We quote the following known results for a later use.

Lemma 1 (Ref. [19]). *For $l \geq 3$ and $u \geq 2$, there exists a C_l-RGDD of type g^u if and only if $g(u-1) \equiv 0 \pmod 2$, $gu \equiv 0 \pmod l$, $l \equiv 0 \pmod 2$ if $u = 2$, and $(g, u, l) \notin \{(2,3,3), (6,3,3), (2,6,3), (6,2,6)\}$.*

Lemma 2 (Ref. [20]). *A $\{C_3, C_4\}$-frame of type g^u exists if and only if $u \geq 3$ and $g \equiv 0 \pmod 2$.*

3. Necessary Conditions and Preliminary Lemmas

Let $n \equiv 0 \pmod 2$, $n \geq 4$. To start with, in this section we will give necessary conditions for the existence of a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ and then we will prove some basic lemmas that are useful for obtaining our main result. Let $p = n(n+1)$.

Lemma 3. *Let $v \equiv 0 \pmod p$. If there exists a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ then $(r, s) \in J(v)$.*

Proof. By the resolvability:

$$\frac{rnv}{n} + \frac{nsv}{n+1} = \frac{v(v-2)}{2},$$

and hence

$$2(n+1)r + 2ns = (n+1)(v-2). \tag{1}$$

Denote by R the set of r C_n-factors and by S the set of s $K_{1,n}$-factors. Since the factors of R are regular of degree 2, every vertex of K_v-I is incident to r C_n-factors in R and $(v-2) - 2r$ edges in S. Assume that any fixed vertex appears in x factors of S with degree n and in y factors of S with degree 1. Since

$$x + y = s \text{ and } nx + y = v - 2 - 2r,$$

equality (1) gives us:

$$(n+1)(v-2-nx-y) + 2n(x+y) = (n+1)(v-2),$$

which implies $y = nx$ and $s = (n+1)x$. Replacing $s = (n+1)x$ in Equation (1) provides $r = \frac{v-2}{2} - nx$, where $x < \frac{v-2}{2n}$ (because r is a positive integer) and so $0 \leq x \leq \lfloor \frac{v-2}{2n} \rfloor$. □

In what follows, we will denote by $(a_1, a_2, \ldots, a_n)$ the n-cycle on $\{a_1, a_2, \ldots, a_n\}$ with edge-set $\{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, a_1\}\}$, and by $(a; a_1, a_2, \ldots, a_n)$ the graph $K_{1,n}$ on the vertex-set $\{a, a_1, a_2, \ldots, a_n\}$ with edge-set $\{\{a, a_1\}, \{\{a, a_2\}, \ldots, \{a, a_n\}\}$. If G is a graph whose vertices belong to $\mathbb{Z}_v$, then we call *orbit of B under Z_v* the set $(G) = \{G + i : i \in \mathbb{Z}_v\}$, where $G + i$ is the graph with $V(G+i) = \{a + i : a \in V(G)\}$ and $E(G+i) = \{\{a+i, b+i\} : \{a, b\} \in V(G)\}$.

Lemma 4. *A $(C_n, K_{1,n})$-URD(r, s) of $C_{n(t)}$ where $t = n+1$ exists for $(r, s) = (n+1, 0), (1, n+1)$.*

Proof. Start from the cycle $C = (0, 1, \ldots, n-1)$ on $\mathbb{Z}_n$ and expand it $t = n+1$ times. For the case $(r, s) = (1, n+1)$, take the following factors:

$$F = \{(0_j, 1_j, \ldots, (n-1)_j) : j = 0, 1, \ldots, n\},$$

$$F_j = \{(i_j; (1+i)_{j+1}, (1+i)_{j+2}, \ldots, (1+i)_{j+n}) : i \in \mathbb{Z}_n\},\ j \in \mathbb{Z}_{n+1}.$$

For the case $(r, s) = (n+1, 0)$, take the following C_n-factors:

$$F'_j = \{(0_i, 1_{i+j}, 2_i, 3_{i+j}, \ldots, (n-2)_i, (n-1)_{i+j}) : i \in \mathbb{Z}_{n+1}\},\ j \in \mathbb{Z}_{n+1}. \quad \square$$

Lemma 5. *A G-factorization of $G_{(n)}$ exists for $G = C_n, K_{1,n}$.*

Proof. For $G = C_n$, start from the n-cycle $(1, 2, \ldots, n)$ and on $\{1, 2, \ldots, n\} \times \mathbb{Z}_n$ consider the following C_n-factors:

$$F_i = \{(1_i, 2_{i+j}, 3_i, 4_{i+j}, \ldots, n_{i+j}) : j \in \mathbb{Z}_n\},\ i \in \mathbb{Z}_n.$$

For $G = K_{1,n}$, start from $(0; 1, 2, \ldots, n)$ and on $\{0, 1, 2, \ldots, n\} \times \mathbb{Z}_n$ consider the following $K_{1,n}$-factors:

$$F'_i = \{(0_i; 1_0, 1_1, \ldots, 1_{n-1}), (0_{i+1}; 2_0, 2_1, \ldots, 2_{n-1}), \ldots, (0_{i-1}; n_0, n_1, \ldots, n_{n-1})\},\ i \in \mathbb{Z}_n. \quad \square$$

Lemma 6. *There exists a $(C_n, K_{1,n})$-URD$(n, n(n+1))$ of $C_{n(p)}$.*

Proof. Let F_i, $i = 1, 2, \ldots, n$ be the C_n-factorization of $C_{n(n)}$ given by Lemma 5. Expand each vertex $t = n+1$ times. For $i = 1, 2, \ldots, n$, for each n-cycle C of F_i on $V(C) \times \mathbb{Z}_t$ place a copy of a $(C_n, K_{1,n})$-URD$(1, n+1)$ of $C_{n(t)}$ (given by Lemma 4) to get a $(C_n, K_{1,n})$-URD$(n, n(n+1))$ of $C_{n(p)}$. $\square$

It is not difficult to generalize Lemma 4.8 of [17] so as to obtain a more general result that holds for any even n.

Lemma 7. *A $(C_n, K_{1,n})$-URD$(0, (n+1)^2)$ of $C_{m(p)}$ exists for every $m \geq 3$.*

Lemma 8. *There exists a $(C_n, K_{1,n})$-URD$^*(2(n+1); 0, n+1)$.*

Proof. The orbit of $B = (0; 1, 2, \ldots, n)$ under $\mathbb{Z}_{2(n+1)}$ is the block set of a $K_{1,n}$-decomposition of $K_{2(n+1)} - I$ and can be partitioned into the $n+1$ factors $F_i = \{B + i + (n+1)j, j = 0, 1\}$, for $i = 0, 1, \ldots, n$, to obtain the required design. $\square$

Lemma 9. *Let $v = pk$, $k \geq 1$. A $(C_n, K_{1,n})$-URD$^*(v; r, s)$ exists for every $(r, s) \in \{(\frac{v-2}{2} - nx, (n+1)x) : x = 0, 1, \ldots, \frac{kn}{2}\}$.*

Proof. Start from a C_n-RGDD of type $2^{\frac{nk}{2}}$, which exists by Lemma 1 and has $\alpha = \frac{nk}{2} - 1$ factors. Applying the GDD-construction with $t = n+1$ gives a $(C_n, K_{1,n})$-URGDD$(\bar{r}, \bar{s})$ of type $[2(n+1)]^{\frac{nk}{2}}$ for each $(\bar{r}, \bar{s}) \in (\frac{nk}{2} - 1) * \{(n+1, 0), (1, n+1)\}$ (the input designs are given by Lemma 4). Now fill the groups with copies of a $(C_n, K_{1,n})$-URD$^*(2(n+1); 0, n+1)$ from Lemma 8 to get a $(C_n, K_{1,n})$-URD$^*(pk; r, s)$ for each $(r, s) \in \{(0, n+1)\} + (\frac{nk}{2} - 1) * \{(n+1, 0), (1, n+1)\} = \left\{\frac{v-2}{2} - nx, (n+1)x) : x = 1, \ldots, \frac{nk}{2}\right\}$. The missing case $(x = 0)$ corresponds to a C_n-factorization of $K_{pk} - I$, which is known to exist (see [21]). $\square$

Lemma 10. *A $(C_n, K_{1,n})$-URD$^*(p; r, s)$ exists for every $(r, s) \in J(p)$.*

Proof. It follows by Lemma 9 for $k = 1$. □

Lemma 11. *A $(C_n, K_{1,n})$-URD*$^*(2p; r, s)$ *exists for every* $(r, s) \in J(2p)$.

Proof. It follows by Lemma 9 for $k = 2$. □

Lemma 12. *A $(C_n, K_{1,n})$-URGDD(r, s) of type p^{1+2k}, $k \geq 1$, exists for every* $(r, s) \in \{kp - nx, (n+1)x) : x = 0, 1, \ldots, kn, k(n+1)\}$.

Proof. Applying the GDD-construction with $t = n + 1$ to a C_n-RGDD of type n^{1+2k} (which exists by Lemma 1 and has $\alpha = nk$ factors) gives a $(C_n, K_{1,n})$-URGDD$(\bar{r}, \bar{s})$ of type p^{1+2k} for each $(\bar{r}, \bar{s}) \in nk * \{(n+1, 0), (1, n+1)\} = \{pk - nx, (n+1)x) : x = 0, 1, \ldots, nk\}$ (the input designs are given by Lemmas 4). For $(r, s) = (0, k(n+1)^2)$, apply the GDD-construction with $t = p$ to a C_{1+2k}-RGDD of type 1^{1+2k}, which exists by Lemma 1 and has $\alpha = k$ factors (the input designs are given by Lemma 7). □

Lemma 13. *Let $v = p + 2pk$, $k > 0$. A $(C_n, K_{1,n})$-IURD*$^*(p + 2pk, p; [r', s'], [r, s])$ *exists for each* $(r', s') \in J(p)$ *and* $(r, s) \in \{pk - nx, (n+1)x) : x = 0, 1, \ldots, nk, (n+1)k\}$. *In addition, if* $k \leq \frac{n}{2} + 1$, *then a* $(C_n, K_{1,n})$-*URD*$^*(v; r, s)$ *exists for every* $(r, s) \in J(v)$.

Proof. It follows by applying the filling construction to the GDD from Lemma 12 and using copies of a $(C_n, K_{1,n})$-URD$^*(p; r, s)$ from Lemma 10 as input designs. □

As a consequence of the previous lemma we have the following two lemmas.

Lemma 14. *A $(C_n, K_{1,n})$-IURD$(3p, p; [r', s'], [r, s])$ exists for each* $(r', s') \in J(p)$ *and* $(r, s) \in \{(p - nx, (n+1)\,x), x = 0, 1, \ldots, n+1\}$.

Lemma 15. *A $(C_n, K_{1,n})$-URD*$^*(3p; r, s)$ *exists for every* $(r, s) \in J(3p)$.

Lemma 16. *A $(C_n, K_{1,n})$-URGDD(r, s) of type $(2p)^k$, $k \geq 2$, exists for every* $(r, s) \in \{((k-1)p - nx, (n+1)x) : x = 0, 1, \ldots, n(k-1), (n+1)(k-1)\}$.

Proof. Applying the GDD-construction with $t = n + 1$ to a C_n-RGDD of type $(2n)^k$, $k \geq 2$, (which exists by Lemma 1 and has $\alpha = n(k-1)$ factors) gives a $(C_n, K_{1,n})$-URGDD$(\bar{r}, \bar{s})$ of type $(2p)^k$ for each $(\bar{r}, \bar{s}) \in (k-1)n * \{(n+1, 0), (1, n+1)\} = \{p(k-1) - nx, (n+1)x) : x = 0, 1, \ldots, n(k-1)$ (the input designs are given by Lemmas 4). For $(r, s) = (0, (k-1)(n+1)^2)$, apply the GDD-construction with $t = p$ to a C_{2k}-RGDD of type 2^k, $k \geq 2$, which exists by Lemma 1 and has $\alpha = k - 1$ factors (the input designs are given by Lemma 7) □

Lemma 17. *A $(C_n, K_{1,n})$-URGDD(r, s) of type $(2p)^2$ exists for every* $(r, s) \in \{(p - nx, (n+1)x) : x = 0, 1, \ldots, n+1\}$.

Proof. It follows by Lemma 16 for $k = 2$. □

Lemma 18. *A $(C_n, K_{1,n})$-URGDD(r, s) of $C_{4(p)}$ exists for every* $(r, s) \in \{p - nx, (n+1)x) : x = 0, 1, \ldots, n+1\}$.

Proof. It follows by Lemma 17 because the graph $K_{2p,2p}$ is isomorphic to $C_{4(p)}$. □

Lemma 19. *A $(C_n, K_{1,n})$-URGDD(r, s), $n \neq 6$, of type p^2 exists for every* $(r, s) \in \{\frac{p}{2} - nx, (n+1)x) : x = 0, 1, \ldots, \frac{n}{2}\}$.

Proof. For $n \neq 6$, applying the GDD-construction with $t = n+1$ to a C_n-RGDD of type n^2 (which exists by Lemma 1 and has $\alpha = \frac{n}{2}$ factors) gives a $(C_n, K_{1,n})$-URGDD$(\bar{r}, \bar{s})$ of type p^2 for each $(\bar{r}, \bar{s}) \in \frac{n}{2} * \{(n+1, 0), (1, n+1)\} = \{\frac{p}{2} - nx, (n+1)x) : x = 0, 1, \dots, \frac{n}{2}\}$ (the input designs are given by Lemmas 4). □

Lemma 20. *A $(C_n, K_{1,n})$-URGDD(r,s) of type $(2p)^3$ exists for every $(r,s) \in \{(2p - nx, (n+1)x) : x = 0, 1, \dots, 2(n+1)\}$.*

Proof. By Lemma 16 a $(C_n, K_{1,n})$-URGDD(r,s) of type $(2p)^3$ exists for every $(r,s) \in \{(2p - nx, (n+1)x) : x = 0, 1, \dots, 2n, 2n+2\}$. We need to solve the case for $x = 2n+1$. For $n \neq 6$, apply the GDD-construction with $t = p$ to a (C_6, K_2)-URGDD$(1,2)$ of type 2^3 (which can be obtained from a C_6-RGDD of type 2^3 by replacing one 6-cycle with two 1-factors) and get a $(C_n, K_{1,n})$-URGDD(r,s) of type $(2p)^3$ with $(r,s) = 2(\frac{n}{2}, \frac{n}{2}(n+1)) + (0, (n+1)^2) = (n, (n+1)(2n+1))$ (the input designs are two copies of a $(C_n, K_{1,n})$-URGDD$(\frac{n}{2}, \frac{n}{2}(n+1))$ of type p^2 given by Lemma 19, and a copy of a $(C_n, K_{1,n})$-URD$(0, (n+1)^2)$ of $C_{6(p)}$ from Lemma 7). For $n = 6$, apply the GDD-construction with $t = p = 42$ to a C_6-RGDD of type 2^3 and get a $(C_6, K_{1,6})$-URGDD$(6, 91)$ of type 84^3 (the input designs are given by Lemmas 6 and 7). □

By Lemmas 11 and 20, and the filling constructions the following two lemmas follow.

Lemma 21. *A $(C_n, K_{1,n})$-IURD$(6p, 2p; [r', s'], [r, s])$ exists for each $(r', s') \in J(2p)$ and $(r,s) \in \{(2p - nx, (n+1)x), x = 0, 1, \dots, 2(n+1)\}$.*

Lemma 22. *A $(C_n, K_{1,n})$-URD$^*(6p; r, s)$ exists for each $(r,s) \in J(6p)$.*

Lemma 23. *A $(C_n, K_{1,n})$-URD$^*(10p; r, s)$ exists for every $(r,s) \in J(10p)$.*

Proof. Apply the filling construction to a $(C_n, K_{1,n})$-URGDD(r,s) of type $(2p)^5$ with $(r,s) \in \{(4p - nx, (n+1)x) : x = 0, 1, \dots, 4n, 4(n+1)\}$ (given by Lemma 16 for $k = 5$) by using copies of a $(C_n, K_{1,n})$-URD$^*(2p; r, s)$ from Lemma 11 as input designs. □

4. The Main Result

Lemma 24. *Let $v \equiv 0 \pmod{4p}$. Then a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ exists for every $(r,s) \in J(v)$.*

Proof. Let $v = 4pk$, $k \geq 1$. Applying the GDD-construction with $t = 2p$ to a 2-RGDD of type 1^{2k} (i.e., a 1-factorization of K_{2k}, which is known to have $\alpha = 2k - 1$ 1-factors) gives a $(C_n, K_{1,n})$-URGDD$(\bar{r}, \bar{s})$ of type $(2p)^{2k}$ for each $(\bar{r}, \bar{s}) \in (2k-1) * \{(p - nx, (n+1)x) : x = 0, 1, \dots, n+1\}$ (the input designs are given by Lemma 17). Now fill the groups with copies of a $(C_n, K_{1,n})$-URD$^*(2p; r', s')$ with $(r', s') \in J(2p)$ (from Lemma 11) to get a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ for each $(r,s) \in J(p) + (2k-1) * \{(p - nx, (n+1)x) : x = 0, 1, \dots, n+1\} = J(4pk)$. □

Lemma 25. *Let $v \equiv 2p \pmod{4p}$. Then a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ exists for every $(r,s) \in J(v)$.*

Proof. Let $v = 2p + 4pk$, $k \geq 0$. The cases $v = 2p, 6p$ and $10p$ follow by Lemmas 11, 22 and 23, respectively. For $k \geq 3$, applying the frame-construction with $t = 2p$ and $h = 2p$ to a 2-frame of type 2^k (see [22]) gives a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ for each $(r,s) \in J(2p) + 2k * \{(p - nx, (n+1)x) : x = 0, 1, \dots, n+1\} = J(2p + 4pk)$ (the input designs are given by Lemmas 11, 17 and 21). □

Lemma 26. *Let $v \equiv p \pmod{2p}$. Then a $(C_n, K_{1,n})$-URD$^*(v; r, s)$ exists for every $(r,s) \in J(v)$.*

Proof. Let $v = p + 2pk$, $k \geq 0$. The cases $v = p, 3p$ and $5p$ follow by Lemmas 10, 13 and 15, respectively. For $l \geq 3$, apply the frame-construction with $t = p$ and $h = p$ to a $\{C_3, C_4\}$-frame of type 2^l, which is known to exist ([20]) and have $\alpha = 1$ factor missing in any fixed group, and get a $(C_n, K_{1,n})$-URD$(v; r, s)$ for each $(r, s) \in J(p) + k * \{(p - mx, (m+1)x), x = 0, 1, \ldots, m+1\} = J(p + 2pk)$ (the input designs are given by Lemmas 10, 12, 14 and 18). □

As a consequence of Lemmas 24–26, our main result immediately follows.

Theorem 2. *Let* $v \equiv 0 \pmod{n(n+1)}$. *There exists a* $(C_n, K_{1,n})$-*URD*$^*(v; r, s)$ *if and only if* $(r, s) \in J(v)$.

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